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# On KMS weights of $C^*$-algebras associated to topological graphs

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# Abstract

In this thesis we give a complete description of the extremal $\beta$-KMS weights for the gauge-action on the $C^*$-algebra associated to a second-countable topological graph. We give a description in terms of ergodic measures $\nu$ on the boundary path space $\partial E$ satisfying $\sigma^*\nu = e^\beta \nu$ on $\partial E \setminus E^0$. And a description in terms of extremal $\beta$-sub-invariant measures $\mu$ on the vertex space $E^0$. We also develop some theory about regular Borel measure using sheaf-theory that has been useful for comparing different measures and gives a new description of the pullback of a regular Borel measure along a local homeomorphism.

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# Contents

# Chapter 1

# Introduction

Topological graphs give a setting for studying a large family of $C^*$-algebras, among others it generalizes the existing theory of graph algebras and gives a new setting to study homeomorphism $C^*$-algebras. A lot of the theory was first developed by Katsura in [Kat04]. Katsura associated a $C^*$-algebra to the quadruple $E = (E^0, E^1, r, s)$ where $E^0$ and $E^1$ are locally compact Hausdorff spaces, $s : E^1 \to E^0$ is a local homeomorphism and $r : E^1 \to E^0$ is continuous. It was shown by Yeend in [Yee07] that the $C^*$-algebras constructed from topological graphs admit a groupoid model. This is done by constructing the boundary path space $\partial E$ and constructing a monoidal action on this space from the natural numbers $\mathbb{N}$ via the *backwards shift map* $\sigma : \partial E \setminus E^0 \to \partial E$. With this one can consider the Deaconu-Renault groupoid of this action.

Groupoid $C^*$-algebras have been a fruitful class of $C^*$-algebras when studying dynamical systems and $\beta$-KMS states. Given a groupoid $\mathcal{G}$ and a continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$, one is able to construct a $C^*$-dynamical system $(C^*(\mathcal{G}), \mathbb{R}, \alpha^c)$ using Pontryagin duality. A lot of work was initiated by Renault in [Ren80] where he is able to give a description of the $\beta$-KMS states for $\alpha^c$ in the case where $\mathcal{G}$ is a locally compact Hausdorff étale groupoid. He shows that each $\beta$-KMS state for $\alpha^c$ restricts to a quasi-invariant probability measure on the unit space $\mathcal{G}^{(0)}$ with Radon-Nikodym derivative $e^{-\beta c}$. When the groupoid is principal he is also able to show that these $\beta$-KMS states are in fact all the $\beta$-KMS states for $\alpha^c$.

In [Nes13] Neshveyev is able to generalize Renault's description of $\beta$-KMS states for $\alpha^c$ on principal étale groupoids to the non-principal case. Christensen is able to further generalize Neshveyev's results to give a description of $\beta$-KMS *weights* for $\alpha^c$ in [Chr23]. In particular, Christensen is able to give a representation result, akin to Riesz representation Theorem, for a class of $\beta$-KMS weights for $\alpha^c$ on *injectively graded* étale groupoids, cf. Theorem 3.5.14. This result by Christensen is fundamental for a lot of the work done in this thesis.

Schafhauser initiated the study of tracial states on $C^*$-algebras associated to topological graphs in [Sch18]. The $C^*$-algebra of a topological graph comes equipped with a natural action from the circle group, generally called the *gauge-action*. Schafhauser was in particular interested in the tracial states which are gauge-invariant, and he was able to give a description of these tracial states both in terms of invariant probability measures on the boundary path space $\partial E$ and vertex-invariant probability measures on the vertex space $E^0$. Schafhauser was unable to completely characterize the gauge-invariant tracial states but conjectured that for free topological graphs, every tracial state should be gauge-invariant.

In [Chr22] Christensen uses the program developed by Schafhauser along with his results from [Chr23] to study tracial *weights* on $C^*$-algebras associated to *second-countable* topological graphs. He is in particular able to give a description of the gauge-invariant tracial weights, see Theorem 5.3.4, and proves Schafhauser's conjecture to be true.

We realized that the techniques developed by Schafhauser in [Sch18] and Christensen in [Chr22] can be used to study $\beta$-KMS weights for the gauge-action on $C^*$-algebras associated to second-countable topological graphs. In particular, Christensen's description of gauge-invariant tracial weights allows us to give a complete description of the extremal $\beta$-KMS weights for the gauge-action in terms of ergodic measures $\nu$ on the boundary path space $\partial E$ satisfying $\sigma^*\nu = e^\beta \nu$ on $\partial E \setminus E^0$. The work done by Schafhauser further allowed us to give a description of these measures $\nu$ on the boundary path space $\partial E$ in terms of $\beta$-sub-invariant measures on the vertex space $E^0$. Our main result is stated in Theorem 6.0.12.

## Outline

- In chapter 2 we study the structure of regular Borel measures on second-countable locally compact Hausdorff spaces using sheaf-theory. This approach allows us to give a description of the pullback of a regular Borel measure along a local homeomorphism that does not rely on the duality between measures and linear functionals. We also show that this description of the pullback is the same as the description given by Riesz representation Theorem for second-countable spaces, c.f. Proposition 2.0.16.

- In chapter 3 we give an introduction to étale groupoids and construct the full groupoid $C^*$-algebra. We define $C^*$-dynamical systems and weights, and give enough background theory to understand and state Theorem 3.5.14, which gives a description of the $\beta$-KMS weights on the full groupoid $C^*$-algebra.

- In chapter 4 we study topological graphs and construct the associated boundary path space. We give a detailed proof showing that the boundary path space admits a locally compact Hausdorff topology and that there is a monoidal action from the natural numbers $\mathbb{N}$ on the boundary path space via the backwards shift map $\sigma : \partial E \setminus E^0 \to \partial E$. With this action we construct the associated Deaconu-Renault groupoid giving us a groupoid model of the graph $C^*$-algebra. The graph $C^*$-algebra is equipped with a natural action from the circle group, called the gauge-action. We restate Theorem 3.5.14 in the context of topological graphs and the gauge-action, c.f. Theorem 4.4.5.

- In chapter 5 we study the loop structure of a second-countable topological graph and relate it to the gauge-invariance of tracial weights on the graph $C^*$-algebra. We are able to state a refinement of Theorem 4.4.5, c.f. Theorem 5.3.7, which gives a description of the extremal $\beta$-KMS weights on the graph $C^*$-algebra in terms of ergodic measures $\nu$ on the boundary path space satisfying $\sigma^*\nu = e^\beta \nu$ on $\partial E \setminus E^0$.

- In chapter 6 we study $\beta$-sub-invariant measures on the vertex space of a second-countable topological graph. The main result in this chapter gives a bijection between the regular Borel measures $\nu$ on the boundary path space satisfying $\sigma^*\nu = e^\beta \nu$ on $\partial E \setminus E^0$ and the $\beta$-sub-invariant measures on the vertex space, c.f. Theorem 6.0.11.

Chapter 2

# Sheaf of measures

We will at multiple occasions need to be able to pullback and pushforward measures between spaces. The pullback is usually defined using Riesz representation Theorem, which gives a description of the pullback in terms of an integral. We would however prefer to describe the pullback without invoking any integrals, this led us towards sheaf-theory.

We ended up developing enough theory in complete generality that we felt we are justified in dedicating an entire chapter to it. We start by defining the pushforward of a measure.

**Definition 2.0.1.**
Let $X$ and $Y$ be topological spaces, $f : X \to Y$ be continuous and $\mu$ be a Borel measure on $X$. We get a Borel measure $f_*\mu$ on $Y$, the *pushforward of* $\mu$, defined by $f_*\mu(B) = \mu(f^{-1}(B))$ for Borel subsets $B \subset Y$. ♡

Note that the pushforward of a *regular* Borel measure isn't necessarily regular. At the end of this chapter we give criteria for when the pushforward of a regular Borel measure is regular.

Let $f : X \to Y$ be a local homeomorphism, $\mu$ be a Borel measure on $Y$ and $U$ be an open subset of $X$ such that $f|_U$ is injective. We define the pullback $f|_U^*\mu$ of $\mu$ on $U$ by setting $f|_U^*\mu(B) = \mu(f(B))$ for all Borel subsets $B$ in $U$. If we let $\mathcal{U}$ be a cover of $X$ with open subsets $U$ in $X$ such that $f|_U$ is injective, we get a family of Borel measures $\{f|_U^*\mu\}_{U\in\mathcal{U}}$. We would like to be able to glue these measures to obtain a global measure on $X$, and we would like the resulting measure to be independent of our choice of open cover of $X$.

Sheaf-theory is precisely the tool that axiomatizes such problems and gives us a clear road map for things we need to prove in order to define the pullback of a measure. For an introduction in sheaf-theory see [Har77]. We nevertheless present the necessary definitions here. To our knowledge using sheaf-theory to study measures in this context is all original work.

**Definition 2.0.2.**
Let $X$ be a topological space. A *presheaf of sets* $\mathcal{F}$ on $X$ is a collection of the following data:

- For each open subset $U \subset X$ we have a set $\mathcal{F}(U)$, where the elements of this set is called the *sections* of $\mathcal{F}$ over $U$.
- For each inclusion of open subsets $V \subset U$ in $X$ we have a function

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$$

called the *restriction morphism.* For sections $s \in \mathcal{F}(U)$ we write $\rho^U_V(s) = s|_V$.

The restriction morphisms have to satisfy the functorial properties:

(1) For every open subset $U \subset X$ the restriction morphism $\rho^U_U$ must be the identity function on $\mathcal{F}(U)$.

(2) For inclusions of open subsets $W \subset V \subset U$ in $X$ we have that the composition of restriction morphisms $\rho^V_W \circ \rho^U_V = \rho^U_W$. ♡

**Definition 2.0.3.**
Let $\mathcal{F}$ be a presheaf of sets on $X$. We say that $\mathcal{F}$ is a *sheaf of sets* if the following axioms hold for any open subset $W \subset X$ and any open cover $\mathcal{U}$ of $W$:

(*Locality*) Suppose we have sections $s, t \in \mathcal{F}(W)$ such that $s|_U = t|_U$ for all $U \in \mathcal{U}$. Then $s = t$.

(*Gluing*) Suppose we have a family of sections $\{s_U\}_{U \in \mathcal{U}}$ with $s_U \in \mathcal{F}(U)$, such that $s_U|_{U \cap V} = s_V|_{U \cap V}$ for all $U, V \in \mathcal{U}$. Then there exists a global section $s \in \mathcal{F}(W)$ such that $s|_U = s_U$ for all $U \in \mathcal{U}$. ♡

Note that if $\mathcal{F}$ is a sheaf of sets on $X$, and we have a family of sections that agree on intersections, the sheaf axioms give the existence of a unique global section that restricts to each of the local sections. What we now want to show is that for suitable topological spaces we can give sets of regular Borel measures a sheaf structure. With that in place we get an obvious way of defining the pullback of a regular Borel measure. At the end of this section we give an alternate way of defining the pullback of a regular Borel measure using Riesz representation Theorem.

**Definition 2.0.4.**
Let $X$ be a topological space. For each open subset $U \subset X$, let

$$\mathcal{M}(U) = \{\mu : \mathcal{B}(U) \to [0, \infty] \mid \mu \text{ is a Borel measure}\},$$

where $\mathcal{B}(U)$ denotes the Borel $\sigma$-algebra of $U$. For inclusion of open subsets $V \subset U$ in $X$, we define the restriction map $\rho^U_V : \mathcal{M}(U) \to \mathcal{M}(V)$ by $\mu|_V(B) = \mu(B)$ where $\mu \in \mathcal{M}(U)$ and $B \subset V$ is a Borel subset. This is well-defined since any Borel subset in $V$ is automatically a Borel subset in $U$, since $V$ is open in $U$. We denote the collection of this data, namely the sets $\mathcal{M}(U)$ and the restriction maps $\rho^U_V : \mathcal{M}(U) \to \mathcal{M}(V)$, by $\mathcal{M}$. ♡

**Lemma 2.0.5.**
*Let $X$ be a topological space, then $\mathcal{M}$ is a presheaf of sets on $X$.*

*Proof.*
Fix an open subset $U \subset X$ and let $\mu \in \mathcal{M}(U)$. Clearly $\mu|_U = \mu$, so $\rho^U_U = id_{\mathcal{M}(U)}$. If we have inclusions of open subsets $W \subset V \subset U$ in $X$ we have for all Borel subsets $B \subset W$ that

$$(\mu|_V)|_W(B) = \mu|_V(B) = \mu(B) = \mu|_W(B).$$

Hence, $\rho^V_W \circ \rho^U_V = \rho^U_W$. □

For general topological spaces $X$ we won't get that $\mathcal{M}$ is a sheaf. To prove the locality axiom we require that any open cover $\mathcal{U}$ of $X$ has a countable subcover, which isn't guaranteed. The need for this comes from the fact that the measure of a disjoint union equals the sum of the respective measures only if the union is at most countable. We do however get a sheaf for second-countable spaces.

**Lemma 2.0.6.**
*Let $X$ be a second-countable space, then $\mathcal{M}$ is a sheaf of sets on $X$.*

*Proof.*
Note first that second-countability of $X$ implies that any subspace of $X$ is also second-countable. Hence, it suffices to prove that locality and gluing holds for $X$ to conclude that they hold for any open subset of $X$.

Let $\mathcal{U}$ be an open cover of $X$. Since $X$ is second-countable we get a countable sub-cover $\mathcal{V}$ of $\mathcal{U}$. Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{N}}$ be an indexing of $\mathcal{V}$.

For subsets $S \subset X$ we introduce the notation $S(i) = S \cap (V_i \setminus (V_1 \cup \ldots \cup V_{i-1}))$. Clearly $S = \bigcup_{i\in\mathbb{N}} S(i)$ with $S(i) \cap S(j) = \varnothing$ for $i \neq j$ and each $S(i) \subset V_i$.
*Locality:*
Suppose we have Borel measures $\mu, \nu \in \mathcal{M}(X)$ such that $\mu|_U = \nu|_U$ for all $U \in \mathcal{U}$. Then for any Borel subset $B \subset X$ we have that

$$\mu(B) = \mu\left(\bigcup_{i\in\mathbb{N}} B(i)\right) = \sum_{i\in\mathbb{N}} \mu(B(i)) = \sum_{i\in\mathbb{N}} \mu|_{V_i}(B(i)) = \sum_{i\in\mathbb{N}} \nu|_{V_i}(B(i)) = \nu(B).$$

*Gluing:*
Suppose we have a Borel measures $\mu_U \in \mathcal{M}(U)$ for each $U \in \mathcal{U}$ such that $\mu_U|_{U\cap V} = \mu_V|_{U\cap V}$ for all $U, V \in \mathcal{U}$. Define $\mu_i = \mu_{V_i}$. We define a global Borel measure $\mu \in \mathcal{M}(X)$ by

$$\mu(B) = \sum_{i\in\mathbb{N}} \mu_i(B(i)), \tag{2.1}$$

where $B \subset X$ is a Borel subset. This is indeed a measure:

$$\begin{aligned}
\mu(\varnothing) &= \sum_{i\in\mathbb{N}} \mu_i(\varnothing) = 0,\\
\mu(B) &= \sum_{i\in\mathbb{N}} \mu_i(B(i)) \geq 0,\\
\mu\left(\bigcup_{j\in\mathbb{N}} B_j\right) &= \sum_{i\in\mathbb{N}} \mu_i\left(\bigcup_{j\in\mathbb{N}} B_j(i)\right) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \mu_i(B_j(i)) = \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} \mu_i(B_j(i)) = \sum_{j\in\mathbb{N}} \mu(B_j),
\end{aligned}$$

where $B \subset X$ is a Borel subset, and $\{B_j\}_{j\in\mathbb{N}}$ is a collection of mutually disjoint Borel subsets in $X$. Note that we can interchange the sums in the last line because we are summing positive numbers.

Now let $U \in \mathcal{U}$ and $B \subset U$ be a Borel subset. Note that each $B(i) \subset U \cap V_i$. Then we have that

$$\begin{aligned}
\mu|_U(B) &= \mu(B)\\
&= \sum_{i\in\mathbb{N}} \mu_i(B(i))\\
&= \sum_{i\in\mathbb{N}} \mu_i|_{U\cap V_i}(B(i))\\
&= \sum_{i\in\mathbb{N}} \mu_U|_{U\cap V_i}(B(i))\\
&= \sum_{i\in\mathbb{N}} \mu_U(B(i))\\
&= \mu_U\left(\bigcup_{i\in\mathbb{N}} B(i)\right)\\
&= \mu_U(B).
\end{aligned}$$

□

We now give an example of a (not second-countable) space for which locality doesn't hold.

**Example 2.0.7.**
Consider $\mathbb{R}$ with the discrete topology. Let $\mathcal{U} = \{\{x\}\}_{x\in\mathbb{R}}$ be an open covering of $\mathbb{R}$. Let $\mu$ be the measure defined by

$$\mu(B) = \begin{cases} 0 & \text{if } B \text{ is countable or finite,} \\ \infty & \text{if } B \text{ is uncountable.} \end{cases}$$

This clearly defines a measure. It is clear that $\mu$ agrees with the zero measure on all restrictions to $\{x\} \in \mathcal{U}$, however $\mu$ is not the zero measure, so locality doesn't hold. ◇

We are however mostly interested in *regular* Borel measures. Some authors give different definitions of a regular measure, we will use the definition used by Christensen from the book Measure Theory by Cohn. Point (1) in this definition is not included by some.

**Definition 2.0.8** ([Coh13] page 189-190)**.**
Let $X$ be a topological space. A Borel measure $\mu$ on $X$ is *regular* if

(1) for each compact subset $K \subset X$ we have that $\mu(K) < \infty$,

(2) for each Borel subset $B \subset X$ we have that

$$\mu(B) = \inf\{\mu(U) \mid U \supset B \text{ with } U \text{ open in } X\},$$

(3) and for each open subset $U \subset X$ we have that

$$\mu(U) = \sup\{\mu(K) \mid K \subset U \text{ with } K \text{ compact}\}.$$ ♡

We now want to find the requirements on our topological space for which $\mathcal{M}_{\text{reg}}$, the family of regular Borel measures, becomes a sheaf.

**Lemma 2.0.9.**
*Let $X$ be a topological space, then $\mathcal{M}_{reg}$ is a sub-presheaf of $\mathcal{M}$ on $X$.*

*Proof.*
We only need to check that the restriction of a regular Borel measure is again a regular Borel measure. Fix open subsets $U, V \subset X$ such that $V \subset U$ and let $\mu \in \mathcal{M}_{\text{reg}}(U)$.

(1) Let $K \subset V$ be compact, then $K$ is compact in $U$, hence

$$\mu|_V(K) = \mu(K) < \infty.$$

(2) Let $B \subset V$ be a Borel subset, then

$$\begin{aligned} \mu|_V(B) &= \mu(B) \\ &= \inf\{\mu(W) \mid W \supset B \text{ with } W \text{ open in } U\} \\ &\leq \inf\{\mu(W) \mid W \supset B \text{ with } W \text{ open in } V\} \\ &= \inf\{\mu|_V(W) \mid W \supset B \text{ with } W \text{ open in } V\}. \end{aligned}$$

We get the inequality in the third line because every open subset of $V$ is also an open subset of $U$. To see that $\mu|_V(B) \geq \inf\{\mu|_V(W) \mid W \supset B \text{ with } W \text{ open in } V\}$ we notice that for any open subset $W \subset U$ containing $B$ we get that $W \cap V$ is open in $V$ and contains $B$ with $\mu(W \cap V) \leq \mu(W)$. Hence,

$$\mu|_V(B) = \inf\{\mu|_V(W) \mid W \supset B \text{ with } W \text{ open in } V\}.$$

(3) Let $W \subset V$ be an open subset, then

$$\begin{aligned}\mu|_V(W) &= \mu(W) \\ &= \sup\{\mu(K) \mid K \subset W \text{ with } K \text{ compact in } U\} \\ &= \sup\{\mu(K) \mid K \subset W \text{ with } K \text{ compact in } V\} \\ &= \sup\{\mu|_V(K) \mid K \subset W \text{ with } K \text{ compact in } V\}.\end{aligned}$$

Thus, the restriction of a regular Borel measure is still regular. The fact that $\mathcal{M}_{\text{reg}}$ is a presheaf now follows from the fact that $\mathcal{M}$ is a presheaf. □

To make life simpler for ourselves we choose to work with second-countable locally compact Hausdorff spaces. For such spaces we have the following result which we state without proof.

**Lemma 2.0.10** ([Coh13] Proposition 7.2.3)**.**
*Let $X$ be a second-countable locally compact Hausdorff space and $\mu$ be a Borel measure on $X$. If $\mu$ is finite on compact sets we have that $\mu$ is regular.* □

**Proposition 2.0.11.**
*Let $X$ be a second-countable locally compact Hausdorff space. Then $\mathcal{M}_{reg}$ is a sub-sheaf of $\mathcal{M}$ on $X$.*

*Proof.*
We have that any subspace of $X$ is second-countable locally compact and Hausdorff, hence it suffices to prove that locality and gluing holds for $X$ to conclude that it holds for any open subset of $X$.

Let $\mathcal{U}$ be an open cover of $X$. Locality for $\mathcal{M}_{\text{reg}}$ follows by the same argument as for $\mathcal{M}$. To prove gluing, suppose we have regular Borel measures $\mu_U \in \mathcal{M}_{\text{reg}}(U)$ for each $U \in \mathcal{U}$ such that $\mu_U|_{U\cap V} = \mu_V|_{U\cap V}$ for all $U, V \in \mathcal{U}$. We get a global measure $\mu \in \mathcal{M}(X)$ by equation (2.1). Since $\mathcal{M}$ is a sheaf we get that $\mu$ restricts to each $\mu_U$ for $U \in \mathcal{U}$. It remains to check that $\mu$ is in fact regular. Let

$$\mathcal{V} = \{V \subset X \mid V \text{ is open in } X \text{ and } \overline{V} \subset U \text{ for some } U \in \mathcal{U}\}.$$

This is an open cover of $X$: For any $x \in X$ let $U \in \mathcal{U}$ be a neighborhood about $x$. Then $X \setminus U$ is closed, and since $X$ is locally compact Hausdorff, $X$ is in particular regular, so we can find disjoint open subsets $V$ and $W$ containing $x$ and $X \setminus U$ respectively. Then $\overline{V} \subset U$, showing that $V \in \mathcal{V}$.

Let now $K \subset X$ be compact, then $\mathcal{V}$ is an open cover of $K$, so it admits a finite sub-cover $\{V_i\}_{i=1}^n$ of $K$. Let $U_i \in \mathcal{U}$ be such that $V_i \subset \overline{V_i} \subset U_i$. Then $K \cap \overline{V_i} \subset U_i$ is compact. Thus,

$$\mu(K) \leq \sum_{i=1}^n \mu(K \cap \overline{V_i}) = \sum_{i=1}^n \mu|_{U_i}(K \cap \overline{V_i}) = \sum_{i=1}^n \mu_{U_i}(K \cap \overline{V_i}) < \infty.$$

Hence, by Lemma 2.0.10 we have that $\mu$ is in fact regular, proving that gluing holds for $\mathcal{M}_{\text{reg}}$. □

We now show that we can pullback measures.

**Proposition 2.0.12.**
*Let $f : X \to Y$ be a local homeomorphism between second-countable locally compact Hausdorff spaces and $\mu$ be a regular Borel measure on $Y$. Then there exists a unique regular Borel measure $f^*\mu$ on $X$, the* pullback of $\mu$, *such that $f^*\mu(U) = \mu(f(U))$ for all open subsets $U \subset X$ where $f|_U$ is injective.*

*Proof.*
Let

$$\mathcal{U} = \{U \subset X \mid U \text{ is open in } X \text{ and } f|_U \text{ is injective}\}.$$

Since $f$ is a local homeomorphism this is an open cover of $X$. For each $U \in \mathcal{U}$ we get a regular Borel measure $f^*\mu_U \in \mathcal{M}_{\text{reg}}(U)$ by the equation $f^*\mu_U(B) = \mu(f(B))$, where $B \subset U$ is a Borel subset. The fact that this defines a Borel measure is obvious. Regularity follows as such: Let $K \subset U$ be compact, then $f^*\mu_U(K) = \mu(f(K)) < \infty$ by compactness of $f(K)$ which follows by continuity of $f$. By Lemma 2.0.10 $f^*\mu_U$ is regular.

Now we need to show that these measures agree on intersections. Let $U, V \in \mathcal{U}$. Then for any Borel subset $B \subset U \cap V$ we have that

$$f^*\mu_U|_{U\cap V}(B) = f^*\mu_U(B) = \mu(f(B)) = f^*\mu_V|_{U\cap V}(B).$$

Since $\mathcal{M}_{\text{reg}}$ is a sheaf we then get that there exists a unique regular Borel measure $f^*\mu$ on $X$ such that for any $U \in \mathcal{U}$

$$f^*\mu(U) = f^*\mu|_U(U) = f^*\mu_U(U) = \mu(f(U)).$$ □

We now give an alternate description of the pullback of a regular Borel measure using Riesz representation Theorem, which we recall here without proof.

**Theorem 2.0.13** ([Coh13] Theorem 7.2.8, Riesz Representation Theorem)**.**
*Let $X$ be a locally compact Hausdorff space and $l : C_c(X) \to \mathbb{C}$ be a positive bounded linear functional. Then there exists a unique regular Borel measure $\mu$ on $X$ such that*

$$l(f) = \int_X f d\mu$$

*for all $f \in C_c(X)$.* □

**Lemma 2.0.14.**
*Let $X$ and $Y$ be topological spaces, $\varphi : X \to Y$ a local homeomorphism and $f \in C_c(X)$. Then the function*

$$y \mapsto \sum_{x\in\varphi^{-1}(y)} f(x)$$

*from $Y$ to the complex numbers $\mathbb{C}$ is well-defined and continuous with compact support.*

*Proof.*
For each $x \in \text{supp}(f)$ let $U_x$ be an open neighborhood in $X$ about $x$ such that $\varphi|_{U_x}$ is injective. Then $\{U_x\}_{x\in\text{supp}(f)}$ is an open cover of $\text{supp}(f)$, and since $f$ is compactly supported we get a finite subcover $\{U_{x_i}\}_{i=1}^n$ of $\text{supp}(f)$. Then we may write for each $y \in Y$

$$\sum_{x\in\varphi^{-1}(y)} f(x) = \sum_{i=1}^n f \circ \varphi|_{U_{x_i}}^{-1}(y).$$

This equality makes the above assertions obvious. □

**Proposition 2.0.15.**
*Let $\varphi : X \to Y$ be a local homeomorphism between locally compact Hausdorff spaces and $\mu$ be a regular Borel measure on $Y$. Then there exists a unique regular Borel measure $\varphi^*\mu$ on $X$ such that*

$$\int_X f d\varphi^*\mu = \int_Y \sum_{x\in\varphi^{-1}(y)} f(x) d\mu(y),$$

*where $f \in C_c(X)$.*

*Proof.*
Define a linear functional $l : C_c(X) \to \mathbb{C}$ by

$$l(f) = \int_Y \sum_{x \in \varphi^{-1}(y)} f(x) d\mu(y).$$

By Lemma 2.0.14 this is indeed a bounded linear functional. Positivity follows by positivity of the measure $\mu$. By Riesz representation Theorem, there exists a unique regular Borel measure $\varphi^*\mu$ on $X$ such that for any $f \in C_c(X)$

$$\int_X f d\varphi^*\mu = l(f) = \int_Y \sum_{x \in \varphi^{-1}(y)} f(x) d\mu(y). \qquad \square$$

We end this section by proving that for second-countable locally compact Hausdorff spaces, these two ideas of pullbacks are the same. Hence, when we work with second-countable locally compact Hausdorff spaces we will simply refer to the pullback of a measure as *the* pullback of a measure. We also show that the pushforward of regular Borel measure is again regular if the function in question is proper.

**Proposition 2.0.16.**
*Let $\varphi : X \to Y$ be a local homeomorphism between second-countable locally compact Hausdorff spaces and $\mu$ be a regular Borel measure on $Y$. Then the pullback of $\mu$ as in Proposition 2.0.12 equals the one in Proposition 2.0.15.*

*Proof.*
Let $\lambda$ be the pullback of $\mu$ as in Proposition 2.0.12 and $\nu$ be the pullback of $\mu$ as in Proposition 2.0.15. Let

$$\mathcal{U} = \{U \subset X \mid U \text{ is open in } X \text{ and } \varphi|_U \text{ is injective}\}.$$

This is an open cover of $X$ since $\varphi$ is a local homeomorphism. By Proposition 2.0.11 we know that $\mathcal{M}_{\text{reg}}$ is a sheaf on $X$, so we need only check that $\lambda|_U = \nu|_U$ for each $U \in \mathcal{U}$ to conclude that $\lambda = \nu$. Let $U \in \mathcal{U}$ and $B \subset U$ be a Borel subset. Then

$$\begin{aligned}
\nu|_U(B) &= \nu(B) \\
&= \int_X \chi_B d\nu \\
&= \int_Y \sum_{x \in \varphi^{-1}(y)} \chi_B(x) d\mu(y) \\
&= \int_Y \chi_B(\varphi|_U^{-1}(y)) d\mu(y) \\
&= \int_Y \chi_{\varphi(B)} d\mu \\
&= \mu(\varphi(B)) \\
&= \lambda|_U(B).
\end{aligned} \qquad \square$$

**Proposition 2.0.17.**
*Let $f : X \to Y$ be a proper continuous function between second-countable locally compact Hausdorff spaces and $\mu$ be a regular Borel measure on $X$. Then the pushforward $f_*\mu$ of $\mu$ is regular.*

*Proof.*
Let $K \subset Y$ be compact, then

$$f_*\mu(K) = \mu(f^{-1}(K)) < \infty$$

since $f^{-1}(K)$ is compact in $X$. By Lemma 2.0.10 we have that $f_*\mu$ is regular. □

Note that in the proofs of Proposition 2.0.12 and Proposition 2.0.16, most of the work was done by finding a suitable open cover of our space. This will in general be the case. Using sheaf theory reduces a lot of the proofs in this thesis to the problem of finding a suitable open cover of our space.

Chapter 3

# Étale groupoids and their $C^*$-algebras

A lot of theory about groupoid $C^*$-algebras was developed by Renault and can be found in [Ren80]. Sections one through four in this chapter will loosely follow the structure and presentation done in the lecture notes by Sims on Hausdorff étale groupoids and their $C^*$-algebras, cf. [Sim17]. It will be mentioned when original proofs are presented.

The last section of this chapter gives an outline of the theory needed to understand Theorem 3.5.14 ([Chr23] Theorem 7.4) which gives a description of the $\beta$-KMS weights on a groupoid in terms of ergodic measures on the unit space of this groupoid.

## 3.1 Definition and basic results

One elegant way of defining a groupoid $\mathcal{G}$ is to say that it is a small category where each morphism is invertible. This definition is a great way of visualizing a lot of the algebra that a groupoid describes. We will however start this section by giving a more direct definition and work our way towards the category theoretic definition from this.

**Definition 3.1.1.**
A *groupoid* is a set $\mathcal{G}$ with a distinguished subset $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ with a multiplication map $\mathcal{G}^{(2)} \to \mathcal{G}$ where $(x,y) \mapsto xy$, and an inverse map $\mathcal{G} \to \mathcal{G}$ where $x \mapsto x^{-1}$. These maps have to satisfy the following three axioms:

*(Associativity)* If $(x,y),(y,z) \in \mathcal{G}^{(2)}$ then both $(xy,z),(x,yz) \in \mathcal{G}^{(2)}$ and $(xy)z = x(yz)$,

*(Inverse 1)* $(x^{-1})^{-1} = x$ for any $x \in \mathcal{G}$,

*(Inverse 2)* $(x,x^{-1}) \in \mathcal{G}^{(2)}$ for any $x \in \mathcal{G}$ and for any $(x,y) \in \mathcal{G}^{(2)}$ we have that $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$. ♡

This definition only mentions the algebra that is present in a groupoid. We will eventually think of the multiplication map as composition of morphisms. In this context it is useful to define what will eventually become the objects of our category.

**Definition 3.1.2.**
Let $\mathcal{G}$ be a groupoid. The *unit space* of $\mathcal{G}$ is the set $\mathcal{G}^{(0)} = \{x^{-1}x \mid x \in \mathcal{G}\}$. The elements of $\mathcal{G}^{(0)}$ are called *units*. Notice that since for any $x \in \mathcal{G}$ we have that $(x^{-1})^{-1} = x$, so we can also describe $\mathcal{G}^{(0)}$ as the set $\{xx^{-1} \mid x \in \mathcal{G}\}$.

Associated to the unit space we define two maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ by the equations

$$r(x) = xx^{-1}, \; s(x) = x^{-1}x,$$

for an element $x \in \mathcal{G}$. These definitions are well-defined by the inverse axioms, ensuring that both $(x, x^{-1}), (x^{-1}, x) \in \mathcal{G}^{(2)}$. We call $r$ the *range* map and $s$ the *source* map. ♡

We will throughout this thesis use the letters $x, y, z$ to denote general elements in the groupoid and $u, v, w$ to denote units.

**Lemma 3.1.3.**
*Let $\mathcal{G}$ be a groupoid and $x \in \mathcal{G}$, then the following holds.*

*(i)* $(r(x), x), (x, s(x)) \in \mathcal{G}^{(2)}$ *and* $r(x)x = x = xs(x)$,

*(ii)* $r(x^{-1}) = s(x)$ *and* $s(x^{-1}) = r(x)$,

*(iii)* $x^{-1} \in \mathcal{G}$ *is the unique element such that* $(x, x^{-1}) \in \mathcal{G}^{(2)}$ *and* $xx^{-1} = r(x)$. *Similarly,* $x^{-1} \in \mathcal{G}$ *is the unique element such that* $(x^{-1}, x) \in \mathcal{G}^{(2)}$ *and* $x^{-1}x = s(x)$.

*Proof.*

(i) Associativity gives $(r(x), x), (x, s(x)) \in \mathcal{G}^{(2)}$ since both $(x, x^{-1}), (x^{-1}, x) \in \mathcal{G}^{(2)}$. The second inverse axiom gives us that

$$r(x)x = (xx^{-1})x = x = x(x^{-1}x) = xs(x).$$

(ii) By the first inverse axiom the following calculations hold:

$$r(x^{-1}) = x^{-1}(x^{-1})^{-1} = x^{-1}x = s(x)$$
$$s(x^{-1}) = (x^{-1})^{-1}x^{-1} = xx^{-1} = r(x).$$

(iii) Suppose $(x, y) \in \mathcal{G}^{(2)}$ and $xy = r(x) = xx^{-1}$. Since $(x^{-1}, x) \in \mathcal{G}^{(2)}$ associativity says that $(x^{-1}x, y), (x^{-1}, xy) \in \mathcal{G}^{(2)}$. The second inverse axiom then gives us that

$$y = (x^{-1}x)y = x^{-1}(xy) = x^{-1}r(x) = x^{-1}(xx^{-1}) = x^{-1}.$$

The second part of this statement is similarly proven. □

The following lemma shows that we have left- and right-cancelation in a groupoid.

**Lemma 3.1.4.**
*Let $\mathcal{G}$ be a groupoid. Suppose $(x, y), (z, y) \in \mathcal{G}^{(2)}$ and $xy = zy$. Then $x = z$. Similarly, if $(y, x), (y, z) \in \mathcal{G}^{(2)}$ and $yx = yz$. Then $x = y$.*

*Proof.*
Associativity gives us that $(x, yy^{-1}), (z, yy^{-1}) \in \mathcal{G}^{(2)}$, and the second inverse axiom allows the following calculation to hold:

$$x = x(yy^{-1}) = (xy)y^{-1} = (zy)y^{-1} = z(yy^{-1}) = z.$$

The second part of this statement follows by a similar argument. □

**Lemma 3.1.5.**
*Let $\mathcal{G}$ be a groupoid. Then $(x, y) \in \mathcal{G}^{(2)}$ if and only if $s(x) = r(y)$. We also have that*

*(i)* $r(xy) = r(x)$ *and* $s(xy) = s(y)$ *for any* $(x, y) \in \mathcal{G}^{(2)}$,

*(ii)* $(xy)^{-1} = y^{-1}x^{-1}$ *for any* $(x, y) \in \mathcal{G}^{(2)}$ *and,*

*(iii)* $r(u) = u = s(u)$ *for any* $u \in \mathcal{G}^{(0)}$.

*Proof.*
Suppose first that $s(x) = r(y)$, then by definition we have that $x^{-1}x = yy^{-1}$. Then $(x, yy^{-1}) = (x, x^{-1}x) \in \mathcal{G}^{(2)}$, and since $(yy^{-1}, y) \in \mathcal{G}^{(2)}$, we get by the second inverse axiom and associativity that $(x, y) = (x, yy^{-1}y) \in \mathcal{G}^{(2)}$.

Now suppose $(x, y) \in \mathcal{G}^{(2)}$, then we have that $(x^{-1}, xy) \in \mathcal{G}^{(2)}$, so we may perform the following calculation:

$$s(x)y = x^{-1}xy = y = r(y)y,$$

where the last equality follows by Lemma 3.1.3 (i). By Lemma 3.1.4 we get that $s(x) = r(y)$. This completes the first part of the proof.

(i) Let $(x, y) \in \mathcal{G}^{(2)}$, then $(r(x), xy) \in \mathcal{G}^{(2)}$, so by Lemma 3.1.3 (i) we have that

$$r(x)(xy) = (r(x)x)y = xy = r(xy)(xy).$$

Thus, by Lemma 3.1.4 we have that $r(x) = r(xy)$. The statement about the source map is similarly proven.

(ii) Let $(x, y) \in \mathcal{G}^{(2)}$. First we want to make sure that the product $y^{-1}x^{-1}$ is defined. By statement (i) we have that

$$y^{-1}y = s(y) = s(xy) = (xy)^{-1}(xy).$$

By the uniqueness part of Lemma 3.1.3 (iii) we have that $y^{-1} = (xy)^{-1}x$. Then, since $(x, x^{-1}) \in \mathcal{G}^{(2)}$ we get by associativity that $(y^{-1}, x^{-1}) = ((xy)^{-1}x, x^{-1}) \in \mathcal{G}^{(2)}$. It follows by associativity that $(xy, y^{-1}x^{-1}) \in \mathcal{G}^{(2)}$. Hence, by statement (i)

$$xyy^{-1}x^{-1} = xx^{-1} = r(x) = r(xy) = (xy)(xy)^{-1}.$$

The uniqueness part of Lemma 3.1.3 (iii) then gives us that $y^{-1}x^{-1} = (xy)^{-1}$.

(iii) Let $u \in \mathcal{G}^{(0)}$ and choose $x \in \mathcal{G}$ such that $u = x^{-1}x$. By statement (i) we immediately get that $s(u) = s(x) = x^{-1}x = u$. By Lemma 3.1.3 (ii) we then get that

$$r(u) = r(x^{-1}) = s(x) = u.$$ ☐

With these results established we can see that a groupoid is indeed a small category where each morphism is invertible.

**Proposition 3.1.6.**
*Let* $\mathcal{G}$ *be a groupoid. Then* $\mathcal{G}$ *becomes a small category where each morphism is invertible by letting* $\mathcal{G}$ *be the set of morphisms,* $\mathcal{G}^{(0)}$ *be the set of objects,* $s : \mathcal{G} \to \mathcal{G}^{(0)}$ *be the domain map and* $r : \mathcal{G} \to \mathcal{G}^{(0)}$ *be the co-domain map. The composition of morphisms is given by the multiplication map* $\mathcal{G}^{(2)} \to \mathcal{G}$.

*Proof.*
Note first that the composition is well-defined by Lemma 3.1.5 and associativity follows by the associativity axiom for groupoids. To show the existence of identity morphisms we first note that since $\mathcal{G}^{(0)} \subset \mathcal{G}$ we have that any object $u \in \mathcal{G}^{(0)}$ is also a morphism. The morphism $u$ is the clear candidate for the identity on the object $u$. First we note that $u$ is indeed a morphism from $u$ to $u$ by Lemma 3.1.5 (iii) saying that $r(u) = u = s(u)$. Now suppose that $x, y \in \mathcal{G}$ are such that $r(x) = u$ and $s(y) = u$. By Lemma 3.1.3 (i) we have that $ux = x$ and $yu = y$, hence $u$ is the identity morphism from $u$ to $u$.

The fact that each morphism is invertible follows by the fact that we have an inverse map $\mathcal{G} \to \mathcal{G}$ and the inverse axioms for groupoids. ☐

The fact that the converse is also true, that each small category where each morphism is invertible gives a groupoid, is obvious. Hence, one is justified in defining groupoids as small categories with invertible morphisms.

With this in place we introduce a very important example for this thesis.

**Example 3.1.7** (Deaconu-Renault groupoids)**.**
Let $X$ be a set, $G$ an abelian group and $S \subset G$ a submonoid of $G$. In a lot of cases $G = \mathbb{Z}$ and $S = \mathbb{N}$. Suppose $S$ acts on $X$. By this we mean for each $s, t \in S$ we have functions $\varphi_s, \varphi_t : X \to X$ such that

$$\varphi_0 = id_X$$
$$\varphi_t \circ \varphi_s = \varphi_{t+s}.$$

To avoid cluttered notation we will simply denote the functions $\varphi_s$ by the symbol $s$. Let $\mathcal{G}$ be the set

$$\mathcal{G} = \{(x, s-t, y) \in X \times G \times X \mid sx = ty\}.$$

Further we let $\mathcal{G}^{(0)} = \{(x, 0, x) \mid x \in X\}$ and associate this with $X$ in the obvious way. We define the range and source maps, multiplication and the inverse map as follows:

$$\begin{aligned} r(x, g, y) &= x, \\ s(x, g, y) &= y, \\ (x, g, y)(y, h, z) &= (x, g+h, z), \\ (x, g, y)^{-1} &= (y, -g, x). \end{aligned}$$

We easily check that this multiplication is well-defined: Let $s_1, s_2, t_1, t_2 \in S$ such that $s_1 x = t_1 y$ and $s_2 y = t_2 z$, then

$$(s_2 + s_1)x = s_2(s_1 x) = s_2(t_1 y) = (s_2 + t_1)y = t_1(s_2 y) = t_1(t_2 z) = (t_2 + t_1)z,$$

showing that the multiplication map is well-defined.

With these operations $\mathcal{G}$ becomes a groupoid. We will check that this defines a small category with invertible morphisms. $\mathcal{G}$ becomes the set of morphisms and $X$ becomes the set of objects. The multiplication map defines composition of morphisms and $s$ and $r$ gives us domain and co-domain maps.
*Associativity*:
Let $(x, g, y),\ (y, h, z),\ (z, f, w) \in \mathcal{G}$, then

$$\begin{aligned} ((x, g, y)(y, h, z))(z, f, w) &= (x, g+h, z)(z, f, w) \\ &= (x, g+h+f, w) \\ &= (x, g, y)(y, h+f, w) \\ &= (x, g, y)((y, h, z)(z, f, w)). \end{aligned}$$

*Identity*:
We will show that $\mathcal{G}^{(0)}$ becomes the set of identity morphisms. Let $(x, 0, x) \in \mathcal{G}^{(0)}$, $(x, g, y) \in \mathcal{G}$ and $(z, h, x) \in \mathcal{G}$. Clearly $(x, 0, x)$ is a morphism from $x$ to $x$, and

$$\begin{aligned} (x, 0, x)(x, g, y) &= (x, g, y), \\ (z, h, x)(x, 0, x) &= (z, h, x). \end{aligned}$$

*Invertibility*:
Let $(x, g, y) \in \mathcal{G}$, then

$$\begin{aligned} (x, g, y)(x, g, y)^{-1} &= (x, g, y)(y, -g, x) = (x, g-g, x) = (x, 0, x), \\ (x, g, y)^{-1}(x, g, y) &= (y, -g, x)(x, g, y) = (y, -g+g, y) = (y, 0, y). \end{aligned}$$ ◇

We introduce another important example of groupoids, namely equivalence relations. This is perhaps one of the earliest examples of groupoids which was studied in detail. They will in particular give an introduction to *principal* groupoids.

**Example 3.1.8** (Equivalence relations)**.**
Let $R$ be an equivalence relation on a set $X$. Set $R^{(0)} = \{(x,x) \mid x \in X\}$. We may clearly identify $R^{(0)}$ with $X$. We define the range and source maps, multiplication and the inverse map as follows:

$$\begin{aligned} r(x,y) &= x, \\ s(x,y) &= y, \\ (x,y)(y,z) &= (x,z), \\ (x,y)^{-1} &= (y,x), \end{aligned}$$

for $(x,y),(y,z) \in R$. These operations are all well-defined by the axioms for an equivalence relation. With these operations $R$ becomes a groupoid. We will check that this defines a small category with invertible morphisms. $R$ becomes the set of morphisms and $X$ becomes the set of objects. The multiplication map gives us composition of morphisms and $s$ and $r$ gives us domain and co-domain maps.
*Associativity*:
Let $(x,y)$, $(y,z)$, $(z,w) \in R$, then

$$\begin{aligned} ((x,y)(y,z))(z,w) &= (x,z)(z,w) \\ &= (x,w) \\ &= (x,y)(y,w) \\ &= (x,y)((y,z)(z,w)). \end{aligned}$$

*Identity*:
We will show that $R^{(0)}$ becomes the set of identity morphisms. So let $(x,x) \in R^{(0)}$, $(x,y) \in R$ and $(z,x) \in R$. Clearly $(x,x)$ is a morphism from $x$ to $x$, and

$$\begin{aligned} (x,x)(x,y) &= (x,y), \\ (z,x)(x,x) &= (z,x). \end{aligned}$$

*Invertibility*:
Let $(x,y) \in R$, then

$$\begin{aligned} (x,y)(x,y)^{-1} &= (x,y)(y,x) = (x,x), \\ (x,y)^{-1}(x,y) &= (y,x)(x,y) = (y,y). \end{aligned}$$ ◇

We want to define the principal groupoids to be precisely the groupoids which are isomorphic to equivalence relations. It is therefore necessary to define groupoid homomorphisms.

**Definition 3.1.9.**
Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A function $\varphi : \mathcal{G} \to \mathcal{H}$ is a *groupoid homomorphism* if $(\varphi \times \varphi)(\mathcal{G}^{(2)}) \subset \mathcal{H}^{(2)}$ and $\varphi(xy) = \varphi(x)\varphi(y)$ whenever $(x,y) \in \mathcal{G}^{(2)}$. ♡

**Lemma 3.1.10.**
*Let $\mathcal{G}$ be a groupoid. We may associate an equivalence relation to $\mathcal{G}^{(0)}$ by setting*

$$R(\mathcal{G}) = \{(r(x), s(x)) \mid x \in \mathcal{G}\} \subset \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}.$$

*The map $x \mapsto (r(x), s(x))$ is a surjective groupoid homomorphism between $\mathcal{G}$ and $R(\mathcal{G})$.*

*Proof.*
We first show that $R(\mathcal{G})$ is an equivalence relation.
*Reflexivity*:
Let $u \in \mathcal{G}^{(0)}$. Then $(u,u) = (r(u), s(u)) \in R(\mathcal{G})$.
*Symmetry*:
Let $(u,v) \in R(\mathcal{G})$ and choose $x \in \mathcal{G}$ such that $(u,v) = (r(x), s(x))$. Then we have that $(v,u) = (s(x), r(x)) = (r(x^{-1}), s(x^{-1})) \in R(\mathcal{G})$.
*Transitivity*:
Let $(u,v),(v,w) \in R(\mathcal{G})$ and choose $x,y \in \mathcal{G}$ such that $r(x) = u$, $s(x) = v = r(y)$ and $s(y) = w$. Then

$$r(xy) = r(x) = u, \;\; s(xy) = s(y) = w,$$

thus $(u,w) = (r(xy), s(xy)) \in R(\mathcal{G})$.

The map $x \mapsto (r(x), s(x))$ is clearly surjective, and a groupoid homomorphism by

$$(r(xy), s(xy)) = (r(x), s(y)) = (r(x), s(x))(r(y), s(y))$$

for any $(x,y) \in \mathcal{G}^{(2)}$. Note that the last equality follows since $s(x) = r(y)$ by Lemma 3.1.5. □

**Definition 3.1.11.**
We say that a groupoid $\mathcal{G}$ is *principal* if the map $x \mapsto (r(x), s(x))$ from $\mathcal{G}$ to $R(\mathcal{G})$ is injective. ♡

A groupoid is clearly algebraically isomorphic to an equivalence relation if and only if it is principal. We want to make the distinction between equivalence relations and principal groupoids when we introduce a topology to groupoids, as an equivalence relation will generally have the relative topology from the set it is an equivalence relation on, while a principal groupoid might have some other topology.

**Definition 3.1.12.**
Let $\mathcal{G}$ be a groupoid and $u,v \in \mathcal{G}^{(0)}$. We define the sets

$$\begin{aligned}
\mathcal{G}_u &= \{x \in \mathcal{G} \mid s(x) = u\},\\
\mathcal{G}^u &= \{x \in \mathcal{G} \mid r(x) = u\},\\
\mathcal{G}_u^v &= \mathcal{G}_u \cap \mathcal{G}^v = \{x \in \mathcal{G} \mid s(x) = u, r(x) = v\}.
\end{aligned}$$

Note that for any unit $u \in \mathcal{G}^{(0)}$, the set $\mathcal{G}_u^u$ becomes a group with identity element $u$. These groups are called *isotropy subgroups*.

We define the set

$$\operatorname{Iso}(\mathcal{G}) = \bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}_u^u = \{x \in \mathcal{G} \mid r(x) = s(x)\},$$

called the *isotropy subgroupoid* of $\mathcal{G}$. The fact that this is a subgroupoid is clear. We also clearly have that $\mathcal{G}^{(0)} \subset \operatorname{Iso}(\mathcal{G})$. ♡

If we use the category theoretic definition of a groupoid $\mathcal{G}$ we can think of the isotropy subgroupoid $\operatorname{Iso}(\mathcal{G})$ as measuring to what extent the objects in our category have an interesting internal structure. If there is no isotropy, i.e. $\operatorname{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$, then the only morphisms from one object to itself is the identity morphism, which isn't that interesting. However, if a groupoid has a lot of isotropy there will be many morphisms from one object to itself.

We end this section by showing that the principal groupoids are exactly the groupoids with no isotropy.

**Lemma 3.1.13.**
*A groupoid $\mathcal{G}$ is principal if and only if $\mathcal{G}^{(0)} = \mathrm{Iso}(\mathcal{G})$.*

*Proof.*
Suppose $\mathcal{G}$ is principal. Let $x \in \mathrm{Iso}(\mathcal{G})$ and set $u = r(x) = s(x)$. By definition, we have that $u \in \mathcal{G}^{(0)}$ hence $r(u) = u = s(u)$. Then

$$u \mapsto (r(u), s(u)) = (r(x), s(x)) \leftarrow\!\shortmid x,$$

and since $\mathcal{G}$ is principal we have that these maps are injective. In particular, we get that $x = u \in \mathcal{G}^{(0)}$, hence $\mathcal{G}^{(0)} = \mathrm{Iso}(\mathcal{G})$.

Now suppose that $\mathcal{G}^{(0)} = \mathrm{Iso}(\mathcal{G})$. We want to show that the map $x \mapsto (r(x), s(x))$ is injective. Assume $(r(x), s(x)) = (r(y), s(y))$. Since $s(x) = s(y) = r(y^{-1})$, we have that the product $xy^{-1}$ is defined and

$$r(xy^{-1}) = r(x) = r(y) = s(y^{-1}) = s(xy^{-1}).$$

Thus, $xy^{-1} \in \mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$, so

$$xy^{-1} = s(xy^{-1}) = s(y^{-1}) = (y^{-1})^{-1}y^{-1} = yy^{-1}.$$

Hence, by the uniqueness part of Lemma 3.1.3 (iii) we get that $x = y$, meaning that the map $x \mapsto (r(x), s(x))$ is injective, hence $\mathcal{G}$ is principal. □

## 3.2 Topological groupoids

We have now established all the necessary algebraic tools about groupoids, however our goal will be to use these objects to study $C^*$-algebras. Hence, it will be necessary to develop some theory for topological groupoids. In particular some theory about étale groupoids will be established. Étale groupoids can be thought of as the groupoid analogue to discrete groups, and they give rise to some nice computational results.

**Definition 3.2.1.**
A *topological groupoid* is a groupoid $\mathcal{G}$ endowed with a topology such that the range map, source map, multiplication map and inverse map, are continuous. The subsets $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ and $\mathcal{G}^{(0)} \subset \mathcal{G}$ are endowed with the subspace topologies. ♡

Our goal in this section is to show that when a groupoid $\mathcal{G}$ is endowed with a suitably nice topology, the sets $\mathcal{G}_u$ and $\mathcal{G}^u$ are discrete.

**Lemma 3.2.2.**
*If $\mathcal{G}$ is a topological groupoid, then $\mathcal{G}^{(0)}$ is closed if and only if $\mathcal{G}$ is Hausdorff.*

*Proof.*
Suppose $\mathcal{G}$ is Hausdorff and let $(u_\lambda)_{\lambda \in \Lambda} \subset \mathcal{G}^{(0)}$ be a net that converges to an element $u \in \mathcal{G}$. For any index $\lambda \in \Lambda$ we have that $r(u_\lambda) = u_\lambda = s(u_\lambda)$ since $u_\lambda \in \mathcal{G}^{(0)}$. Then by continuity of the range and source maps we have that

$$\lim u_\lambda = \lim r(u_\lambda) = r(\lim u_\lambda) = r(u).$$

Similarly, $\lim u_\lambda = s(u)$. Since $\mathcal{G}$ is Hausdorff, limits are unique. We therefore get that $r(u) = u = s(u)$, hence $u \in \mathcal{G}^{(0)}$, meaning that $\mathcal{G}^{(0)}$ is closed.

Now suppose $\mathcal{G}^{(0)}$ is closed. Let $(x_\lambda)_{\lambda \in \Lambda} \subset \mathcal{G}$ be a net and $x, y \in \mathcal{G}$ be such that $\lim x_\lambda = x$ and $\lim x_\lambda = y$. By continuity of the inverse and multiplication maps we have that the net $(x_\lambda^{-1} x_\lambda)_{\lambda \in \Lambda} \subset \mathcal{G}^{(0)}$ converges to the element $x^{-1}y$. Since $\mathcal{G}^{(0)}$ is closed we get that $x^{-1}y \in \mathcal{G}^{(0)}$. Thus,

$$x^{-1}y = r(x^{-1}y) = r(x^{-1}) = x^{-1}x,$$

so $y = x$ by Lemma 3.1.4, meaning that $\mathcal{G}$ is Hausdorff. □

**Definition 3.2.3.**
A topological groupoid $\mathcal{G}$ is *étale* if the range map $r : \mathcal{G} \to \mathcal{G}$ is a local homeomorphism. ♡

Note that for an étale groupoid $\mathcal{G}$ we have that the source map $s : \mathcal{G} \to \mathcal{G}$ is also a local homeomorphism by the fact that we may write $s(x) = r(x^{-1})$ for any $x \in \mathcal{G}$. This uses the fact that the inverse map is a homeomorphism, which is clear.

**Lemma 3.2.4.**
*Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.*

*Proof.*
For each $x \in \mathcal{G}$ let $U_x$ be an open neighborhood in $X$ about $x$ such that $r|_{U_x}$ is injective. Then each $r(U_x)$ is open in $\mathcal{G}$ and a subset of $\mathcal{G}^{(0)}$. Hence,

$$\mathcal{G}^{(0)} = \bigcup_{x \in \mathcal{G}} r(U_x),$$

showing that $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$. □

**Definition 3.2.5.**
Let $\mathcal{G}$ be an étale groupoid. A subset $B \subset \mathcal{G}$ is called a *bisection* if there exists an open subset $U \subset X$ containing $B$ such that both $r|_U$ and $s|_U$ are injective. ♡

**Lemma 3.2.6.**
*Let $\mathcal{G}$ be a second-countable étale groupoid. Then $\mathcal{G}$ has a countable base of open bisections.*

*Proof.*
Since $\mathcal{G}$ is second-countable we can find a countable dense subset $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{G}$. For each $n \in \mathbb{N}$ choose countable neighborhood bases $\{U_{n,i}\}_{i\in\mathbb{N}}$, $\{V_{n,i}\}_{i\in\mathbb{N}}$ about $x_n$ such that both $r|_{U_{n,i}}$ and $s|_{V_{n,i}}$ are injective for every $i \in \mathbb{N}$. Then $\{U_{n,i} \cap V_{n,i}\}_{n,i\in\mathbb{N}}$ is a countable base of open bisections. □

**Corollary 3.2.7.**
*Let $\mathcal{G}$ be a second-countable Hausdorff étale groupoid. Then $\mathcal{G}_u$ and $\mathcal{G}^u$ are discrete.*

*Proof.*
Let $x \in \mathcal{G}_u$, then by definition $s(x) = u$. By Lemma 3.2.6 we may find a basic open bisection $U$ about $x$, then the restricted source map $s|_U$ is injective. In particular $s|_U^{-1}(u) = x$. Hence, $\{x\} = \mathcal{G}_u \cap U$ is open in $\mathcal{G}_u$. Since $\mathcal{G}$ is Hausdorff, $\{x\}$ is also closed in $\mathcal{G}_u$, thus $\mathcal{G}_u$ is discrete.

The proof that $\mathcal{G}^u$ is discrete is completely analogues to this one. □

We end this section by further developing some theory about Deaconu-Renault groupoids. Some details presented here can be found in Sims' notes, but most of it is original work.

**Example 3.2.8** (Deaconu-Renault groupoids continued)**.**
Let $X$ be a second-countable locally compact Hausdorff space. We will for simplicity consider the Deaconu-Renault groupoid associated to $X$ where we have a monoidal action from $\mathbb{N}$ as a submonoid of $\mathbb{Z}$. So suppose $\mathbb{N}$ acts on $X$ by local homeomorphisms and denote by $\mathcal{G}$ the associated Deaconu-Renault groupoid. For open subsets $U, V \subset X$ and natural numbers $m, n \in \mathbb{N}$, we define the set $Z(U, m, n, V)$ to be

$$Z(U, m, n, V) = \{(x, m-n, y) \mid x \in U,\ y \in V,\ mx = ny\}.$$

Then the collection

$$\mathcal{B} = \{Z(U, m, n, V) \mid U, V \text{ open in } X,\ m, n \in \mathbb{N}\}$$

form a base making $\mathcal{G}$ a second-countable locally compact Hausdorff étale groupoid, which we will prove in the following lemmas. ♢

**Lemma 3.2.9.**
*$\mathcal{B}$ is a base.*

*Proof.*
*$\mathcal{G}$ is covered by basic opens*:
It is easily seen that

$$\mathcal{G} = \bigcup_{m,n\in\mathbb{N}} Z(X, m, n, X).$$

*The intersection of two basic opens is precisely covered by basic opens*:
Fix open subsets $U_1, U_2, V_1, V_2 \subset X$ and natural numbers $m_1, m_2, n_1, n_2 \in \mathbb{N}$. Define

$Z_1 = Z(U_1, m_1, n_1, V_1)$ and $Z_2 = Z(U_2, m_2, n_2, V_2)$ and let $(x, g, y) \in Z_1 \cap Z_2$. Since $m_1 - n_1 = g = m_2 - n_2$ we get that $m_1 - m_2 = n_1 - n_2$. By possibly relabeling we may assume that $m_1 \geq m_2$, and $n_1 \geq n_2$. Let $r \in \mathbb{N}$ be such that $m_1 - m_2 = r = n_1 - n_2$. Define $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. Then we have that $(x, g, y) \in Z(U_3, m_2, n_2, V_3)$.

Now we just need to show that $Z(U_3, m_2, n_2, V_3) \subset Z_1 \cap Z_2$. It is clear that $Z(U_3, m_2, n_2, V_3) \subset Z_2$. To see that $Z(U_3, m_2, n_2, V_3) \subset Z_1$, let $(z, h, w) \in Z(U_3, m_2, n_2, V_3)$. Then we get that

$$m_1 z = (r + m_2) z = r(m_2 z) = r(n_2 w) = (r + n_2) w = n_1 w,$$

hence $(z, h, w) \in Z_1$. □

**Lemma 3.2.10.**
*$\mathcal{G}$ is second-countable.*

*Proof.*
Since $X$ is second-countable we may choose a countable base for $X$: $\{U_k\}_{k \in \mathbb{N}}$. Then the collection $\{Z(U_k, m, n, U_l)\}_{k,m,n,l \in \mathbb{N}}$ is a countable base for $\mathcal{G}$. □

**Lemma 3.2.11.**
*$\mathcal{G}$ is Hausdorff.*

*Proof.*
Let $(x_1, m_1 - n_1, y_1), (x_2, m_2 - n_2, y_2) \in \mathcal{G}$ such that $(x_1, m_1 - n_1, y_1) \neq (x_2, m_2 - n_2, y_2)$. There are three cases to consider: $x_1 \neq x_2$, $y_1 \neq y_2$ and $m_1 - n_1 \neq m_2 - n_2$.

If $x_1 \neq x_2$ we may choose disjoint open neighborhoods $U_1, U_2 \subset X$ about $x_1, x_2$ respectively since $X$ is Hausdorff. Then $(x_1, m_1 - n_1, y_1) \in Z(U_1, m_1, n_1, X)$, $(x_2, m_2 - n_2, y_2) \in Z(U_2, m_2, n_2, X)$ and $Z(U_1, m_1, n_1, X) \cap Z(U_2, m_2, n_2, X) = \emptyset$. A similar argument holds for when $y_1 \neq y_1$.

Finally, if $m_1 - n_1 \neq m_2 - n_2$ we notice that $Z(X, m_1, n_1, X) \cap Z(X, m_2, n_2, X) = \emptyset$. This completes the proof, since clearly $(x_1, m_1 - n_1, y_1) \in Z(X, m_1, n_1, X)$ and $(x_2, m_2 - n_2, y_2) \in Z(X, m_2, n_2, X)$. □

To prove that $\mathcal{G}$ is locally compact we will use a description of compact sets that uses universal nets. For a quick introduction in universal nets see [Kjæ95]. We will here only state the definition and main result that we will use.

**Definition 3.2.12.**
Let $X$ be a set. A net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ is *universal* if for every subset $Y \subset X$ we have that the net is either eventually in $Y$ or eventually in $X \setminus Y$. ♡

**Theorem 3.2.13** ([Kjæ95] Theorem 1.6.2)**.**
*Let $X$ be a topological space. Then $X$ is compact if and only if every universal net in $X$ converges.* □

With this we can prove that $\mathcal{G}$ is locally compact.

**Lemma 3.2.14.**
*$\mathcal{G}$ is locally compact.*

*Proof.*
For this we will prove that if $K, L \subset X$ are compact, we have for any natural numbers $m, n \in \mathbb{N}$ that $Z(K, m, n, L)$ is compact. Let $((x_\lambda, m-n, y_\lambda))_{\lambda\in\Lambda}$ be a universal net in $Z(K, m, n, L)$. Then $(x_\lambda)_{\lambda\in\Lambda}$ is a universal net in $K$, and since $K$ is compact we have that this net converges to an element $x \in K$. Similarly, we have that $(y_\lambda)_{\lambda\in\Lambda}$ converges to an element $y \in L$. Since $\mathbb{N}$ acts on $X$ by local homeomorphisms and $X$ is Hausdorff we have that

$$mx = \lim_\lambda mx_\lambda = \lim_\lambda ny_\lambda = ny,$$

so $(x, m-n, y) \in Z(K, m, n, L)$. It remains to show that $((x_\lambda, m-n, y_\lambda))_{\lambda\in\Lambda}$ converges to $(x, m-n, y)$.

Let $Z(U, p, q, V)$ be an open neighborhood about $(x, m-n, y)$. Since the net $(x_\lambda)_{\lambda\in\Lambda}$ converges to $x$ we have that it is eventually in $U$. Similarly, $(y_\lambda)_{\lambda\in\Lambda}$ is eventually in $V$. By passing to a subnet we may assume that $x_\lambda \in U$ and $y_\lambda \in V$ for all indices $\lambda \in \Lambda$. Since both $mx = ny$ and $px = qy$ we have that $m-n = p-q$, so $m-p = n-q$.

In the case where $m \leq p$, let $r = p-m$. Then we have for all indices $\lambda \in \Lambda$ that

$$px_\lambda = (r+m)x_\lambda = r(mx_\lambda) = r(ny_\lambda) = (r+n)y_\lambda = qy_\lambda.$$

Hence, $(x_\lambda, m-n, y_\lambda) \in Z(U, p, q, V)$ for all indices $\lambda \in \Lambda$.

In the case where $m > p$, let $s = m-p$. By continuity of the action on $X$ by $\mathbb{N}$ we have that the nets $(px_\lambda)_{\lambda\in\Lambda}$ and $(qy_\lambda)_{\lambda\in\Lambda}$ converges to $px$ and $qy$ respectively. By assumption, $px = qy$. Let $W \subset X$ be an open neighborhood about $px = qy$ such that action by $b|_W$ is injective. Then we have that $(px_\lambda)_{\lambda\in\Lambda}$ and $(qy_\lambda)_{\lambda\in\Lambda}$ are eventually in $W$. For large enough indices $\lambda$ we have that the equalities

$$b(px_\lambda) = (b+p)x_\lambda = mx_\lambda = ny_\lambda = (b+q)y_\lambda = b(qy_\lambda)$$

imply that $px_\lambda = qy_\lambda$ by injectivity of $b|_W$. Hence, $((x_\lambda, m-n, y_\lambda))_{\lambda\in\Lambda}$ is eventually in $Z(U, p, q, V)$ proving that this net converges to $(x, m-n, y)$.

To see that $\mathcal{G}$ is locally compact let $(x, m-n, y) \in \mathcal{G}$. Choose relatively compact open neighborhoods $U, V \subset X$ about $x$ and $y$ respectively. Then $(x, m-n, y) \in Z(U, m, n, V) \subset Z(\overline{U}, m, n, \overline{V})$, where $Z(\overline{U}, m, n, \overline{V})$ is compact. Since $\mathcal{G}$ is Hausdorff this shows that $\mathcal{G}$ is locally compact. □

**Lemma 3.2.15.**
*$\mathcal{G}$ is étale.*

*Proof.*
Let $(x, m-n, y) \in \mathcal{G}$ and $U, V \subset X$ be open neighborhoods about $x, y \in X$ respectively such that $m|_U$ and $n|_V$ are injective. Let $W = (mU) \cap (nV)$, $U' = m|_U^{-1}(W)$ and $V' = n|_V^{-1}(W)$. Since $\mathbb{N}$ acts on $X$ by local homeomorphisms we get that all these sets are open in $X$. We claim that the range map restricted to $Z(U', m, n, V')$,

$$r : Z(U', m, n, V') \to Z(U', 0, 0, U')$$

is a homeomorphism. By definition, we have that $r$ is continuous, and by construction of $U'$ and $V'$ we have that $r$ is bijective. To check that $r$ is open we check instead that its inverse is continuous. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in $U'$ converging to $x \in U'$. By construction, we have for each index $\lambda \in \Lambda$ that there exists a unique $y_\lambda \in V'$ such that $mx_\lambda = ny_\lambda$. There is also a unique $y \in V'$ such that $mx = ny$. Since $X$ is Hausdorff we get that

$$\lim_\lambda ny_\lambda = \lim_\lambda mx_\lambda = mx = ny.$$

So since $n|_{V'}$ is injective we get that $(y_\lambda)_{\lambda\in\Lambda}$ converges to $y$. We now have a net $((x_\lambda, m-n, y_\lambda))_{\lambda\in\Lambda}$ in $Z(U', m, n, V')$ such that $(x_\lambda)_{\lambda\in\Lambda}$ converges to $x \in U'$ and $(y_\lambda)_{\lambda\in\Lambda}$ converges to $y \in V'$. However, this is precisely the situation we had in the proof showing that $\mathcal{G}$ is locally compact, so we can conclude that $((x_\lambda, m-n, y_\lambda))_{\lambda\in\Lambda}$ converges to $(x, m-n, y)$. Hence, $r$ is open. □

## 3.3 Continuous functions on groupoids

If we have a locally compact Hausdorff groupoid, $\mathcal{G}$, we get a commutative $C^*$-algebra by looking at $C_0(\mathcal{G})$, where the algebra structure on $C_0(\mathcal{G})$ is simply given by the point-wise product. However, we would very much like to preserve the non-commutative structure that we get from a groupoid. To this end we define a convolution product on $C_c(\mathcal{G})$ and see that we get a *-algebra which will in general be non-commutative.

**Definition 3.3.1.**
Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. We want to make the complex vector space $C_c(\mathcal{G})$ into a *-algebra. This is done by defining multiplication using the convolution product: For $f, g \in C_c(\mathcal{G})$ and $x \in \mathcal{G}$ we define $f * g \in C_c(\mathcal{G})$ by the equation

$$(f * g)(x) = \sum_{yz=x} f(y)\, g(z),$$

where the sum ranges over $(y,z) \in \mathcal{G}^{(2)}$ such that $x = yz$. For $f \in C_c(\mathcal{G})$ and $x \in \mathcal{G}$ we define the involution of $f$ by the equation

$$f^*(x) = \overline{f(x^{-1})}. \qquad \heartsuit$$

**Proposition 3.3.2.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Let $f, g \in C_c(\mathcal{G})$ and $x \in \mathcal{G}$. Then the set $\{(y,z) \in \mathcal{G}^{(2)} \mid yz = x,\ f(y)\, g(z) \neq 0\}$ is finite.*

*Proof.*
If $yz = x$ we have that $r(x) = r(yz) = r(y)$ and $s(x) = s(yz) = s(z)$, so $y \in \mathcal{G}^{r(x)}$ and $z \in \mathcal{G}_{s(x)}$. By Corollary 3.2.7 these are discrete sets, thus the intersections $\mathcal{G}^{r(x)} \cap \operatorname{supp}(f)$ and $\mathcal{G}_{s(x)} \cap \operatorname{supp}(g)$ are finite since $f$ and $g$ have compact support. This completes the proof. □

With this the convolution product on $C_c(\mathcal{G})$ is well-defined, justifying our use of it in Definition 3.3.1. Note that the convolution product may equivalently be defined by the equations

$$(f * g)(x) = \sum_{y \in \mathcal{G}^{r(x)}} f(y)\, g(y^{-1}x) \tag{3.1}$$

$$= \sum_{y \in \mathcal{G}_{s(x)}} f(xy^{-1})\, g(y). \tag{3.2}$$

Note that in the proof of Proposition 3.3.2 it was not required that the functions were continuous, hence the convolution product may be extended to any function with compact support. In particular the functions $\delta_a : \mathcal{G} \to \{0,1\}$ where $a \in \mathcal{G}$ defined by

$$\delta_a(x) = \begin{cases} 1, & x = a, \\ 0, & x \neq a, \end{cases}$$

will be useful when constructing an injective *-representation of $C_c(\mathcal{G})$.

We will see that the convolution products becomes particularly simple when the functions we are convolving are supported on bisections. It is useful to know that these functions span the entire *-algebra.

**Lemma 3.3.3.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Then*

$$C_c(\mathcal{G}) = \operatorname{span}\{f \in C_c(\mathcal{G}) \mid \operatorname{supp}(f) \text{ is a bisection}\}.$$

*Proof.*
By Lemma 3.2.6 we may choose a countable base of open bisections for $\mathcal{G}$. Let $f \in C_c(\mathcal{G})$ and $\{U_i\}_{i\in\mathbb{N}}$ be an open cover of $\operatorname{supp}(f)$ with each $U_i$ a bisection. By compactness of $\operatorname{supp}(f)$ we may pass to a finite sub-cover $\{U_i\}_{i=1}^n$. Let $\{h_i\}_{i=1}^n$ be a partition of unity subordinate to $\{U_i\}_{i=1}^n$. Define functions $f_i \in C_c(\mathcal{G})$ by the point-wise products $f_i = f \cdot h_i$. Then $\operatorname{supp}(f_i) \subset U_i$, and

$$f = \sum_{i=1}^n f_i. \qquad \square$$

**Lemma 3.3.4.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Suppose $f, g \in C_c(\mathcal{G})$ are supported on bisections. Then for any $x \in \operatorname{supp}(f * g)$ such that $x = yz$ with $y \in \operatorname{supp}(f)$ and $z \in \operatorname{supp}(g)$ we have that*

$$(f * g)(x) = f(y)\, g(z).$$

*Proof.*
Let $U, V \subset \mathcal{G}$ be open bisections such that $\operatorname{supp}(f) \subset U$ and $\operatorname{supp}(g) \subset V$. Then $r|_U$ and $s|_V$ are injective. Let $x \in \operatorname{supp}(f * g)$ be such that $x = yz$ with $y \in U$ and $z \in V$. By injectivity of $r|_U$ we have that $y$ is the unique element in $U$ such that $r(x) = r(yz) = r(y)$. Similarly, $z$ is the unique element in $V$ such that $s(x) = s(yz) = s(z)$. Hence, $(y, z) \in \mathcal{G}^{(2)}$ is the only pair in $\mathcal{G}^{(2)}$ that satisfies $yz = x$ and $f(y)\, g(z) \neq 0$. Thus, $(f * g)(x) = f(y)\, g(z)$. $\square$

**Lemma 3.3.5.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Then $C_c(\mathcal{G}^{(0)})$ is a *-subalgebra of $C_c(\mathcal{G})$.*

*Furthermore, for $f \in C_c(\mathcal{G})$ supported on a bisection we have that $f^* * f \in C_c(\mathcal{G}^{(0)})$ is supported on $s(supp(f))$ and $(f^* * f)(s(x)) = |f(x)|^2$ for any $x \in \operatorname{supp}(f)$.*

*Similarly, $f * f^* \in C_c(\mathcal{G}^{(0)})$ is supported on $r(supp(f))$ and $(f * f^*)(r(x)) = |f(x)|^2$ for any $x \in \operatorname{supp}(f)$.*

*Finally, for $f \in C_c(\mathcal{G})$ and $h \in C_c(\mathcal{G}^{(0)})$ we have that*

$$(h * f)(x) = h(r(x))\, f(x) \text{ and}$$
$$(f * h)(x) = f(x)\, h(s(x))$$

*for any $x \in \operatorname{supp}(f)$.*

*Proof.*
The fact that $C_c(\mathcal{G}^{(0)})$ is a *-subalgebra of $C_c(\mathcal{G})$ follows immediately by the fact that $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$, which follows by Lemma 3.2.4 since $\mathcal{G}$ is étale.

For $f \in C_c(\mathcal{G})$ supported on a bisection we have that $f^*$ is supported on $(\operatorname{supp}(f))^{-1}$. By the equations, $r(x^{-1}) = s(x)$ and $s(x^{-1}) = r(x)$ for any $x \in \mathcal{G}$, we get that $(\operatorname{supp}(f))^{-1}$ is a bisection. Hence, by Lemma 3.3.4, we have that for any $x = y^{-1}z \in \operatorname{supp}(f^* * f)$ with $y, z \in \operatorname{supp}(f)$,

$$(f^* * f)(x) = f^*(y^{-1})\, f(z) = \overline{f(y)}\, f(z).$$

Since the product $y^{-1}z$ is defined we have that $r(z) = s(y^{-1}) = r(y)$, and since $\operatorname{supp}(f)$ is a bisection we get that $y = z$ by injectivity of $r|_{\operatorname{supp}(f)}$. Hence, any $x \in \operatorname{supp}(f^* * f)$ is of the form $y^{-1}y = s(y)$ with $y \in \text{supp}(f)$. So $f^* * f$ is indeed supported on $s(\text{supp}(f))$, and the equation

$$(f^* * f)(s(x)) = |f(x)|^2$$

holds. The statement about $f * f^*$ is similarly proven.

Let $f \in C_c(\mathcal{G})$ and $h \in C_c(\mathcal{G}^{(0)})$. By equation (3.1) we have that

$$(f * h)(x) = \sum_{y \in \mathcal{G}^{r(x)}} f(y)\, h(y^{-1}x)$$

for any $x \in \mathcal{G}$. Since $h \in C_c(\mathcal{G}^{(0)})$ we have that $h(y^{-1}x) \neq 0$ only if $y^{-1}x \in \mathcal{G}^{(0)}$, so assume it is. Then

$$y^{-1}x = s(y^{-1}x) = s(x) = x^{-1}x,$$

hence $y = x$. Thus,

$$(f * h)(x) = f(x)\, h(s(x)).$$

The equation for $h * f$ is similarly proven to be true. □

This lemma also establishes that $C_c(\mathcal{G}^{(0)})$ is a *commutative* *-subalgebra of $C_c(\mathcal{G})$ and the convolution product simplifies to the point-wise product: If $f, g \in C_c(\mathcal{G}^{(0)})$ we have that

$$(f * g)(x) = f(x)\, g(s(x)) = f(x)\, g(x) = g(x)\, f(x) = g(x)\, f(s(x)) = (g * f)(x)$$

for any $x \in \mathcal{G}^{(0)}$. Thus, $C_c(\mathcal{G}^{(0)})$ sits inside the commutative $C^*$-algebra $C_0(\mathcal{G}^{(0)})$.

## 3.4 The full groupoid $C^*$-algebra

To give $C_c(\mathcal{G})$ the structure of a $C^*$-algebra we will construct a $C^*$-norm by looking at all the *-representations of $C_c(\mathcal{G})$. For this to work we need to make sure that the representations are suitably bounded. In the non-second-countable case this is quite tricky. However, Sims is able to give an easy proof in the second-countable case, which we present here.

**Proposition 3.4.1.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Then for any $f \in C_c(\mathcal{G})$ there exists a constant $K_f \geq 0$ such that $\|\pi(f)\| \leq K_f$ for all *-representations $\pi : C_c(\mathcal{G}) \to B(H)$ on a Hilbert space $H$. If $\operatorname{supp}(f)$ is a bisection, $K_f$ may be chosen to be equal to $\|f\|_\infty$.*

*Proof.*
Suppose $\pi : C_c(\mathcal{G}) \to B(H)$ is a *-representation on a Hilbert space $H$. Then the restriction $\pi|_{C_c(\mathcal{G}^{(0)})}$ uniquely extends to a *-homomorphism between $C_0(\mathcal{G}^{(0)})$ and $B(H)$. As a *-homomorphism between $C^*$-algebras it is in particular norm-decreasing. So for $f \in C_c(\mathcal{G})$ supported on a bisection we have by Lemma 3.3.5 that $f^* * f \in C_c(\mathcal{G}^{(0)})$ and $\|f^* * f\|_\infty = \|f\|_\infty^2$. Hence,

$$\|\pi(f)\|^2 = \|\pi(f^* * f)\| \leq \|f^* * f\|_\infty = \|f\|_\infty^2.$$

The general case follows by Lemma 3.3.3, allowing us to write any $f \in C_c(\mathcal{G})$ as a linear combination $f = \sum_{i=1}^n f_i$ with each $f_i \in C_c(\mathcal{G})$ supported on a bisection. Set $K_f = \sum_{i=1}^n \|f_i\|_\infty$. Then the triangle inequality gives us that

$$\|\pi(f)\| \leq \sum_{i=1}^n \|\pi(f_i)\| \leq \sum_{i=1}^n \|f_i\|_\infty = K_f.$$ □

With this we want to say that we get a $C^*$-norm on $C_c(\mathcal{G})$ by looking at the equation

$$\|f\|_{\max} = \sup\{\|\pi(f)\| \mid \pi : C_c(\mathcal{G}) \to B(H) \text{ is a *-representation}\},$$

where $f \in C_c(\mathcal{G})$. A priori we don't know that this is a norm, it could be the case that this only defines a semi-norm on $C_c(\mathcal{G})$. To show that this is indeed a norm we will construct a *-representation that is injective.

**Definition 3.4.2.**
Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. For a unit $u \in \mathcal{G}^{(0)}$ we define a *-representation $\pi_u : C_c(\mathcal{G}) \to B(\ell^2(\mathcal{G}_u))$ by the equation

$$\pi_u(f)\delta_a = \sum_{x \in \mathcal{G}_{r(a)}} f(x)\delta_{xa},$$

with $f \in C_c(\mathcal{G})$ and $a \in \mathcal{G}_u$. $\pi_u$ is called the *regular representation* of $C_c(\mathcal{G})$ associated to $u$. Note that it is sufficient to define the operator $\pi_u(f)$ on the subspace $\operatorname{span}\{\delta_a\}_{a \in \mathcal{G}_u}$ since it is dense in $\ell^2(\mathcal{G}_u)$ (this follows by Corollary 3.2.7 ensuring that $\mathcal{G}_u$ is discrete). The fact that $\pi_u$ is a continuous *-homomorphism follows by the fact that the above formula is simply the convolution product between $f$ and $\delta_a$. ♢

We give a straight forward proof of the next fact.

**Proposition 3.4.3.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. Then the *-representation $\bigoplus_{u\in\mathcal{G}^{(0)}} \pi_u$ is injective.*

*Proof.*
Let $f \in C_c(\mathcal{G})$ and assume that $\pi_u(f) = 0$ for all units $u \in \mathcal{G}^{(0)}$. Then for any $a \in \mathcal{G}$ we have that

$$0 = \langle \pi_{s(a)}(f)\delta_a, \delta_{s(a)} \rangle = \sum_{x\in\mathcal{G}_{r(a)}} f(x)\langle \delta_{xa}, \delta_{s(a)} \rangle = f(a^{-1}),$$

where the last equality follows from the fact that $\langle \delta_{xa}, \delta_{s(a)} \rangle \neq 0$ if and only if $xa = s(a)$, implying that $x = a^{-1}$. Thus, $f(\mathcal{G}) = f((\mathcal{G})^{-1}) = \{0\}$, meaning $f = 0$. Hence, $\bigoplus_{u\in\mathcal{G}^{(0)}} \pi_u$ is injective. □

With this we can give an explicit construction of a $C^*$-algebra associated to a groupoid $\mathcal{G}$.

**Definition 3.4.4.**
Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. The *reduced $C^*$-algebra* of $\mathcal{G}$, denoted by $C_r^*(\mathcal{G})$, is the completion of

$$\left(\bigoplus_{u\in\mathcal{G}^{(0)}} \pi_u\right)(C_c(\mathcal{G})) \subset \bigoplus_{x\in\mathcal{G}^{(0)}} B(\ell^2(\mathcal{G}_u)).$$ ♡

Although this $C^*$-algebra is useful in many situations it is not the $C^*$-algebra we will be working with. Thanks to Proposition 3.4.3 we know that there exists an injective *-representation of $C_c(\mathcal{G})$, so we are justified in making the following definition.

**Definition 3.4.5.**
Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. We get a $C^*$-algebra, which we denote by $C^*(\mathcal{G})$, by taking the completion of $C_c(\mathcal{G})$ in the $C^*$-norm

$$\|f\|_{\max} = \sup\{\|\pi(f)\| \mid \pi : C_c(\mathcal{G}) \to B(H) \text{ is a *-representation}\},$$

where $f \in C_c(\mathcal{G})$. We call $C^*(\mathcal{G})$ the *full $C^*$-algebra* of $\mathcal{G}$. This construction identifies $C_c(\mathcal{G})$ as a dense subspace of $C^*(\mathcal{G})$. ♡

## 3.5 $C^*$-dynamical systems, weights and a characterization of $\beta$-KMS weights

With the necessary preliminaries about groupoids established we look to introduce enough theory to understand Theorem 3.5.14 ([Chr23] Theorem 7.4). First we need to define what a $C^*$-dynamical system is and what a $\beta$-KMS weight is.

**Definition 3.5.1** ([BR87] Definition 2.7.1)**.**
A $C^*$*-dynamical system* is a triple $(\mathcal{A}, G, \alpha)$ with $\mathcal{A}$ being a $C^*$-algebra, $G$ a locally compact group and $\alpha$ a strongly continuous map $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ satisfying

$$\alpha_e = id_{\mathcal{A}}$$
$$\alpha_g \alpha_h = \alpha_{gh},$$

where $e \in G$ is the identity element and $g, h \in G$. By strongly continuous we mean that the map $g \mapsto \alpha_g(a)$ is norm-continuous for all $a \in \mathcal{A}$. ♡

We will in particular be interested in the case where $G = \mathbb{R}$. In this case it will be possible to analytically continue the dynamics of $\alpha$ from $\mathbb{R}$ to $\mathbb{C}$.

**Definition 3.5.2** ([BR87] Definition 2.5.20)**.**
Let $(\mathcal{A}, \mathbb{R}, \alpha)$ be a $C^*$-dynamical system. For $z \in \mathbb{C}$ with $\mathrm{Im}(z) \geq 0$ we define the set $S(z)$ to be the horizontal strip $S(z) = \{w \in \mathbb{C} \mid \mathrm{Im}(w) \in [0, \mathrm{Im}(z)]\}$. When $\mathrm{Im}(z) \leq 0$ we define $S(z)$ similarly, $S(z) = \{w \in \mathbb{C} \mid \mathrm{Im}(w) \in [\mathrm{Im}(z), 0]\}$.

We define the set $D(\alpha_z) \subset \mathcal{A}$ to be the elements $a \in \mathcal{A}$ such that there exists a continuous function $f : S(z) \to \mathcal{A}$ which is *analytic* on the interior of $S(z)$ such that $f(t) = \alpha_t(a)$ for all $t \in \mathbb{R}$. We define $\alpha_z(a) = f(z)$ for $z \in S(z)$. Analytic in this context means that the composition $\varphi \circ f$ is holomorphic on the interior of $S(z)$ for all $\varphi \in \mathcal{A}^*$. We say that an element $a \in \mathcal{A}$ is *analytic for* $\alpha$ if there exists an entire function $f : \mathbb{C} \to \mathcal{A}$ such that $f(t) = \alpha_t(a)$ for all $t \in \mathbb{R}$. ♡

Next we define what a *weight* is. We may think of weights as unbounded positive functionals, and in some way a generalization of regular measures.

**Definition 3.5.3** ([Ped79] Definition 5.1.1)**.**
Let $\mathcal{A}$ be a $C^*$-algebra and denote by $\mathcal{A}_+$ the convex cone of positive elements in $\mathcal{A}$. A *weight* on $\mathcal{A}$ is a map $\psi : \mathcal{A}_+ \to [0, \infty]$ such that

$$\psi(a + b) = \psi(a) + \psi(b)$$
$$\psi(\lambda a) = \lambda \psi(a)$$

for all $a, b \in \mathcal{A}_+$ and $\lambda \geq 0$. We denote by $\mathcal{A}_+^{\psi}$ the set of all positive elements $a \in \mathcal{A}_+$ for which $\psi(a) < \infty$. Since the positive elements of a $C^*$-algebra span the entire $C^*$-algebra we define $\mathcal{A}^{\psi} = \mathrm{span}(\mathcal{A}_+^{\psi})$, and we may extend $\psi$ uniquely to a positive functional on $\mathcal{A}^{\psi}$ in the obvious way. (This is done in detail in [Ped79] lemma 5.1.2).

We call a weight $\psi$

- *densely defined* if $\mathcal{A}_+^{\psi}$ is dense in $\mathcal{A}_+$,
- *lower semi-continuous* if $\{a \in \mathcal{A}_+ \mid \psi(a) \leq \lambda\}$ is closed for all $\lambda \geq 0$, and
- *proper* if it is densely defined and lower semi-continuous. ♡

**Definition 3.5.4** ([Kus97] Definition 2.8)**.**
Let $(\mathcal{A}, \mathbb{R}, \alpha)$ be a $C^*$-dynamical system and $\beta \in \mathbb{R}$. We call a weight $\psi$ on $\mathcal{A}$ a *$\beta$-KMS weight for $\alpha$* if it is a proper weight satisfying

(1) $\psi \circ \alpha_t = \psi$ for all $t \in \mathbb{R}$, and

(2) for every $a \in D(\alpha_{-\beta i/2})$ we have that

$$\psi(a^*a) = \psi(\alpha_{-\beta i/2}(a)\,\alpha_{-\beta i/2}(a)^*). \qquad \heartsuit$$

The theory of dynamical systems and weights is vast and there are a lot of things that is worth mentioning. However, for the purposes of this thesis we will have to settle with only familiarizing ourselves with these definitions.

Note that for groupoids, if we have a continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ we get a $C^*$-dynamical system $(C^*(\mathcal{G}), \mathbb{R}, \alpha^c)$ by using Pontryagin duality as follows:

**Definition 3.5.5.**
Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism. Let $t \in \mathbb{R}$ and $\chi_t : \mathbb{R} \to \mathbb{T}$ be the character associated to $t$. Then we get a continuous *-homomorphism $\alpha_t^c : C_c(\mathcal{G}) \to C_c(\mathcal{G})$ defined by

$$\alpha_t^c(f)(x) = (\chi_t \circ c)(x)\,f(x) = e^{itc(x)}\,f(x), \text{ for } x \in \mathcal{G} \text{ and } f \in C_c(\mathcal{G}).$$

By continuity this extends to a *-homomorphism $\alpha_t^c : C^*(\mathcal{G}) \to C^*(\mathcal{G})$. It is clear that $\alpha_{-t}^c$ is an inverse to $\alpha_t^c$, so we get a map $\alpha^c : \mathbb{R} \to \mathrm{Aut}(C^*(\mathcal{G}_E))$ defined by $t \mapsto \alpha_t^c$. $\heartsuit$

Note that if $z \in \mathbb{C}$ we have that $\alpha_z^c(f) \in C_c(\mathcal{G})$ for every $f \in C_c(\mathcal{G})$, hence every $f \in C_c(\mathcal{G})$ is analytic for $\alpha^c$. We will also need to know that $\beta$-KMS weights for $\alpha^c$ will be finite on $C_c(\mathcal{G})$. We state this result by Christensen without proof.

**Proposition 3.5.6** ([Chr23] Proposition 6.1)**.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid, $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R}$. If $\psi$ is a $\beta$-KMS weight on $C^*(\mathcal{G})$ for $\alpha^c$, then $C_c(\mathcal{G}) \subset C^*(\mathcal{G})^{\psi}$.* $\square$

Now we turn our attention towards studying measures on the unit space of a groupoid.

**Definition 3.5.7** ([Ren80] Definition 3.1)**.**
Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and $\mu$ be a regular Borel measure on the unit space $\mathcal{G}^{(0)}$. Denote by $\mu_r = r^*\mu$ and $\mu_s = s^*\mu$ the pullbacks of $\mu$ along $r$ and $s$. $\heartsuit$

When the measures $\mu_r$ and $\mu_s$ were first defined, they were defined as in Proposition 2.0.15, using Riesz representation Theorem. We will also be using the definition of the pullback as in Proposition 2.0.12, hence the fact that these are equal (by Proposition 2.0.16) is extremely valuable.

We will be interested in when these measures are equivalent, i.e. they are absolutely continuous with respect to each other. In this case we can compute their respective Radon-Nikodym derivatives.

**Definition 3.5.8** ([Ren80] Definition 3.2 and Definition 3.4)**.**
Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid, $\mu$ a regular Borel measure on the unit space $\mathcal{G}^{(0)}$ and $c : \mathcal{G} \to \mathbb{R}$ be a Borel function. We say that $\mu$ is *quasi-invariant with Radon-Nikodym cocycle $c$* if $\mu_r$ and $\mu_s$ are equivalent and $\frac{d\mu_r}{d\mu_s} = c$. We denote the set of all quasi-invariant measures with Radon-Nikodym cocycle $c$ by $\Delta(c)$. $\heartsuit$

Note that in the above definition we may assume that $c$ is positive everywhere since $\mu_r$ and $\mu_s$ are equivalent. Then $\frac{d\mu_s}{d\mu_r} = 1/c$.

Now we can show that every $\beta$-KMS weight for $\alpha^c$ gives rise to a quasi-invariant measure with Radon-Nikodym cocycle $e^{-\beta c}$. Thanks to Proposition 3.5.6, Renault's proof of these results generalizes without any problems, c.f. [Ren80] Proposition 5.4.

**Proposition 3.5.9.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid, $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and $\beta \in \mathbb{R} \setminus \{0\}$. If $\psi$ is a $\beta$-KMS weight on $C^*(\mathcal{G})$ for $\alpha^c$ then $\psi$ restricts to a quasi-invariant measure $\mu \in \Delta(e^{-\beta c})$ on the unit space $\mathcal{G}^{(0)}$.*

*Proof.*
By Proposition 3.5.6 we have that $\psi|_{C_c(\mathcal{G})}$ is bounded, in particular $\psi|_{C_c(\mathcal{G}^{(0)})}$ is bounded. Thus, by Riesz representation Theorem, c.f. Theorem 2.0.13, we get a regular Borel measure $\mu$ on the unit space $\mathcal{G}^{(0)}$ such that

$$\psi(h) = \int_{\mathcal{G}^{(0)}} h d\mu$$

for every $h \in C_c(\mathcal{G}^{(0)})$. Note that this requires that the convolution product on $C_c(\mathcal{G}^{(0)})$ reduces to the point-wise product. Now we need to check that $\mu \in \Delta(e^{-\beta c})$.

Let $f \in C_c(\mathcal{G})$ and $u \in \mathcal{G}^{(0)}$. By equation (3.2) we get that

$$(f^* * f)(u) = \sum_{y \in \mathcal{G}_u} f^*(y^{-1}) f(y) = \sum_{y \in \mathcal{G}_u} |f(y)|^2.$$

Since $f$ is analytic for $\alpha^c$ we also get by equation (3.1) that

$$\begin{aligned}(\alpha^c_{-\beta i/2}(f) * \alpha^c_{-\beta i/2}(f)^*)(u) &= \sum_{y \in \mathcal{G}^u} e^{\beta/2c(y)} f(y) \, (e^{\beta/2c} f)^*(y^{-1}) \\ &= \sum_{y \in \mathcal{G}^u} e^{\beta/2c(y)} f(y) \, e^{\beta/2c(y)} f^*(y^{-1}) \\ &= \sum_{y \in \mathcal{G}^u} e^{\beta c(y)} |f(y)|^2.\end{aligned}$$

By Lemma 2.0.14 we get that the maps

$$\begin{aligned} u &\mapsto \sum_{y \in \mathcal{G}_u} |f(y)|^2 \\ u &\mapsto \sum_{y \in \mathcal{G}^u} e^{\beta c(y)} |f(y)|^2 \end{aligned}$$

are in $C_c(\mathcal{G}^{(0)})$ for every $f \in C_c(\mathcal{G})$. So we get, by using the definition of pullback in Proposition 2.0.15, that

$$\begin{aligned}\int_{\mathcal{G}} |f|^2 d\mu_s &= \int_{\mathcal{G}^{(0)}} \sum_{y \in \mathcal{G}_u} |f(y)|^2 d\mu(u) \\ &= \psi(f^* * f) \\ &= \psi(\alpha^c_{-\beta i/2}(f) * \alpha^c_{-\beta i/2}(f)^*) \\ &= \int_{\mathcal{G}^{(0)}} \sum_{y \in \mathcal{G}_u} e^{\beta c(y)} |f(y)|^2 d\mu(u) \\ &= \int_{\mathcal{G}} e^{\beta c} |f|^2 d\mu_r\end{aligned}$$

for every $f \in C_c(\mathcal{G})$. In particular, if $f \in C_c(\mathcal{G})$ is positive we have that there exists a unique positive function $g \in C_c(\mathcal{G})$ such that $g^2 = f$. Thus,

$$\int_{\mathcal{G}} f d\mu_s = \int_{\mathcal{G}} |g|^2 d\mu_s = \int_{\mathcal{G}} e^{\beta c}|g|^2 d\mu_r = \int_{\mathcal{G}} e^{\beta c} f d\mu_r.$$

So $\mu \in \Delta(e^{-\beta c})$. □

**Definition 3.5.10** ([Ren80] Definition 3.5)**.**
Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. We say that a subset $B \subset \mathcal{G}^{(0)}$ is *invariant* if $r(s^{-1}(B)) = B$.

Let $c : \mathcal{G} \to \mathbb{R}$ be a Borel function. We say that $\mu \in \Delta(c)$ is *extremal* when for any $\mu_1, \mu_2 \in \Delta(c)$ that satisfies $\mu = t\mu_1 + (1-t)\mu_2$ for $0 < t < 1$, we have that $\mu_1 = \mu = \mu_2$.

We say that a Borel measure $\nu$ on $\mathcal{G}^{(0)}$ is *ergodic* if for any $B \subset \mathcal{G}^{(0)}$, that is invariant, we have that

$$\nu(B) = 0 \text{ or } \nu(\mathcal{G}^{(0)} \setminus B) = 0.$$ ♡

We state the next theorem without proof.

**Theorem 3.5.11** ([Chr23] Theorem 5.5)**.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid and $c : \mathcal{G} \to \mathbb{R}$ be a positive Borel function. A measure $\mu \in \Delta(c)$ is extremal if and only if it is ergodic.* □

These measure-theoretic definitions and results lay the foundations for the dynamics of the groupoid. Christensen is able to use this to give a description of the $\beta$-KMS weights of groupoids with the additional structure of being *injectively graded* by an abelian group.

**Definition 3.5.12** ([Chr23] Definition 7.1)**.**
Let $A$ be a discrete countable abelian group and $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. We say that $\mathcal{G}$ is *injectively graded by $A$* if there is a continuous groupoid homomorphism $\Phi : \mathcal{G} \to A$ satisfying $\ker \Phi \cap \mathcal{G}_x^x = \{x\}$ for all $x \in \mathcal{G}^{(0)}$. This is equivalent to $\Phi$ being injective on all isotropy subgroups of $\mathcal{G}$. ♡

In most cases the group $A$ in the above definition will be the integers $\mathbb{Z}$.

Now we have defined everything we need to state Christensen's main theorems.

**Theorem 3.5.13** ([Chr23] Theorem 7.3 (1) and (2))**.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid injectively graded by a discrete countable abelian group $A$ via a map $\Phi : \mathcal{G} \to A$. Let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism and let $\beta \in \mathbb{R}$. If $\mu \in \Delta(e^{-\beta c}) \setminus \{0\}$ is ergodic then:*

*(1) The subset*

$$X(C) = \{x \in \mathcal{G}^{(0)} \mid \Phi(\mathcal{G}_x^x) = C\}$$

*is Borel and invariant for each subgroup $C \subset A$.*

*(2) There exists a unique subgroup $B \subset A$ with $\mu(\mathcal{G}^{(0)} \setminus X(B)) = 0$.* □

**Theorem 3.5.14** ([Chr23] Theorem 7.4)**.**
*Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid injectively graded by a discrete countable abelian group $A$ via a map $\Phi : \mathcal{G} \to A$. Let $c : \mathcal{G} \to \mathbb{R}$ be a continuous groupoid homomorphism, $\beta \in \mathbb{R} \setminus \{0\}$ and $\mu \in \Delta(e^{-\beta c}) \setminus \{0\}$ be ergodic. Denote by $B$ the unique subgroup of $A$ associated to $\mu$ given by Theorem 3.5.13 (2). There exists an affine bijection from the state-space of the group $C^*$-algebra $C^*(B)$ to the $\beta$-KMS weights for*

*$\alpha^c$ on $C^*(\mathcal{G})$ that restricts to the functional on $C_c(\mathcal{G}^{(0)})$ corresponding to $\mu$. A state $\varphi$ maps to the $\beta$-KMS weight $\psi_\varphi$ given by*

$$\psi_\varphi(f) = \int_{X(B)} \sum_{g \in \mathcal{G}_x^x} f(g)\, \varphi(u_{\Phi(g)})\, d\mu(x)$$

*for $f \in C_c(\mathcal{G})$.* □

Note that since the subgroup $B$ is a discrete abelian group we have that its $C^*$-algebra is isomorphic to $C(\hat{B})$ where $\hat{B}$ is the Pontryagin dual to $B$. The state space of $C^*(B)$ is then given by the probability measures on $\hat{B}$. So given a second-countable locally compact Hausdorff étale groupoid $\mathcal{G}$ injectively graded by a discrete countable abelian group $A$ via a map $\Phi : \mathcal{G} \to A$, a continuous groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$, we get a $\beta$-KMS weight by the following procedure:

(1) Choose $\mu \in \Delta(e^{-\beta c}) \setminus \{0\}$ ergodic and let $B$ be the unique subgroup of $A$ associated to $\mu$.

(2) Choose $\nu$ a probability measure on $\hat{B}$.

Then we get a $\beta$-KMS weight for $\alpha^c$ by the equation

$$\psi_{\mu,\nu}(f) = \int_{X(B)} \sum_{g \in \mathcal{G}_x^x} f(g) \left( \int_{\hat{B}} \chi(\Phi(g)) d\nu(\chi) \right) d\mu(x) \tag{3.3}$$

where $f \in C_c(\mathcal{G})$.

The $\beta = 0$ case is special. If we have a 0-KMS weight $\psi$, point (2) in Definition 3.5.4 says that $\psi(a^*a) = \psi(aa^*)$ for all $a \in D(\alpha_0)$. This is precisely what it means to be tracial. One could define a *tracial weight* to be a 0-KMS weight, but it is more useful to think of tracial weights as $\beta$-KMS weights for the trivial dynamics.

**Definition 3.5.15.**
Let $\mathcal{A}$ be a $C^*$-algebra and $\psi$ be a proper weight on $\mathcal{A}$. We say that $\psi$ is *tracial* if it satisfies $\psi(a^*a) = \psi(aa^*)$ for all $a \in \mathcal{A}$. ♡

If we now replace the groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ in Theorem 3.5.14 with the identity homomorphism we get that this theorem gives a description of the tracial weights. Note that we will in this case be interested in the ergodic measures $\mu \in \Delta(1)$.

# Chapter 4

# Topological graphs and a groupoid model for their $C^*$-algebras

In this chapter we define locally compact Hausdorff graphs $E = (E^0, E^1, r, s)$ and the associated boundary path space $\partial E$. We construct a locally compact Hausdorff topology on the boundary path space and use this to construct a locally compact Hausdorff étale groupoid $\mathcal{G}_E$. We further give a description of the dynamics of the graph $C^*$-algebra in terms of the dynamics on the boundary path space and restate Theorem 3.5.14 in light of these new descriptions.

## 4.1 Topological graphs

The definition of a topological graph is inspired by directed graphs $(E^0, E^1, r, s)$ where $E^0$ is the discrete (usually at-most countable) set of vertices, and $E^1$ is the discrete (usually at-most countable) set of edges, and $r, s : E^1 \to E^0$ are functions defining the direction of each edge. One usually calls $r$ the range map and $s$ the source map. Making this topological, one usually generalizes and say that $E^0$ and $E^1$ are topological spaces where now $r$ and $s$ are continuous. However, for our purposes this is too general. We therefore make the following definition.

**Definition 4.1.1.**
A *topological graph* $E$ is a quadruple $(E^0, E^1, r, s)$ where $E^0$ and $E^1$ are locally compact Hausdorff spaces, $s : E^1 \to E^0$ is a local homeomorphism, called the *source map*, and $r : E^1 \to E^0$ is a continuous map called the *range map*. We call $E^0$ and $E^1$ the *vertex* space and *edge* space respectively. In the case where both $E^0$ and $E^1$ are second-countable we will say that the topological graph $E$ is second-countable. ♡

Note that we now have range and source maps for both graphs and groupoids. We hope it will be clear from context which objects we are working with.

We now want to describe paths in the graph. There are multiple conventions used for describing a path, we will use the one that is used by Christensen in his paper [Chr22].

**Definition 4.1.2.**
Let $E$ be a topological graph. For edges $\{e_1, \ldots, e_n\} \subset E^1$ satisfying $s(e_k) = r(e_{k+1})$ we define a path of length $n$ to be the composition

$$e = e_1 e_2 \cdots e_n,$$

and we write $|e|$ for the length of $e$. We denote by $E^n$ the set of all paths of length $n$, we may consider it as a subspace of $\prod_{k=1}^n E^1$ giving it the structure of a locally compact Hausdorff space. We define the *finite path space* $E^*$ by

$$E^* = \bigsqcup_{n=0}^{\infty} E^n.$$

Similarly, we define the *infinite path space* $E^\infty$ as the set of all infinite compositions $e_1 e_2 \cdots$.

It is possible to extend the range and source maps to $E^*$ by setting

$$r(e) = r(e_1), \; s(e) = s(e_{|e|}).$$

For $v \in E^0$ we set $r(v) = v = s(v)$. It is also possible to extend the range map to $E^\infty$ by the same equation as above. ♡

We will generally use the letters $u$, $v$ and $w$ to denote vertices, the letters $e$, $f$ and $g$ to denote edges and $a$, $b$ and $c$ to denote paths. We would very much like to talk about paths without having to constantly refer to its length, we therefore make the following definition.

**Definition 4.1.3.**
Let $E$ be a topological graph and $a \in E^* \sqcup E^\infty$. Write $a = a_1 a_2 \cdots a_{|a|}$. For $k, n \in \mathbb{N}$ such that $k \leq n \leq |a|$ we will write

$$\begin{aligned} a(n) &= a_1 a_2 \cdots a_n, \\ a(k,n) &= a_k a_{k+1} \cdots a_n, \\ a(0) &= r(a). \end{aligned}$$

If we want to refer to a specific edge in the path we will simply use the subscript notation $a_n$ to denote the $n$-th edge of $a$. ♡

**Lemma 4.1.4.**
*Let $E$ be a topological graph and $a \in E^* \sqcup E^\infty$. Then the map $a \mapsto a(n)$ is continuous for every $n \leq |a|$.*

*Proof.*
If $a \in E^*$, suppose $k \geq n$ and let

$$p_n^k : \prod_{i=1}^{k} E^1 \to \prod_{i=1}^{n} E^1$$

be the projection onto the first $n$ entries. By universality of the product topology, $p_n^k$ is continuous and restricts to a continuous map

$$q_n^k : E^k \to E^n.$$

Then the map $a \mapsto a(n)$ is simply

$$\bigsqcup_{k=n}^{\infty} q_n^k : \bigsqcup_{k=n}^{\infty} E^k \to E^n,$$

which is continuous by universality of the disjoint union topology.

If $a \in E^\infty$ the map $a \mapsto a(n)$ is simply the restriction of the projection

$$p_n : \prod_{i=1}^{\infty} E^1 \to \prod_{i=1}^{n} E^1$$

to $E^\infty$, which is continuous by universality of the product topology. □

Next we want to introduce the *regular* vertices. We want to capture the vertices that have edges pointing to them, but not too many edges. In the discrete graph case this is defined by saying that the inverse image $r^{-1}(v)$ of a vertex $v$ is finite and non-empty. For the general case we have to be a bit more careful.

**Definition 4.1.5.**
Let $E$ be a topological graph and $v \in E^0$ be a vertex. We say that $v$ is *regular* if there exists a relatively compact open neighborhood $U \subset E^0$ about $v$ such that the inverse image $r^{-1}(\overline{U}) \subset E^1$ is compact, and $r(r^{-1}(U)) = U$. We denote the set of regular vertices as $E^0_{\text{reg}}$.

A vertex $v$ is *singular* if it is not regular, and we denote the set of singular vertices as $E^0_{\text{sng}} = E^0 \setminus E^0_{\text{reg}}$.

Finally, we define a finite path to be *singular* if the source of the path is a singular vertex. We denote the set of singular finite paths as $E^*_{\text{sng}} = E^* \cap s^{-1}(E^0_{\text{sng}})$. ♡

**Lemma 4.1.6.**
*Let $E$ be a topological graph. Then the set of regular vertices is open.*

*Proof.*
If $E^0_{\text{reg}}$ is empty it is open, so assume it is not empty. Then for $v \in E^0_{\text{reg}}$ there exists an open neighborhood $U_v \subset E^0$ about $v$ that satisfies all the criteria making $v$ regular. For any $w \in U_v$ we have that $U_v$ is an open neighborhood about $w$ making $w$ a regular vertex. Hence, we may write

$$E^0_{\text{reg}} = \bigcup_{v \in E^0_{\text{reg}}} U_v$$

showing that $E^0_{\text{reg}}$ is open. □

Note that topological graphs gives a general setting for quite a lot of different structures. If we consider a topological graph with the discrete topology we simply get a directed graph. Another nice example is the following.

**Example 4.1.7.**
Let $Z$ be a locally compact Hausdorff space and $h : Z \to Z$ be a local homeomorphism. We get a topological graph by setting $E^0 = E^1 = Z$, the range map $r = id_Z$ and the source map $s = h$. ◇

With this example in mind a lot of the intuition one would hope to have about graphs disappear. It would be tempting to use a lot of the theory for countable directed graphs and try to generalize them to the topological case, but this example might display why one needs to be extra careful when working with topological graphs. Unfortunately this means that the proofs we present in this chapter are quite detail oriented, making them quite cumbersome to read.

## 4.2 The boundary path space of a topological graph

The boundary path space $\partial E$ was first shown to have a natural locally compact Hausdorff topology by Yeend, c.f. [Yee07]. In this section we give a proof of the same fact following the outline of a proof given by Schafhauser c.f. [Sch18] Proposition 3.2.

**Definition 4.2.1.**
Let $E$ be a topological graph, we define the *boundary path space* to be the disjoint union

$$\partial E = E^*_{\text{sng}} \sqcup E^\infty. \qquad \heartsuit$$

**Proposition 4.2.2.**
*Let $E$ be a topological graph. For subsets $S \subset E^*$ we define*

$$Z(S) = \{a \in \partial E \mid a(n) \in S \text{ for some } n \in \mathbb{N} \text{ with } n \leq |a|\}.$$

*Then the collection*

$$\mathcal{B} = \{Z(U) \setminus Z(K) \mid U \text{ is open in } E^*,\ K \text{ is compact in } E^*\}$$

*form a base for a locally compact Hausdorff topology on $\partial E$.*

To prove that this gives a locally compact topology turns out to be extremely complicated due to some subtleties regarding the singular paths. If we, for a moment, grant the fact that $\mathcal{B}$ is a base for a topology on $\partial E$ we can prove the following useful fact. Note that Schafhauser's proof of this is done making some simplifications which we felt weren't justified.

**Lemma 4.2.3.**
*Let $E$ be a topological graph and $K \subset E^*$ be compact. Then the set $Z(K)$ is compact in $\partial E$.*

*Proof.*
Let $K \subset E^*$ be compact and $(a^\lambda)_{\lambda\in\Lambda} \subset Z(K)$ be a universal net. We want to construct a path in $Z(K)$ that we can show is the limit point of $(a^\lambda)_{\lambda\in\Lambda}$.

The first reduction we make is to establish that $(a^\lambda)_{\lambda\in\Lambda}$ has a subnet consisting of boundary paths that are extensions of paths in $K$ of a specific length $k$. To do this we use the fact that $K$ is compact, so we have that the set

$$N = \{k \in \mathbb{N} \mid K \cap E^k \neq \varnothing\}$$

is finite. For each index $\lambda \in \Lambda$ let $k_\lambda \in N$ be the largest integer such that $a^\lambda(k_\lambda) \in K$. Thus, the net $(k_\lambda)_{\lambda\in\Lambda}$ is a universal net in $N$, so it is eventually constant, and we let $k$ be that constant. By passing to a subnet we may assume that $a^\lambda(k) \in K$ for every index $\lambda \in \Lambda$. Then $(a^\lambda(k))_{\lambda\in\Lambda}$ is a universal net in the compact set $K$ thus it converges to some finite path $a \in K$ of length $k$.
*The first induction argument*:
If $s(a) \in E^0_{\text{sng}}$ we are done since this implies that $a \in Z(K)$. It might still be the case that $a$ is too short and won't be a limit point of $(a^\lambda)_{\lambda\in\Lambda}$. But since $s(a) \in E^0_{\text{sng}}$ there is no obvious way for us to extend $a$ yet.

If $s(a) \in E^0_{\text{reg}}$ we have that there exists a relatively compact open neighborhood $U \subset E^0_{\text{reg}}$ about $s(a)$ such that $r^{-1}(\overline{U}) \subset E^1$ is compact and $r(r^{-1}(U)) = U$. By continuity of the source map we have that the net $(s(a^\lambda(k)))_{\lambda\in\Lambda}$ converges to $s(a)$, so we have that $(s(a^\lambda(k)))_{\lambda\in\Lambda}$ is eventually in $U$. By passing to a subnet we may assume that $s(a^\lambda(k)) \in U$ for all indices $\lambda \in \Lambda$. Since $U \subset E^0_{\mathrm{reg}}$ we have that $a^\lambda(k) \notin \partial E$. In particular, we have that $|a^\lambda| > k$ for all indices $\lambda \in \Lambda$. Let $a^\lambda_{k+1}$ be the $(k+1)$th edge of $a^\lambda$. Then $a^\lambda_{k+1} \in r^{-1}(U) \subset r^{-1}(\overline{U})$ and $(a^\lambda_{k+1})_{\lambda\in\Lambda}$ is a universal net in the compact set $r^{-1}(\overline{U})$, so it converges to an edge $a_{k+1} \in r^{-1}(\overline{U})$. By continuity of the range and source maps, and since $E^0$ is Hausdorff, we have that

$$r(a_{k+1}) = \lim r(a^\lambda_{k+1}) = \lim s(a^\lambda_k) = \lim s(a^\lambda(k)) = s(a).$$

Thus, the composition $a' = aa_{k+1} \in E^{k+1}$ is defined and the net $(a^\lambda(k+1))_{\lambda\in\Lambda}$ converges to $a'$ by universality of the product topology. We may continue this argument inductively to construct a path $\tilde{a} \in Z(K)$, either of infinite length or of finite length with a singular source. This completes the first induction argument.

With this in place we may assume that we have a universal net $(a^\lambda)_{\lambda\in\Lambda} \subset Z(K)$ and a path $a \in Z(K)$ such that for any $n \in \mathbb{N}$ with $n \le |a|$, the net $(a^\lambda(n))_{\lambda\in\Lambda}$ converges to $a(n)$. Furthermore, there exists a $k \in \mathbb{N}$ with $k \le |a|$ such that $a^\lambda(k) \in K$ for all indices $\lambda \in \Lambda$. In the case where $a$ has finite length it might turn out that $a$ is not a limit point of the net $(a^\lambda)_{\lambda\in\Lambda}$. We will in this case show that we can extend $a$ in another inductive argument to guarantee ourselves a limit point of the net $(a^\lambda)_{\lambda\in\Lambda}$.

*The second induction argument*:
If $(a^\lambda)_{\lambda\in\Lambda}$ converges to $a$ we are done, since this is what we want to achieve. So assume that $(a^\lambda)_{\lambda\in\Lambda}$ doesn't converge to $a$. Then there exists an open subset $U \subset E^*$ and a compact set $L \subset E^*$ such that $a \in Z(U) \setminus Z(L)$ and $(a^\lambda)_{\lambda\in\Lambda}$ is *not* eventually in this basic open set in $\partial E$. Since $a \in Z(U)$ there exists $n \in \mathbb{N}$ such that $a(n) \in U$. By construction of $a$ we have that $(a^\lambda(n))_{\lambda\in\Lambda}$ converges to $a(n)$, so $(a^\lambda(n))_{\lambda\in\Lambda}$ is eventually in $U$. By passing to a subnet we may assume that $a^\lambda(n) \in U$ for all indices $\lambda \in \Lambda$, so $a^\lambda \in Z(U)$ for all indices $\lambda \in \Lambda$. Since we are assuming that $(a^\lambda)_{\lambda\in\Lambda}$ is *not* eventually in $Z(U) \setminus Z(L)$ we get that $(a^\lambda)_{\lambda\in\Lambda}$ must eventually be in $Z(L)$. By passing to a subnet we may therefore assume that $a^\lambda \in Z(L)$ for all indices $\lambda \in \Lambda$.

Since $L$ is compact we have that the set

$$M = \{l \in \mathbb{N} \mid L \cap E^l \neq \varnothing\}$$

is finite. For each index $\lambda \in \Lambda$ let $l_\lambda \in M$ be the largest integer such that $a^\lambda(l_\lambda) \in L$. Thus, the net $(l_\lambda)_{\lambda\in\Lambda}$ is a universal net in $M$, so it is eventually constant. Let $l$ be that constant. For our purposes we may assume that $L \cap E^n = \varnothing$ for all $n > l$ since any $a^\lambda(n)$ will in this case not be in $L$ by maximality of $l$. So we may assume that $l = \max M$.

We want to argue that $l > |a|$. If $l \le |a|$ we have that the map $a \mapsto a(m)$ is defined for all $m \in M$ and $a(m) \notin L$ since $a \in Z(U) \setminus Z(L)$. For $m \in M$ we have that $E^m$ is Hausdorff and $L$ is compact, hence the set $E^m \setminus L$ is an open neighborhood about $a(m)$. Since the net $(a^\lambda(m))_{\lambda\in\Lambda}$ converges to $a(m)$ we have that $(a^\lambda(m))_{\lambda\in\Lambda}$ is eventually in $E^m \setminus L$ for all $m \in M$, in particular for $m = l$. This contradicts the fact that $(a^\lambda)_{\lambda\in\Lambda}$ is eventually in $Z(L)$. Hence, $l > |a|$.

By compactness of $L$ we have that the universal net $(a^\lambda(l))_{\lambda\in\Lambda}$ converges to a path $b' \in L$ of length $l$. By the first induction argument applied to $b'$ we get a path $b \in Z(L)$. We claim that $b$ is an extension of $a$. Indeed, by the properties of $a$ and $b$ we have for every $n \in \mathbb{N}$ with $n \le |a|$ that the net $(a^\lambda(n))_{\lambda\in\Lambda}$ converges to both $a(n)$ and $b(n)$. Since $E^n$ is Hausdorff we get that $a(n) = b(n)$, in particular we have that $b(|a|) = a(|a|) = a$. And finally since $|a| < l$ and $l \le |b|$ we get that $b$ is a true extension of $a$. So even though $s(a) \in E^0_{\mathrm{sng}}$ we have now managed to extend $a$. Furthermore, since $b(k) = a(k) \in K$

we have that $b \in Z(K)$, so we may continue this argument inductively until we have a boundary path in $Z(K)$ that is the limit point of $(a^\lambda)_{\lambda\in\Lambda}$. This completes the second induction argument and proves that for any $K \subset E^*$ that is compact, we have that $Z(K)$ is compact. □

*Proof of Proposition 4.2.2.*
We first show that $\mathcal{B}$ is a base.
*$\partial E$ is covered by basic opens*:
This is achieved by noticing that $E^0$ is open in $E^*$ and the empty set is compact, so we may write $\partial E = Z(E^0) \setminus Z(\varnothing)$.
*The intersection of two basic opens is precisely covered by basic opens*:
Fix open subsets $U, V \subset E^*$ and compact subsets $K, L \subset E^*$ and let

$$a \in (Z(U) \setminus Z(K)) \cap (Z(V) \setminus Z(L)) = (Z(U) \cap Z(V)) \setminus (Z(K) \cup Z(L)).$$

Then there exists $k, l \in \mathbb{N}$ such that $a(k) \in U$ and $a(l) \in V$. Without loss of generality we may assume that $k \leq l$. Set

$$W = \{b \in V \mid |b| \geq k \text{ and } b(k) \in U\}.$$

Notice that we can write

$$W = (V \cap (\bigsqcup_{n=k}^{\infty} E^n)) \cap (k)^{-1}(U \cap E^k),$$

where $(k)^{-1}$ denotes the inverse image of the map $b \mapsto b(k)$. By Lemma 4.1.4 this map is continuous, hence $W$ is open in $E^*$. Clearly $a \in Z(W)$ and $Z(W) \subset Z(U) \cap Z(V)$. Furthermore, $K \cup L$ is compact and $Z(K) \cup Z(L) = Z(K \cup L)$, so

$$a \in Z(W) \setminus Z(K \cup L) \subset (Z(U) \setminus Z(K)) \cap (Z(V) \setminus Z(L)).$$

Hence, $\mathcal{B}$ is a base.
*$\partial E$ is Hausdorff*:
Let $a, b \in \partial E$ with $a \neq b$. Let $|a| = k$ and $|b| = l$ with $k$ and $l$ possibly infinite.

We first consider the case when $k = l$. Then there exists $m \in \mathbb{N}$ such that $a(m) \neq b(m)$. Note that for finite paths, $m$ may be chosen to be equal to $k$ and $l$. Since $E^m$ is Hausdorff we may find disjoint open neighborhoods $U, V \subset E^m$ about $a(m)$ and $b(m)$ respectively. By definition, $U$ and $V$ are also open in $E^*$, and we have that $Z(U) \cap Z(V) = \varnothing$. Clearly $a \in Z(U)$ and $b \in Z(V)$.

For the case with $k < l$ and $a(k) \neq b(k)$ we may choose disjoint open neighborhoods as we did in the case when $k = l$.

Finally, when $k < l$ and $a(k) = b(k)$ we can do as follows: Since $E^{k+1}$ is locally compact we may choose a relatively compact open neighborhood $U \subset E^{k+1}$ about $b(k+1)$. Then $Z(U)$ and $\partial E \setminus Z(\overline{U})$ are disjoint open neighborhoods about $b$ and $a$ respectively. Hence, $\partial E$ is Hausdorff.
*$\partial E$ is locally compact*:
Let $a \in \partial E$. Choose a relatively compact open neighborhood $U \subset E^0$ about $r(a)$, then $a \in Z(U) \subset Z(\overline{U})$ where $Z(\overline{U})$ is compact by Lemma 4.2.3. □

## 4.3 Deaconu-Renault groupoids of topological graphs

With the work in the previous sections we have constructed the boundary path space of a topological graph and endowed it with a suitable topology. Now we will show that there exists a natural action from $\mathbb{N}$ on the boundary path space allowing us to form the associated Deaconu-Renault groupoid.

**Definition 4.3.1.**
Let $E$ be a topological graph and let $a \in \partial E \setminus E^0$. We define a map $\sigma : \partial E \setminus E^0 \to \partial E$ by

$$\sigma(a) = a(2, |a|), \text{ if } |a| \geq 2,$$
$$\sigma(a) = s(a), \text{ if } |a| = 1.$$

We call $\sigma$ the *backwards shift map.* ♡

**Proposition 4.3.2** ([Sch18] proposition 3.5)**.**
*Let $E$ be a topological graph. Then the backwards shift map $\sigma : \partial E \setminus E^0 \to \partial E$ is a local homeomorphism.*

*Proof.*
We want to show that $\sigma$ is open, continuous and locally injective, proving that $\sigma$ is a local homeomorphism. We start with proving that $\sigma$ is open. To do this we want to relate the backwards shift of boundary paths with the backwards shift of finite paths. We therefore define maps for $n > 1$: $\sigma_n : E^n \to E^{n-1}$ by $\sigma_n(a) = a(2, n)$. Since we consider $E^n$ as a subspace of the product space $\prod_{k=1}^{n} E^1$, we have that $\sigma_n$ is the projection onto the last $n-1$ coordinates showing that $\sigma_n$ is continuous. For $n = 1$ we define $\sigma_1 = s$, the source map. Then we define the map

$$\sigma_* = \bigsqcup_{n=1}^{\infty} \sigma_n : E^* \setminus E^0 \to E^*.$$

The fact that $\sigma$ is open will now follow from the equality

$$\sigma((Z(U) \setminus Z(K)) \setminus E^0) = Z(\sigma_*(U \setminus E^0)) \setminus Z(\sigma_*(K \setminus E^0)),$$

where $U \subset E^*$ is an open subset and $K \subset E^*$ is compact. If we can prove that $\sigma_*$ is open and continuous we get that $\sigma_*(U \setminus E^0)$ is open in $E^*$ and $\sigma_*(K \setminus E^0)$ is compact in $E^*$, showing that $\sigma((Z(U) \setminus Z(K)) \setminus E^0)$ is indeed open.

To prove that $\sigma_*$ is continuous and open we first notice that by universality of the disjoint union topology it is enough to show that $\sigma_n$ is continuous and open for all $n \geq 1$. We have already stated that continuity follows by universality of the product topology.

We now show that $\sigma_n$ is open for each $n \geq 1$. For $n = 1$ we have that $\sigma_1$ is the source map, which is a local homeomorphism, hence it is open. Fix $n \geq 2$, an open subset $U \subset E^n$ and $a \in U$. We want to show that $\sigma_n(U) \subset E^{n-1}$ is an open subset. Let $(b^\lambda)_{\lambda \in \Lambda}$ be a net in $E^{n-1}$ that converges to $\sigma_n(a)$. Let $V \subset E^1$ be an open neighborhood about $a_1$ such that the restricted source map $s|_V$ is injective. By continuity of the range map we have that

$$\lim r(b^\lambda) = r(\sigma_n(a)) = s(a_1),$$

which is in $s(V)$, so we have that the net $(r(b^\lambda))_{\lambda \in \Lambda}$ is eventually in $s(V)$. By passing to a subnet we may assume that $r(b^\lambda) \in s(V)$ for all indices $\lambda \in \Lambda$. Injectivity of $s|_V$ gives that for each index $\lambda \in \Lambda$ there exists a unique edge $e^\lambda \in V$ such that $s(e^\lambda) = r(b^\lambda)$. Furthermore, we have that the net $(e^\lambda)_{\lambda \in \Lambda}$ converges to $a_1$ since $(s(e^\lambda))_{\lambda \in \Lambda}$ converges to

$s(a_1)$. Then the net $(e^\lambda b^\lambda)_{\lambda\in\Lambda}$ converges to $a$ in $U$, hence this net is eventually in $U$. As a consequence we get that $(b^\lambda)_{\lambda\in\Lambda} = (\sigma_n(e^\lambda b^\lambda))_{\lambda\in\Lambda}$ is eventually in $\sigma_n(U)$. Thus, $\sigma_n(U)$ is open in $E^{n-1}$ proving that $\sigma_n$ is open. This proves that $\sigma$ is open.

To prove that $\sigma$ is continuous let $a \in \partial E \setminus E^0$ and $(a^\lambda)_{\lambda\in\Lambda}$ be a net in $\partial E \setminus E^0$ that converges to $a$. Let $U \subset E^*$ be an open subset and $K \subset E^*$ be compact such that $\sigma(a) \in Z(U) \setminus Z(K)$. Then there exists an $m \in \mathbb{N}$ such that $\sigma_*(a(m)) \in U$. In particular this means that $|a| \geq m$, so $Z(E^m)$ is an open neighborhood about $a$. Hence, the net $(a^\lambda)_{\lambda\in\Lambda}$ is eventually in $Z(E^m)$, thus $|a^\lambda| \geq m$ for large enough indices $\lambda$. By Lemma 4.1.4 the map $a \mapsto a(m)$ is continuous, so by continuity of $\sigma_*$ we get that the net $(\sigma_*(a^\lambda(m)))_{\lambda\in\Lambda}$ converges to $\sigma_*(a(m))$, hence this net is eventually in $U$. This shows that $(\sigma(a^\lambda))_{\lambda\in\Lambda}$ is eventually in $Z(U)$. One similarly proves that $(\sigma(a^\lambda))_{\lambda\in\Lambda}$ is eventually *not* in $Z(K)$, hence $(\sigma(a^\lambda))_{\lambda\in\Lambda}$ is eventually in $Z(U) \setminus Z(K)$. This shows that $(\sigma(a^\lambda))_{\lambda\in\Lambda}$ converges to $\sigma(a)$, proving that $\sigma$ is continuous.

We finally prove that $\sigma$ is locally injective. Fix $a \in \partial E \setminus E^0$ and let $V \subset E^1$ be an open neighborhood about $a_1$ such that $s|_V$ is injective. Then $Z(V)$ is an open neighborhood about $a$. Let $b, b' \in Z(V)$ with $\sigma(b) = \sigma(b')$. Then $b_1, b_1' \in V$ and

$$s(b_1) = r(\sigma(b)) = r(\sigma(b')) = s(b_1').$$

Since $s|_V$ is injective we get that $b_1 = b_1'$. Thus,

$$b = b_1\sigma(b) = b_1'\sigma(b') = b'.$$

This completes the proof. □

**Definition 4.3.3.**
Let $E$ be a topological graph. By Proposition 4.3.2 we get a monoidal action from $\mathbb{N}$ on $\partial E$ by the map $(k, a) \mapsto \sigma^k(a)$ from $\mathbb{N} \times \partial E$ to $\partial E$ where $\sigma^k = \sigma \circ \ldots \circ \sigma$ is repeated composition of $\sigma$, $k$ times. We let

$$\mathcal{G}_E = \{(a, k-l, b) \in \partial E \times \mathbb{Z} \times \partial E \mid k, l \geq 0,\ k \leq |a|,\ l \leq |b|,\ \sigma^k(a) = \sigma^l(b)\}.$$

Notice that this is the Deaconu-Renault groupoid associated to $\partial E$ with a monoidal action from $\mathbb{N}$ as a submonoid of $\mathbb{Z}$. Thanks to Example 3.2.8 we know that $\mathcal{G}_E$ is a locally compact Hausdorff étale groupoid. If we further have that $E$ is second-countable we also get that $\mathcal{G}_E$ is second-countable, so we may form the full $C^*$-algebra $C^*(\mathcal{G}_E)$ in this case. ♡

## 4.4 The dynamics of a topological graph

In this section we give a description of Christensen's theorem, Theorem 3.5.14, in the setting of the Deaconu-Renault groupoid associated to the boundary path space of a second-countable topological graph. In particular, we give a description of the invariant sets of the unit space and a description of the quasi-invariant measures on the unit space.

We start by showing that there is a natural action from the circle group on $C^*(\mathcal{G}_E)$, generally called the *gauge-action*. We will do this using Pontryagin duality, as in Definition 3.5.5.

**Definition 4.4.1.**
Let $E$ be a second-countable graph. Let $\Phi : \mathcal{G}_E \to \mathbb{Z}$ be the homomorphism given by $\Phi(a,k,b) = k$. We get a gauge-action on $C^*(\mathcal{G}_E)$ by dualizing $\Phi$ as follows: Let $z \in \mathbb{T} \subset \mathbb{C}$ and $\chi_z : \mathbb{Z} \to \mathbb{T}$ be the character associated to $z$, i.e. for $k \in \mathbb{Z}$,

$$\chi_z(k) = z^k.$$

Then we get a continuous *-homomorphism $\gamma_z : C_c(\mathcal{G}_E) \to C_c(\mathcal{G}_E)$ defined by

$$\gamma_z(f)(a,k,b) = (\chi_z \circ \Phi)(a,k,b)\, f(a,k,b) = z^k f(a,k,b).$$

By continuity this extends to a *-homomorphism $\gamma_z : C^*(\mathcal{G}_E) \to C^*(\mathcal{G}_E)$. It is clear that $\gamma_{\bar{z}}$ is an inverse to $\gamma_z$, so we get a map $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(\mathcal{G}_E))$ defined by $z \mapsto \gamma_z$.

We can use this action to get an action from $\mathbb{R}$ as well, by simply letting $\alpha^\Phi : \mathbb{R} \to \mathrm{Aut}(C^*(\mathcal{G}_E))$ be defined by $\alpha^\Phi_t = \gamma_{e^{it}}$. This describes the dynamics of the graph, so it makes sense to talk about $\beta$-KMS weights for $\alpha^\Phi$.

We say that a tracial weight $\psi$ on $C^*(\mathcal{G}_E)$ is *gauge-invariant* if it is invariant under the map $\gamma$, i.e. for all $A \in C^*(\mathcal{G}_E)_+$ and $z \in \mathbb{T}$ we have that

$$\psi(\gamma_z(A)) = \psi(A).$$

Note that every $\beta$-KMS weight for $\alpha^\Phi$ is necessarily gauge-invariant by definition. ♡

The statements in the next two lemmas can be found without proof within the text on page 6 of [Chr22] for the case $\beta = 0$. Lemma 4.4.3 is a generalization from the $\beta = 0$ case to the $\beta \neq 0$ case. A proof of the $\beta = 0$ case can be found within the proof of lemma 3.2 in [Tho14]. We will present our own proofs of these facts.

Note also that we only work with the graph groupoid in this section so all range and source maps are the ones from the groupoid $\mathcal{G}_E$.

**Lemma 4.4.2.**
*Let $E$ be a second-countable topological graph. Then $B \subset \mathcal{G}_E^{(0)} \cong \partial E$ is invariant (as in Definition 3.5.10) if and only if $\sigma^{-1}(B) = B \setminus E^0$.*

*Proof.*
Let $B \subset \partial E$ be invariant, i.e. $r(s^{-1}(B)) = B$. We will prove that $\sigma^{-1}(B) = B \setminus E^0$. Let $a \in \sigma^{-1}(B)$. In particular, we have that $(a, 1, \sigma(a)) \in s^{-1}(B)$, so

$$a = r(a,1,\sigma(a)) \in r(s^{-1}(B)) = B.$$

Hence $\sigma^{-1}(B) \subset B \setminus E^0$.

Now assume $a \in B \setminus E^0$. Then $(\sigma(a), -1, a) \in s^{-1}(B)$, so

$$\sigma(a) = r(\sigma(a), -1, a) \in r(s^{-1}(B)) = B,$$

hence $B \setminus E^0 \subset \sigma^{-1}(B)$. Thus, $\sigma^{-1}(B) = B \setminus E^0$.

For the converse assume that $\sigma^{-1}(B) = B \setminus E^0$. Clearly $B \subset r(s^{-1}(B))$ since for any $a \in B$ we have that $(a, 0, a) \in s^{-1}(B)$, hence

$$a = r(a, 0, a) \in r(s^{-1}(B)).$$

Assume that $a \in r(s^{-1}(B))$, then there exists $(a, k-l, b) \in \mathcal{G}_E$ such that $b \in B$. Since $\sigma^k(a) = \sigma^l(b)$ we get that $|b| \geq l$. In particular, we have that $b \in B \setminus E^0$, so $b \in \sigma^{-1}(B)$ implying that $\sigma(b) \in B$. Since $|b| \geq l$ we get that $|\sigma(b)| \geq l-1$, so we may repeatedly apply $\sigma$ to $b$ at least $l$ times to conclude that $\sigma^l(b) \in B$. Hence,

$$\sigma^k(a) = \sigma^l(b) \in B.$$

Thus, $\sigma^{k-1}(a) \in \sigma^{-1}(B) = B \setminus E^0$, so in particular $\sigma^{k-1}(a) \in B$. We can repeat this $k$ times to get that $a \in B$, which implies that $r(s^{-1}(B)) \subset B$. Hence, $r(s^{-1}(B)) = B$ completing the proof. □

**Lemma 4.4.3.**
*Let $E$ be a second-countable topological graph, $\beta \in \mathbb{R} \setminus \{0\}$, $\Phi : \mathcal{G}_E \to \mathbb{Z}$ be as in Definition 4.4.1 and $\nu$ be a regular Borel measure on $\partial E$. Then $\nu \in \Delta(e^{-\beta\Phi})$ if and only if $\sigma^*\nu = e^\beta \nu$ on $\partial E \setminus E^0$.*

*Proof.*
Suppose that $\nu \in \Delta(e^{-\beta\Phi})$. Let $B \subset \mathcal{G}_E$ be a Borel subset. Then we have that $B \cap \Phi^{-1}(\{k\})$ is a Borel subset for any $k \in \mathbb{Z}$ since $\Phi$ is continuous. Hence,

$$\begin{aligned}
\nu_r(B) &= \int_B e^{-\beta\Phi} d\nu_s \\
&= \sum_{k \in \mathbb{Z}} \int_{B \cap \Phi^{-1}(\{k\})} e^{-\beta\Phi} d\nu_s \\
&= \sum_{k \in \mathbb{Z}} \int_{B \cap \Phi^{-1}(\{k\})} e^{-\beta k} d\nu_s \\
&= \sum_{k \in \mathbb{Z}} e^{-\beta k} \int_{B \cap \Phi^{-1}(\{k\})} d\nu_s \\
&= \sum_{k \in \mathbb{Z}} e^{-\beta k} \nu_s(B \cap \Phi^{-1}(\{k\})).
\end{aligned}$$

So we have that $\nu \in \Delta(e^{-\beta\Phi})$ if and only if

$$r^*\nu = \sum_{k \in \mathbb{Z}} e^{-\beta k} (s^*\nu)|_{\Phi^{-1}(\{k\})}, \tag{4.1}$$

where the notation $(s^*\nu)|_{\Phi^{-1}(\{k\})}$ simply means that

$$(s^*\nu)|_{\Phi^{-1}(\{k\})}(B) = s^*\nu(B \cap \Phi^{-1}(\{k\}))$$

for Borel subsets $B \subset \mathcal{G}_E$. Note that if $W \subset \mathcal{G}_E$ is an open bisection we have that equation (4.1) takes the form

$$\nu(r(W)) = \sum_{k \in \mathbb{Z}} e^{-\beta k} \nu(s(W \cap \Phi^{-1}(\{k\}))).$$

Now consider the collection

$$\mathcal{U} = \{U \subset \partial E \setminus E^0 \mid U \text{ is open in } \partial E \text{ and } \sigma|_U \text{ is injective}\}.$$

By Proposition 4.3.2 we have that $\sigma$ is a local homeomorphism, so $\mathcal{U}$ is an open cover of $\partial E \setminus E^0$. By Proposition 2.0.11, $\mathcal{M}_{\text{reg}}$ is a sheaf on $\partial E \setminus E^0$, so it suffices to check that $\sigma^*\nu = e^\beta \nu$ on each $U \in \mathcal{U}$ to conclude that they are equal on $\partial E \setminus E^0$.

We claim that for any $U \in \mathcal{U}$ the basic open set $Z(U,1,0,\sigma(U))$ is a bisection. Indeed, any element in $Z(U,1,0,\sigma(U))$ is of the form $(a,1,\sigma(a))$ by injectivity of $\sigma|_U$. Hence, the equations

$$a = r(a,1,\sigma(a)) = r(b,1,\sigma(b)) = b,$$
$$\sigma(a) = s(a,1,\sigma(a)) = s(b,1,\sigma(b)) = \sigma(b),$$

show that $r|_{Z(U,1,0,\sigma(U))}$ and $s|_{Z(U,1,0,\sigma(U))}$ are injective.

Fix $U \in \mathcal{U}$ and let $B \subset U$ be a Borel subset. It is clear that $Z(B,1,0,\sigma(B))$ is a Borel subset of $Z(U,1,0,\sigma(U))$ and that $Z(B,1,0,\sigma(B)) \cap \Phi^{-1}(\{k\})$ is nonempty if and only if $k = 1$, and in this case $Z(B,1,0,\sigma(B)) \cap \Phi^{-1}(\{1\}) = Z(B,1,0,\sigma(B))$. Hence, by equation (4.1) we get that

$$\begin{aligned}\nu(B) &= r^*\nu(Z(B,1,0,\sigma(B)))\\ &= e^{-\beta} s^*\nu(Z(B,1,0,\sigma(B)))\\ &= e^{-\beta}\nu(\sigma(B)).\end{aligned}$$

For the converse we will show that equation (4.1) holds. By Proposition 2.0.11, $\mathcal{M}_{\text{reg}}$ is a sheaf on $\mathcal{G}_E$, so it suffices to check that equation (4.1) holds on each open subset in some suitable open cover of $\mathcal{G}_E$. The open cover we will use is the following:

$$\mathcal{V} = \{Z(U,m,n,V) \mid m,n \in \mathbb{N}, U \in \mathcal{U}_m \text{ and } V \in \mathcal{U}_n\},$$

where

$$\mathcal{U}_k = \{U \subset \partial E \setminus E^{k-1} \mid U \text{ is open in } \partial E \setminus E^{k-1} \text{ such that } \sigma^k|_U \text{ is injective}\}.$$

$\mathcal{V}$ is an open cover of $\mathcal{G}_E$: Let $(a, m-n, b) \in \mathcal{G}_E$, then $|a| \geq m$ and $|b| \geq n$, so $a \in \partial E \setminus E^{m-1}$ and $b \in \partial E \setminus E^{n-1}$. For every $k \in \mathbb{N}$ we have that $\sigma^k$ is a local homeomorphism by Proposition 4.3.2, so we have that there exists a $U \in \mathcal{U}_m$ and a $V \in \mathcal{U}_n$ such that $a \in U$ and $b \in V$. Hence, $(e, m-n, b) \in Z(U,m,n,V)$ where $Z(U,m,n,V) \in \mathcal{V}$.

Next we claim that each $Z(U,m,n,V) \in \mathcal{V}$ is a bisection. Indeed, let $(a_1, m-n, b_1), (a_2, m-n, b_2) \in Z(U,m,n,V)$ and suppose $r(a_1, m-n, b_1) = r(a_2, m-n, b_2)$. By definition of $r$ we get that $a_1 = a_2$. Then we get that

$$\sigma^n(b_1) = \sigma^m(a_1) = \sigma^m(a_2) = \sigma^n(b_2).$$

By injectivity of $\sigma^n|_V$ we get that $b_1 = b_2$, hence $r|_{Z(U,m,n,V)}$ is injective. Injectivity of $s|_{Z(U,m,n,V)}$ is similarly proven to be true.

Fix $Z(U,m,n,V) \in \mathcal{V}$ and let $B \subset Z(U,m,n,V)$ be a Borel subset. It is clear that $B \cap \Phi^{-1}(\{k\})$ is nonempty if and only if $k = m-n$, and in this case $B \cap \Phi^{-1}(\{m-n\}) = B$. Also since $r|_B$ and $s|_B$ are injective, we have that

$$B = Z(r(B), m, n, s(B)).$$

By linearity of the pullback $\sigma^*$ we have that $e^{\beta k}\nu = (\sigma^k)^*\nu$ on $\partial E \setminus E^{k-1}$ for every $k \in \mathbb{N}$, hence

$$\begin{aligned} e^{\beta m}\nu(r(B)) &= (\sigma^m)^*\nu(r(B)) \\ &= \nu(\sigma^m(r(B))) \\ &= \nu(\sigma^n(s(B))) \\ &= (\sigma^n)^*\nu(s(B)) \\ &= e^{\beta n}\nu(r(B)). \end{aligned}$$

Note that equality two and four follows by injectivity of $\sigma^m|_U$ and $\sigma^n|_V$. Hence,

$$\nu(r(B)) = e^{-\beta(m-n)}\nu(s(B)),$$

so equation (4.1) is satisfied, completing the proof. □

We now want to describe the isotropy subgroups of $\mathcal{G}_E$.

**Definition 4.4.4.**
Let $E$ be a second-countable topological graph. For $a \in \partial E$ we define the *periodicity group of $a$* to be the group

$$Per(a) = \{k - l \in \mathbb{Z} \mid k, l \in \mathbb{N}, k, l \leq |a| \text{ and } \sigma^k(a) = \sigma^l(a)\}.$$

It is clear that this is the isotropy subgroup $(\mathcal{G}_E)_a^a$. ♡

Finally, we notice that the map $\Phi : \mathcal{G}_E \to \mathbb{Z}$ defined in Definition 4.4.1 gives us an injective grading of the groupoid, since $\Phi$ clearly becomes injective when restricted to the different periodicity groups.

We are now able to rephrase Theorem 3.5.14 and equation (3.3) in terms of these periodicity groups. It is worth noting that due to Lemma 4.4.2 and Lemma 4.4.3 we are able to make sense of this result by only having data from the graph. Note that for $\beta = 0$ this reduces to [Chr22] Theorem 1.3 which gives a description of the tracial weights on the graph $C^*$-algebra.

**Theorem 4.4.5.**
*Let $E$ be a second-countable topological graph, $\beta \in \mathbb{R} \setminus \{0\}$, $\Phi : \mathcal{G}_E \to \mathbb{Z}$ be as in Definition 4.4.1 and $\nu$ be an ergodic measure on $\partial E$ such that $\sigma^*\nu = e^\beta\nu$ on $\partial E \setminus E^0$. Then there exists a unique subgroup $H$ of $\mathbb{Z}$ with*

$$\nu(\partial E \setminus \{a \in \partial E \mid Per(a) = H\}) = 0,$$

*and there is an affine bijection between the probability measures on the Pontryagin dual $\hat{H}$ of $H$ and the set of $\beta$-KMS weights for $\alpha^\Phi$ on $C^*(\mathcal{G}_E)$ restricting to the functional on $C_c(\partial E)$ corresponding to $\nu$. A probability measure $\lambda$ on $\hat{H}$ maps to a $\beta$-KMS weight $\psi_{\nu,\lambda}$ given by*

$$\psi_{\nu,\lambda}(f) = \int_{\partial E} \sum_{n \in H} f(a, n, a) \left( \int_{\hat{H}} \chi(n) d\lambda(\chi) \right) d\nu(a),$$

*for $f \in C_c(\mathcal{G}_E)$.* □

In Chapter 5 we achieve a small refinement of this theorem where we show that the subgroup $H$ of $\mathbb{Z}$ is always trivial in the $\beta \neq 0$ case.

Chapter 5

# Gauge invariance of tracial weights on $C^*$-algebras of topological graphs

In this chapter we show that the question of whether or not the $C^*$-algebra of a second-countable topological graph $E$ admits a gauge-invariant tracial weight, depends on the loop structure of the graph. In particular, we will show that $C^*(\mathcal{G}_E)$ admits a tracial weight which is not gauge-invariant if and only if the graph $E$ contains a *summable loop*.

We also show that if $\nu$ is an ergodic measure on $\partial E$ satisfying $\sigma^*\nu = e^\beta \nu$ on $\partial E \setminus E^0$ with $\beta \neq 0$, then the associated subgroup $H$ of $\mathbb{Z}$ given by Theorem 4.4.5 is trivial.

## 5.1 Loops in topological graphs

In this section we develop the terminology needed to study loops in topological graphs. The important concepts will be the idea of an *isolated loop*, a *summable loop*, and the notion of a *free topological graph*. We will also prove some basic results about these concepts.

First we introduce some useful notation.

**Definition 5.1.1.**
Let $E$ be a topological graph and $S, R \subset E^0$ be subsets. We define the following sets:

$$\begin{aligned} E^*S &= \{a \in E^* \mid s(a) \in S\}, \\ RE^* &= \{a \in E^* \mid r(a) \in R\}, \\ RE^*S &= \{a \in E^* \mid r(a) \in R,\ s(a) \in S\}. \end{aligned}$$ ♡

**Definition 5.1.2.**
Let $E$ be a topological graph. We say that a path $c \in E^*$ is a *loop* if $|c| \geq 1$ and $r(c) = s(c)$. We denote the set of loops on $E$ as $\Omega E$. Every loop $c \in \Omega E$ gives rise to a well-defined infinite path $c^\infty = cc\cdots$

If $c$ is a loop we say that $a \in E^*$ is an *exit of* $c$ if $a \in E^* r(c)$ and not of the form $a = bc$ for any path $b \in E^*$. We denote the set of exits of $c$ by $E^*_c$. ♡

**Definition 5.1.3.**
Let $E$ be a topological graph. We say that a loop $c \in \Omega E$ is a *simple* loop if $r(c_i) \neq r(c_j)$ for any $1 \leq i, j \leq |c|$ with $i \neq j$. ♡

**Definition 5.1.4.**
Let $E$ be a topological graph. We say that a loop $a \in \Omega E$ is an *isolated* loop if it satisfies

(1) $a \neq c^n$ for any loop $c \in \Omega E$ and $n \geq 2$.

(2) There exists an open neighborhood $V_{r(a)} \subset E^0$ about $r(a)$ with $|V_{r(a)} E_a^*| < \infty$.

Note that (2) essentially says that the number of exit paths of $a$ that end close to $a$ is finite. ♡

**Definition 5.1.5.**
Let $E$ be a topological graph. We say that a loop $a \in \Omega E$ is a *summable* loop if it satisfies

(1) $a \neq c^n$ for any loop $c \in \Omega E$ and $n \geq 2$.

(2) For all $w \in E^0$ there exists an open neighborhood $V_w \subset E^0$ about $w$ with $|V_w E_a^*| < \infty$.

Note that (2) essentially says that the number of exit paths of $a$ to an arbitrary vertex is finite. ♡

**Lemma 5.1.6** ([Chr22] Lemma 4.2)**.**
*Let $E$ be a topological graph and $a \in \Omega E$ be an isolated loop. Then every loop $c \in r(a)E^*$ is of the form $a^n$ for some $n \in \mathbb{N}$. In particular $a$ is a simple loop.*

*Proof.*
By contradiction assume that there exists a loop $c \in r(a)E^*$ that is not of the form $a^n$ for any $n \in \mathbb{N}$. Choose $n$ as large as possible such that $c = c'a^n$. Then $c'$ is a loop with $|c'| > 0$. Since $n$ was chosen to be as large as possible we get that $c' \in E_a^*$. It might be the case that $r(c'_i) = r(a)$ for $i \neq 1$. In this case we may decompose $c' = c''d$ into two loops $c''$ and $d$ with $r(d_i) = r(a)$ if and only if $i = 1$. Since $c' \in E_a^*$ we get that $d \in E_a^*$. We now want to show that $d^m \in E_a^*$ for all $m \in \mathbb{N}$.

If $|d| \geq |a|$ we immediately get that $d^m \in E_a^*$ for all $m \in \mathbb{N}$. So suppose that $|d| < |a|$. Choose $k \in \mathbb{N}$ minimally such that $|d^k| \geq |a|$. If $d^k \in E_a^*$ we again get that $d^m \in E_a^*$ for all $m \in \mathbb{N}$, so suppose that $d^k = ba$ for some loop $b$. Since $a$ is isolated we have by Definition 5.1.4 (1) that $a \neq d^k$, so $|b| > 0$. Since $k$ was chosen minimally we get that $|b| < |d|$, hence we get that $d = bd'$ for some loop $d'$, which contradicts the fact that $r(d_i) = r(a)$ if and only if $i = 1$ since $r(d') = r(a)$ and $d \neq d'$.

Hence, we get that $\{d^m\}_{m \in \mathbb{N}} \subset V_{r(a)} E_a^*$, which contradicts Definition 5.1.4 (2).

If $a$ is not simple then there exists $i, j$ distinct such that $r(a_i) = r(a_j)$. Then, by assuming that $i < j$ we can construct a loop $a' = a(i-1)a(j, |a|)$, which is in $r(a)E^*$, that is not of the form $a^n$ for any $n \in \mathbb{N}$, contradicting the first part of this proof. □

**Definition 5.1.7** ([Kat04] Definition 5.4)**.**
Let $E$ be a topological graph. Then $E$ is *free* if there does not exist a vertex $v \in E^0$ that satisfies the following:

(F1) There exists a simple loop $c = l_1 ... l_n \in \Omega E$ such that $r(c) = v$.

(F2) If $e \in E^1$ with $s(e) \in \{r(b) \mid b \in E^* v\}$ and $r(e) = r(l_k)$ for some $k$, then $e = l_k$.

(F3) $v$ is isolated in $\{r(b) \mid b \in E^* v\}$. ♡

**Lemma 5.1.8** ([Chr22] Lemma 4.4)**.**
*Let $E$ be a second-countable topological graph. Then $E$ is free if and only if there does not exist an isolated loop $c \in \Omega E$.*

*Proof.*
For the if statement assume that there does not exist an isolated loop in $\Omega E$. We will, by contradiction, assume that $E$ is not free. Then there exists a vertex $v \in E^0$ that satisfies (F1), (F2) and (F3) in Definition 5.1.7. Let $a \in vE^*$ be the simple loop from (F1). Since $a$ is simple it satisfies (1) in Definition 5.1.4. By (F3) there exists an open neighborhood $V_v \subset E^*v$ about $v$ such that for any $b \in E^*v$ with $r(b) \in V_v$ we have that $r(b) = v$. We will show that $V_v E_a^* = \{v\}$. In this case we have that the loop $a$ satisfies (2) in Definition 5.1.4, hence $a$ is isolated giving us our desired contradiction. By (F3) we have that any path in $V_v E_a^*$ must be a loop, so suppose $c \in V_v E_a^*$ is a loop. Then

$$r(c_1) = r(c) = v = r(a) = r(a_1),$$

and

$$s(c_1) = r(\sigma(c)) \in \{r(b) \mid b \in E^*v\},$$

so by (F2) we have that $c_1 = a_1$. If $|c| > 1$ we see that

$$r(c_2) = s(c_1) = s(a_1) = r(a_2)$$

and

$$s(c_2) = r(\sigma^2(c)) \in \{r(b) \mid b \in E^*v\},$$

so again by (F2) we get that $c_2 = a_2$. We may inductively continue this argument to get that

$$c = a^n a(k)$$

for some $n \in \mathbb{N}$ and $0 \leq k < |a|$. But $s(c) = v = s(a)$, so we have that $k = 0$ since $a$ is a simple loop. Then $c \in E_a^*$ is of the form $ba$ for some path $b$, a contradiction with the definition of $E_a^*$. This proves the if statement.

For the only if statement assume that $E$ is free. We will, by contradiction, assume that there exists an isolated loop $a \in \Omega E$. Set $v = r(a)$. By Lemma 5.1.6 we get that $a$ is simple, so $v$ satisfies (F1). Let $e \in E^1$ be such that $s(e) = r(b)$ for some $b \in E^*v$ and $r(e) = r(a_k)$ for some $k$. Then $a(k-1)eb \in r(a)E^*$ is a loop, so by Lemma 5.1.6 we have that $a(k-1)eb = a^n$ for some $n \geq 1$. Hence, $e = a_k$ so (F2) holds for $v$. Since $a$ is isolated there exists an open neighborhood $V_v \subset E^0$ about $v$ such that $|V_v E_a^*| < \infty$. Hence,

$$N = \{r(b) \mid b \in V_v E_a^* \text{ with } r(b) \neq v\}$$

is a finite set in $E^0$. Since $E^0$ is Hausdorff we get that $N$ is closed, so $V_v \setminus N$ is open in $E^0$, and it contains $v$. Hence, $V_v \setminus N$ shows that $v$ is isolated in $\{r(b) \mid b \in E^*v\}$, thus $v$ satisfies (F3). So, $v$ is a vertex showing that $E$ is not free, a contradiction since we assumed $E$ to be free. □

**Definition 5.1.9.**
Let $E$ be a topological graph. We say that two loops $a, b \in \Omega E$ are *equivalent* if $|a| = |b|$ and there exists a $k \in \mathbb{N}$ with $0 \leq k < |a|$ such that $\sigma^k(a^\infty) = b^\infty$. ♡

**Lemma 5.1.10.**
*Let $E$ be a topological graph. Then the relation defined in Definition 5.1.9 is an equivalence relation on $\Omega E$.*

*Proof.*
Let $a, b, c \in \Omega E$. Clearly $a$ is equivalent to $a$ by choosing $k = 0$ in the definition. Assume that $a$ is equivalent to $b$, let $k$ be such that $\sigma^k(a^\infty) = b^\infty$ and define $l = |a| - k$. Then we have that

$$a^\infty = \sigma^{|a|}(a^\infty) = \sigma^{k+l}(a^\infty) = \sigma^l(b^\infty).$$

Hence, $b$ is equivalent to $a$. Now assume further that $b$ is equivalent to $c$ and let $m$ be such that $\sigma^m(b^\infty) = c^\infty$. Then we have that

$$\sigma^{k+m}(a^\infty) = \sigma^m(b^\infty) = c^\infty.$$

If $k + m \geq |a|$ choose $n = k + m - |a|$ to see that $a$ is indeed equivalent to $c$. This completes the proof. □

**Lemma 5.1.11** ([Chr22] Lemma 4.5)**.**
*Let $E$ be a topological graph. Suppose that $a, b \in \Omega E$ are two equivalent loops and $V \subset E^0$ is a subset such that $|VE_b^*| < \infty$, then $|VE_a^*| < \infty$. In particular this implies that $a$ is summable if and only if $b$ is summable.*

*Proof.*
Set $n = |a| = |b|$ and $0 \leq k < |a|$ such that $\sigma^k(a^\infty) = b^\infty$. Then $b = a(k+1,n)a(k)$. If $c \in VE_a^*$ we get that $ca(k) \in VE^*r(b)$. We will show that the map $c \mapsto ca(k)$ from $VE_a^*$ has codomain $VE_b^* \sqcup \{db \mid d \in VE_b^*\}$. Indeed, if $c = db^2$ for some path $d$. Then

$$ca(k) = da(k+1,n)a(k)a(k+1,n)a(k) = da(k+1,n)aa(k),$$

hence $c = da(k+1,n)a$, contradicting the fact that $c \in E_a^*$. We then get an injective map

$$VE_a^* \to VE_b^* \sqcup \{db \mid d \in VE_b^*\}.$$

So if $|VE_b^*| < \infty$ we get by injectivity of this map that $|VE_a^*| < \infty$. □

**Definition 5.1.12.**
Let $E$ be a topological graph. We say that a path $b \in \partial E$ is *eventually cyclic* if there exists a loop $c \in \Omega E$ and an exit $a \in E_c^*$ such that $b = ac^\infty$. We denote the set of all eventually cyclic paths by $\rho E$.

For $S \subset E^0$ we define

$$\begin{aligned}\rho ES &= \{b \in \rho E \mid s(b_i) \in S \text{ for infinitely many } i \geq 1\} \\ &= \{b \in \rho E \mid b = ac^\infty \text{ with } s(c_i) \in S \text{ for some } 1 \leq i \leq |c|\}.\end{aligned}$$ ♡

**Lemma 5.1.13** ([Chr22] Lemma 4.6)**.**
*Let $E$ be a topological graph and $S \subset E^0$ be Borel. Then $\rho ES$ is an invariant Borel subset of $\partial E$.*

*Proof.*
$\rho ES$ is obviously invariant under $\sigma$. For $i \in \mathbb{N}$ the map $r \circ \sigma^i : \partial E \setminus E_{\text{sng}}^* \to E^0$ is continuous by Proposition 4.3.2. Hence, the sets

$$M_n = (\sigma^n)^{-1}(E^\infty) \cap \bigcup_{i=1}^{n} (r \circ \sigma^i)^{-1}(S)$$

are Borel in $\partial E$. Thus,

$$\rho ES = \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbb{N}} (\sigma^m)^{-1}(M_n)$$

is Borel in $\partial E$. □

## 5.2 Ergodic measures on eventually cyclic paths

In this section we show that there is a bijection between rays of non-zero ergodic measures $\nu$ on $\partial E$ satisfying $\sigma^* \nu = \nu$ on $\partial E \setminus E^0$ that are concentrated on the eventually cyclic paths and equivalence classes of summable loops.

The following fact is extracted from [Chr22] proposition 5.1, which we will need in its full generality.

**Lemma 5.2.1.**
*Let $E$ be a second-countable topological graph. If $\nu$ is an ergodic measure on $\partial E$ such that $\nu(\partial E \setminus \rho E) = 0$, then there exists a loop $c \in \Omega E$ such that $c \neq a^n$ for any loop $a \in \Omega E$ and $\nu(\partial E \setminus \{bc^\infty \mid b \in E_c^*\}) = 0$.*

*Proof.*
We will construct the set $\{bc^\infty \mid b \in E_c^*\}$ by inductively showing that $\nu$ is concentrated on smaller and smaller sets. First we consider the family

$$\mathcal{V} = \{V \subset E^0 \mid V \text{ is relatively compact in } E^0\}.$$

This is an open cover of $E^0$ since $E^0$ is locally compact and Hausdorff. Since $E^0$ is also second-countable we get that $\mathcal{V}$ admits a countable subcover $\{V_i\}_{i\in\mathbb{N}}$. We have that $\rho E = \bigcup_{i\in\mathbb{N}} \rho E\overline{V_i}$ with each $\rho E\overline{V_i}$ Borel and invariant by Lemma 5.1.13. Since $\nu$ is ergodic we have that for each $i \in \mathbb{N}$ that either

$$\nu(\rho E\overline{V_i}) = 0 \text{ or } \nu(\partial E \setminus \rho E\overline{V_i}) = 0.$$

Since $\nu$ is concentrated on $\rho E$ and $\nu \neq 0$ we get that there exists a $j \in \mathbb{N}$ such that

$$\nu(\partial E \setminus \rho E\overline{V_j}) = 0.$$

We now want to show that there exists a vertex $v \in \overline{V_j}$ such that $\nu$ is concentrated on $\rho E v$. First we notice that $\overline{V_j}$ as a subspace of $E^0$ is second-countable compact and Hausdorff. In particular $\overline{V_j}$ is metrizable by Urysohn's metrization Theorem. So we may assume that the topology on $\overline{V_j}$ is given by a metric. We may then cover $\overline{V_j}$ with open sets of diameter less than one, and in the same way we found the set $\overline{V_j}$, we may conclude that there exists an open set $W_1$ of diameter less than one such that

$$\nu(\partial E \setminus \rho E W_1) = 0.$$

Then we may cover $W_1$ of open sets of diameter less than $1/2$ and again conclude that there exists an open set $W_2 \subset W_1$ of diameter less than $1/2$ such that

$$\nu(\partial E \setminus \rho E W_2) = 0.$$

Continuing inductively we get a sequence of open sets $\{W_i\}_{i=1}^\infty$ such that for each $i$ we have that $W_{i+1} \subset W_i$, where $W_i$ has diameter less than $1/i$ and

$$\nu(\partial E \setminus \rho E W_i) = 0.$$

Then

$$0 = \nu\left(\bigcup_{i=1}^\infty \partial E \setminus \rho E W_i\right) = \nu\left(\partial E \setminus \left(\bigcap_{i=1}^\infty \rho E W_i\right)\right) = \nu\left(\partial E \setminus \rho E\left(\bigcap_{i=1}^\infty W_i\right)\right).$$

Since $\nu \neq 0$ we get in particular that $\bigcap_{i=1}^{\infty} W_i \neq \emptyset$. Hence, $\bigcap_{i=1}^{\infty} W_i = \{v\}$ for some $v \in E^0$ since the diameter of these sets approach zero. Thus, $\nu$ is concentrated on $\rho E v$.

Next we would like to show that this set is at most countable. So consider the family

$$\mathcal{U} = \{U \subset E^* \mid U \text{ is open in } E^* \text{ and } s|_U \text{ is injective}\}.$$

Since $s$ is a local homeomorphism we get that this is an open cover of $E^*$, and since $E$ is second-countable we get that $E^*$ is second-countable, hence $\mathcal{U}$ admits a countable subcover $\{U_i\}_{i\in\mathbb{N}}$. Writing

$$s^{-1}(\{v\}) = \bigcup_{i\in\mathbb{N}} s|_{U_i}^{-1}(\{v\})$$

we see that $s^{-1}(\{v\})$ is at most countable, hence $\rho E v = \rho E \cap s^{-1}(\{v\})$ is at most countable. Writing $\{c^i\}_{i\in I} = \Omega E v$ where $I$ is an at most countable set, we get that

$$\rho E v = \bigcup_{i\in I} \{b(c^i)^\infty \mid b \in E^* v\},$$

where each of the sets $\{b(c^i)^\infty \mid b \in E^* v\}$ are invariant Borel sets. The fact that these are Borel follows from the fact that $\partial E$ is Hausdorff, making singleton sets Borel. These sets are countable, so they can be written as a countable union of singleton sets, making them Borel. Again since $\nu$ is ergodic we get that either

$$\nu(\{b(c^i)^\infty \mid b \in E^* v\}) = 0 \text{ or } \nu(\partial E \setminus \{b(c^i)^\infty \mid b \in E^* v\}) = 0.$$

And since $\nu \neq 0$ we get that there exists a loop $c \in \Omega E$ with $s(c) = v$ such that

$$\nu(\partial E \setminus \{bc^\infty \mid b \in E^* v\}) = 0.$$

It might be the case that $c = a^n$ for some loop $a \in \Omega E$ and $n \geq 2$. In this case $c^\infty = a^\infty$, so we might assume that this is not the case, i.e. $c \neq a^n$ for any loop $a \in \Omega E$ and $n \geq 2$.

It only remains to prove that $E_c^*$ is in bijection with $\{bc^\infty \mid b \in E^* v\}$. We will do this via the map $b \mapsto bc^\infty$.

To prove surjectivity consider the element $bc^\infty$ with $b \in E^* v$. Either $b \in E_c^*$, in this case $b \mapsto bc^\infty$, or $b = b'c^n$ for some $n \geq 1$ and $b' \in E_c^*$. Then

$$b' \mapsto b'c^\infty = b'c^n c^\infty = bc^\infty.$$

To prove injectivity suppose $bc^\infty = ac^\infty$ with $b, a \in E_c^*$. If $|b| = |a|$ we get that $b = a$ by direct comparison. If $|b| > |a|$ we get that $b = ab'$ with $b' \in E_c^*$. Then we have that

$$ab'c^\infty = bc^\infty = ac^\infty,$$

so $b'c^\infty = c^\infty$. Since $b' \in E_c^*$ we get that $b' = c^k d$ for some loop $d \in E_c^*$ with $0 < |d| < |c|$. But then $c = d^n$ for some $n \geq 2$, a contradiction. Hence, $b = a$ proving injectivity. □

**Proposition 5.2.2** ([Chr22] Proposition 5.1).
*Let $E$ be a second-countable topological graph. Assume that $\nu \in \Delta(1)$ is a non-zero ergodic measure on $\partial E$ such that $\nu(\partial E \setminus \rho E) = 0$. Then there exists a summable loop $c \in \Omega E$ such that $\nu$ is concentrated on the at most countable set*

$$\{bc^\infty \mid b \in E_c^*\}.$$

*Hence, $\nu$ is of the form*

$$\nu = r \sum_{b \in E_c^*} \delta_{bc^\infty}$$

*for some $r > 0$.*

*Proof.*
By Lemma 5.2.1 we get a loop $c \in \Omega E$ such that $c \neq a^n$ for any loop $a \in \Omega E$ and $\nu(\partial E \setminus \{bc^\infty \mid b \in E_c^*\}) = 0$. Define $r = \nu(\{c^\infty\})$. Since $\nu \in \Delta(1)$ we get by Lemma 4.4.3 that $\sigma^*\nu = \nu$ on $\partial E \setminus E^0$, hence

$$\begin{aligned}\nu &= \sum_{b \in E_c^*} \nu(\{bc^\infty\})\delta_{bc^\infty} \\ &= \sum_{b \in E_c^*} \nu(\sigma^{|b|}(\{bc^\infty\}))\delta_{bc^\infty} \\ &= \sum_{b \in E_c^*} \nu(\{c^\infty\})\delta_{bc^\infty} \\ &= \nu(\{c^\infty\}) \sum_{b \in E_c^*} \delta_{bc^\infty} \\ &= r \sum_{b \in E_c^*} \delta_{bc^\infty}.\end{aligned}$$

It only remains to show that $c$ is indeed summable. We have chosen $c$ such that Definition 5.1.5 (1) is satisfied. Let $w \in E^0$ and $V_w \subset E^0$ be an open neighborhood about $w$ such that $\overline{V_w}$ is compact. By regularity of $\nu$ and Lemma 4.2.3 we get that

$$r|V_w E_c^*| = r \sum_{b \in E_c^*} \delta_{bc^\infty}(Z(V_w)) = \nu(Z(V_w)) \leq \nu(Z(\overline{V_w})) < \infty.$$

Hence, $c$ satisfies Definition 5.1.5 (2), so $c$ is summable. □

**Theorem 5.2.3** ([Chr22] Theorem 5.2)**.**
*Let $E$ be a second-countable topological graph. Then there is a bijection between the rays of non-zero ergodic measures $\nu \in \Delta(1)$ concentrated on $\rho E$ and equivalence classes of summable loops $c \in \Omega E$. The ray corresponding to the loop $c$ is $\{r\nu \mid r \geq 0\}$, where*

$$\nu = \sum_{b \in E_c^*} \delta_{bc^\infty}. \tag{5.1}$$

*Proof.*
Let $c \in \Omega E$ be a summable loop. We want to define a measure by equation (5.1). To do this we first check that the sum in this equation is at most countable to make sure that this is indeed a measure. Consider the family

$$\mathcal{U} = \{U \subset E^* \mid U \text{ is open in } E^* \text{ and } s|_U \text{ is injective}\}.$$

Since $s$ is a local homeomorphism, this is an open cover of $E^*$. Furthermore, since $E$ is second-countable, we get that $E^*$ is second-countable, hence $\mathcal{U}$ admits a countable subcover $\{U_i\}_{i \in \mathbb{N}}$. By writing

$$s^{-1}(\{s(c)\}) = \bigcup_{i \in \mathbb{N}} s|_{U_i}^{-1}(\{s(c)\})$$

we see that $s^{-1}(\{s(c)\})$ is at most countable, hence $E_c^*$ is at most countable. Thus, the measure $\nu$ given by equation (5.1) is indeed a measure.
*$\nu$ is non-zero*:

$$\nu(\{c^\infty\}) = \sum_{b \in E_c^*} \delta_{bc^\infty}(\{c^\infty\}) = 1.$$

*$\nu$ is regular*:
Since $\partial E$ is second-countable locally compact Hausdorff by Proposition 4.2.2 we have by

Lemma 2.0.10 that it is enough to show that $\nu(K) < \infty$ for any $K \subset \partial E$ that is compact. So let $K \subset \partial E$ be a compact set. By continuity of $r : \partial E \to E^0$ we get that $r(K) \subset E^0$ is compact. Since $c$ is summable we can for each $w \in r(K)$ find an open neighborhood $V_w \subset E^0$ about $w$ such that $|V_w E_c^*| < \infty$. Then the collection $\{V_w\}_{w \in r(K)}$ is an open cover of $r(K)$, hence it admits a finite subcover $\{V_i\}_{i=1}^n$. Then

$$K \subset Z(V_1) \cup \ldots \cup Z(V_n).$$

It follows that

$$\begin{aligned}
\nu(K) &\le \nu(Z(V_1) \cup \ldots \cup Z(V_n)) \\
&\le \nu(Z(V_1)) + \ldots + \nu(Z(V_n)) \\
&= \sum_{b \in E_c^*} \delta_{bc^\infty}(Z(V_1)) + \ldots + \sum_{b \in E_c^*} \delta_{bc^\infty}(Z(V_n)) \\
&= |V_1 E_c^*| + \ldots + |V_n E_c^*| \\
&< \infty.
\end{aligned}$$

*$\nu$ is ergodic*:
To show that $\nu$ is ergodic let $B \subset \partial E$ be invariant. To say that either

$$\nu(B) = 0 \text{ or } \nu(\partial E \setminus B) = 0$$

is equivalent to saying that

$$B \cap \{bc^\infty \mid b \in E_c^*\} = \varnothing \text{ or } B \cap \{bc^\infty \mid b \in E_c^*\} = \{bc^\infty \mid b \in E_c^*\}.$$

Equivalently if $ac^\infty \in B$ for one $a \in E_c^*$ we have that $bc^\infty \in B$ for all $b \in E_c^*$. So assume that $ac^\infty \in B$ for one $a \in E_c^*$. Since $B$ is invariant we get that $c^\infty = \sigma^{|a|}(ac^\infty) \in B$. Then for any $b \in E_c^*$ we get that $bc^\infty \in (\sigma^{|b|})^{-1}(B) \subset B$, so $bc^\infty \in B$.
$\nu \in \Delta(1)$:
Let $B \subset \partial E$ be Borel such that $\sigma|_B$ is injective. To see that $\nu(\sigma(B)) = \nu(B)$ it is enough to see that $|B \cap \{bc^\infty \mid b \in E_c^*\}| = |\sigma(B) \cap \{bc^\infty \mid b \in E_c^*\}|$ which is obvious by injectivity of $\sigma|_B$. Hence, $\sigma^*\nu = \nu$, so by Lemma 4.4.3 $\nu \in \Delta(1)$.

Finally, we show that equivalent summable loops maps to the same measures. So assume that $c'$ is equivalent to $c$. We want to show that

$$\{bc^\infty \mid b \in E_c^*\} = \{a(c')^\infty \mid a \in E_{c'}^*\}.$$

By symmetry, it is enough to show that

$$\{bc^\infty \mid b \in E_c^*\} \subset \{a(c')^\infty \mid a \in E_{c'}^*\}.$$

Let $b \in E_c^*$ and $0 \le k < |c|$ be such that $\sigma^k((c')^\infty) = c^\infty$, then we have that $b\sigma^k(c')$ is either in $E_{c'}^*$ or of the form $ac'$ with $a \in E_{c'}^*$. In either case we get that

$$\begin{aligned}
bc^\infty &= b\sigma^k(c')(c'^\infty), \; b\sigma^k(c') \in E_{c'}^* \\
bc^\infty &= b\sigma^k(c')(c'^\infty) = ac'(c'^\infty) = a(c')^\infty, \; a \in E_{c'}^*.
\end{aligned}$$

Hence, we get a well-defined map

$$\begin{gathered}
\{\text{Equivalence classes of summable loops}\} \\
\downarrow \\
\{\text{Rays of non-zero ergodic measures } \nu \in \Delta(1) \text{ concentrated on } \rho E\}
\end{gathered}$$

given by equation (5.1). This map is surjective by Proposition 5.2.2. To see that this map is injective let $\nu_1$ and $\nu_2$ be two measures given by the loops $c^1$ and $c^2$ as in equation (5.1). Assume that $\nu_1 = \nu_2$. Then

$$1 = \nu_2(\{(c^2)^\infty\}) = \nu_1(\{(c^2)^\infty\}) = \sum_{b \in E^*_{c^1}} \delta_{bc^1}(\{(c^2)^\infty\}).$$

Hence, there exists an exit $b \in E^*_{c^1}$ such that $(c^2)^\infty = b(c^1)^\infty$. Then the equation

$$\sigma^{|b|}((c^2)^\infty) = (c^1)^\infty$$

show that $c^1$ is equivalent to $c^2$. Note that the fact $|c^1| = |c^2|$ also follows from this equality since both loops are summable. This completes the proof. □

## 5.3 Gauge-invariant tracial weights

With the analysis of measures concentrated on eventually cyclic paths from the previous section we are able to give criteria for when there exists non gauge-invariant tracial weights. We are also able to show that for $\beta \in \mathbb{R} \setminus \{0\}$ and ergodic measures $\nu$ on $\partial E$ satisfying $\sigma^* \nu = e^{\beta} \nu$ on $\partial E \setminus E^0$ that the associated subgroup $H$ of $\mathbb{Z}$ given by Theorem 4.4.5 is trivial. We start by making an observation about the periodicity subgroups $Per(a)$.

**Lemma 5.3.1.**
*Let $E$ be a second-countable topological graph and $a \in \partial E$. Then $Per(a) \neq \{0\}$ if and only if $a \in \rho E$.*

*Proof.*
Suppose $Per(a) \neq \{0\}$. Then there exists $k, l \in \mathbb{N}$ such that $k > l$ and $\sigma^k(a) = \sigma^l(a)$. Define $c = \sigma^l(a)$. Then we have that

$$\sigma^{k-l}(c) = \sigma^{k-l}(\sigma^l(a)) = \sigma^k(a) = \sigma^l(a) = c,$$

so $c(k-l)$ is a loop in $\Omega E$. Write $b = a(l)$ and $d = c(k-l)$. Then

$$a = bd^{\infty}.$$

If $k$ and $l$ were chosen minimally we can conclude that $b \in E_d^*$ showing that $a \in \rho E$.

Now suppose that $a \in \rho E$, and let $c \in \Omega E$ and $b \in E_c^*$ be such that $a = bc^{\infty}$. Set $k = |b| + |c|$ and $l = |b|$. Then $k > l$ since $|c| \geq 1$ and

$$\sigma^k(a) = \sigma^l(a).$$

Hence, $k - l \in Per(a)$, showing that $Per(a) \neq \{0\}$. In fact $Per(a) = (k-l)\mathbb{Z} = |c|\mathbb{Z}$. □

**Proposition 5.3.2** ([Chr22] Proposition 6.1)**.**
*Let $E$ be a second-countable topological graph. If there exists a tracial weight $\omega$ on $C^*(\mathcal{G}_E)$ which is not gauge-invariant, then there exists a summable loop in $\Omega E$.*

*Sketch of proof.*
Suppose $\omega$ is a tracial weight on $C^*(\mathcal{G}_E)$ that is not gauge-invariant. By Proposition 3.5.9 we have that there exists a measure $\nu \in \Delta(1)$ on $\partial E$ such that

$$\omega(h) = \int_{\partial E} h d\nu$$

for all $h \in C_c(\partial E)$. We would like to use Theorem 4.4.5 to describe $\omega$, but we are not guaranteed that $\nu$ is ergodic. Christensen is able to show in his proof that there must exist an extremal tracial weight on $C^*(\mathcal{G}_E)$ that is not gauge-invariant using convexity theory. We will in this sketch skip this part of the proof for the sake of brevity. See [Chr22] proposition 6.1 for the full details.

Having found an extremal tracial weight $\psi$ on $C^*(\mathcal{G}_E)$, which is not gauge-invariant, we get by Theorem 4.4.5 a unique invariant ergodic measure $\nu_\psi$ on $\partial E$, a unique subgroup $H_\psi$ of $\mathbb{Z}$ and a unique character $\chi_\psi \in \hat{H}_\psi$ such that for any $f \in C_c(\mathcal{G}_E)$ we have that

$$\psi(f) = \int_{\partial E} \sum_{n \in H_\psi} f(a, n, a) \chi_\psi(n) d\nu_\psi(a),$$

where $\nu_\psi$ is concentrated on the set $\{a \in \partial E \mid Per(a) = H_\psi\}$. Since $\psi$ is non-zero we get that $\nu_\psi$ is non-zero. We want to show that $\nu_\psi$ is concentrated on $\rho E$, then we get by

Proposition 5.2.2 that there exists a summable loop in $\Omega E$. By Lemma 5.3.1 it is enough to show that $H_\psi$ is non-trivial since in this case

$$\{a \in \partial E \mid Per(a) = H_\psi\} \subset \rho E.$$

Since $\psi$ is not gauge-invariant we have by Proposition 3.5.6 that there exists a function $f \in C_c(\mathcal{G}_E)$ and $z \in \mathbb{T}$ such that $\psi(\gamma_z(f)) \neq \psi(f)$. In particular this implies that

$$\begin{aligned}\psi(\gamma_z(f)) &= \int_{\partial E} \sum_{n \in H_\psi} z^n f(a,n,a)\chi_\psi(n) d\nu_\psi(a)\\ &\neq \int_{\partial E} \sum_{n \in H_\psi} f(a,n,a)\chi_\psi(n) d\nu_\omega(a) = \psi(f).\end{aligned}$$

If $H_\psi = \{0\}$ these expressions are necessarily equal, contradicting the fact that $\psi$ is not gauge-invariant. Hence, $H_\psi \neq \{0\}$ completing the proof. □

**Proposition 5.3.3** ([Chr22] Proposition 6.2).
*Let $E$ be a second-countable topological graph. If there exists a summable loop $c \in \Omega E$, then there exists a non gauge-invariant tracial weight on $C^*(\mathcal{G}_E)$.*

*Proof.*
Let $c$ be a summable loop in $\Omega E$. By Theorem 5.2.3 we get a non-zero ergodic measure $\nu \in \Delta(1)$ such that

$$\nu = \sum_{b \in E_c^*} \delta_{bc^\infty}.$$

By Lemma 5.3.1 the unique subgroup $H \subset \mathbb{Z}$ associated to $\nu$ given by Theorem 3.5.13 (2) is $H = |c|\mathbb{Z}$. Then we get a tracial weight by Theorem 4.4.5 where for $f \in C_c(\mathcal{G}_E)$ we have that

$$\psi(f) = \int_{\partial E} \sum_{n \in H} f(a,n,a) d\nu(a).$$

Now let $f \in C_c(\mathcal{G}_E)$ be a positive function such that $\mathrm{supp}(f) \subset \Phi^{-1}(\{|c|\})$ and $f(c^\infty, |c|, c^\infty) = 1$. The existence of such a function $f$ is guaranteed by for example considering the characteristic function

$$\chi_{\Phi^{-1}(\{|c|\}) \cap V},$$

where $V \subset \mathcal{G}_E$ is a relatively compact open neighborhood about $(c^\infty, |c|, c^\infty)$. The function $\chi_{\Phi^{-1}(\{|c|\}) \cap V}$ is then $\nu$ integrable, so we can approximate it with continuous functions with compact support. Choosing one of these approximations gives us our desired $f$, after possibly rescaling. Then $\psi(f) > 0$. Let $z \in \mathbb{T}$ be such that $z^{|c|} \neq 1$, then

$$\begin{aligned}\psi(f) &= \int_{\partial E} f(a,|c|,a) d\nu(a)\\ &\neq \int_{\partial E} z^{|c|} f(a,|c|,a) d\nu(a) = \psi(\gamma_z(f)).\end{aligned}$$

Hence, $\psi$ is not gauge-invariant, completing the proof. □

By Proposition 5.3.2 and Proposition 5.3.3 we get the following result.

**Theorem 5.3.4** ([Chr22] Theorem 6.4).
*Let $E$ be a second-countable topological graph. All tracial weights on $C^*(\mathcal{G}_E)$ are gauge-invariant if and only if there does not exist a summable loop $a \in \Omega E$.* □

**Corollary 5.3.5** ([Chr22] Theorem 6.5)**.**
*Let $E$ be a second-countable topological graph. If $E$ is free then every tracial weight on $C^*(\mathcal{G}_E)$ is gauge-invariant.*

*Proof.*
If there exists a non gauge-invariant tracial weight on $C^*(\mathcal{G}_E)$, then by Theorem 5.3.4 there exists a summable loop in $\Omega E$. Summable loops are clearly isolated, so by Lemma 5.1.8, $E$ is not free, a contradiction. □

We end this chapter by showing that if $\nu$ is an ergodic measure on $\partial E$ satisfying $\sigma^*\nu = e^\beta\nu$ on $\partial E \setminus E^0$ with $\beta \neq 0$, then the associated subgroup $H$ of $\mathbb{Z}$ given by Theorem 4.4.5 is trivial.

**Proposition 5.3.6.**
*Let $E$ be a second-countable topological graph, $\beta \in \mathbb{R} \setminus \{0\}$, $\Phi : \mathcal{G}_E \to \mathbb{Z}$ be as in Definition 4.4.1 and $\nu$ be an ergodic measure on $\partial E$ such that $\sigma^*\nu = e^\beta\nu$ on $\partial E \setminus E^0$. Then the unique subgroup $H$ of $\mathbb{Z}$ associated to $\nu$ given by Theorem 4.4.5 is trivial.*

*Proof.*
Suppose, to get a contradiction, that $H \neq \{0\}$. Since $\nu$ is concentrated on the set $\{a \in \partial E \mid Per(a) = H\}$, we get by Lemma 5.3.1 that $\nu$ is concentrated on $\rho E$. Since $\nu$ is ergodic we get by Lemma 5.2.1 that there exists a loop $c \in \Omega E$ such that $\nu(\partial E \setminus \{bc^\infty \mid b \in E_c^*\}) = 0$. So we have that

$$\begin{aligned}\nu &= \sum_{b\in E_c^*} \nu(\{bc^\infty\})\delta_{bc^\infty} \\ &= \sum_{b\in E_c^*} e^{-\beta|b|}\nu(\{c^\infty\})\delta_{bc^\infty} \\ &= \nu(\{c^\infty\}) \sum_{b\in E_c^*} e^{-\beta|b|}\delta_{bc^\infty}.\end{aligned}$$

This also shows us that $H = |c|\mathbb{Z}$. As in the proof of Proposition 5.3.3 we can find a positive function $f \in C_c(\mathcal{G})$ with $\operatorname{supp}(f) \subset \Phi^{-1}(|c|)$ such that $f(c^\infty, |c|, c^\infty) = 1$. Let $z \in \mathbb{T}$ be such that $z^{|c|} = i$. Let $\psi_\nu$ be the $\beta$-KMS weight given by Theorem 4.4.5 associated to $\nu$. Then

$$\psi(f) = \int_{\partial E} \sum_{n\in H} f(a,n,a)d\nu(a) = \nu(\{c^\infty\}) \sum_{b\in E_c^*} e^{-\beta|b|} f(bc^\infty, |c|, bc^\infty) > 0,$$

and

$$\psi(\gamma_z(f)) = i\nu(\{c^\infty\}) \sum_{b\in E_c^*} e^{-\beta|b|} f(bc^\infty, |c|, bc^\infty).$$

Hence, $\psi$ is not gauge-invariant, a contradiction since $\psi$ is by definition invariant under the dynamics $\alpha^\Phi$. So $H$ must be the trivial group. □

**Theorem 5.3.7.**
*Let $E$ be a second-countable topological graph, $\beta \in \mathbb{R} \setminus \{0\}$ and $\Phi : \mathcal{G}_E \to \mathbb{Z}$ be as in Definition 4.4.1. There is a bijection between the set of ergodic measures $\nu$ on $\partial E$ such that $\sigma^*\nu = e^\beta\nu$ and the extremal $\beta$-KMS weights for $\alpha^\Phi$ on $C^*(\mathcal{G}_E)$. The extremal $\beta$-KMS weight associated to the measure $\nu$ is given by the equation*

$$\psi_\nu(f) = \int_{\partial E} f(a,0,a)d\nu(a), \tag{5.2}$$

*for $f \in C_c(\mathcal{G}_E)$.*

*Proof.*
Let $\psi$ be an extremal $\beta$-KMS weight on $C^*(\mathcal{G}_E)$ for $\alpha^\Phi$. The associated measure $\nu$ given by Proposition 3.5.9 must necessarily be extremal, which by Theorem 3.5.11 means that $\nu$ is ergodic. Hence, $\psi$ satisfies the criteria for Theorem 4.4.5. By Proposition 5.3.6 we have that the subgroup $H$ of $\mathbb{Z}$ associated to $\nu$ is trivial, so $\psi$ takes the form

$$\psi(f) = \int_{\partial E} f(a, 0, a) d\nu(a),$$

for $f \in C_c(\mathcal{G}_E)$. This establishes the bijection since any ergodic measure $\nu$ on $\partial E$ satisfying $\sigma^*\nu = e^\beta \nu$ on $\partial E \setminus E^0$ gives us an extremal $\beta$-KMS weight by equation (5.2). □

# Chapter 6

# Sub-invariant measures

The results in Chapter 5 are of theoretical use, but given a second-countable topological graph $E$ it will be difficult to use these results to make concrete computations. In this chapter we show that there is an affine bijection between the set of quasi-invariant measures on the boundary space $\partial E$ and the set of sub-invariant measures on the vertex space $E^0$. The sub-invariant measures can in some cases be used to easily compute the KMS-spectra of certain topological graphs.

The results in this chapter are proved using the same techniques found in [Chr22] and [Sch18].

**Definition 6.0.1.**
Let $E$ be a second-countable topological graph and $\beta \in \mathbb{R}$. We define the map $T : \mathcal{M}_{\text{reg}}(E^0) \to \mathcal{M}(E^0)$ by $T = r_* s^*$ where $r, s : E^1 \to E^0$ are the range and source maps respectively. We say that a regular Borel measure $\mu$ on the vertex space $E^0$ is *$\beta$-sub-invariant* if

$$T\mu \le e^{\beta}\mu \text{ on } E^0$$

with equality on $E^0_{\text{reg}}$. Denote the set of all $\beta$-sub-invariant measures by $\mathcal{M}^{\beta}_{\text{sub}}(E^0)$. ♡

**Lemma 6.0.2.**
*Let $E$ be a second-countable topological graph, $\beta \in \mathbb{R}$ and $\Phi : \mathcal{G}_E \to \mathbb{Z}$ be as in Definition 4.4.1. Then the range map $r : \partial E \to E^0$ induces a map*

$$r_* : \Delta(e^{-\beta\Phi}) \to \mathcal{M}^{\beta}_{sub}(E^0).$$

*Proof.*
Let $\nu \in \Delta(e^{-\beta\Phi})$. To show that $r_*\nu$ is regular it is enough, by Proposition 2.0.17, to show that $r : \partial E \to E^0$ is proper. Let $K \subset E^0$ be compact. Then $r^{-1}(K) = Z(K)$ which is compact in $\partial E$ by Lemma 4.2.3. Hence, $r$ is proper.

Let

$$\mathcal{U} = \{U \subset E^1 \mid U \text{ is open in } E^1 \text{ and } s|_U \text{ is injective}\}.$$

Since $s : E^1 \to E^0$ is a local homeomorphism, $\mathcal{U}$ is an open cover of $E^1$. Since $E^1$ is second-countable, $\mathcal{U}$ admits a countable sub-cover $\{U_i\}_{i \in \mathbb{N}}$. Fix a Borel subset $B \subset E^0$.

Then

$$\begin{aligned}
T(r_*\nu)(B) &= \sum_{i\in\mathbb{N}} r_*\nu(s(r^{-1}(B)\cap(U_i\smallsetminus(U_1\cup\ldots\cup U_{i-1})))) \\
&= \sum_{i\in\mathbb{N}} \nu(Z(s(r^{-1}(B)\cap(U_i\smallsetminus(U_1\cup\ldots\cup U_{i-1}))))) \\
&= \sum_{i\in\mathbb{N}} \nu(\sigma(Z(r^{-1}(B)\cap(U_i\smallsetminus(U_1\cup\ldots\cup U_{i-1}))))) \\
&= e^{\beta}\sum_{i\in\mathbb{N}} \nu(Z(r^{-1}(B)\cap(U_i\smallsetminus(U_1\cup\ldots\cup U_{i-1})))) \\
&= e^{\beta}\nu\left(\bigcup_{i\in\mathbb{N}} Z(r^{-1}(B)\cap(U_i\smallsetminus(U_1\cup\ldots\cup U_{i-1})))\right) \\
&= e^{\beta}\nu\left(Z\left(\bigcup_{i\in\mathbb{N}} r^{-1}(B)\cap(U_i\smallsetminus(U_1\cup\ldots\cup U_{i-1}))\right)\right) \\
&= e^{\beta}\nu(Z(r^{-1}(B))) \\
&\le e^{\beta}\nu(Z(B)) \\
&= e^{\beta}(r_*\nu)(B),
\end{aligned}$$

where the fourth equality follows by Lemma 4.4.3 and the inequality in the penultimate line is an equality if $B\subset E^0_{\text{reg}}$. Thus, $r_*\nu\in\mathcal{M}^{\beta}_{\text{sub}}(E^0)$. □

To prove injectivity of $r_*$ we will use some theory about $\pi$-systems. See Chapter 1.6 in [Coh13] for a thorough description of $\pi$-systems and Dynkin classes. We will here only state the definition of a $\pi$-system and the main result that we will need.

**Definition 6.0.3.**
Let $X$ be a set. A $\pi$-system on $X$ is a family of subsets of $X$ that is stable under finite intersections. ♡

**Lemma 6.0.4** ([Coh13] Corollary 1.6.4)**.**
*Let $(X,\mathcal{A})$ be a measurable space and $\mathcal{C}$ be a $\pi$-system such that $\sigma(\mathcal{C})=\mathcal{A}$. If $\mu$ and $\nu$ are measures on $(X,\mathcal{A})$ that agree on $\mathcal{C}$ and there exists a countable increasing sequence of sets $\{C_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that $\mu(C_n)<\infty$ for all $n\in\mathbb{N}$ and $X=\bigcup_{n\in\mathbb{N}}C_n$, then $\mu=\nu$.* □

Now we give description of a $\pi$-system that will be quite useful.

**Lemma 6.0.5.**
*Let $E$ be a second-countable topological graph. Then the family*

$$\mathcal{C}=\{Z(V)\subset\partial E \mid V\subset E^n \text{ is a Borel subset and } n\in\mathbb{N}\} \tag{6.1}$$

*is a $\pi$-system that generates the Borel $\sigma$-algebra of $\partial E$.*

*Proof.*
Let $U\subset E^*$ be open and $K\subset E^*$ be compact. It is enough to show that $Z(U)\smallsetminus Z(K)\in\sigma(\mathcal{C})$ to show that $\mathcal{C}$ generates the Borel $\sigma$-algebra. Define, for $n\in\mathbb{N}$, $U_n=U\cap E^n$ and $K_n=K\cap E^n$. Then we have that the following equality

$$Z(U)=Z\left(\bigcup_{n\in\mathbb{N}}U_n\right)=\bigcup_{n\in\mathbb{N}}Z(U_n)$$

shows that $Z(U)\in\sigma(\mathcal{C})$. Similarly, we get that $Z(K)\in\sigma(\mathcal{C})$. Hence, $Z(U)\smallsetminus Z(K)\in\sigma(\mathcal{C})$.

Now we need to show that $\mathcal{C}$ is a $\pi$-system. Let $U \subset E^n$ and $V \subset E^m$ be Borel sets. We may assume that $n \geq m$. We define

$$W = \{e \in U \mid e(m) \in V\}.$$

By Lemma 4.1.4 we have that $W$ is Borel since we can write $W = U \cap (m)^{-1}(V)$ where $(m)^{-1}$ denotes the inverse image of the map $e \mapsto e(m)$. Clearly

$$Z(U) \cap Z(V) = Z(W),$$

which shows that $\mathcal{C}$ is a $\pi$-system. □

**Lemma 6.0.6.**
*Let $E$ be a second-countable topological graph, $\beta \in \mathbb{R}$ and $\Phi : \mathcal{G}_E \to \mathbb{Z}$ be as in Definition 4.4.1. Then the map*

$$r_* : \Delta(e^{-\beta\Phi}) \to \mathcal{M}^{\beta}_{sub}(E^0),$$

*induced by the range map $r : \partial E \to E^0$, is injective.*

*Proof.*
Fix $\nu_1, \nu_2 \in \Delta(e^{-\beta\Phi})$ such that $r_*\nu_1 = r_*\nu_2$. We want to check that $\nu_1 = \nu_2$ on the $\pi$-system (6.1) in the previous lemma. Suppose $V \subset E^n$ is Borel, let

$$\mathcal{U} = \{U \subset E^n \mid U \text{ is open in } E^n \text{ and } \sigma^n|_U \text{ is injective}\}.$$

This is an open cover of $V$ since $\sigma^n$ is a local homeomorphism, and since $E^n$ is second-countable we get that $\mathcal{U}$ admits a countable subcover, $\{U_i\}_{i\in\mathbb{N}}$. So to check that $\nu_1(Z(V)) = \nu_2(Z(V))$ it is enough to check that $\nu_1(Z(U_i)) = \nu_2(Z(U_i))$ for each $i \in \mathbb{N}$ by disjointing $V$ with the $U_i$'s. By Lemma 4.4.3 we get that

$$\begin{aligned}
\nu_1(Z(U_i)) &= e^{-\beta n}\nu_1(\sigma^n(Z(U_i))) \\
&= e^{-\beta n}\nu_1(Z(\sigma^n(U_i))) \\
&= e^{-\beta n}(r_*\nu_1)(\sigma^n(U_i)) \\
&= e^{-\beta n}(r_*\nu_2)(\sigma^n(U_i)) \\
&= \nu_2(Z(U_i)).
\end{aligned}$$

Hence, $\nu_1 = \nu_2$ on the $\pi$-system (6.1). To conclude that $\nu_1 = \nu_2$ on the Borel $\sigma$-algebra of $\partial E$ we need to show that we can cover $\partial E$ with a sequence of increasing sets from the $\pi$-system (6.1), with each set having finite measure. We can do this as follows: Consider the collection

$$\mathcal{V} = \{V \subset E^0 \mid V \text{ is relatively compact in } E^0\}.$$

Since $E^0$ is locally compact Hausdorff this is an open cover of $E^0$, and since $E^0$ is second-countable $\mathcal{V}$ admits a countable subcover $\{V_i\}_{i\in\mathbb{N}}$. Define $W_i = \bigcup_{j=0}^{i} V_j$. Then each $W_i$ is relatively compact in $E^0$ and $W_i \subset W_{i+1}$, so we have that the sequence $\{Z(W_i)\}_{i\in\mathbb{N}}$ is an increasing sequence such that

$$\bigcup_{i\in\mathbb{N}} Z(W_i) = Z\left(\bigcup_{i\in\mathbb{N}} W_i\right) = Z(E^0) = \partial E.$$

Finally, we have that

$$\nu_1(Z(W_i)) \leq \nu_1(Z(\overline{W_i})) < \infty$$

since $\nu_1$ is a regular measure and $Z(\overline{W_i})$ is compact by Lemma 4.2.3. Hence, $\nu_1 = \nu_2$ by Lemma 6.0.4, which proves that $r_*$ is injective. □

Proving surjectivity of $r_*$ turns out to be quite complicated. The $\beta = 0$ case is done by Schafhauser, c.f. [Sch18] Proposition 4.4, and it turns out that his proof generalizes to all values of $\beta$ quite easily. The main idea will be to show that the boundary space $\partial E$ is the limit of spaces $\partial E_n$, where each space $\partial E_n$ only contains paths of length less than or equal to $n$. In this case we may use the backwards shift map to pullback a measure $\mu$ on $E^0$ to each $\partial E_n$ and get a measure defined on $\partial E$ by making some limit arguments.

The spaces $\partial E_n$ are defined analogous to $\partial E$, namely:

**Definition 6.0.7.**
Let $E$ be a topological graph. We define the *n-th boundary path space* $\partial E_n$ by

$$\partial E_n = E^0_{\text{sng}} \sqcup \ldots \sqcup E^{n-1}_{\text{sng}} \sqcup E^n.$$

For subsets $S \subset E^0 \sqcup \ldots \sqcup E^n$ we define the set

$$Z_n(S) = \{a \in \partial E_n \mid a(k) \in S \text{ for some } k \text{ satisfying } 1 \le k \le |a|\}.$$

$\partial E_n$ becomes a locally compact Hausdorff space with the topology generated by sets $Z_n(U) \setminus Z_n(K)$ where $U \subset E^0 \sqcup \ldots \sqcup E^n$ is an open subset and $K \subset E^0 \sqcup \ldots \sqcup E^n$ is compact. The proof of this is completely the same as the one for $\partial E$ done in Section 4.2. Of course if $E$ is second-countable, $\partial E_n$ becomes second-countable as well. Note that for $n = 0$ we have that $\partial E_0 = E^0$.

For $n \ge 1$ we define maps $\rho_n : \partial E_n \to \partial E_{n-1}$ by

$$\rho_n(a) = \begin{cases} a(n-1), & |a| = n, \\ a, & |a| < n, \end{cases}$$

as well as maps $\rho_{n,\infty} : \partial E \to \partial E_n$ by

$$\rho_{n,\infty}(a) = \begin{cases} a(n), & |a| > n, \\ a, & |a| \le n. \end{cases}$$

Note that $\rho_{0,\infty}$ is the range map $r : \partial E \to E^0$. ♡

**Lemma 6.0.8** ([Sch18] Proposition 3.3)**.**
*Let $E$ be a topological graph. Then the maps $\rho_n$ and $\rho_{n,\infty}$ are continuous, proper and surjective for each $n \ge 1$.*

*Proof.*
*Surjectivity*:
Let $a \in \partial E_{n-1}$. If $s(a) \in E^0_{\text{sng}}$ we have that $a \in E^k_{\text{sng}}$ for some $k$ satisfying $1 \le k \le n-1$, hence $a \in \partial E_n$ and $\rho_n(a) = a$.

If $s(a) \in E^0_{\text{reg}}$ we have that $|a| = n-1$. By definition of regular vertices there exists an edge $e \in E^1$ such that $r(e) = s(a)$, hence the path $ae \in E^n \subset \partial E_n$ and $\rho_n(ae) = a$.

Inductively applying this argument gives us that $\rho_{n,\infty}$ is also surjective.
*Continuity*:
For subsets $S \subset E^0 \sqcup \ldots \sqcup E^{n-1}$ it is clear that $\rho_n^{-1}(Z_{n-1}(S)) = Z_n(S)$. So if $U \subset E^0 \sqcup \ldots \sqcup E^{n-1}$ is an open subset and $K \subset E^0 \sqcup \ldots \sqcup E^{n-1}$ is compact, then

$$\rho_n^{-1}(Z_{n-1}(U) \setminus Z_{n-1}(K)) = Z_n(U) \setminus Z_n(K).$$

Hence, $\rho_n$ is continuous. Basically the same argument show that $\rho_{n,\infty}$ is continuous.

*Proper*:
Let $K \subset \partial E_{n-1}$ be compact and consider the family

$$\mathcal{U} = \{U \subset E^0 \sqcup \ldots \sqcup E^{n-1} \mid U \text{ is relatively compact in } E^0 \sqcup \ldots \sqcup E^{n-1}\}.$$

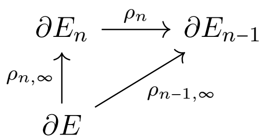
Since $E^0 \sqcup \ldots \sqcup E^{n-1}$ is locally compact Hausdorff, $\mathcal{U}$ is an open cover of $E^0 \sqcup \ldots \sqcup E^{n-1}$. In particular, the collection $\{Z_{n-1}(U)\}_{U \in \mathcal{U}}$ is an open cover of $K$, so there exists a finite subset of $\mathcal{U}$, $\{U_i\}_{i=1}^k$, such that $K \subset Z_{n-1}(U_1) \cup \ldots \cup Z_{n-1}(U_k)$. Then

$$\rho_n^{-1}(K) \subset \bigcup_{i=1}^k \rho_n^{-1}(Z_{n-1}(U_i)) \subset \bigcup_{i=1}^k \rho_n^{-1}(Z_{n-1}(\overline{U_i})) \subset \bigcup_{i=1}^k Z_n(\overline{U_i}).$$

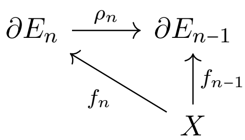
By Lemma 4.2.3, $\bigcup_{i=1}^k Z_n(\overline{U_i})$ is compact in $\partial E_n$. Since $\rho_n$ is continuous we have that $\rho_n^{-1}(K)$ is closed in $\bigcup_{i=1}^k Z_n(\overline{U_i})$, hence $\rho_n^{-1}(K)$ is compact. The proof that $\rho_{n,\infty}$ is proper follows by the same argument. ☐

**Lemma 6.0.9** ([Sch18] Proposition 3.3)**.**
*Let $E$ be a topological graph. Then $\partial E$ is homeomorphic to* $\lim_n(\partial E_n, \rho_n)$.

*Proof.*
Note first that by Lemma 6.0.8 the maps $\rho_n$ are continuous. It therefore makes sense to consider the limit $\lim_n(\partial E_n, \rho_n)$ in the category of topological spaces. It is clear that we are in the situation where each of the diagrams



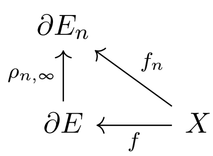

commute. We want to show that if $X$ is a topological space with continuous maps $f_n : X \to \partial E_n$ such that the diagrams




commute, then there exists a continuous map $f : X \to \partial E$ such that $f_n = \rho_{n,\infty} \circ f$ for all $n \in \mathbb{N}$. We construct this function $f$ as follows: Fix $x \in X$. There are two cases to consider. First if there exists an $m \in \mathbb{N}$ such that $a = f_m(x)$ satisfies $|a| < m$ then $a \in E^*_{\text{sng}} \subset \partial E$, so we define $f(x) = a$. The second case is if $a^n = f_n(x)$ satisfies $|a^n| = n$ for all $n \in \mathbb{N}$. Then we have that

$$a^{n-1} = f_{n-1}(x) = \rho_n(f_n(x)) = \rho_n(a^n) = a^n(n-1),$$

so each path $a^n$ is an extension of the path $a^{n-1}$. Thus, we may take the limit to get an infinite path $a \in E^\infty \subset \partial E$. We define $f(x) = a$, where $a(n) = a^n$ for each $n \in \mathbb{N}$. It is clear by this definition of $f$ that the diagrams



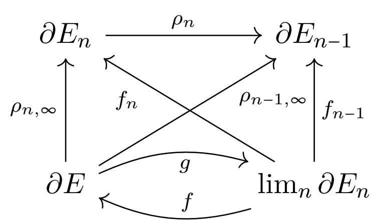
commute. We now need to show that $f$ is continuous.

Since $E^*$ is locally compact Hausdorff we have that any open subset $U \subset E^*$ can be written as a union of relatively compact subsets of $E^*$. So we need only check that $f^{-1}(Z(U) \setminus Z(K))$ is open in $X$ for a relatively compact subset $U \subset E^*$ and a compact set $K \subset E^*$. For such sets we have that they can only intersect $E^n$ for finitely many $n \in \mathbb{N}$. Hence, there exists a $k \in \mathbb{N}$ such that $U \cap E^n = K \cap E^n = \varnothing$ for all $n \geq k$. Thus, $Z(U) \setminus Z(K) = \rho_{k,\infty}^{-1}(Z_k(U) \setminus Z_k(K))$, so

$$f^{-1}(Z(U) \setminus Z(K)) = f^{-1}(\rho_{k,\infty}^{-1}(Z_k(U) \setminus Z_k(K))) = f_k^{-1}(Z_k(U) \setminus Z_k(K)),$$

which is open in $X$ by continuity of $f_k$. Thus, $f$ is continuous.

By universality of the limit $\lim_n \partial E_n$ we have that there exists a unique continuous map $g : \partial E \to \lim_n \partial E_n$, and by the discussion earlier in this proof there exists a continuous map $f : \lim_n \partial E_n \to \partial E$, making all the triangles in the following diagram commute:




We need to show that $f$ is the inverse of $g$. We start by showing that $g \circ f$ is the identity on $\lim_n \partial E_n$. Let $x \in \lim_n \partial E_n$ and define $y = g \circ f(x)$. By commutativity of the bottom right triangle we get that $f_{n-1} \circ g \circ f = f_{n-1}$, so $f_{n-1}(y) = f_{n-1}(x)$ for all $n \geq 1$. By universality of the limit we get that $x = y$.

Now we show that $f \circ g$ is the identity on $\partial E$. Let $a \in \partial E$ and define $b = f \circ g(a)$. By commutativity of the bottom left triangle we get that $\rho_{n,\infty} \circ f \circ g = \rho_{n,\infty}$, so $\rho_{n,\infty}(b) = \rho_{n,\infty}(a)$ for all $n \in \mathbb{N}$. It is clear from the definition of $\rho_{n,\infty}$ that $a = b$. This finishes the proof. □

**Lemma 6.0.10** ([Sch18] Proposition 4.4)**.**
*Let $E$ be a second-countable topological graph, $\beta \in \mathbb{R}$ and $\Phi : \mathcal{G}_E \to \mathbb{Z}$ be as in Definition 4.4.1. Then the map*

$$r_* : \Delta(e^{-\beta\Phi}) \to \mathcal{M}^{\beta}_{sub}(E^0)$$

*induced by the range map $r : \partial E \to E^0$ is surjective.*

*Proof.*
Let $\mu \in \mathcal{M}^{\beta}_{\text{sub}}(E^0)$. We want to define regular Borel measures $\mu_n$ on $\partial E_n$ for each $n \in \mathbb{N}$ such that $(\rho_n)_*\mu_n = \mu_{n-1}$. We will then use universality of the limit to show that there exists a unique regular Borel measure $\nu$ on $\partial E$ such that $r_*\nu = \mu$.

First we set $\mu_0 = \mu$ as $\partial E_0 = E^0$. We recursively define $\mu_n$ by the equation

$$\mu_n = e^{-\beta}\sigma^*\mu_{n-1} + \mu|_{E^0_{\text{sng}}} - e^{-\beta}(T\mu)|_{E^0_{\text{sng}}}.$$

For Borel subsets $B \subset \partial E_n$ we read this equation as follows:

$$\mu_n(B) = e^{-\beta}(\sigma^*\mu_{n-1})(B \setminus E^0) + \mu(B \cap E^0_{\text{sng}}) - e^{-\beta}(T\mu)(B \cap E^0_{\text{sng}}). \tag{6.2}$$

Just as in Proposition 4.3.2 we have that $\sigma : \partial E_n \setminus E^0 \to \partial E_{n-1}$ is a local homeomorphism, so $\sigma^*\mu_{n-1}$ becomes a regular Borel measure on $\partial E_n \setminus E^0$ by Proposition 2.0.12.

Furthermore, since $\mu \in \mathcal{M}^{\beta}_{\text{sub}}(E^0)$ we have that $\mu - e^{-\beta}(T\mu)$ is a regular Borel measure by the inequality $T\mu \leq e^{\beta}\mu$. Hence, $\mu_n$ is a regular Borel measure.

We will prove by induction that $(\rho_n)_*\mu_n = \mu_{n-1}$. For $n = 1$ we have that $\rho_1 : \partial E_1 \to E^0$ is the map $r \sqcup id_{E^0} : E^1 \sqcup E^0_{\text{sng}} \to E^0$ and $\sigma : \partial E_1 \setminus E^0 \to E^0$ is the source map $s : E^1 \to E^0$. Hence,

$$\begin{aligned}(\rho_1)_*\mu_1 &= e^{-\beta}(r_*s^*\mu) + \mu|_{E^0_{\text{sng}}} - e^{-\beta}(T\mu)|_{E^0_{\text{sng}}}\\ &= e^{-\beta}(T\mu) + \mu|_{E^0_{\text{sng}}} - e^{-\beta}(T\mu)|_{E^0_{\text{sng}}}\\ &= e^{-\beta}(T\mu)|_{E^0_{\text{reg}}} + \mu|_{E^0_{\text{sng}}}\\ &= \mu|_{E^0_{\text{reg}}} + \mu|_{E^0_{\text{sng}}}\\ &= \mu\\ &= \mu_0.\end{aligned}$$

So assume that $(\rho_k)_*\mu_k = \mu_{k-1}$ for all $k \leq n$. To show that $(\rho_{n+1})_*\mu_{n+1} = \mu_n$ we will consider the collection

$$\mathcal{C}_n = \{Z_n(V) \subset \partial E^n \mid V \subset E^k \text{ is a Borel subset and } 0 \leq k \leq n\}.$$

Just as in Lemma 6.0.5 this is a $\pi$-system that generates the Borel $\sigma$-algebra of $\partial E_n$. The case with $V \subset E^0$ is special, so we will tackle this last.

Fix a Borel subset $V \subset E^k$ where $1 \leq k \leq n$. Note that for the set $Z_n(V)$ equation (6.2) simplifies to

$$\mu_n(Z_n(V)) = e^{-\beta}(\sigma^*\mu_{n-1})(Z_n(V)). \tag{6.3}$$

Consider the family

$$\mathcal{U}_k = \{U \subset E^k \mid U \text{ is open in } E^k \text{ and } \sigma|_U \text{ is injective}\}.$$

Since $\sigma$ is a local homeomorphism on $E^k$ we get that $\mathcal{U}_k$ is an open cover of $E^k$, in particular it is an open cover of $V$. Since $E^k$ is second-countable we get that $\mathcal{U}_k$ admits a countable subcover of $V$, denote it by $\{U_i\}_{i\in\mathbb{N}}$. By disjointing $V$ we get that $(\rho_{n+1})_*\mu_{n+1}(Z_n(V)) = \mu_n(Z_n(V))$ if $(\rho_{n+1})_*\mu_{n+1}(Z_n(U_i)) = \mu_n(Z_n(U_i))$ for all $i \in \mathbb{N}$. So we compute:

$$\begin{aligned}(\rho_{n+1})_*\mu_{n+1}(Z_n(U_i)) &= \mu_{n+1}(\rho_{n+1}^{-1}(Z_n(U_i)))\\ &= \mu_{n+1}((Z_{n+1}(U_i)))\\ &= e^{-\beta}(\sigma^*\mu_n)(Z_{n+1}(U_i))\\ &= e^{-\beta}\mu_n(\sigma(Z_{n+1}(U_i)))\\ &= e^{-\beta}\mu_n(Z_n(\sigma(U_i)))\\ &= e^{-\beta}\mu_n(\rho_n^{-1}(Z_{n-1}(\sigma(U_i))))\\ &= e^{-\beta}(\rho_n)_*\mu_n(Z_{n-1}(\sigma(U_i)))\\ &= e^{-\beta}\mu_{n-1}(Z_{n-1}(\sigma(U_i)))\\ &= e^{-\beta}\mu_{n-1}(\sigma(Z_n(U_i)))\\ &= e^{-\beta}(\sigma^*\mu_{n-1})(Z_n(U_i))\\ &= \mu_n(Z_n(U_i)).\end{aligned}$$

When $V \subset E^0$ we have that $Z_n(V) \setminus E^0 = Z_n(r^{-1}(V))$. By disjointing $r^{-1}(V)$ with sets $U \in \mathcal{U}_1$ we may assume that $\sigma|_{r^{-1}(V)}$ is injective. The same calculation as above can now be carried out, where we also have to drag along the terms involving $\mu|_{E^0_{\mathrm{sng}}}$ and $(T\mu)|_{E^0_{\mathrm{sng}}}$. So we have that $(\rho_{n+1})_*\mu_{n+1} = \mu_n$ on the $\pi$-system $\mathcal{C}_n$. Making the same argument as we did in Lemma 6.0.6, we get by Lemma 6.0.4, that $(\rho_{n+1})_*\mu_{n+1} = \mu_n$ on $\partial E_n$.

Having defined the measures $\mu_n$ we proceed to construct the measure $\nu$ on $\partial E$. Gelfand duality establishes an equivalence between the category of locally compact Hausdorff spaces and the category of commutative $C^*$-algebras. Since $\partial E$ is the limit of the directed system given by $(\partial E_n, \rho_n)_{n\in\mathbb{N}}$ by Lemma 6.0.9, we get that $C_0(\partial E) = \mathrm{colim}_n\, C_0(\partial E_n)$ by Gelfand duality. Note that the map $C_0(\partial E_{n-1}) \to C_0(\partial E_n)$ is given by precomposing with $\rho_n$. We want to show that this map restricts to a map $C_c(\partial E_{n-1}) \to C_c(\partial E_n)$. Indeed, if $f \in C_c(\partial E_{n-1})$ we have that $\rho_n^{-1}(\mathrm{supp}(f))$ is compact, since $\rho_n$ is proper by Lemma 6.0.8. Thus $f \circ \rho_n \in C_c(\partial E_n)$. Note that the same argument shows that $f \circ \rho_{n,\infty} \in C_c(\partial E)$ for every $f \in C_c(\partial E_n)$ and $n \in \mathbb{N}$.

Denote by $l_n : C_c(\partial E_n) \to \mathbb{C}$ the positive bounded linear map

$$l_n(f) = \int_{\partial E_n} f \, d\mu_n.$$

By Lemma 6.0.8 we have that $\rho_n$ is surjective for each $n \in \mathbb{N}$, so we get commutative diagrams

$$\begin{array}{ccc} C_c(\partial E_n) & \longleftarrow & C_c(\partial E_{n-1}) \\ {\scriptstyle l_n}\downarrow & \swarrow{\scriptstyle l_{n-1}} & \\ \mathbb{C} & & \end{array}$$

By continuity of $l_n$ we get that $l_n$ extends uniquely to a positive continuous *-homomorphism, which we also denote by $l_n : C_0(\partial E_n) \to \mathbb{C}$. Hence, we get commutative diagrams

$$\begin{array}{ccc} C_0(\partial E_n) & \longleftarrow & C_0(\partial E_{n-1}) \\ {\scriptstyle l_n}\downarrow & \swarrow{\scriptstyle l_{n-1}} & \\ \mathbb{C} & & \end{array}$$

By universality of the co-limit we get that there exists a unique positive continuous *-homomorphism $l : C_0(\partial E) \to \mathbb{C}$, making the following diagram commute for every $n \in \mathbb{N}$:

$$\begin{array}{ccc} & & C_0(\partial E_n) \\ & \swarrow & \downarrow{\scriptstyle l_n} \\ C_0(\partial E) & \xrightarrow{\ l\ } & \mathbb{C} \end{array}$$

Restricting $l$ to $C_c(\partial E)$ we can use Riesz representation Theorem, c.f. Theorem 2.0.13, to get that there exists a unique regular Borel measure $\nu$ on $\partial E$ such that

$$l(f) = \int_{\partial E} f \, d\nu$$

for every $f \in C_c(\partial E)$. In particular the diagram

$$\begin{array}{ccc} & & C_c(E^0) \\ & \swarrow & \downarrow{\scriptstyle l_0} \\ C_c(\partial E) & \xrightarrow{\ l\ } & \mathbb{C} \end{array}$$

commutes, i.e.

$$l(f\circ r)=\int_{\partial E} f\circ r d\nu=\int_{r(\partial E)} f d(r_*\nu)=\int_{E^0} f d(r_*\nu)=\int_{E^0} f d\mu$$

for all $f\in C_c(E^0)$. The second equality follows by Lemma 6.0.8 saying that $r=\rho_{0,\infty}$ is surjective. Hence, $r_*\nu=\mu$.

It remains to show that $\nu\in\Delta(e^{-\beta\Phi})$, which by Lemma 4.4.3 is equivalent to showing that $\sigma^*\nu=e^\beta\nu$ on $\partial E\setminus E^0$. By universality of the co-limit we have that this holds since equation (6.3) holds.

This completes the proof showing that $r_*$ is surjective. □

We then get the following result.

**Theorem 6.0.11.**
*Let $E$ be a second-countable topological graph, $\beta\in\mathbb{R}$ and $\Phi:\mathcal{G}_E\to\mathbb{Z}$ be as in Definition 4.4.1. The map $r:\partial E\to E^0$ induces an affine bijection between the sets $\Delta(e^{-\beta\Phi})$ and $\mathcal{M}^\beta_{sub}(E^0)$.*

*Proof.*
The bijection between $\Delta(e^{-\beta\Phi})$ and $\mathcal{M}^\beta_{\text{sub}}(E^0)$ is given by Lemma 6.0.6 and Lemma 6.0.10. The fact that this bijection is affine follows by the fact that the pushforward $r_*$ is linear, so convex combinations are preserved. □

Now we get a description of the extremal KMS weights on $C^*(\mathcal{G}_E)$ in terms of the sub-invariant measures on the vertex space.

**Theorem 6.0.12.**
*Let $E$ be a second-countable topological graph, $\beta\in\mathbb{R}\setminus\{0\}$ and $\Phi:\mathcal{G}_E\to\mathbb{Z}$ be as in Definition 4.4.1. There is a bijection between the following three sets:*

*(A) Extremal $\beta$-KMS weights for $\alpha^\Phi$ on $C^*(\mathcal{G}_E)$.*

*(B) Ergodic measures $\nu$ on $\partial E$ satisfying $\sigma^*\nu=e^\beta\nu$ on $\partial E\setminus E^0$.*

*(C) Extremal $\beta$-sub-invariant measures $\mu\in\mathcal{M}^\beta_{sub}(E^0)$.*

*Proof.*
The bijection between $(A)$ and $(B)$ is Theorem 5.3.7. The bijection between $(B)$ and $(C)$ is Theorem 6.0.11 along with Lemma 4.4.3. □

Chapter 7

# Conclusion

In [Sch18] Schafhauser gives a complete description of the gauge-invariant tracial states on the $C^*$-algebra of a topological graph $E$ in terms of vertex-invariant probability measures on the vertex space $E^0$. Christensen is able to use his main result from [Chr23] to generalize Schafhauser's work to give a description of tracial weights on the $C^*$-algebra of a second-countable topological graph $E$ in terms of what he calls harmonic- and boundary-measures on the vertex space $E^0$, c.f. [Chr22]. Christensen is in particular able to give a description of the gauge-invariant tracial weights and is able to prove a conjecture by Schafhauser that every tracial state on the $C^*$-algebra of a free topological graph is gauge-invariant.

We noticed that the techniques developed in these papers, c.f. [Sch18] and [Chr22], generalize to give a description of all extremal $\beta$-KMS weights for the gauge-action on the $C^*$-algebra of a second-countable topological graph. In particular, we noticed that Christensen's proof of gauge invariance of tracial weights c.f. [Chr22] Theorem 6.4, can also be used to show that there is a bijection between the extremal $\beta$-KMS weights for the gauge-action and ergodic measures $\nu$ on the boundary path space $\partial E$ satisfying $\sigma^*\nu = e^{\beta}\nu$ on $\partial E \setminus E^0$, c.f. Theorem 5.3.7. With this result in place, it remained to check that Schafhauser's techniques generalized from the case of tracial weights to the case of $\beta$-KMS weights for the gauge-action. This we did in Chapter 6. Collecting all our results gives us Theorem 6.0.12.

A lot of the proofs we have given are inspired by the existing proofs by Schafhauser and Christensen. We do, however, use the theory we developed in Chapter 2 to great extent. Realizing that sheaf theory would be useful to compare measures on a second-countable space has allowed us to prove many things without invoking the duality between measures and functionals. Furthermore, since many of our results are of the type "two measures are equal", most of the challenge in proving these results is to find a suitable open cover of our space and check if these measures are equal locally.

If time had permitted, it would have been interesting to further study the sheaf structure of regular Borel measures. There are in particular many algebraic invariants one can compute given a sheaf. On first sight, it is a problem that the set of regular Borel measures does not have the structure of an abelian group. It does, however, have the structure of an abelian monoid, so it would be possible to consider the associated Grothendieck group. It would be interesting to see how far one could apply sheaf-theory in this case.

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