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Localized Mixed States

A Characterization of Strictly Localized Mixed States
with Examples in Quantum Optics

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Abstract

When states are generated on-demand in a laboratory, we expect that the states are confined to that laboratory immediately after they are generated. Mathematically, we represent the confined states through strictly localized states. We introduce and use algebraic quantum field theory (QFT) to define and characterize strictly localized mixed states, with specific examples in quantum optics.

In experiments, classical uncertainties or environmental interactions are present. Hence, a suitable representation of a state in an experiment is a mixed state. We present a notion of strict locality for mixed states and characterize the quantum operations that generate strictly localized mixed states from the vacuum. We show that the generalization from pure states is non-trivial by demonstrating that the set of all strictly localized mixed states is larger than the set of states generated by mixes of strictly localized pure states. Additionally, we find a criterion for when strictly localized mixed are expressible as a mixture of strictly localized pure states.

The results mentioned above hold for any QFT that satisfies the usual postulates of a local field theory. One such theory is the quantum optics field theory. After demonstrating this fact, we present two examples of strict locality for mixed states. The first is an explicit example of a strictly localized mixed state that cannot be expressed as a mixture of strictly localized pure states. In the second application, we attempted to create a strictly localized mixed state close to a photon. We get a maximum fidelity of ~ 0.2 between the strictly localized mixed state and photon. In contrast, J. Gulla and J. Skaar [GS21a] achieved a fidelity of ~ 1 using strictly localized pure states.

As an introductory example, we present negative energy density in QFT to generate a state strictly localized with respect to energy density. In particular, we created a state that is a mixture of a photon and squeezed vacuum state to yield a vanishing energy density for negative times.

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Preface

This report represents the work done for a master's degree in theoretical physics. I have aimed to write at a level that is accessible to a second-year master's student with basic knowledge of analysis, while also introducing the topics in a way that encourages readers to explore them further.

The second chapter and the first section of the third chapter provide a summary of existing theory, adapted to fit the context of my main work. The last section of the third chapter and most of the fourth chapter consist of my original research, and any known overlap with existing theory is addressed.

I want to express my deep gratitude to my advisor, Johannes Skaar, for his unwavering support and guidance throughout this project. Johannes has always been available to offer valuable feedback and engage in helpful discussions, while also giving me the freedom to explore topics that interest me. I have always felt respected and supported in his presence and hold him in the highest regard.

I am also immensely grateful to Nils Johannes Mikkelsen, whose assistance was vital to the progress of this work. The mathematical aspects of this thesis were well beyond my level of expertise, and Nils provided me with relevant materials and patiently helped me understand the literature. Our discussions over the past two years have been a source of great joy and inspiration.

I also want to thank Carl Martin Fevang for helping me proofread the thesis — your English skills came in clutch.

Finally, I would like to thank my friends and family for their constant support and humor throughout this journey, with a special shout-out to mamma, pappa, Peder, teoriseksjonen, Blindern Studenterhjem, BSF and Ninjas.

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Contents

Chapter 1

Introduction

1.1 Quantum Physics

Quantum physics is an umbrella term for all physical processes at the atomic scale. The term "quantum" stems from nature tending to operate in discrete quantities — or quanta — at small scales. The study of quantum physics started in the 1900s when Max Planck developed a correct model of black-body radiation by quantizing light. This was the beginning of what many refer to as the first quantization. Further work by Albert Einstein on the photoelectric effect, and Niels Bohr on energy levels of atoms, further supported the quantization of matter. This work sparked significant innovation in the description of quantum physics, with Erwin Schödinger formulating his famous wave equation in 1925. His equation allowed us to describe matter through wave functions and probability. However, the equation was not compatible with Einstein's theory of special relativity. Paul Dirac tried to resolve this in 1928 when he wrote down his relativistic wave equation, which described massive spin-1/2 particles. Following experimental observations, there was a needed have a description of matter with a dynamic number of particles. That led to what many refer to as the second quantization, where we quantize relativistic fields. We call the quantization of relativistic fields quantum field theory (QFT).

Within QFT, we say that there are many different QFTs, each with its purpose. Many of this thesis' results will apply to any QFT defined on a separable Hilbert space. However, when we investigate specific examples, we will work with a QFT associated with quantum optics, in which the electromagnetic field is quantized. This formalism is convenient, as it deals with free and non-charged fields. In addition, we can effectively ignore the polarization in some applications, which makes the formalism effectively scalar. Furthermore, it is well-defined in the sense that operators and the resulting Hilbert space can be mathematically defined. The simple and rigorous nature of this formalism has been beneficial when coming up with examples and applications. Quantum optics is also a highly researched topic with many real-world applications, for instance, within measurement and information technology [GK04].

1.2 Strict Locality

Classical field theory is an example of a local theory, because field configurations at one point are only directly influenced by the immediate surroundings. This behavior contrasts Newtonian mechanics, where objects separated in space can interact instantaneously. Most modern theories are local because they satisfy causality. The common characteristic of causal theories is that information cannot travel faster than the speed of light in a vacuum. In particular, causality means that two spacetime regions that are space-like separated cannot interact.

Imagine we have a causal theory and some on-demand source that generates a state. This source could for instance generate light. If somebody turns the source on, then the state generated should not be measurable outside its light cone. Otherwise, we would violate causality. Such states are what we call strictly localized. In other words, we argue that any on-demand generated state should be strictly localized. Several experiments involve generating states on-demand, which is why it is important to characterize such states in a local theory.

Quantum field theory is another local theory, and the local structure is often captured through an equal-time commutator relation. The commutator relation guarantees that space-like separated field configurations commute. Usually, one treats the field operators as building blocks of the theory. Hence, it might seem intuitive that this would manifest as a local structure. We will explicitly demonstrate this by defining what we mean by local operations and measurements.

Fields are local; however, they are not valid operators on a Hilbert space before we integrate them against a suitable test function. Strictly speaking, this means that the field operators are neither observable nor represent operations which an experimentalist can perform. That is why the first part of this thesis aims to introduce QFT formulated in terms of local operators and observables, sometimes referred to as algebraic QFT.

With local operators and observables defined, we can define what one means by strict locality in QFT. For now, view a strictly localized state as something that looks like vacuum outside its localization region. In other words, one cannot distinguish the strictly localized state from vacuum with local measurements outside the localization region. Our mission is to describe the structure of such states and develop examples which help concretize the formalism. Finding examples is also relevant for experimental verification. Although people have done this for pure states [Kni61][Lic63][GS21a], we will extend the formalism to mixed states.

When we first started to investigate strict locality for mixed states, we wondered if one could generate localized mixes that consist of non-local states. To capture this idea, we wanted to see if having a localized mixed state with respect to one observable is possible, even though it is a mix of non-localized states with respect to the same observable. We chose the observable associated with energy density. This choice is because energy density is an observable often used in quantum optics. In addition, since the associated operator is normal-ordered, we can make the energy density

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negative. The hypothesis was that we could have some state with positive energy density and compensate by mixing in a state with negative energy density.

1.3 Structure

This thesis contains four main chapters:

- Chapter 2: Background. Here we cover preliminary material, often not taught in master-level courses. We define our units and some relevant notation (Section 2.1). Following this, we address the quantization process, define operators and operator-valued distributions, and use this formalism to generate local operators and algebras (Section 2.2). Last, we briefly introduce mixed states and characterize operations on such states (Section 2.3).

- Chapter 3: Negative Energy Density. This chapter explores negative energy density in quantum optics (Section 3.1) and mixed states localized with respect to energy density (Section 3.2).

- Chapter 4: Strict Localization. In this chapter, we give a brief introduction on strict localization for pure states (Section 4.1) and discuss more in-depth the importance of the formalism (Section 4.2). Following this, we extend the formalism to mixed states (Section 4.3) and briefly address the overlap with existing literature. Last but not least, we present two example states in the quantum optics formalism (Section 4.4 and 4.5).

- Chapter 5: Epilogue. The purpose of this chapter is to give a summary of the results and their importance (Section 5.1) and list some avenues for further work (Section 5.2).

Chapter 2

Background

In this chapter, we introduce the theory used throughout the thesis. We have focused on topics not commonly taught in master-level courses. In particular, the local algebra formulation of QFT was unfamiliar to the author before this project began. The literature on the said topic is highly mathematical, so this chapter aims to introduce the topic to someone with rudimentary analysis knowledge. To better understand this formalism, we found it helpful to concretize the space of states and the corresponding operators. Hence, the section on quantization and operators is quite detailed.

First, we briefly define the basic conventions and notation. Following this, we have a section on quantization. Specifically, we construct the bosonic Fock space, define operators, operator-valued distributions, field operators and local operator algebras. We conclude the background chapter by covering mixed states and quantum operations for infinite dimensional Hilbert spaces.

2.1   Units and Notation

The following are some basic notations and conventions. In this project, we use natural units:

$$c = \hbar = \epsilon_0 = \mu_0 = 1 \text{ and } e = \sqrt{4\pi\alpha}, \tag{2.1}$$

where μ_0 is the vacuum permeability, α the fine-structure constant, c the speed of light in vacuum, $\hbar$ the reduced Planck constant and ϵ_0 the vacuum permittivity. The following are common notation:

- $\mathcal{H}$ is a separable complex Hilbert space.

- H is the Fock space generated from $\mathcal{H}$.

- If $a \in \mathbb{C}$, then a^* is its complex conjugate.

- $\mathcal{O}$ denotes a bounded open region in Minkowski spacetime.

- We will make use of the Dirac notation. In this formalism, states and the inner products are denoted as $|\psi\rangle$ and $\langle\psi|\phi\rangle$ respectively.

- The Fourier transform of a function f is defined as

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt, \qquad (2.2)$$

and the factor $1/2\pi$ appears in the inverse Fourier transform.

2.2   Quantization, Operators and States

There are many ways of constructing quantum field theories. Introductory textbooks usually quantize through canonical quantization. Roughly speaking, this involves imposing a commutation relation on the classical fields, promoting them to operator-valued distributions and demanding that they create "particles" when acting on some arbitrary vacuum state. This section aims to demystify many of these notions. Specifically, we want to address what space our particles live in, operator-valued distributions and how this relates to the formalism used in quantum optics, where one treats *free* massless spin-1 bosons, namely photons.[1] We will also define what we mean by local operations, which will be helpful when we define strict locality.

The structure will be as follows: First, we construct the bosonic Fock space using single-particle Hilbert spaces. Second, we define operators on Fock space and use them to interpret operator-valued distributions. We base these subsections on the book [Tal22] by Michel Talagrand, and one can find all the proofs there.[2] Third, we impose the canonical commutation relation to promote the electric- and magnetic field to operator-valued distributions. Last but not least, we will briefly cover local operators and operator algebras, and describe their connection to local operations.

2.2.1   Bosonic Fock Space

We need a theory that can handle a dynamic number of particles. The usual way of doing this is through the procedure known as second quantization to get the so-called Fock space. This Hilbert space allows us to create and annihilate particles using ladder operators. Furthermore, because it is a Hilbert space, we retain many of the properties that we enjoy in non-relativistic quantum mechanics. Constructing the Fock space is technical; however, in this subsection, we will do it more intuitively by omitting mathematical subtleties that add unnecessary complexity.

We begin by constructing an n-particle Hilbert space, namely the n-fold tensor product $\mathcal{H}^{\otimes n}$. For $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle \in \mathcal{H}$, we want a multilinear tensor product $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \in \mathcal{H}^{\otimes n}$. By multilinear, we mean that it is linear in each factor. This construction can be created in a

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[1]Purists might argue that free theories are unphysical. However, free theories are used to describe actual experiments. In addition, when evaluating scattering in interaction theories, the asymptotic in- and out states behave as states in a free theory.

[2]The book by Talagrand is highly recommended for people who take issue with authors "hiding" mathematical issues and subtleties. In the book, Talagrand points out what is rigorous and not, and when we perform mathematical errors.

coordinate-independent way; however, we do this on a given basis.[3] Let $\{|e_i\rangle\}$ be an orthonormal basis of $\mathcal{H}$. We consider the set of all possible linear combinations

$$\sum_{i_1,\ldots,i_n} \alpha_{i_1\ldots i_n} |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_n}\rangle, \quad \sum_{i_1,\ldots,i_n} |\alpha_{i_1\ldots i_n}|^2 < \infty. \qquad (2.3)$$

On these linear combinations, we define addition and multiplication in the natural way. Moreover, we define an inner product

$$\begin{aligned}
&\left(\sum \alpha_{i_1\ldots i_n} |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_n}\rangle, \sum \beta_{i_1\ldots i_n} |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_n}\rangle\right) \\
&:= \sum \alpha^*_{i_1\ldots i_n} \beta_{i_1\ldots i_n}.
\end{aligned} \qquad (2.4)$$

The set of such states, with the inner product defined above, forms a Hilbert space. Hence, the map $(|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle) \mapsto |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \in \mathcal{H}^{\otimes n}$, is well-defined.

Although there are two kinds of elementary particles — bosons and fermions — we will only explicitly deal with bosons in this thesis. Bosons follow Bose-Einstein statistics, so we need to symmetrize the space to get rid of unphysical states. We denote the space $\mathcal{H}^{\otimes n}_{\text{sym}}$ to be all elements of the form (2.3) such that

$$\alpha_{i_1\ldots i_n} = \alpha_{i_{\sigma(1)}\ldots i_{\sigma(n)}}, \qquad (2.5)$$

for all $i_1 \ldots i_n$ and $\sigma \in S_n$. Here S_n is the group of all permutations of $1, \ldots, n$ and the parenthesis denotes elements within σ.[4] Physically, this means that we do not distinguish particles, and thus not permutations of said particles either. We have that $\mathcal{H}^{\otimes n}_{\text{sym}}$ is a closed subspace of $\mathcal{H}^{\otimes n}$ and therefore also a Hilbert space.

For later convenience, we can define a basis of $\mathcal{H}^{\otimes n}_{\text{sym}}$. Let $\{|e_i\rangle\}$ be a basis of $\mathcal{H}$ and $m_1, m_2, \ldots, m_n$ a sequence of numbers between one and n, with the possibility of repeating numbers. Also, define n_i as the number m_js that are equal to i (this gives $\sum_i n_i = n$). Then, the set of all states of the form:

$$|n_1, n_2, \ldots\rangle := C(n) \sum_{\sigma \in S_n} \left|e_{m_{\sigma(1)}}\right\rangle \otimes \cdots \otimes \left|e_{m_{\sigma(n)}}\right\rangle, \quad \sum_i n_i = n, \qquad (2.6)$$

where $C(n)$ is some normalization constant which one can find through combinatorics — forms an orthonormal basis of $\mathcal{H}^{\otimes n}_{\text{sym}}$.

So far, we have introduced abstract notation and concepts. To make sense of this construction, let us give an example. Let $\mathcal{H}$ be a single-particle Hilbert space, with some basis $\{|i\rangle\}$.[5] Then, the state $|n_1, n_2, \ldots\rangle \in \mathcal{H}^{\otimes n}_{\text{sym}}$ represents having n_1 bosons in state $|0\rangle$, n_2 bosons in the state $|1\rangle$ and so

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[3]To show this in a coordinate-independent way, we can use isomorphisms between Hilbert spaces. That is possible because all separable and infinite-dimensional Hilbert spaces are isomorphic.

[4]For example $\sigma = \{1, 3, 2\}$ is an element in S_3 with $\sigma(2) = 3$.

[5]These states could for instance be photon pulses or energy eigenstates in regular non-relativistic quantum mechanics.

forth. In other words, the numbers n_i represent the number of particles, and their position represents each particle's state.

Now we are ready to define the bosonic Fock space. Let H be the set of sequences $(|\psi_n\rangle)_{n=0}^{\infty}$ where for each n we have $|\psi_n\rangle \in \mathcal{H}_{\text{sym}}^{\otimes n}$, and the sequence satisfies $\sum_n \||\psi_n\rangle\|^2 < \infty$. Moreover, define the inner product on H as

$$\langle \psi | \phi \rangle := \sum_n \langle \psi_n | \phi_n \rangle. \tag{2.7}$$

This inner product makes H a Hilbert space and is what we refer to as the bosonic Fock space. In this space we interpret $\|\psi_i\|^2 / \sum_n \|\psi_n\|$ as the probability of having i particles. Additionally, since we have a basis for each $\mathcal{H}_{\text{sym}}^{\otimes n}$, this naturally induces a basis of H. We can denote those basis-elements

$$|n_1, n_2, \ldots\rangle, \quad \sum_{i=1}^{\infty} n_i < \infty, \tag{2.8}$$

with the natural understanding that

$$|n_1, n_2, \ldots\rangle = (0, \ldots, \underbrace{|n_1, n_2, \ldots\rangle}_{\text{position } \Sigma_i n_i}, 0, \ldots). \tag{2.9}$$

Note that this basis depends on an initial choice of basis for $\mathcal{H}$. Also, different Hilbert spaces $\mathcal{H}$ will induce a different Fock spaces H. This difference comes up in Section 2.2.4 when we perform a choice of gauge.

Before defining operators on Fock space, we will quickly define the vacuum state. Let $\mathcal{H}^{\otimes 0} := \mathbb{C}$ be the zero-particle space with the number one as the basis. Formally speaking, the vacuum in H then is $(1, 0, 0, \ldots)$; however, we will just write $|0\rangle$.

2.2.2 Operators on Fock Space

Now that we have the Fock space for bosons, we will define the corresponding operators. First and foremost, since the operators are going to be unbounded, we will define them on a dense subspace. This unboundedness is a subtle but significant detail, as unbounded operators cannot be defined on the entire Hilbert space.[6] Let H_0 be the dense subspace defined by

$$H_0 := \{(|\psi_n\rangle)_{n=0}^{\infty} \in H : |\psi_n\rangle \neq 0 \text{ for a finite number of } n\}. \tag{2.10}$$

To define the operators on H_0 it is sufficient to look at how they act on the corresponding basis. H_0 shares the same basis as H (2.9).

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[6]This is not limited to QFT. Let $\mathcal{H}$ be the Hilbert space for the harmonic oscillator and $\{|n\rangle\}$ its energy eigenstate basis. For a state $|\psi\rangle = \sum_n c_n |n\rangle$, if c_n tends slowly enough to zero, then the Hamiltonian will map $|\psi\rangle$ outside $\mathcal{H}$.

Definition 2.2.1. Fix a basis for $\mathcal{H}$ and let H_0 be defined by (2.10). For each $k \geq 1$ define the linear operators $a_k^\dagger : H_0 \to H$ and $a_k : H_0 \to H$ by

$$a_k^\dagger \left| n_1, n_2, \ldots \right\rangle = \sqrt{n_k + 1} \left| n_1, n_2, \ldots, n_k + 1, \ldots \right\rangle, \tag{2.11a}$$

$$a_k \left| n_1, n_2, \ldots \right\rangle = \begin{cases} \sqrt{n_k} \left| n_1, n_2, \ldots, n_k - 1, \ldots \right\rangle & \text{if } n_k \geq 1, \\ 0 & \text{if } n_k = 0. \end{cases} \tag{2.11b}$$

The following theorem let us characterize the operators defined in Definition 2.2.1:

Theorem 2.2.1. *The operators $a_k^\dagger$ and a_k satisfy*

$$\text{For } n \geq 0, \quad a_k^\dagger \text{ maps } \mathcal{H}_{sym}^{\otimes n} \text{ into } \mathcal{H}_{sym}^{\otimes n+1}. \tag{2.12a}$$

$$\text{For } n \geq 1, \quad a_k \text{ maps } \mathcal{H}_{sym}^{\otimes n} \text{ into } \mathcal{H}_{sym}^{\otimes n-1}. \tag{2.12b}$$

$$\text{For all } \left| \psi \right\rangle, \left| \phi \right\rangle \in H_0, \quad \langle \psi a_k^\dagger | \phi \rangle = \langle \psi | a_k \phi \rangle \tag{2.12c}$$

$$[a_k, a_l^\dagger] = \delta_{kl} \tag{2.12d}$$

$$[a_k, a_l] = 0 \tag{2.12e}$$

Proof. See [Tal22]. $\qquad\square$

In Theorem 2.2.1, the equations (2.12a) and (2.12b) indicate that $a_k^\dagger$ and a_k have the physical interpretation of creating and annihilating particles respectively. Equation (2.12c) makes a_k the adjoint of $a_k^\dagger$, thus making sense of the notation. Lastly, the commutation relations (2.12d) and (2.12e) are similar to those of the harmonic oscillator, giving us the popular notion of this being a space consisting of infinite harmonic oscillators.

2.2.3 Operator-Valued Distributions

Now we are ready to define operator-valued distributions. When integrated against a suitable test function, operator-valued distributions yield a linear operator on H_0. Again, it is possible to do this for general Hilbert spaces and in a coordinate-independent way; however, we will fix $\mathcal{H}$ as the space of square-integrable functions ($\mathcal{H} = L^2(\mathbb{R})$).[7] Let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}$. For any square-integrable function $f \in \mathcal{H}$, we can write it in terms of the basis: $f = \sum_i \alpha_i e_i$. Define the linear operators

$$A(f) = \sum_{i=1}^\infty \alpha_i^* a_i \text{ and } A^\dagger(f) = \sum_{i=1}^\infty \alpha_i a_i^\dagger. \tag{2.13}$$

These operators satisfy Theorem 2.2.1, replacing a_k with $A(f)$, $a_k^\dagger$ with $A^\dagger(f)$ and $[A(f), A^\dagger(g)] = \langle f | g \rangle$. Note that for $f = e_k$ we recover $A(f) = a_k$.

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[7]Some readers might recognize this as the space of wave-functions — or states if you will — used in non-relativistic quantum mechanics.

Now we will define the operator-valued distributions $a(k)$ and $a^\dagger(k)$:

$$\int dk\, f(k)^* a(k) := A(f) \text{ and } \int dk\, f(k) a^\dagger(k) := A^\dagger(f). \qquad (2.14)$$

Strictly speaking, this means that it does not make sense to treat $a(k)$ and $a^\dagger(k)$ as singular objects as one cannot separate them from their definition. However, we still use them as notational devices, understanding that one should integrate them later. Using the definition (2.14) and $[A(f), A^\dagger(g)] = \langle f|g \rangle$ we get

$$\int dk\, f(k)^* g(k) = \int dk\, dk'\, f(k)^* g(k')[a(k), a^\dagger(k')], \qquad (2.15)$$

re-discovering the popular notion that $[a(k), a^\dagger(k')] = \delta(k - k')$.

2.2.4 Quantizing the Fields

The solution of Maxwell's equations in vacuum, using the Coulomb gauge is (we derive this in Appendix A)

$$\mathbf{E}(\mathbf{r}, t) = \int_{\text{all } k\text{-space}} d^3\mathbf{k}\, \mathcal{E}(\omega) \sum_{l=1}^{2} \left[a_l(\mathbf{k})\mathbf{e}_l(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{h.c.} \right], \qquad (2.16a)$$

$$\mathbf{B}(\mathbf{r}, t) = \int_{\text{all } k\text{-space}} d^3\mathbf{k}\, \mathcal{E}(\omega) \sum_{l=1}^{2} \left[a_l(\mathbf{k})\hat{\mathbf{k}} \times \mathbf{e}_l(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{h.c.} \right] \qquad (2.16b)$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively, $\mathcal{E}$ is for normalization, ω the angular frequency, $\mathbf{k}$ the wave vector, $\mathbf{e}_l$ polarization vectors and h.c. denotes the hermitian conjugate. Fixing the normalization $\mathcal{E}(\omega) = K\sqrt{\omega}$, where $K > 0$ is real, we can impose the canonical commutation relation [CDG97]

$$[a_i(\mathbf{k}), a_j^\dagger(\mathbf{k}')] = \delta_{ij}\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \qquad (2.17a)$$

$$[a_i(\mathbf{k}), a_j(\mathbf{k}')] = [a_i^\dagger(\mathbf{k}), a_j^\dagger(\mathbf{k}')] = 0. \qquad (2.17b)$$

This promotes a_i and $a_i^\dagger$ to operator-valued distributions as defined by (2.14).

Before continuing, we need to address the choice of gauge. The Coulomb gauge does not leave our equations manifestly Lorentz invariant. We accept this because optical experiments often are described using a single reference frame. Furthermore, when choosing the Coulomb gauge, we set the longitudinal components of the classical fields to zero. If we were to choose another gauge — say Lorenz gauge — we would also get longitudinal ladder operators. The consequence of this, is that the resulting Fock space would contain more states and operators than in Coulomb gauge, and give us a seemingly different physical theory. However, this goes against the notion that choosing a gauge should not change the physics. It is possible to show that observables are independent of said gauge transformations [LIS17].

The expressions in (2.16a) and (2.16b) are cumbersome. In optical experiments, the states often travel in a fixed direction — say x-direction —

and with a fixed polarization. This means that we do not expect the presence of states moving in y- or z-direction, nor have any other polarization present in the states. We say that modes not present in a state are in the vacuum state.

When an operator is normal ordered (or Wick ordered), all creation operators are to the left of all annihilation operators. The physical interpretation is that normal ordering removes vacuum contribution. Hence, modes that are in the vacuum state does not contribute to the expectation value of a normal ordered operator. This means that although the field operators are given by (2.16a) and (2.16b), we can effectively use ones without any y- or z-dependence and remove one of the polarization directions. We emphasize that this only works when the operators are normal ordered, which removes all the vacuum contributions. Hence, for states moving in positive x-direction and with a fixed polarization, the effective operators become (splitting them into a positive- and negative frequency part):

$$E = E^+(x,t) + E^-(x,t), \quad B = B^+(x,t) + B^-(x,t), \qquad (2.18)$$

where

$$E^+(x,t) = \int_0^\infty d\omega \, \mathcal{E}(\omega) a(\omega) e^{-i\omega(t-x)} \text{ and} \qquad (2.19a)$$

$$B^+(x,t) = \int_0^\infty d\omega \, \mathcal{E}(\omega) a(\omega) e^{-i\omega(t-x)}. \qquad (2.19b)$$

Here we removed the vector notation, with the understanding that the E-field is perpendicular to the B-field, and the integration is from zero to infinity because we only allow positive frequencies and modes moving in positive x-direction.

We can also fix an effective underlying Hilbert space. Let $\mathcal{H} = L^2(0,\infty)$ and choose an orthonormal basis $\{\xi_i(\omega)\}_i$ that satisfies [Blo+90]

$$\sum_i \xi_i^*(\omega)\xi_i(\omega') = \delta(\omega - \omega'). \qquad (2.20)$$

Then, we can introduce the operators $A(\xi_k) \equiv a_k$ and $A^\dagger(\xi_k) \equiv a_k^\dagger$ given by equation (2.14); which also satisfy Theorem 2.2.1. We will refer to this as pulse mode formalism. Note that in this formalism, the state $a_1^\dagger |0\rangle = |1_1\rangle$ denotes an arbitrary pulse. This is because we can choose ξ_1 to be any normalized function in L^2, and iteratively find the basis by an orthogonalization process. This formalism also allows us to write (2.19a) in a much simpler form. If we fix $x = 0$, E^+ becomes:

$$E^+(t) = \sum_i E_i(t) a_i, \qquad (2.21)$$

where

$$E_i(t) = \int_0^\infty d\omega \, \mathcal{E}(\omega)\xi_i(\omega) e^{-i\omega t}. \qquad (2.22)$$

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2.2.5 Local Algebra of Operators

In this subsection, we define local operators and algebras thereof following the convention in Haag's book on local quantum physics [Haa96]. We also state some mathematical properties of said algebras. For mathematicians, since they have to deal with subtle details, this subject is complicated and technical. However, for physicists it is relatively simple. We want a mathematical structure where we can add and multiply operators together, which is why we want an algebra. Also, we want those operators to be associated with some local operation, hence local operators. Combining those two properties, we get an algebra of local operators. The note [Wit18] by Edward Witten provides an excellent introduction to this subject with fewer mathematical details.

As mentioned, $\mathcal{O}$ will always denote a bounded open region in Minkowski spacetime. The following defines what we mean by local operators:

Definition 2.2.2. Let Φ be an arbitrary field operator. We refer to the smeared field,

$$\Phi_f = \int d^4x \, f(x)\Phi(x), \qquad (2.23)$$

as a *local operator* — with localization region $\mathcal{O}$ — if f vanish whenever x lies outside $\mathcal{O}$. In other words, f has *compact support* in $\mathcal{O}$.

The idea is that these operators represent local operations we can perform inside the region $\mathcal{O}$. It would be convenient to consider the algebra generated by products and sums of local operators — this is called a polynomial algebra. However, since the local operators can be unbounded, we run into some technical problems. As we mentioned when defining operators on Fock space, unbounded operators can only be defined on a dense subset. Moreover, different operators need not be defined on the same subset. This leads to issues when we want to find the domain after multiplying and adding unbounded operators. Because of this, we want to consider bounded operators instead.

We can restrict the spectrum of an operator using a bounded function. Given an observable, mathematically represented by a self-adjoint operator A, with a possibly unbounded spectrum $\sigma(A)$, and with a decomposition

$$A = \int_{\sigma(A)} d\lambda \, \lambda \, |\lambda\rangle \langle\lambda| . \qquad (2.24)$$

The claim is that we can, *without loss of generality*, consider the bounded version of the operator

$$f(A) := \int_{\sigma(A)} d\lambda \, f(\lambda) \, |\lambda\rangle \langle\lambda| , \qquad (2.25)$$

where f is some bounded function. The physical argument here is that this is only a relabel of the possible measurement outcomes. Therefore, we should not expect this to affect any physical theory. In fact, we do not expect to ever

measure an unbounded quantity in practice; hence, this is not only practical, but something we should demand from a physical theory.

With bounded versions of local operators, we are ready to define local algebras:

Definition 2.2.3. Associated with any region $\mathcal{O}$, there is a *local algebra* $\mathcal{A}(\mathcal{O})$. This is the *weakly closed polynomial algebra*, generated by the *bounded* versions of *local operators*. We say a sequence $\{A_i\}$ converge *weakly* if all matrix elements $|\langle\psi| A_i |\phi\rangle|$ converge.

Remark. *There are several ways of defining algebras consisting of bounded operators on a Hilbert space. The choice of algebra also has physical significance. The details are not important for our purposes, nor something the writer is familiar with. Also, it is worth mentioning that this structure forms a C*-algebra, and is a widely studied structure in mathematics.*

We will discuss the physical intuition of the local algebras further when we introduce the concept of strict locality. For now, we want to state some properties of local algebras. First and foremost, from Definition 2.2.3 we get isotony ($\mathcal{O}_1 \subseteq \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$) and causality ($\mathcal{O}_1$ is space-like separated from $\mathcal{O}_2 \Rightarrow [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$). These properties are typical postulates in an algebraic QFT. Also, the following two theorems state what mathematicians call the *cyclic* and *separating* quality of the vacuum.

Theorem 2.2.2. *[The Reeh-Schlieder theorem] The vacuum state $|0\rangle$ is a* cyclic *vector for the local algebra $\mathcal{A}(\mathcal{O})$. In other words, the subspace generated by using the entire algebra on the vacuum, $\mathcal{A}(\mathcal{O})|0\rangle$, is dense in H.*

Proof. This was originally proved by Reeh and Schlieder [RS61] using fields; however, this has been extended by H. Araki [Ara99] to general QFTs that satisfy the local algebra formulation. $\square$

Theorem 2.2.3. *The vacuum state $|0\rangle$ is* separating *for $\mathcal{A}(\mathcal{O})$. This means that for $L \in \mathcal{A}(\mathcal{O})$, then $L|0\rangle = 0$ if and only if $L = 0$.*

Proof. See [Ara99]. $\square$

The non-local effects of Theorem 2.2.2 might seem paradoxical. The fact that we can get arbitrarily close to *any* state by using operators in a local algebra seems to violate causality. It turns out that this does not violate causality, see e.g. [Wit18] for a discussion of the non-local effects of Theorem 2.2.2.

2.3 Mixed States and Quantum Operations

The main subject of the thesis will be strict localization for mixed states. Mixed states represent classical uncertainties in quantum systems or, equivalently, entanglement with an auxiliary system. Hence, we will need some formalism related to quantum information theory. Although this subject is vast, we will only touch on a few aspects in this section. Specifically, we will define the space of mixed states, what they represent physically and valid operations on said states. A lot of the literature on this subject treats finite-dimensional Hilbert spaces. When one moves to the infinite-dimensional case, one must be careful about the convergence of limits. However, we will see that most of the results that we will use, extends from finite- to infinite dimensions. The primary source for the following two subsections is the book [NC00].

2.3.1 Mixed States

As mentioned above, mixed states describe systems whose state is not entirely known. For example, assume a system could be in any number of states $\{|\psi_i\rangle\}$, and that it is possible to assign each state with a probability p_i. This would be a classical mixture of quantum states and is what we refer to as mixed states. We represent such states through the density operator:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|, \quad \sum_i p_i = 1. \tag{2.26}$$

Hence, for each mixed state, there is a corresponding density operator representation ρ. However, the representation $\{|\psi_i\rangle\}$ of ρ is not unique, as we will demonstrate in Theorem 2.3.1.

The reason that this representation is convenient is that we can calculate the expectation value of an operator using the trace. Given any orthonormal basis $\{|n\rangle\}$, we define the trace of an operator as

$$\mathrm{tr}[A] := \sum_n \langle n| A |n\rangle. \tag{2.27}$$

Then, given a system represented by a mixed state ρ, the expectation value of an observable B is

$$\langle B\rangle = \sum_i p_i \langle\psi_i| B |\psi_i\rangle = \mathrm{tr}[\rho B]. \tag{2.28}$$

Mixed states are what we call *trace class* — i.e. they have a well-defined trace — with a trace equal to one. The set of density operators is a subset of bounded linear operators on H, closed under *convex* sums. That is, for mixed states ρ_1 and ρ_2, then

$$a\rho_1 + b\rho_2, \text{ where } a, b > 0 \text{ and } a + b = 1, \tag{2.29}$$

is a mixed state. For those interested in the mathematical structure of trace class operators on infinite-dimensional Hilbert spaces, we recommend Christa

Hawthorne's note "A brief introduction to trace class operators", which can be found on her website.[8]

As mentioned, a mixed state does not have a unique representation. In finite dimensions, it is well known that we can relate any representation of a mixed state through a unitary matrix. In infinite dimensions, one can relate density operators through partial isometries.[9] This is captured through the following theorem:

Theorem 2.3.1. *Let $\rho = \sum_i p_i |g_i\rangle \langle g_i|$ be the spectral decomposition of a mixed state. We have that $\rho = \sum_i q_i |f_i\rangle \langle f_i|$ if and only if*

$$\sqrt{q_i} |f_i\rangle = \sum_j u_{ij}^* \sqrt{p_j} |g_j\rangle, \quad u_{ij}^* = \langle g_i| U |g_j\rangle^* = \langle g_j| U^\dagger |g_i\rangle, \quad (2.30)$$

where U is a partial isometry such that the range of $U^\dagger U$ contains the range of ρ.

Remark. *If the range of ρ is finite, one can choose U to be unitary. Also, note that we do* not *require $\langle f_i|f_j\rangle = \delta_{ij}$.*

Proof. This is a rewrite of the result presented in [Had81]. $\qquad\square$

2.3.2 Quantum Operations

The quantum operation formalism is a powerful tool used to describe general transformations and evolution of mixed states. We will need quantum operations to define specific maps of mixed states. Using the axiomatic approach, we have that a quantum operation $\mathcal{E}$ satisfies the following axioms:[10]

Axiom 1. $\mathcal{E}$ maps mixed states to trace-class operators, and satisfy $0 \leq \mathrm{tr}[\mathcal{E}(\rho)] \leq 1$ for all mixed states ρ.

Axiom 2. $\mathcal{E}$ is a *convex-linear map*. That is, $\mathcal{E}$ acts linearly on *convex* sums of mixed states:

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i). \quad (2.31)$$

Axiom 3. The map $\mathcal{E}$ is *completely positive*.

Remark. *Complete positivity means that if we introduce a second, finite-dimensional system R of arbitrary dimensionality, then the map $1_R \otimes \mathcal{E}$ is positive on the joint system $R \otimes H$, where 1_R is the identity on R. A map is called positive if it maps positive operators to positive operators.*

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[8]https://cdchawthorne.com/writings/

[9]We remind the reader that an operator U is a partial isometry if it is an isometry ($U^\dagger U = 1$) on the orthogonal complement of its kernel ($\ker(U)^\perp$).

[10]We explain the physical interpretation below.

In the first axiom we demand that $0 \leq \mathrm{tr}[\mathcal{E}(\rho)] \leq 1$ rather than $\mathcal{E}$ being *trace-preserving* $(\mathrm{tr}[\mathcal{E}(\rho)] = 1)$. This axiom is convenient for several reasons. For example, if we want $\mathcal{E}$ to represent some measurement, then $\mathrm{tr}[\mathcal{E}(\rho)]$ is interpreted as the probability of measurement outcome given the mixed state ρ. By not normalizing the post-measurement state, the operation can be linear.

Axiom 2 follows from classical probability theory. Suppose a mixed state ρ is obtained by randomly selecting other mixed states. We can represent such states through a convex sum: $\rho = \sum_i p_i \rho_i$, where p_i is the probability that we picked state ρ_i. If we were to perform a quantum operation $\mathcal{E}$ on ρ we expect to get the state $\mathcal{E}(\rho_i)/\mathrm{tr}[\mathcal{E}(\rho_i)]$ with a corresponding probability $p(i|\mathcal{E})$. Here $\mathrm{tr}[\mathcal{E}(\rho_i)]$ is due to normalization, and $p(i|\mathcal{E})$ is the probability that we are in state ρ_i given that we performed the quantum operation $\mathcal{E}$. In other words

$$\frac{\mathcal{E}(\rho)}{\mathrm{tr}[\mathcal{E}(\rho)]} = \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\mathrm{tr}[\mathcal{E}(\rho_i)]}. \tag{2.32}$$

If we insert Bayes' theorem for $p(i|\mathcal{E})$,[11] then eq. (2.32) reduces to

$$\mathcal{E}(\rho) = \sum_i p_i \mathcal{E}(\rho_i). \tag{2.33}$$

The condition of complete positivity also makes sense. First and foremost, complete positivity implies positivity, which guarantees that $\mathcal{E}$ maps mixed states to mixed states — up to normalization. Also, if one has a combined system $A \otimes B$ and performs an operation on one of the systems, then this yields a valid mixed state up to normalization. One can prove that positivity on one of the systems is not sufficient to guarantee positivity on the combined system [NC00].

The axiomatic definition of quantum operations is cumbersome, hence we would like a more straightforward formulation. The following theorem lets us relate the axioms to an operator-sum representation:

Theorem 2.3.2. *The map $\mathcal{E}$ satisfies axioms 1-3 if and only if*

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \tag{2.34}$$

for some set of operators $\{E_i\}$ which map the input Hilbert space to the output Hilbert space, and $\sum_i E_i^\dagger E_i \leq 1$.

Proof. We begin by showing sufficiency. Let $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$ with $\sum_i E_i^\dagger E_i \leq 1$. Then $\mathcal{E}$ is obviously linear and satisfies $0 \leq \mathrm{tr}[\mathcal{E}(\rho)] \leq 1$. Hence, it remains to show complete positivity. Let A be any positive operator, then for any state $|\psi\rangle$ of the extended system $R \otimes H$ we have

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[11]$p(A|B) = p(B|A)p(A)/p(B)$

that

$$\langle\psi|\,(1_R\otimes\mathcal{E})(A)\,|\psi\rangle = \sum_i \langle\psi|\,(1_R\otimes E_i)A(1_R\otimes E_i^\dagger)\,|\psi\rangle$$
$$= \sum_i \langle\phi_i|\,A\,|\phi_i\rangle \geq 0, \tag{2.35}$$

where $|\phi\rangle_i = (\mathcal{I}\otimes E_i)\,|\psi\rangle$ and the inequality follows from the positivity of A. Hence, $\mathcal{E}$ is a quantum operation.

Necessity follows directly from Kraus' representation theorem [Kra83]. Let $\mathcal{E}$ be any completely positive convex-linear map, and maps trace-class operators onto themselves, and satisfies $\mathrm{tr}[\mathcal{E}(\rho)] \leq 1$ for density operators. Then, the theorem states that $\mathcal{E}$ can be represented by a finite or countably infinite sequence of bounded operators $\{E_i\}$. The mapping is given by

$$\mathcal{E}(\rho) = \sum_i E_i\rho E_i^\dagger, \tag{2.36}$$

and the operators satisfy

$$\sum_i E_i^\dagger E_i \leq 1, \tag{2.37}$$

concluding the proof. $\qquad\square$

Chapter 3

Negative Energy Density

The main topic for this thesis is strict locality for mixed states. We have yet to define what we mean by strict locality. However, as will become clear in Chapter 4, we may think of a strictly localized state as something that looks like the vacuum outside its localization region.[1] When we first studied this topic, we wondered whether we could generate mixed states consisting of non-localized states, that results in a localized mixed state. That is, does there exist a mixed state

$$\rho = \sum_i p_i \left| \psi_i \right\rangle \left\langle \psi_i \right|, \tag{3.1}$$

and some local observable A, such that

$$\mathrm{tr}(\rho A) = \left\langle 0 \right| A \left| 0 \right\rangle, \text{ but } \left\langle \psi_i \right| A \left| \psi_i \right\rangle \neq \left\langle 0 \right| A \left| 0 \right\rangle. \tag{3.2}$$

We chose to look at the observable corresponding to normal-ordered energy density. There is existing interest in localizing photon-states with respect to energy density [Bia98]. However, there are practical reasons for choosing normal ordered energy density as well. First, Epstein et al. [EGJ65] have shown that a positive definite local energy density is incompatible with any usual postulates of local field theory (in [Wit18] E. Witten argues that this follows directly from Theorem 2.2.3). Second, this negative energy density is not bounded, meaning it can be made arbitrarily negative. These qualities indicate that one could create a state with positive energy density, such as a single photon, and compensate by using a state with negative energy density.

The concept of negative energy density might be unsettling, as it violates classical energy conditions. An example is the weak energy condition, which states that every observer corresponding to a time-like vector will observe a non-negative energy density. Even still, experimentalists claim to have detected effects explained by negative energy density [Lam04][Spa58], through the Casimir effect [Cas48] — though some dispute that the Casimir effect is explained by negative energy density [Jaf05]. The phrase "negative energy" might also be misleading, as we can interpret it as a suppression of

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[1]This means the vacuum-state itself is a strictly-localized state, strictly localized outside any region.

vacuum fluctuations. This is because the energy density operator is normal ordered, which means that we have subtracted vacuum contributions.

These sub-vacuum effects can have macroscopic consequences if they are unrestricted. L. H. Ford and T. A. Roman have written papers on the potential consequences and restrictions of negative energy density in different QFTs. The articles [For10] and [Rom06] give an excellent overview. To summarize the articles, unrestricted negative energy densities indicate the existence of traversable wormholes, warp drive, time travel and more. Although interesting, some might find this unsettling. Luckily (or unluckily), QFTs seem to put bounds on these effects through so-called quantum inequalities (QIs), first introduced by Ford in [For78]. Specifically, QIs limit the magnitude of negative energy one can measure in a given temporal interval. These QIs have been calculated in different QFTs and spacetime geometries. Given some smearing function g, we can define the average energy measured as

$$U = \int_{-\infty}^{\infty} dt \, \langle u \rangle g(t), \text{ where } \int_{-\infty}^{\infty} dt \, g(t) = 1, \tag{3.3}$$

and $\langle u \rangle$ is the average energy density for some state. If g has the characteristic width τ, then for a massless field we get [For10]:

$$U = \int_{-\infty}^{\infty} dt \, g(t, \tau) \langle u \rangle \geq -\frac{C}{\tau^d}, \tag{3.4}$$

where d is the spacetime dimension and C is a positive constant. This equation tells us that although we, in theory, can measure arbitrarily negative energy densities, it is only possible for short periods in time. In other words, the more negative energy we want, the shorter the time window we have to measure it. These conditions seriously restrict the macroscopic effects listed above.

In a formalism similar to the pulse mode formalism (as defined in Section 2.2.4), Korolov and Ford investigated the maximal possible sub-vacuum effect one can achieve [KF18]. It turns out that the maximal effect can be found in squeezed vacuum states [Sto70]. This is convenient, as squeezed states are widely studied in quantum optics, and people have produced such states in a laboratory [GK04][Lin+86]. Furthermore, it makes squeezed states a natural candidate to compensate for a state with positive energy density. Therefore, in the subsequent sections (Section 3.1 and 3.2), we first demonstrate the presence of negative energy density in squeezed vacuum states. After that, we create a mixed state, localized with respect to energy density, even though it consists of non-local states.

3.1 Negative Energy Density in Squeezed Vacuum States

In this section, we will demonstrate the presence of negative energy density in squeezed vacuum states. They are defined in terms of a complex variable

$\zeta = re^{i\theta}$ [Sto70]

$$|\zeta\rangle = S(\zeta)|0\rangle, \qquad (3.5)$$

where r is called the squeeze parameter and

$$S(\zeta) = e^{\frac{1}{2}(\zeta^* a^2 - \zeta a^{\dagger 2})}. \qquad (3.6)$$

Here we have used pulse mode formalism and put $a_1 = a$. This operator is unitary and satisfies $S(-\zeta) = S^\dagger(\zeta) = S^{-1}(\zeta)$. The resulting state is an excitation of all possible even number particle eigenstates [GK04]

$$|\zeta\rangle = \frac{1}{\sqrt{\cosh(r)}} \sum_{n=0}^{\infty} (-e^{i\theta}\tanh(r))^n \frac{\sqrt{(2n)!}}{2^n n!} |2n\rangle. \qquad (3.7)$$

The energy density for the electromagnetic field is

$$\rho_E = \frac{1}{2}\left(E^2 + B^2\right). \qquad (3.8)$$

Hence, we can let the energy density operator u be defined by the squared electric field, normal-ordered:[2]

$$u =: E^2 := E^+(t)^2 + E^-(t)^2 + 2E^-(t)E^+(t), \qquad (3.9)$$

with $E^- = (E^+)^\dagger$ and E^+ given by eq. (2.19a). Because we are working with one pulse (the other modes are in the vacuum state) and u is normal ordered, we can effectively reduce eq. (3.9) to

$$u = E(t)^2 a^2 + E^*(t)^2 a^{\dagger 2} + 2|E(t)|^2 a^\dagger a, \qquad (3.10)$$

where $E(t) = E_1(t)$ is given by eq. (2.22).[3] Using the identity [GK04]

$$S^\dagger a S = a\cosh(r) - a^\dagger e^{i\theta}\sinh(r), \qquad (3.11)$$

we can calculate

$$\langle\zeta|\, a^2\, |\zeta\rangle = \langle 0|\, Sa\, \underbrace{S^\dagger S}_{1}\, aS^\dagger\, |0\rangle = -e^{i\theta}\sinh(r)\cosh(r), \qquad (3.12)$$

and the mean number of particles

$$\langle\zeta|\, a^\dagger a\, |\zeta\rangle = \langle 0|\, (S^\dagger a S)^\dagger SaS^\dagger\, |0\rangle = \sinh^2(r). \qquad (3.13)$$

Moreover, with $\langle\zeta|\, a^\dagger a^\dagger\, |\zeta\rangle = \langle\zeta|\, aa\, |\zeta\rangle^*$ we get the expected energy density

$$\langle u\rangle = \sinh(r)\left(2|E(t)|^2\sinh(r) - 2\cosh(r)\,\mathrm{Re}[e^{i\theta}E(t)^2]\right). \qquad (3.14)$$

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[2]Here we use that the contribution from the magnetic field is equal to that of the electric field.

[3]Here we drop the subindex for readability. The function $E(t)$ should not be confused with the field operator. It should be evident from the context whether E is a function or operator.

We have that $\cosh(r),\ \sinh(r) > 0$ for $r > 0$. Furthermore, since $\cosh(r) \approx 1$ and $\sinh \approx 0$ for small r, we get that $\langle u \rangle$ can be negative for small r.

If we let the squeeze parameter r be small, we can use the Taylor expansion of tanh and cosh

$$\tanh(r) = r + \mathcal{O}(r^3),\ \ \cosh(r) = 1 + \mathcal{O}(r^2), \tag{3.15}$$

to rewrite eq. (3.7) into

$$|\zeta\rangle \approx |0\rangle + \frac{r}{\sqrt{2}}e^{i\theta}|2\rangle =: |\psi\rangle. \tag{3.16}$$

Here we have absorbed the minus sign into $e^{i\theta}$. Using eq. (3.10) we can calculate the expected energy density of $|\psi\rangle$:

$$\langle\psi| u |\psi\rangle = 2r\,\mathrm{Re}(e^{i\theta}E(t)^2) + \mathcal{O}(r^2). \tag{3.17}$$

$\mathrm{Re}(z)$ denotes the real part of a complex number z. If we write $E(t) = E_r(t) + iE_i(t)$, keep only first-order terms in r and fix $\theta = 0$ we get

$$\langle\psi| u |\psi\rangle = 2r(E_r(t)^2 - E_i(t)^2). \tag{3.18}$$

This energy density is negative for $|E_r(t)| < |E_i(t)|$.

3.2   Creating Local Density Matrices from Non-Local States

In this section, we create a strictly localized state with respect to energy density. In particular, we want to create a mixed state localized to positive times with respect to energy density. In addition, we want the mixed state to consist of states that are not localized to positive times. The physical system could, for instance, be an experimentalist pressing a button at $t = 0$, generating the desired state on-demand.

Arguably, the simplest state with positive energy density is a single photon ($|1\rangle = a^\dagger |0\rangle$). Hence, define ρ as the mix of a single photon and the state defined by a small squeeze parameter (3.16):

$$\rho = \alpha |1\rangle\langle 1| + \beta |\psi\rangle\langle\psi|,\ \ \alpha + \beta = 1. \tag{3.19}$$

Although one could in principle produce such a state, the practical applications are not obvious. Therefore, we remind the reader that this serves as a proof of concept rather than for practical applications.

We can calculate the expected energy density of ρ

$$\langle u \rangle = \mathrm{tr}[\rho u] = \alpha \langle 1| :E^2: |1\rangle + \beta \langle\psi| :E^2: |\psi\rangle. \tag{3.20}$$

Here $\mathrm{tr}[\cdot]$ denotes the trace. It is easy to show that $\langle 1| :E^2: |1\rangle = 2|E(t)|^2$; moreover, if we fix $\theta = 0$ and keep only first-order terms in r we can reuse eq. (3.18) to get

$$\begin{aligned}\langle u \rangle &= 2\alpha|E(t)|^2 + 2r\beta(E_r(t)^2 - E_i(t)^2)\\ &= (2\alpha + 2r\beta)E_r(t)^2 + (2\alpha - 2r\beta)E_i(t)^2.\end{aligned} \tag{3.21}$$

We want this to be zero for negative times. One way of doing this is by demanding $E_r = 0$ for $t < 0$ and setting the coefficient in front of E_i to zero. The latter is possible by demanding

$$2\alpha - 2r\beta = 0 \Rightarrow \alpha = r\beta. \tag{3.22}$$

Inserting eq. (3.22) into (3.21) and using $\alpha + \beta = 1$ we get

$$\langle u \rangle = \frac{4r}{r+1} E_r(t)^2 = [4r + \mathcal{O}(r^2)]E_r(t)^2. \tag{3.23}$$

Now the question is if there exists a $\xi \in L^2(0, \infty)$ such that

$$E_r(t) = \mathrm{Re}\left[\int_0^\infty d\omega\, \sqrt{\omega}\xi(\omega)e^{-i\omega t}\right] \tag{3.24}$$

is zero for $t < 0$ and non-zero otherwise. The above expression is from eq. (2.22) where we have absorbed K into ξ making it non-normalized.

We want the expressions that follow to be independent of frequency scale. Usually this is done by introducing some arbitrary reference ω_0 and demand $\omega \mapsto \omega/\omega_0$, making ω dimensionless.

We recognize eq. (3.24) as almost being a positive frequency inverse Fourier transformation, as defined in eq. (2.2).[4] Assuming ξ to be zero for negative ω we can extend the integral to minus infinity and obtain

$$E = E_r + iE_i = 2\pi\mathcal{F}^{-1}\left[\sqrt{\omega}\xi\right] \tag{3.25}$$

This implies that

$$\xi(\omega) = \frac{1}{2\pi\sqrt{\omega}}\mathcal{F}[E](\omega). \tag{3.26}$$

Hence, the question is if we can find an E such that ξ and E_r are zero for negative ω and t respectively, and non-zero otherwise, keeping in mind that ξ has to be in L^2.

We can fix E_r to be zero for negative times, so let us address ξ being zero for negative ω. Titchmarsh theorem [Tit48] tells us that if a function f has an imaginary part f_i that is the Hilbert transforms the real part ($f_i = H(f_r)$), then this is *equivalent* to the Fourier transform $\mathcal{F}[f]$ vanishing for negative ω. In fact, the Hilbert transform is a *multiplier operator* that satisfies [Gra14]

$$\mathcal{F}[H(f)](\omega) = -i\,\mathrm{sign}(\omega)\mathcal{F}[f](\omega). \tag{3.27}$$

Here $\mathrm{sign}(\cdot)$ is the signum function, and returns the sign of the given argument. Hence, if we choose E_r to be any function that is zero for negative times and demand that $E_i = H(E_r)$, then we get that

$$\mathcal{F}[E](\omega) = (1 + \mathrm{sign}(\omega))\mathcal{F}[E_r](\omega). \tag{3.28}$$

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[4]Keeping with the convention that the Fourier transform is going from time- to frequency domain.

This means that ξ would be zero for negative ω, which is exactly what we wanted.

As mentioned, we must demand that $\xi \in L^2$. If $E_r \in L^2$, then $\mathcal{F}[E_r] \in L^2$. However, after dividing by $\sqrt{\omega}$ to get ξ, we may get a singularity at $\omega = 0$, which can push ξ out of L^2. To eliminate this problem, we want a well-behaved function that tends to zero as $\omega \to 0$ such that the singularity at $\omega = 0$ is regularized with respect to integration. For an $\epsilon > 0$, we have that $1/\omega^p$ is integrable from 0 to ϵ for any powers $p < 1$. From this we see that $|f(\omega)|^2/\omega$ is integrable, if for small ω, $|f(\omega)| \propto \omega^q$ for any positive number q.[5]

Instead of searching for functions that are proportional to some power of ω around the origin, for simplicity, we will just search for an $\mathcal{F}[E_r]$ that vanishes at $\omega = 0$. Since E_r is purely real, we have that if it is odd, then $\mathcal{F}[E_r]$ is purely imaginary and odd. This would imply $\mathcal{F}[E_r](\omega = 0) = 0$. However, the only function that is odd and zero for negative t is the zero function. Luckily, we have that the Fourier transform is translation invariant up to a local phase:

> **Proposition 3.2.1.** *Let $f : \mathbb{R} \to \mathbb{C}$, $\mathcal{F}$ denote the Fourier transform and $\mathcal{T}_a$ be the translation operator (defined by $\mathcal{T}_a f(t) = f(t+a)$). Then, $\mathcal{F}[\mathcal{T}_a f](\omega) = e^{-ia\omega} \mathcal{F}[f](\omega)$.*

Proof. See [Gra14]. $\qquad\square$

Proposition 3.2.1 tells us that if E_r is antisymmetric around some point $t = a$, then

$$\mathcal{F}[E_r] = e^{-ia\omega} \mathcal{F}[\mathcal{T}_a E_r] \tag{3.29}$$

yields an $\mathcal{F}[\mathcal{T}_a E_r]$ that is purely real and odd. Hence, $\mathcal{F}[E_r](\omega = 0) = 0$ as desired. An example of this is $E_r = \chi_{[0,1]} - \chi_{[1,2]}$, where χ is the indicator function:

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \in \mathbb{R} \setminus A. \end{cases} \tag{3.30}$$

The Fourier transform of $\chi_{[-1/2,1/2]}$ is

$$\mathcal{F}[\chi_{[-1/2,1/2]}](\omega) = \int_{-\infty}^{\infty} dt \, \chi_{[-1/2,1/2]}(t) e^{i\omega t} = \text{sinc}\left(\frac{\omega}{2}\right). \tag{3.31}$$

Then, by the linearity of Fourier transforms and Proposition 3.2.1 we get that

$$\mathcal{F}[E_r](\omega) = \text{sinc}\left(\frac{\omega}{2}\right)\left(e^{\frac{i\omega}{2}} - e^{\frac{3i\omega}{2}}\right)$$
$$= i\omega + \mathcal{O}(\omega^2). \tag{3.32}$$

---

[5]One could think that f continuously tending to zero ($\lim_{x\to 0} f(x) = 0$) is enough. That can be shown *not* to be the case: Take $f(x) = 1/\log(x)$ and an $\epsilon > 0$. Then, $\lim_{x\to 0} f(x) = 0$ and $f \in L(0, \epsilon)$, but $f(x)/\sqrt{x} \notin L^2(0, \epsilon)$.

The function above is proportional to ω around $\omega = 0$, hence we have found an example. Define

$$\xi(\omega) = \frac{1 + \text{sign}(\omega)}{2\pi\sqrt{\omega}} \, \text{sinc}\left(\frac{\omega}{2}\right) \left[e^{\frac{i\omega}{2}} - e^{\frac{3i\omega}{2}}\right]. \qquad (3.33)$$

This ξ is zero for $\omega < 0$ and in L^2. Furthermore, $E = 2\pi\mathcal{F}^{-1}\left[\sqrt{\omega}\xi\right]$ has a real part that is zero for negative times, as desired.

In Figure 3.1, we have plotted the different spectra with squeeze-parameter $r = 0.1$.[6] In the bottom plot we have displayed the real and imaginary part of ξ as a function of ω. In the plot we see that most of the frequency spectrum is around $\omega = 1$.

In the other plot we have displayed the energy density of $|1\rangle$, $|\psi\rangle$ and ρ as a function of time. This plot clearly demonstrates that the mix of two non-local states can give rise to local effects. In particular, we notice that the energy density of $|\psi\rangle$ and $|1\rangle$ is non-zero everywhere, with the exception of discrete points. However, the energy density of ρ is localized to positive times. The singularities are not problematic as they are regularized with respect to integration. If we were to go out and measure the energy, we would have to integrate the spectrum against a suitable function, which would yield a finite result.

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[6]We remind the reader of the map $\omega \mapsto \omega/\omega_0$.
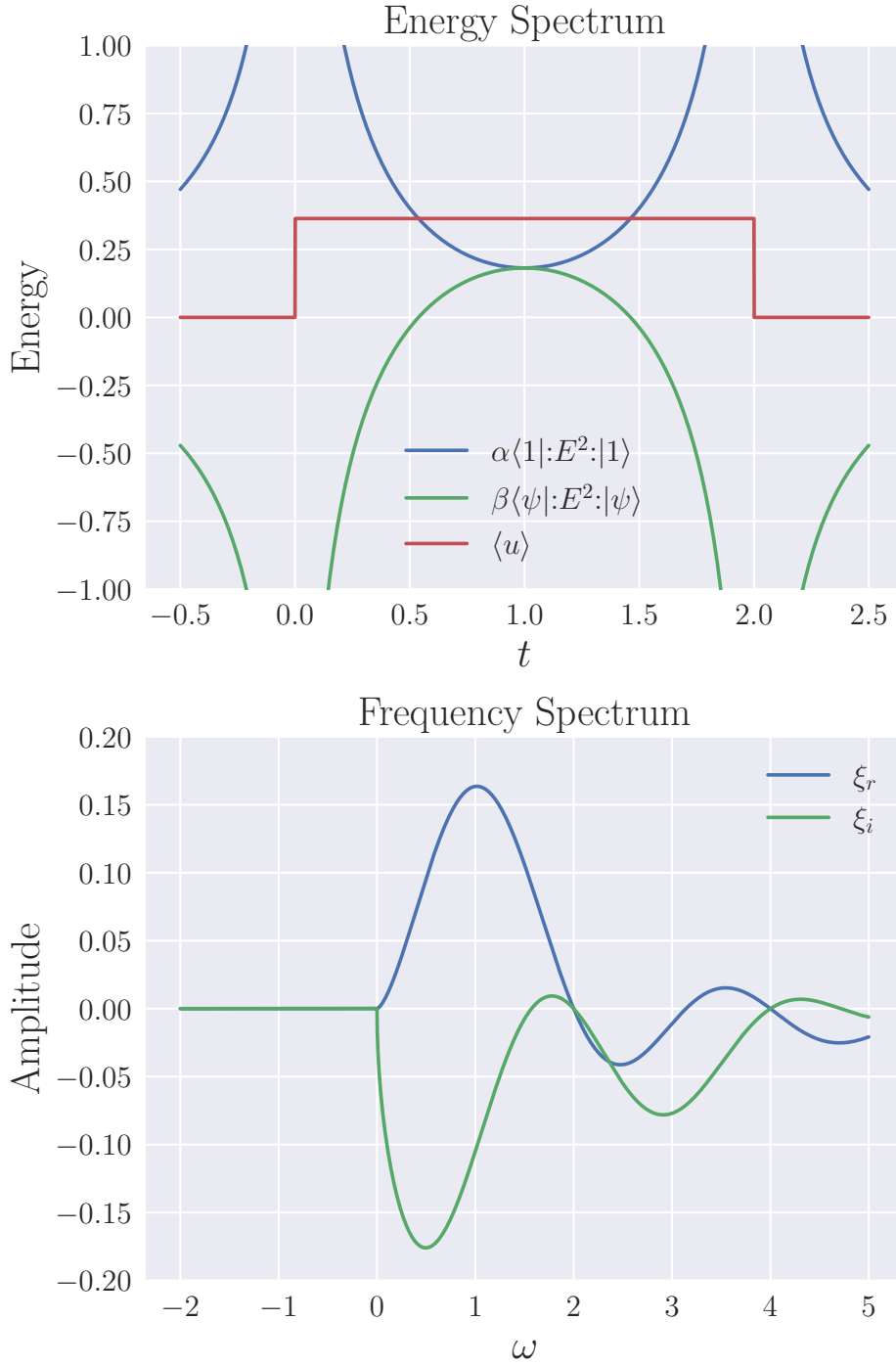
Figure 3.1: Here we have plotted different spectra associated with the mixed state ρ (3.19), generated using the squeeze parameter $r = 0.1$ and the spectra ξ defined by (3.33). The bottom plot shows the real (ξ_r) and imaginary (ξ_i) part of ξ as a function of angular frequency ω. The top plot displays the total energy density $\langle u \rangle$ (3.23) (using first-order terms in r). Also, we have displayed the energy density (scaled using α and β defined by (3.22)) corresponding to the states $|\psi\rangle$ and $|1\rangle$ — as a function of time t.

Chapter 4

Strict Localization

This chapter contains the lion's share of the work done in this thesis. Section 4.1 serves as a brief recapitulation on some work done by J. M. Knight [Kni61] and A. L. Licht [Lic63] on strict locality for pure states. The subsequent section (Section 4.2) attempts to explain why the formalism related to strict locality is essential. In Section 4.3, we extend the formalism set by Knight and Licht to mixed states. All the content in that section is original work; however, we later discovered that there is overlap with existing theory [Lic66]. Alas, we have a small subsection (Section 4.3.1) addressing this.

Last but not least, we have two sections (Section 4.4 and 4.5) that contain one example each in the quantum optics formalism. The first section has a strictly localized state which cannot be written in terms of pure states that are strictly localized. In the second section, we attempt to create a strictly localized mixed state close to a single photon.

4.1   Strict Localization for Pure States

Before we motivate the formalism and delve into strict locality for mixed states, we will briefly cover the existing formalism for pure states. This section is based on the papers [Kni61] and [Lic63] from J. M. Knight and A. L. Licht respectively. We denote $\mathcal{O}$ as any bounded and open region in Minkowski spacetime. Also, $\mathcal{A}(\mathcal{O})$ is the associated local algebra, following Definition 2.2.3.

From Knight, we have the definition of strict locality:

Definition 4.1.1. We say a state $|\psi\rangle \in H$ is *strictly localized outside* $\mathcal{O}$ if

$$\langle\psi| L |\psi\rangle = \langle 0| L |0\rangle \text{, for all } L \in \mathcal{A}(\mathcal{O}). \tag{4.1}$$

Remark. *Here we have modified Knight's definition of strict localization to be outside $\mathcal{O}$ instead of inside. It might be more intuitive to define it in terms of the latter; however, it turns out that the mathematics is more straightforward with the former definition.*

If $\mathcal{A}(\mathcal{O})$ contains all possible measurements in $\mathcal{O}$, then Definition 4.1.1 says that strictly localized states are indistinguishable from the vacuum outside the localization region. This definition is precisely the notion we introduced in the introduction. Licht analyzed the formalism presented by Knight and introduced what we will refer to as Licht operators:

Definition 4.1.2. A linear operator W, is called a *Licht operator* localized *outside* $\mathcal{O}$ if

$$[W, L] = 0 \text{ for all } L \in \mathcal{A}(\mathcal{O}) \text{ and } W^\dagger W = 1. \qquad (4.2)$$

It turns out that Licht operators not only generate strictly localized states from the vacuum, but that *every* strictly localized state can be generated from the vacuum by using Licht operators:

Theorem 4.1.1. *For any region $\mathcal{O}$, the following holds:*

- *Let W be a Licht operator localized outside $\mathcal{O}$, then the state $|\psi\rangle = W |0\rangle$ is strictly localized outside $\mathcal{O}$.*

- *For very state $|\psi\rangle$, strictly localized outside the region $\mathcal{O}$, there exists a* unique *Licht operator W such that $W |0\rangle = |\psi\rangle$.*

Licht also characterized when the superposition of strictly localized states yields a strictly localized state:

Theorem 4.1.2. *Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be strictly localized outside $\mathcal{O}$. Furthermore, let W_1 and W_2 be the corresponding Licht operators that generate $|\psi_1\rangle$ and $|\psi_2\rangle$ respectively. Then the superposition of $|\psi_1\rangle$ and $|\psi_2\rangle$ is also strictly localized outside $\mathcal{O}$ if and only if $W_2^\dagger W_1$ is proportional to the identity ($W_2^\dagger W_1 \propto 1$).*

Remark. *This means that the set of states strictly localized outside of $\mathcal{O}$ is not a subspace of H.*

4.2 Why Care About Strict Localization?

In order to address why we should care about strict localization, we first have to discuss what types of states we can generate in a given region $\mathcal{O}$. Previously, we introduced $\mathcal{A}(\mathcal{O})$ as the set of possible operations in $\mathcal{O}$. For example, if the initial state is the vacuum, then the Reeh-Schlieder theorem (Theorem 2.2.2) tells us that we can get arbitrarily close to any state. However, not every operator in $\mathcal{A}(\mathcal{O})$ represent a reasonable physical process. Some operations might require more energy than is present in our galaxy. Therefore, the idea is that $\mathcal{A}(\mathcal{O})$ contains all possible local operations, not that all operators in $\mathcal{A}(\mathcal{O})$ are physical operations.

In addition, strictly speaking we should only be able to perturb the Hamiltonian and let time evolve [Wit18]. The mathematical representation

of time evolution is unitary. Hence, we argue that the only operations we can perform in $\mathcal{O}$ are represented through unitary operators in $\mathcal{A}(\mathcal{O})$. In that case, any state generated would be strictly localized outside the causal complement of $\mathcal{O}$.[1] However, in praxis, we work with simplified theories, and therefore a suitable operator representation might not exist in the local algebra. For instance, imagine we have an instrument that generates a photon. We can describe the photon using pulse mode formalism; however, photons are usually created by moving charges. Because of this — since the pulse mode formalism contains no charges — we have no reason to expect a unitary operator representation of the instrument to exist in the local algebra. Hence, in the theories we use in practice, we cannot exclude any of the operators in $\mathcal{A}(\mathcal{O})$.

So what role does strict locality make in practical applications? J. Gulla and J. Skaar argue that if states can be generated on-demand, then the resulting state should be strictly localized [GS21a]. They also demonstrated, that one cannot necessarily approximate a given state using strictly localized states. Mathematically, this means that the set of strictly localized states is *not* dense in H. Physically, it means that we cannot create whichever state we want if we have an on-demand source. This property applies in a general theory:

Proposition 4.2.1. *The set of strictly localized states outside $\mathcal{O}$ is* not *dense in H.*

Proof. It is sufficient to demonstrate this in the *weak* sense.[2] Let $L \in \mathcal{A}(\mathcal{O})$ and $|\psi\rangle \in H$ be such that $\langle\psi| L |\psi\rangle \neq \langle 0| L |0\rangle$. Also, let $\{|\phi_n\rangle\}$ be *any* sequence consisting of strictly localized states, localized outside $\mathcal{O}$. Then

$$\lim_n \langle\phi_n| L |\phi_n\rangle = \lim_n \langle 0| L |0\rangle = \langle 0| L |0\rangle \neq \langle\psi| L |\psi\rangle. \qquad (4.3)$$

$\square$

In the algebraic formulation of quantum physics set by R. Haag, D. Kastler and I. E. Segal [HK64][Seg47], states are statistical ensembles of physical systems — or mixed states. Also in the conventional formulation of quantum mechanics, the states of physical systems are mixed whenever classical uncertainties or entanglement with the environment is present. Because of this, we argue that — apart from generalization for its own sake — there is a clear motivation to generalize the results in Section 4.1 to mixed states.

4.3 Strict Locality for Mixed States

We begin by expanding the definition of *strict locality* for pure states (Definition 4.1.1) to mixed states:

---

[1]The causal complement of $\mathcal{O}$ is the set of all points space-like separated from $\mathcal{O}$.
[2]If something does *not* converge weakly, then it does *not* converge strongly.

Definition 4.3.1. We say a mixed state ρ is *strictly localized outside* $\mathcal{O}$ if

$$\mathrm{tr}(L\rho) = \langle 0| L |0\rangle, \text{ for all } L \in \mathcal{A}(\mathcal{O}). \tag{4.4}$$

Remark. *We immediately note that the set of strictly localized mixed states is closed under convex superpositions. This is in contrast to strictly localized pure states (Theorem 4.1.2).*

Following Axioms 1 - 3, we extend the notion of a *Licht operator* to a *Licht map* for mixed states.

Definition 4.3.2. A quantum operation W with corresponding operators $\{W_i\}$, is called a *Licht map* localized *outside* $\mathcal{O}$ if

$$\sum_i W_i^\dagger W_i = 1 \text{ and } [W_i, L] = 0 \text{ for all } i \text{ and every } L \in \mathcal{A}(\mathcal{O}). \tag{4.5}$$

Remark. *Note that this is a valid quantum operation as it satisfies Theorem 2.3.2.*

From Definition 4.3.2, we observe that $W_i^\dagger W_i \propto 1$ implies that W_i is a scaled *Licht operator*. Hence, W_i would generate non-normalized *strictly localized* states from the vacuum. This means that we recover every mixed state generated by an ensemble of *strictly localized* pure states. However, we expect more strictly localized mixed states than the ones generated by mixes of strictly localized pure states. This is because $W_i^\dagger W_i$ does not need to be proportional to the identity. We demonstrate this further down.

The following theorem gives us an equivalent description of Definition 4.3.2:

Theorem 4.3.1. *A quantum operation W described by a sequence of operators $\{W_i\}$ is a Licht map localized outside $\mathcal{O}$ if and only if*

$$\sum_i W_i^\dagger L W_i = L, \text{ for all } L \in \mathcal{A}(\mathcal{O}). \tag{4.6}$$

Proof. The necessity follows readily. Let W be a Licht map with corresponding operators $\{W_i\}$, then $\sum_i W_i^\dagger L W_i = \sum_i W_i^\dagger W_i L = L$. Next we will prove the sufficiency. Let $\{W_i\}$ be an ensemble of operators that satisfy $\sum_i W_i^\dagger L W_i = L$. From $L = 1$ we recover $\sum_i W_i^\dagger W_i = 1$. Therefore, it remains to show that each W_i commutes with all $L \in \mathcal{A}(\mathcal{O})$. By assumption, we get

$$\sum_i W_i^\dagger L W_i = \sum_i W_i^\dagger [L, W_i] + L = \sum_i [W_i^\dagger, L] W_i + L = L, \tag{4.7}$$

which implies

$$\sum_i W_i^\dagger [L, W_i] = \sum_i [W_i^\dagger, L] W_i = 0. \tag{4.8}$$

Moreover, we can calculate

$$\begin{aligned}
&\sum_i W_i^\dagger L^\dagger L W_i \\
=&\sum_i \left([W_i^\dagger, L^\dagger][L, W_i] + [W_i^\dagger, L^\dagger] W_i L + L^\dagger W_i^\dagger [L, W_i] \right) + L^\dagger L \\
=&L^\dagger L
\end{aligned} \tag{4.9}$$

which by inserting (4.8) gives

$$\sum_i [L, W_i]^\dagger [L, W_i] = 0. \tag{4.10}$$

Since this is a sum of positive operators, we must demand that the commutator is zero for all i, concluding the proof. $\qquad\square$

Similar to Licht operators, the Licht maps generate strictly localized mixed states from the vacuum and is illustrated in the following theorem:

Theorem 4.3.2. *Let W be a Licht map localized outside $\mathcal{O}$. Then*

$$\rho = W(|0\rangle \langle 0|) \tag{4.11}$$

is strictly localized outside $\mathcal{O}$.

Proof. For all $L \in \mathcal{A}(\mathcal{O})$, we have $[L, W_i] = 0$ by definition. Using this, the proof is straightforward: Let $\rho = W(|0\rangle \langle 0|)$, then

$$\begin{aligned}
\text{tr}(L\rho) &= \sum_i \text{tr}\left(L W_i |0\rangle \langle 0| W_i^\dagger \right) = \sum_i \text{tr}\left(W_i^\dagger W_i L |0\rangle \langle 0| \right) \\
&= \text{tr}\left(\underbrace{\sum_i W_i^\dagger W_i}_{1} L |0\rangle \langle 0| \right) = \langle 0| L |0\rangle .
\end{aligned} \tag{4.12}$$

Hence, W generates a strictly localized mixed state from the vacuum. $\qquad\square$

We want to fully extend Theorem 4.1.1 and show that *any* strictly localized mixed state can be generated by using a Licht map on the vacuum. To do that, we will need the following theorem:

Theorem 4.3.3. *Assume H_0 is a dense subspace of H. Let L_0 be a bounded operator on H_0. Then L_0 extends uniquely to an operator L on H. It satisfies $\|L_0\| = \|L\|$.*

Remark. *The full proof requires a lot of mathematical details and can be found for Banach spaces here [Ped89].[3] However, it is worth mentioning how one extends L_0: Let $|\psi\rangle \in H$, and $\{|\psi_n\rangle\}$ be a sequence in H_0 converging to $|\psi\rangle$. Then L is defined as $L|\psi\rangle := \lim_n L_0|\psi_n\rangle$. This definition works due to the continuous nature of bounded operators.*

With Theorem 4.3.3, we are ready to prove existence:

Theorem 4.3.4. *For all mixed states ρ strictly localized outside $\mathcal{O}$, there exists a Licht map localized outside $\mathcal{O}$ that generates ρ from the vacuum*

$$W(|0\rangle\langle 0|) = \sum_i W_i |0\rangle\langle 0| W_i^\dagger = \rho. \tag{4.13}$$

Proof. Let ρ be strictly localized outside $\mathcal{O}$, written in diagonal form:

$$\rho = \sum_i p_i |i\rangle\langle i|. \tag{4.14}$$

Define $H_0 := \mathcal{A}(\mathcal{O})|0\rangle$. From the Reeh-Schlieder theorem (Theorem 2.2.2) we have that H_0 is dense in H. Let $|\phi\rangle$ be any state in H_0 and $L_\phi \in \mathcal{A}(\mathcal{O})$ such that $L_\phi |0\rangle = |\phi\rangle$. Define the map

$$\Psi_i(|\phi\rangle) = \sqrt{p_i} L_\phi |i\rangle. \tag{4.15}$$

We need to check that this map is well-defined. Let Ψ_i' be a different map defined in terms of another $L_\phi' \in \mathcal{A}(\mathcal{O})$:

$$\left\| \Psi_i(|\phi\rangle) - \Psi_i'(|\phi\rangle) \right\|^2 = p_i \left\| (L_\phi - L_\phi') |i\rangle \right\|^2 = 0. \tag{4.16}$$

The second equality follows from Theorem 2.2.3.[4] Hence, we have that the map is well-defined. Theorem 2.2.3 also implies linearity

$$\begin{aligned}
\Psi_i(\alpha|\phi_1\rangle + \beta|\phi_2\rangle) &= \sqrt{p_i} L_{\alpha\phi_1 + \beta\phi_2} |i\rangle \\
&= \sqrt{p_i}(\alpha L_{\phi_1} + \beta L_{\phi_2})|i\rangle \\
&= \alpha\Psi_i(|\phi_1\rangle) + \beta\Psi_i(|\phi_2\rangle).
\end{aligned} \tag{4.17}$$

Hence, Ψ_i defines a unique operator $\widetilde{W}_i$ on H_0:

$$\widetilde{W}_i |\phi\rangle := \Psi_i(|\phi\rangle). \tag{4.18}$$

Using strict locality, we demonstrate that $\widetilde{W}_i$ is bounded:

$$\begin{aligned}
\left\| \widetilde{W}_i |\phi\rangle \right\|^2 &= \|\sqrt{p_i} L_\phi |i\rangle\|^2 \\
&= \|L_\phi |0\rangle\|^2 - \sum_{k\neq i} \|\sqrt{p_k} L_\phi |k\rangle\|^2 \\
&\leq \|L_\phi |0\rangle\|^2 = \||\phi\rangle\|^2.
\end{aligned} \tag{4.19}$$

---

[3]A Hilbert space is a Banach space by definition.
[4]Specifically, Theorem 2.2.3 implies $L_\phi = L_\phi'$.

Since $\widetilde{W_i}$ is bounded on a dense subspace, by Theorem 4.3.3, it extends uniquely to a operator W_i defined on the entire Hilbert space H.

Next we check the sequence of operators $\{W_i\}$ against condition (4.6). We do this by calculating its matrix elements. Any dense subspace contains at least one basis [Mer86]. Let $\{|j\rangle\}$ denote a basis of H, consisting of states in H_0. Then we get that

$$\langle j| \sum_i W_i^\dagger L W_i |j'\rangle = \sum_i \langle j| \widetilde{W_i} L \widetilde{W_i} |j'\rangle$$
$$= \sum_i p_i \langle i| L_j L L_{j'} |i\rangle \qquad (4.20)$$
$$= \langle 0| L_j L L_{j'} |0\rangle = \langle j| L |j'\rangle.$$

In other words, the sequence $\{W_i\}$ defines a Licht map:

$$W(|0\rangle \langle 0|) = \sum_i W_i |0\rangle \langle 0| W_i^\dagger = \rho, \qquad (4.21)$$

concluding the proof. $\qquad \square$

Introducing this new formalism would be unnecessary if mixes of strictly localized pure states could generate the entire space of strictly localized mixed states. However, that is not the case, and to demonstrate this we will need a couple of results:

Proposition 4.3.1. *Let W be a* possibly unbounded *operator on $\mathcal{A}(\mathcal{O}) |0\rangle$ satisfying $[W, L] = 0$ for all $L \in \mathcal{A}(\mathcal{O})$. Then $W |0\rangle$ is strictly localized outside $\mathcal{O}$ if and only if $W^\dagger W = 1$.*

Remark. *In other words, for an operator W that commutes with all operators $L \in \mathcal{A}(\mathcal{O})$, we have that $W |0\rangle$ is strictly localized outside $\mathcal{O}$ if and only if W is a Licht operator.*

Proof. Sufficiency follows readily. Let W be an isometry and commute with all operators $L \in \mathcal{A}(\mathcal{O})$. Then W satisfies Theorem 4.1.1 and generates a strictly localized state from the vacuum, localized outside $\mathcal{O}$. Next we prove necessity. Let W commute with all operators $L \in \mathcal{A}(\mathcal{O})$ and generate a strictly localized state from the vacuum. We can calculate the matrix elements of $W^\dagger W$ using operators $L_i \in \mathcal{A}(\mathcal{O})$,

$$(W^\dagger W)_{ij} = \langle 0| L_i^\dagger W^\dagger W L_j |0\rangle. \qquad (4.22)$$

Since $[L_i, W] = 0$ for all i, we get that

$$[L_i^\dagger, W^\dagger] = [W, L_i]^\dagger = 0 \text{ for all } i. \qquad (4.23)$$

Hence, eq. (4.22) reduces to

$$(W^\dagger W)_{ij} = \langle 0| W^\dagger L_i^\dagger L_j W |0\rangle = \langle 0| L_i^\dagger L_j |0\rangle = \delta_{ij}. \qquad (4.24)$$

In the second equality we used that W generates strictly localized states from the vacuum. However, this demonstrates $W^\dagger W = 1$, concluding the proof. $\qquad \square$

Proposition 4.3.2. *Let ρ be strictly localized outside $\mathcal{O}$. Furthermore, let the Licht map defined by the operators $\{W_i\}$ generate ρ from the vacuum:*

$$\rho = \sum_i W_i \left|0\right\rangle \left\langle 0\right| W_i^\dagger = \sum_i q_i \left|g_i\right\rangle \left\langle g_i\right|. \tag{4.25}$$

Then ρ has a representation consisting of strictly localized states $\left|\psi_i\right\rangle$

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle \psi_i\right|, \tag{4.26}$$

if and only if there exists a partially isometric operator U such that

$$\widetilde{W}_i = \sum_j u_{ij}^* W_i, \ \text{ where } u_{ij}^* = \left\langle g_i\right| U \left|g_j\right\rangle^*, \tag{4.27}$$

satisfies $\widetilde{W}_i^\dagger \widetilde{W}_i \propto 1$.

Remark. *Note that this is a criterion for when strictly localized mixed are expressible as a mixture of strictly localized pure states.*

Proof. From Theorem 2.3.1 we have that

$$\sum_i p_i \left|\psi_i\right\rangle \left\langle \psi_i\right| \tag{4.28}$$

is an alternative representation of ρ if and only if there exists a partially isometric operator U such that

$$\sqrt{p_i} \left|\psi_i\right\rangle = \sum_j u_{ij}^* W_i \left|0\right\rangle, \ \text{ where } u_{ij}^* = \left\langle g_i\right| U \left|g_j\right\rangle^*. \tag{4.29}$$

First, we will demonstrate that

$$\widetilde{W}_i := \sum_j u_{ij}^* W_i \tag{4.30}$$

is a valid operator. Let $\left|\phi\right\rangle \in H_0 := \mathcal{A}(\mathcal{O}) \left|0\right\rangle$ and $L \in \mathcal{A}(\mathcal{O})$ such that $L \left|0\right\rangle = \left|\phi\right\rangle$. Then

$$\lim_{N \to \infty} \sum_{j=1}^N u_{ij}^* W_i \left|\phi\right\rangle = L \lim_{N \to \infty} \sum_{j=1}^N u_{ij}^* W_i \left|0\right\rangle = \sqrt{p_i} L \left|\psi_i\right\rangle, \tag{4.31}$$

where L commutes with the limit because it is bounded [Ped89]. From Theorem 2.2.3 we have that L is the *only* operator in $\mathcal{A}(\mathcal{O})$ that generates $\left|\phi\right\rangle$ from vacuum, which implies well-definedness. Hence, the $\widetilde{W}_i$ defined in terms of eq. (4.30) is a valid operator on H_0. Furthermore, $\widetilde{W}_i$ commutes with all $L \in \mathcal{A}(\mathcal{O})$ since the W_is do. From Proposition 4.3.1 $\widetilde{W}_i$ generates non-normalized strictly localized states if and only if it is a Licht operator, concluding the proof. □

With the two previous results, we are ready to show that the extension from strictly localized pure states is non-trivial. In particular, the following theorem demonstrates that not every strictly localized mixed state can be expressed as a mix of strictly localized pure states.

> **Theorem 4.3.5.** *Let ρ be a strictly localized mixed state, localized outside $\mathcal{O}$. Then there does not necessarily exist a representation $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$, where for all i, $|\psi_i\rangle$ is strictly localized outside $\mathcal{O}$.*

Proof. We prove this with an example. Let W_1 and W_2 commute with all operators $L \in \mathcal{A}(\mathcal{O})$, and satisfy

$$W_1^\dagger W_1 + W_2^\dagger W_2 = 1. \tag{4.32}$$

Then the quantum operation defined by W_1 and W_2 is a valid Licht map, and the mixed state

$$\rho = W_1 |0\rangle \langle 0| W_1^\dagger + W_2 |0\rangle \langle 0| W_2^\dagger \tag{4.33}$$

is strictly localized outside $\mathcal{O}$. Furthermore, since the range of ρ is finite, we have from Theorem 2.3.1 that

$$\rho = \widetilde{W}_1 |0\rangle \langle 0| \widetilde{W}_1^\dagger + \widetilde{W}_2 |0\rangle \langle 0| \widetilde{W}_2^\dagger \tag{4.34}$$

is another representation of ρ if and only if

$$\begin{pmatrix} \widetilde{W}_1 \\ \widetilde{W}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -e^{i\phi}b^* & e^{i\phi}a^* \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad |a|^2 + |b|^2 = 1. \tag{4.35}$$

Here the matrix is just an arbitrary 2×2 unitary matrix. By Proposition 4.3.2, the decomposition (4.34) consists of strictly localized states if and only if $\widetilde{W}_i^\dagger \widetilde{W}_i$ is proportional to the identity. We note that if $\widetilde{W}_1^\dagger \widetilde{W}_1 \propto 1$, then we also have that $\widetilde{W}_2^\dagger \widetilde{W}_2 \propto 1$. This follows from

$$\widetilde{W}_1^\dagger \widetilde{W}_1 + \widetilde{W}_2^\dagger \widetilde{W}_2 = W_1^\dagger W_1 + W_2^\dagger W_2 = 1. \tag{4.36}$$

In other words, if $\widetilde{W}_1$ generates a non-normalized strictly local state, then $\widetilde{W}_2$ also does.

To reiterate, we want to demonstrate that there exists two operators W_1 and W_2, such that the operator

$$\widetilde{W}_1 = aW_1 + bW_2 \tag{4.37}$$

cannot generate non-normalized strictly local states from the vacuum. In other words, it satisfies

$$\widetilde{W}_1^\dagger \widetilde{W}_1 = |a|^2 W_1^\dagger W_1 + |b|^2 W_2^\dagger W_2 + 2\operatorname{Re}\left\{a^* b W_1^\dagger W_2\right\} \not\propto 1, \tag{4.38}$$

for all a, b. Here $\text{Re}(\cdot)$ denotes the real part of an operator.[5] If both W_1 and W_2 are hermitian, then we can readily set the term $\text{Re}\left\{a^*bW_1^\dagger W_2\right\}$ to zero by making a^*b purely imaginary.[6] Then we can easily make $\widetilde{W}_1^\dagger \widetilde{W}_1 \propto 1$, by setting $|a|^2 = |b|^2 = 1/2$. Hence, we need to evaluate a slightly more complicated example. After having little success with complex polynomials of hermitian operators, we found an example by using a normal operator.[7]

Let W_1 be a normal operator that commutes with all operators $L \in \mathcal{A}(\mathcal{O})$, and have the representation:

$$W_1 = \sum_i \alpha_i \left|i\right\rangle \left\langle i\right|. \tag{4.39}$$

Furthermore, let us assume W_1 has at least four distinct eigenvalues $|\alpha_i| \leq 1$.[8] We will explicitly create such an operator later in the proof. We can define W_2 as the self-adjoint operator:

$$W_2 = \sum_i \sqrt{1 - |\alpha_i|^2} \left|i\right\rangle \left\langle i\right|. \tag{4.40}$$

Then we have that

$$W_1^\dagger W_1 + W_2^\dagger W_2 = W_1^\dagger W_1 + W_2^2 = 1, \tag{4.41}$$

and the map defined by W_1 and W_2 is a valid Licht map. For $\widetilde{W}_1 = aW_1 + bW_2$ we get that

$$\widetilde{W}_1^\dagger \widetilde{W}_1$$
$$= |a|^2 W_1^\dagger W_1 + |b|^2 W_2^\dagger W_2 + 2\,\text{Re}\left\{a^*bW_1^\dagger W_2\right\}$$
$$= \sum_i \left[|a|^2|\alpha_i|^2 + |b|^2(1-|\alpha_i|^2) + 2\sqrt{1-|\alpha_i|^2}\,\text{Re}\{a^*b\alpha_i^*\}\right] \left|i\right\rangle \left\langle i\right| \tag{4.42}$$
$$= \sum_i \left[|b|^2 + (1-2|b|^2)|\alpha_i|^2 + 2\sqrt{1-|\alpha_i|^2}|b|\sqrt{1-|b|^2}\,\text{Re}\left\{e^{i\gamma}\alpha_i^*\right\}\right] \left|i\right\rangle \left\langle i\right|.$$

In the last equality we used that $|a|^2 + |b|^2 = 1$ and defined γ as the relative phase between a and b. Stating that $\widetilde{W}_1^\dagger \widetilde{W}_1 \propto 1$ is equivalent to saying

$$|b|^2 + (1-2|b|^2)|\alpha_i|^2 + 2\sqrt{1-|\alpha_i|^2}|b|\sqrt{1-|b|^2}\,\text{Re}\left\{e^{i\gamma}\alpha_i^*\right\} = C, \tag{4.43}$$

for all i, where C is some constant.

Since we only have three degrees of freedom — the constant C, γ and $|b|$ — we cannot necessarily compensate for four unique α_is. To illustrate this, let us give an example. Without loss of generality, assume $i = 1, 2, 3, 4$ are

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[5]For a bounded operator A, its real part is defined as $\text{Re}\{A\} := (A + A^\dagger)/2$.

[6]To see this, one also has to recognize that W_1 and W_2 being hermitian, implies that they commute. This follows from $W_1^2 + W_2^2 = 1$.

[7]A normal operator commutes with its hermitian conjugate ($A^\dagger A = AA^\dagger$).

[8]This guarantees $\|W_2\| \leq 1$.

the indices for which the α_is are unique. Then $\alpha_1 = 0$ fixes the constant $C = |b|^2$, and $\alpha_2 = 1$ yields $|b| = 1/\sqrt{2}$. Inserting this back into eq. (4.43) we get

$$\sqrt{1 - |\alpha_i|^2} \operatorname{Re}\{e^{i\gamma}\alpha_i^*\} = 0, \tag{4.44}$$

for all i. By fixing $\alpha_3 = 1/2$ and $\alpha_4 = i\alpha_3$ we see that this equation cannot hold for a fixed γ. Hence, we have found an example.

It remains to demonstrate existence. In Section 4.4 we give an explicit example in the quantum-optics formalism. However, we want to argue that examples exist in an arbitrary QFT. To do that, we must make some assumptions about the operators in local algebras. Arguably, the simplest possible measurements are yes-no outcomes. Hence, we assume local algebras contain non-trivial orthogonal projections.[9] Also, we demand that someone performing experiments in one spacetime region, is not equivalent to the same experiment performed in another that is causally disjoint from the original region.[10] These assumptions are enough to create an example like the above.

Without further ado, here is the argument: Choose $\mathcal{O}$ and three regions $\mathcal{O}_{i=1,2,3}$ such that they are all causally disjoint from each other. Then we have that

$$[\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_i)] = 0 \text{ for all } i, \text{ and } [\mathcal{A}(\mathcal{O}_i), \mathcal{A}(\mathcal{O}_j)] = 0 \text{ for } i \neq j. \tag{4.45}$$

Take an observable in the theory represented by a non-trivial orthogonal projection. This observable would represent some yes-no experiment. In each region $\mathcal{O}_i$, one has a projection operator $P_i \in \mathcal{A}(\mathcal{O}_i)$ representing the observable.[11] Orthogonal projections are diagonalizable [Ped89]. Also, since $[P_i, P_j] = 0$ for all i and j implies compatibility, they can be diagonalized in the same basis:

$$P_i = \sum_j \chi_{A_i}(j) |j\rangle \langle j|, \tag{4.46}$$

where χ is the indicator function (3.30) and A_i is some subset of $\mathbb{N}$. Define the operator W' as:

$$W' = \sum_{i=1}^3 q_i P_i = \sum_j c_j |j\rangle \langle j|, \quad q_i \in [0, \infty) \text{ and } c_j = \sum_{i=1}^3 q_i \chi_{A_i}(j). \tag{4.47}$$

We argue that with proper choices of q_is, W' will have at least four unique c_js. To see this, we can evaluate what happens iteratively:

- $q_1 \chi_{A_1}(j)$: This function has the domain $\{0, q_1\}$ and follows from P_1 being non-trivial. If the domain was $\{q_1\}$ or $\{0\}$, then this would imply $\chi_{A_1} = 1$ or $= 0$ respectively, and make $P_1 = 1$ or $= 0$ respectively.

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[9]Orthogonal projections satisfy $P^\dagger = P = P^2$.

[10]We say two regions are causally disjoint if there is no light- or time-like path linking them.

[11]Orthogonal projections do not generally commute. By introducing three space-like separated regions, we guarantee that they commute.

- $q_1\chi_{A_1}(j) + q_2\chi_{A_2}(j)$: By assumption we have that $P_2 \neq P_1, 1 - P_1$. Otherwise, P_2 would be the same operator as P_1, but with a changed label of measurement outcome. This scenario contradicts the assumption that measurements in space-like separated regions are not equivalent. These assumptions are enough to generate at least three unique outputs. To see this, one has to recognize that q_2 has to appear in the domain somehow. If we were to have two outputs, this could happen in one of three ways: $\{0, q_1 + q_2\}$, $\{q_2, q_1\}$ or $\{q_2, q_1 + q_2\}$. However, either scenario would imply $A_1 = A_2$, $A_1 \cup A_2 = \mathbb{N}$ or $A_2 = \mathbb{N}$ respectively, which contradicts our previous assumptions.

- $c_j = q_1\chi_{A_1}(j) + q_2\chi_{A_2}(j) + q_3\chi_{A_3}(j)$: Similar to above, we have that P_3 cannot be equal any of the other projections, products/sums of the other projections or the identity minus any of the possibilities listed. Otherwise, this would mean that measuring in region three would be equivalent to measuring in the two other regions. It turns out that this is enough to guarantee that c_j has at least four distinct values for different js. To see this, we use a similar trick as above. The function $q_1\chi_{A_1}(j) + q_2\chi_{A_2}(j)$ has at least three different values in its domain, say $\{a_1, a_2, a_3\}$. We must demand that q_3 appear in the domain of c_j somehow. If c_j only has three distinct values in its domain, that can happen in three non-equivalent ways:

  1. $\{a_1 + q_3, a_2, a_3\}$
  2. $\{a_1 + q_3, a_2 + q_3, a_3\}$
  3. $\{a_1 + q_3, a_2 + q_3, a_3 + q_3\}$

  The third scenario would imply $P_3 = 1$. Also, the first and second scenario is equivalent in the sense that if P_3 yields the first scenario, then $1 - P_3$ produces the second. Hence, it is sufficient to demonstrate that the first or second scenario is impossible, given the assumptions. Looking at the first scenario, we see that this would correspond to P_3 being one of the possibilities $\{P_1, P_2, P_1 P_2, 1 - P_1, 1 - P_2, 1 - P_1 - P_2, \pm(P_2 - P_1)\}$. The easiest way to see this is by taking the case where zero is and is not in the spectrum $\{a_1, a_2, a_3\}$. All the scenarios mentioned above are not allowed by assumption. Hence, we can make the spectrum contain at least four unique c_js.

The rest of the proof follows from functional analysis. We observe that W' is normal and an element in the local algebra $\mathcal{A}(\cup_i \mathcal{O}_i)$. This means that W' commutes with all operators $L \in \mathcal{A}(\mathcal{O})$. We can use continuous functional analysis for normal operators [Ped89]: Given a function f that is continuous on the spectrum of W', we have that $f(W') \in \mathcal{A}(\cup_i \mathcal{O}_i)$. Hence, let f be such that

$$f(W') = \sum_j f(c_j) |j\rangle \langle j| = \sum_j \alpha_j |j\rangle \langle j| = W_1. \qquad (4.48)$$

The function f could for example be a complex-valued polynomial that interpolates the desired values, concluding the proof. $\qquad \square$

Theorem 4.3.5 can have physical significance. Assume we have an on-demand source that generates some mixed state. Since the source is on-demand, the resulting state should be strictly localized. We can have two perspectives:

1. Though the resulting state is unknown — whatever it is — we can assume it to be pure and strictly localized.

2. We cannot make assumptions about the pure states, only that the resulting mixed state is strictly localized.

The first perspective would correspond to a mix of strictly localized pure states, while the second would be any strictly localized mixed state. By Theorem 4.3.5, we know that the two perspectives are different.

Both perspectives can be correct and depend on the context. For example, if there is no entanglement between the environment and the instrument, then whatever state we generate should be pure and strictly localized. Hence, a suitable representation would be a mix of strictly localized pure states. On the other hand, assume there is entanglement with the environment, then by doing a partial trace over the environment, we would be left with a strictly localized mixed state — see, for instance, this article [GS21b]. In this case, we cannot require that the mixed state consists of strictly localized pure states, as we have an effective state and not the "entire" state.

4.3.1   Notes on Licht's result

As mentioned, there is an overlap between the work presented in Section 4.3 and this article [Lic66] by A. L. Licht.[12] This subsection aims to point out what Licht has already covered and briefly relate the different formalisms used. Relating the formalisms is the most challenging part. In particular, Licht did not formulate his findings through quantum operations. Also, he did not express states as mixed states, but rather as sequences of operators in a local algebra on a given background.[13] In hindsight, these sequences of operators — which he often refers to as states — are similar to what we call quantum operations. Because of this, we argue that the overlap still brings value, as it is an existing theory presented in a more modern formalism. In addition, the proofs presented in the previous section are different from Licht's proofs.

Licht discusses the effects that different backgrounds have, which we have not done. It seems that the different backgrounds could be encoded into the initial state in our formalism. Hence, what he refers to as a *uniform* background, is $|0\rangle \langle 0|$ as the initial state in our formalism. Furthermore, he

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[12]We would like to emphasize that the work done in Section 4.3 was completely independent of Licht's article. We derived and proved the formalism *before* we discovered the content of Licht's article. The reason we did not notice it before was because Licht did not utilize the same formalism as us, namely mixed states and quantum operations.

[13]This might not be surprising, as the earliest source we could find on the quantum operation formalism that resembles the one presented here, was introduced by Kraus and Hellwig in 1969. That was after Licht published his results in 1966.

has a notion of *selective* and *non-selective* states, which characterize how one can average different experimental trials. Again, that seems to be encoded in the quantum operation formalism. The quantum operation described by $\{E_i\}$ is called selective and non-selective if $\sum_i E_i^\dagger E_i < 1$ and $= 1$ respectively.

The primary overlap seems to be: The extension of strict locality to mixed states (Definition 4.3.1), that Licht maps generate strictly localized states (Theorem 4.3.2) and that one can generate the strictly localized states by using Licht maps (Theorem 4.3.4). Specifically, Licht points out that non-selective states on a uniform background will be strictly localized. This is similar to Theorem 4.3.2, as he *only* treats states prepared by operators in a local algebra. Following this, he has a theorem which characterizes how strictly localized states can be generated from vacuum, which is similar to Theorem 4.3.4.[14] He also has a generalization of Theorem 4.3.4 to arbitrary backgrounds; however, it does not seem related to strict locality.

4.4 Strictly Localized Mixed State Consisting of Non-Local States

In this section, we will explicitly create an example of a strictly localized mixed state that cannot be transformed to consist of strictly localized pure states. In other words, we will demonstrate Theorem 4.3.5 in the quantum-optics formalism. Specifically, we will show that there exist four commuting projections that also commute with observables outside some localization region. Using those projections, we can follow the proof of Theorem 4.3.5 to create a strictly localized mixed state that does not have a representation consisting of strictly localized pure states.

Let $\mathcal{O}$ be an open and bounded region in time. Also, let f be a function with compact support in another bounded open region $\mathcal{O}' \subset \mathcal{O}^C$.[15] By Definition 2.2.2, we can smear the field using f to create a *local operator* with localization region $\mathcal{O}'$:

$$
\begin{aligned}
E_f &= \int_{-\infty}^{\infty} dt\, f(t) \int_0^\infty d\omega\, \mathcal{E}(\omega) a(\omega) e^{-i\omega t} + h.c. \\
&= \int_0^\infty d\omega\, F(-\omega)\mathcal{E}(\omega)a(\omega) + h.c. =: a_f + a_f^\dagger,
\end{aligned}
\tag{4.49}
$$

where $F(\omega)$ is the Fourier transform of $f(t)$. Since E_f is unbounded, we must bound the spectrum to make it a part of the local algebra $\mathcal{A}(\mathcal{O}')$. One way of doing this is finding the spectral decomposition of E_f and then modifying the spectrum using a bounded function. Finding the spectral decomposition of E_f can be difficult; however, we can utilize a trick. The operator E_f is proportional to one of the quadrature operators, which in turn is isomorphic

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[14]We raise this opportunity to point out a mistake in Licht's version of Theorem 4.3.4. Specifically, one should replace the bounded linear operators with the local algebra corresponding to the spacetime region causally disjoint from α.

[15]Here A^C denotes the complement of A.

to the position operator in the harmonic oscillator

$$E_f = \sqrt{2}X_f, \quad X_f = \frac{1}{\sqrt{2}}(a_f + a_f^\dagger). \qquad (4.50)$$

Hence, following [SM13] we can write down the eigenstates of X_f:

$$|X_f\rangle = \frac{e^{-X^2/2}}{\pi^{1/4}}e^{\frac{a_f^{\dagger 2}}{2}+\sqrt{2}Xa_f^\dagger}|0\rangle = \frac{e^{-X^2/2}}{\pi^{1/4}}\sum_{n=0}^\infty \frac{H_n(X)}{2^{n/2}n!}a^{\dagger n}|0\rangle, \qquad (4.51)$$

where $H_n(X)$ are the Hermite functions. Using the Rodrigues representation, we can write down the Hermite functions [AW05]:

$$H_n(X) = (-1)^n e^{X^2}\frac{d^n}{dX^n}e^{-X^2}. \qquad (4.52)$$

We emphasize that the states $|X_f\rangle$ are *neither* physical nor elements in H. Like operator-valued distributions, they are a practical notational device which only makes sense under integration. Hence, the following expressions and manipulations are purely formal.[16] Analogous to the position eigenstates in non-relativistic quantum mechanics, we demand completeness and Dirac orthonormality:

$$\int dX\, |X_f\rangle\langle X_f| = 1, \quad \langle X_f|X_f'\rangle = \delta(X - X'). \qquad (4.53)$$

The states $|X_f\rangle$ are also eigenstates of E_f, with eigenvalues $X \in \mathbb{R}$. Because of this, the spectral decomposition of E_f is

$$E_f = \sqrt{2}\int_{-\infty}^\infty dX\, X\, |X_f\rangle\langle X_f|. \qquad (4.54)$$

Using the Sigmoid function (σ), we can make the spectrum bounded

$$\sigma(E_f) = \int_{-\infty}^\infty dX\, \sigma(\sqrt{2}X)\, |X_f\rangle\langle X_f|. \qquad (4.55)$$

Since $\sigma(\mathbb{R}) = (-1, 1)$, we get that $\sigma(E_f)$ has operator norm equal to one and from Definition 2.2.3, an element in $\mathcal{A}(\mathcal{O}')$. More importantly, $\sigma(E_f)$ commutes with all operators in $\mathcal{A}(\mathcal{O})$.

The next step is to create four *unique* orthogonal projections that commute with each other and all $L \in \mathcal{A}(\mathcal{O})$. Using spectral theory with Borel functional calculus [Ped89], this is easy. Take four A_i's that are open and disjoint subsets of $(-1, 1)$. Furthermore, define

$$P_i := \chi_{A_i}(\sigma(E_f)), \qquad (4.56)$$

where χ is the indicator function (3.30). Although the P_is are not necessarily in $\mathcal{A}(\mathcal{O}')$, Borel functional calculus guarantees that the P_is commute with

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[16]This does not mean that the following treatment is not rigorous. By introducing spectral measures, one can make the same arguments.

all operators that commute with $\sigma(E_f)$.[17] In other words, P_i commute with all operators in $\mathcal{A}(\mathcal{O})$. The fact that the different P_is are unique follows from the A_is being disjoint. Also, since their eigenvalues are real, they are self-adjoint ($P_i^\dagger = P_i$). Hence, it remains to show that they are projections ($P_i^2 = P_i$) and that they commute. We begin by demonstrating that they are projections:

$$P_i^2 = \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dX' \, \chi_{A_i}(\sigma(\sqrt{2}X))\chi_{A_i}(\sigma(\sqrt{2}X'))\delta(X' - X) \, |X_f\rangle \langle X_f'|$$

$$= \int_{-\infty}^{\infty} dX \, \chi_{A_i}(\sigma(\sqrt{2}X))^2 \, |X_f\rangle \langle X_f| = P_i. \tag{4.57}$$

In the last equality we used that $\chi_A^2 = \chi_A$. Last but not least, since the A_is are disjoint, we have that $\chi_{A_i}\chi_{A_j} = 0$ for $i \neq j$, which implies that $P_i P_j = 0$ for $i \neq j$, hence $[P_i, P_j] = 0$ for all i, j. As in the proof of Theorem 4.3.5 (the part after eq. (4.43)), we can define the Licht map generated by

$$W_1 = q_1 P_1 + q_2 P_2 + q_3 P_3 + q_4 P_4 \text{ and } W_2 = \sqrt{1 - W_1^\dagger W_1}, \tag{4.58}$$

where $q_1 = 0$, $q_2 = 1$, $q_3 = 1/2$ and $q_4 = iq_3$. This map generates a strictly localized mixed state from the vacuum, localized outside of $\mathcal{O}$, and cannot be written in terms of strictly localized pure states.

We can determine the states generated by using W_1 and W_2 on the vacuum. First, we have to simplify how we represent the projections. We begin by observing that

$$P_i = \int_{-\infty}^{\infty} dX \, \chi_{A_i}(\sigma(\sqrt{2}X)) \, |X_f\rangle \langle X_f|$$

$$= \int_{B_i} dX \, |X_f\rangle \langle X_f|, \tag{4.59}$$

where

$$B_i := \{x \in \mathbb{R} : \sigma(\sqrt{2}x) \in A_i\}. \tag{4.60}$$

From eq. (4.51) we get that

$$\langle X_f|0\rangle = \frac{e^{-X^2/2}}{\pi^{1/4}}. \tag{4.61}$$

This gives us

$$P_i|0\rangle = \int_{B_i} dX \, \frac{e^{-X^2/2}}{\pi^{1/4}} \, |X_f\rangle. \tag{4.62}$$

Because σ is injective — or more generally bijective — we have that the B_is are disjoint. Hence, we get the following representation of W_1 acting on the vacuum:

$$W_1|0\rangle = \int_{-\infty}^{\infty} g(X)\frac{e^{-X^2/2}}{\pi^{1/4}} \, |X_f\rangle, \tag{4.63}$$

---

[17]This is because the indicator functions are non-continuous.

where

$$g(X) = \begin{cases} q_i & \text{for } X \in B_i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.64}$$

A similar calculation for W_2 — using the completeness relation (4.53) — yields

$$W_2 \left|0\right\rangle = \int_{-\infty}^{\infty} dX \sqrt{1 - |g(X)|^2} \frac{e^{-X^2/2}}{\pi^{1/4}} \left|X_f\right\rangle. \tag{4.65}$$

We have no physical interpretation of the states (4.63) and (4.65). However, the reader might have better intuition, so we write them down. Even still, we have demonstrated Theorem 4.3.5 with an explicit example in quantum optics.

4.5 Strictly Localized Mixed State Close to Photon

This section will be a continuation of Section 3.2. In particular, we will attempt to create a strictly localized mixed state close to a single photon. As in Section 3.2, we will try to localize the state to positive times ($t > 0$), and could be generated by some on-demand source at $t = 0$.

Let $\mathcal{O}$ denote *any* bounded open subset of $t < 0$, and $\mathcal{O}'$ be *a* bounded open subset of $t > 0$. Ideally, we want to construct a mixed state consisting of a single photon and compensate by using some other state:[18]

$$\rho = \alpha \left|1\right\rangle \left\langle 1\right| + \beta \left|\psi\right\rangle \left\langle \psi\right|. \tag{4.66}$$

It is not evident that such a state exists. We would need an operator W_1 that commutes with $\mathcal{A}(\mathcal{O})$ for all $\mathcal{O}$, has operator norm less than or equal to one, and satisfy $W_1 \left|0\right\rangle = \sqrt{\alpha} \left|1\right\rangle$. From Theorem 2.2.2 we know that we can get arbitrary close by using operators in $\mathcal{A}(\mathcal{O}')$.[19] However, we want an exact match.

We have not been able to create such an operator nor show that it exists. Because of this, we will create something similar. For an f with compact support in $\mathcal{O}'$ we can create a local operator similar to eq. (4.49):

$$E_f = a_f + a_f^\dagger = \sqrt{2} \int_{-\infty}^{\infty} dX \, X \left|X_f\right\rangle \left\langle X_f\right|, \tag{4.67}$$

where the second equality is from eq. (4.54). This will be a local operator — localized to the region $\mathcal{O}'$ — and creates a single photon from the vacuum. We need this operator to satisfy $\|E_f\| \leq 1$. We can easily make spectrum bounded by using functional calculus (like we did in Section 4.4). The following are the most obvious candidates:

$$E_f(b) = \int_{-b}^{b} dX \sqrt{2} X \left|X_f\right\rangle \left\langle X_f\right| \text{ and} \tag{4.68a}$$

$$\sigma(E_f) = \int_{-\infty}^{\infty} dX \, \sigma(\sqrt{2} X) \left|X_f\right\rangle \left\langle X_f\right|. \tag{4.68b}$$

---

[18]In Section 3.2 we compensated by using a squeezed vacuum state.

[19]Although we might not have control over α.

The operator $E_f(b)$ will satisfy $\|E_f(b)\| = \sqrt{2}b$, and $\|\sigma(E_f)\| = 1$. In addition, both will commute with $\mathcal{A}(\mathcal{O})$ for all $\mathcal{O}$. However, since the operators have a modified spectrum with respect to E_f, they will generate non-normalized states. The idea is that the operators can still be used to generate strictly localized mixed states close to the desired photon.

Let us start by investigating the Licht map we can generate from $E_f(b)$ as defined in eq. (4.68a). For the following, we will denote $E_f(b)\left|0\right\rangle = \left|1_f(b)\right\rangle$, which is non-normalized. We can calculate how close $\left|1_f(b)\right\rangle$ is to the desired photon $E_f\left|0\right\rangle = \left|1_f\right\rangle$. To gauge closeness, we use the so-called fidelity measure. The general definition is (following the convention in [NC00])

$$F(\rho, \sigma) = \mathrm{tr}\left[\sqrt{\sqrt{\rho^{1/2}\sigma\rho^{1/2}}}\right], \tag{4.69}$$

where ρ and σ are mixed states. We will only need to measure the closeness between mixed and pure states. For $\sigma = \left|\psi\right\rangle\left\langle\psi\right|$, we get that eq. (4.69) reduces to

$$F(\rho, \left|\psi\right\rangle) = \sqrt{\left\langle\psi\right|\rho\left|\psi\right\rangle}. \tag{4.70}$$

Hence, by eq. (4.70), the fidelity between $\left|1_f\right\rangle$ and $\left|1_f(b)\right\rangle$ is

$$\begin{aligned}
F(\left|1_f(b)\right\rangle, \left|1_f\right\rangle) &= \left|\left\langle 1_f(b)|1_f\right\rangle\right| \\
&= \left|2\int_{-b}^{b}\int_{-\infty}^{\infty} dX\, dX'\, X X' \left\langle 0|X_f\right\rangle\left\langle X_f|X_f'\right\rangle\left\langle X_f'|0\right\rangle\right| \\
&= 2\int_{-b}^{b} dX\, X^2 |\left\langle 0|X_f\right\rangle|^2 \\
&= \left\langle 1_f(b)|1_f(b)\right\rangle.
\end{aligned} \tag{4.71}$$

From eq. (4.71), we see that the fidelity between $\left|1_f\right\rangle$ and the re-normalized state $\left|1_f(b)\right\rangle / \|\left|1_f(b)\right\rangle\|$ is $\sqrt{\left\langle 1_f(b)|1_f(b)\right\rangle}$, which we plot below.

We want to create a strictly localized mixed state containing $\left|1_f(b)\right\rangle$. Define $W_1 = E_f(b)$ and

$$W_2 = \sqrt{1 - W_1^2}. \tag{4.72}$$

Then we have that

$$W_1^\dagger W_1 + W_2^\dagger W_2 = W_1^2 + W_2^2 = 1, \tag{4.73}$$

and

$$[W_i, \mathcal{A}(\mathcal{O})] = 0 \text{ for all } \mathcal{O} \text{ and } i. \tag{4.74}$$

The above implies that W_1 and W_2 defines a Licht map, and the mixed state

$$\rho = W_1\left|0\right\rangle\left\langle 0\right|W_1 + W_2\left|0\right\rangle\left\langle 0\right|W_2 \tag{4.75}$$

defines a strictly localized state, localized outside $t < 0$.

Now, we want to find the fidelity between ρ and $|1_f\rangle$. By using the completeness relation (4.53), we can give an explicit representation of W_2:

$$W_2 = \int_{-\infty}^{\infty} dX \sqrt{1 - 2X^2 \chi_{[-b,b]}(X)} \, |X_f\rangle \langle X_f| , \qquad (4.76)$$

where χ is the indicator function (3.30). Inserting for eq. (4.70), we can calculate the fidelity between ρ and $|1_f\rangle$:

$$\begin{aligned} F(\rho, |1_f\rangle) &= \sqrt{\langle 1_f| \rho |1_f\rangle} \\ &= \sqrt{|\langle 1_f| W_1 |0\rangle|^2 + |\langle 1_f| W_2 |0\rangle|^2}. \end{aligned} \qquad (4.77)$$

If we use the representation defined by eq. (4.76), then it is easy to show that the second addend is zero.[20] Hence, inserting for eq. (4.71), we get that

$$F(\rho, |1_f\rangle) = F(|1_f(b)\rangle , |1_f\rangle) = \langle 1_f(b)|1_f(b)\rangle . \qquad (4.78)$$

By using eq. (4.61) we can evaluate the inner product

$$\begin{aligned} \langle 1_f(b)|1_f(b)\rangle &= 2 \int_{-b}^{b} dX \, X^2 \frac{e^{-X^2}}{\sqrt{\pi}} \\ &= \operatorname{erf}(b) - \frac{2b}{\sqrt{\pi}} e^{-b^2}. \end{aligned} \qquad (4.79)$$

Here $\operatorname{erf}(\cdot)$ denotes the error function.

In Figure 4.1, we have plotted the different fidelities as a function of b. The vertical line denotes the maximum possible b, as we have to demand $\|W_i\| \leq 1$ for all i. We observe that the maximal fidelity between ρ and $|1_f\rangle$ is ~ 0.2, even though the fidelity between the re-normalized state $|1_f(b)\rangle / \||1_f(b)\rangle\|$ and $|1_f\rangle$ can be made arbitrarily close to 1. By going through the same procedure, with $E_f(b) \mapsto \sigma(E_f)$ as defined by eq. (4.68b), we end up with a the same fidelity, $F(\rho, |1_f\rangle) \approx 0.2$.

In [GS21a], J. Gulla and J. Skaar achieved a fidelity arbitrarily close to 1 for specific photon pulses by using strictly localized pure states. That is is a much higher fidelity than what we achieved. The key to achieving a higher fidelity might be to find strictly localized mixed states of the form (4.66). Also, we could maybe get a higher fidelity if we were to compare ρ to other photon pulses.

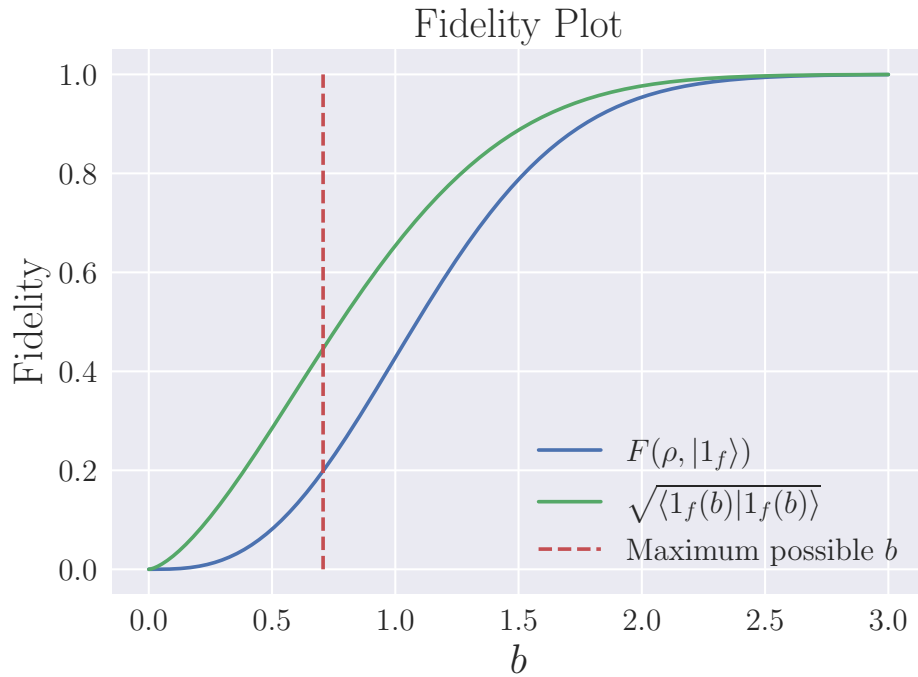[20]The integrand will be odd.

Figure 4.1: In this figure, we have plotted the different fidelities, with different b-values on the x-axis and fidelity on the y-axis. The vertical line represents the maximum b-value that satisfies $\|E_f(b)\| \leq 1$. The blue graph is the fidelity between the mixed state ρ that contains $|1_f(b)\rangle$ and the pure state $|1_f\rangle$. We have also plotted $\sqrt{\langle 1_f(b)|1_f(b)\rangle}$, which represent the fidelity between the normalized state $|1_f(b)\rangle / \||1_f(b)\rangle\|$ and $|1_f\rangle$.

Chapter 5

Epilogue

In this chapter, we discuss some of the results and give some concluding remarks (Section 5.1). In addition, we list some avenues for further work that could be interesting (Section 5.2).

5.1 Discussion and Concluding Remarks

We have introduced the concept of strict locality for mixed states (Definition 4.3.1) and shown that the formalism for strictly localized pure states extends to strictly localized mixed states in a non-trivial way. Specifically, we have demonstrated that Licht operators extend to Licht maps defined by a sequence of operators (Theorem 4.3.4). The defining difference is that the sequence of operators that define Licht maps need not be isometric. This is enough to generate a larger set of strictly localized states that include the set of strictly localized pure states (Theorem 4.3.5); which is why the extension is non-trivial.

We have discussed, from an experimental point of view, why it is important to characterize strict locality for mixed states (Section 4.2). However, it can also be helpful in characterizing the set of strictly localized pure states as well. In particular, relations could sometimes be easier to prove in the Licht map formalism, which would reduce to Licht operators in special cases. Hence, insight into the strictly localized mixed states could give valuable insight into the set of strictly localized pure states.

The operators that generate the Licht map — and Licht operators — have some sense of being local because they commute with all operators outside the localization region. However, we have not determined if the operators are associated with some algebra inside the localization region. In fact, the operators do not necessarily belong to *any* local algebra. This is because we do not necessarily have *Haag duality*. Haag duality states that the set of all bounded operators that commute with $\mathcal{A}(\mathcal{O})$ is its own local algebra associated with the causal complement of $\mathcal{O}$. Duality has been demonstrated for some QFTs and specific shapes of $\mathcal{O}$. However, not for general QFTs, and counterexamples can be found for specific shapes of $\mathcal{O}$ — see for instance this note on Haag duality [GP22].

We have investigated explicit examples of the Licht map formalism

(Section 4.4 and 4.5), but their practical applications still need clarification. The examples, do however, illustrate how the local algebra formulation of QFT can be implemented in a relatively simple QFT. More research is needed to apply the formalism to other QFTs — e.g. charged spinor fields.[1]

We have characterized how one can generate strictly localized states from the vacuum through the Licht map formalism. However, this has yet to yield much insight into the set of strictly localized states themselves. We demonstrated that the set of strictly localized states is not dense in H (Proposition 4.2.1); hence there are many interesting questions related to the structure of strictly localized states. We state some examples in the subsequent section on further work.

Finally, our study of negative energy density has explicitly displayed how mixed states can generate local effects (Section 3.2). While the resulting state may not be strictly localized, we have observed how mixtures of non-strictly localized states can yield strictly localized mixed states with respect to certain observables. There might be some connection between negative energy density and strict locality. As stated, the separating quality of vacuum (Theorem 2.2.3) — which follows from the Reeh-Schlieder theorem — implies the presence of negative energy density [Wit18] and is satisfied by any local algebra formulation of QFT [Ara99]. Hence, there seems to be some connection between negative energy density and localization; however, we have yet to describe such a relation.

5.2 Further Work

There is plenty of interesting avenues for further work. The following are some problems we would like to look at:

- There is a need to further characterize the strictly localized states to better understand their properties. Although we know that specific operators can generate strictly localized states from the vacuum, we still know little about them. For instance, we want to know if the neighborhood of a strictly localized state also consists of strictly localized states, or if some strictly localized states are isolated points in H. The set of strictly localized pure states introduced in [GS21a] has a parametrization; however, that does not mean that it holds for all strictly localized pure states.

- A more detailed characterization of strictly localized states can help us answer other questions related to strict locality. For example, can we determine the set of strictly localized states close to any single photon? Also, if a strictly localized state is the superposition of two states that are not strictly localized, is this superposition fine-tuned? In other words, does a slight perturbation of the superposition yield a non-strictly localized state?

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[1]Nils Johannes Mikkelsen has been working on this in his thesis to be submitted to https://www.duo.uio.no/.

- We want to determine if the set of strictly localized pure states forms a basis, which would imply that any state can be written as a superposition of strictly localized states. This would allow us to express any state as a sum of Licht operators acting on the vacuum. To prove this, one could for instance start by looking at the set of strictly localized coherent states.

- It would be interesting to investigate the set of mixed states generated by quantum operations consisting of operators in a local algebra. While a quantum operation can generate any mixed state, that is not necessarily the case for quantum operations consisting of operators in a local algebra. The first step here could be to extend Proposition 4.2.1 to the space of mixed states — the proof should extend pretty easily. Also, since Theorem 4.3.2 implies that the resulting mixed states will be strictly localized, answering this question would help characterize the set of strictly localized mixed states.

- The previous item could also help us determine if it is possible to create a strictly localized mixed state that consists of a single photon and something else. Although we have tried to create such a state in Section 4.5, it is unclear if we can obtain an exact match.

- We want to better understand the physical properties of states that we cannot represent as a mix of strictly localized pure states. This could help us understand the types of states that we can generate in quantum field theories. Here, one could start out by exploring the properties of the state we introduced in Section 4.4.

- Finally, we would like to find more applications of the Licht map formalism and investigate its use in other QFTs.

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Appendix A

Solving the Free Maxwell's Equations in the Coulomb Gauge

Here, we solve the free Maxwell's equations in the Coulomb gauge. The solutions are relevant when we quantize the electromagnetic field. We begin by writing down Maxwell's equations in the vacuum, using natural units:

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0, \tag{A.1a}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{A.1b}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \tag{A.1c}$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t}. \tag{A.1d}$$

Here $\mathbf{E}$ and $\mathbf{B}$ is the electric- and magnetic field, respectively.

We can represent the fields in terms of the electric scalar potential ϕ and magnetic vector potential $\mathbf{A}$ through:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla \phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \tag{A.2a}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t). \tag{A.2b}$$

Note that these two equations have a gauge invariance. In particular, the transformations

$$\mathbf{A} \to \mathbf{A} + \nabla \psi \text{ and } \phi \to \phi - \frac{\partial \psi}{\partial t}, \tag{A.3}$$

leave the electric- and magnetic field unchanged.

Now we are ready to solve Maxwell's equations in the vacuum. Using the Helmholtz decomposition, we split $\mathbf{A}$ into a longitudinal- and transversal component — in other words, a curl- and divergence-free part. Furthermore, with the gauge freedom, we can fix $\nabla \cdot \mathbf{A} = 0$ (the Coulomb gauge), which sets the longitudinal component to zero. Since we are after vacuum solutions, we

also have $\phi = 0$. By taking the curl of eq. (A.2b) and inserting for eq. (A.2a) and (A.1d), we get[1]

$$\nabla^2 \mathbf{A} = \frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{A.4}$$

This is the wave equation for a vector field, which has well-known solutions [Tor16]

$$\mathbf{A}(\mathbf{r}, t) = \int_{\text{all } k\text{-space}} d^3 k \, \mathcal{E}(k) \sum_{l=1}^{2} \left[a_l(\mathbf{k}) \mathbf{e}_l(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} - kt)} + \text{c.c.} \right] \tag{A.5}$$

Here $k = |\mathbf{k}|$, $\mathcal{E}$ is for normalization, and we have used that $\mathbf{A}$ is transversal, $\mathbf{A} \cdot \mathbf{k} = 0$, to write it in terms of two orthonormal vectors $\mathbf{e}_l$.

Inserting eq. (A.5) into eq. (A.2a) and (A.2b) we find expressions for the $\mathbf{E}$- and $\mathbf{B}$-field:

$$\mathbf{E}(\mathbf{r}, t) = \int_{\text{all } k\text{-space}} d^3 k \, \mathcal{E}(k) \sum_{l=1}^{2} \left[a_l(\mathbf{k}) \mathbf{e}_l(\mathbf{k})(ik) e^{i(\mathbf{k} \cdot \mathbf{r} - kt)} + \text{c.c.} \right], \tag{A.6}$$

$$\mathbf{B}(\mathbf{r}, t) = \int_{\text{all } k\text{-space}} d^3 k \, \mathcal{E}(k) \sum_{l=1}^{2} \left[a_l(\mathbf{k}) i\mathbf{k} \times \mathbf{e}_l(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} - kt)} + \text{c.c.} \right] \tag{A.7}$$

Finally, we use the dispersion relation to write k in terms of angular frequency ($k = \omega$), absorb i into a_l and k into $\mathcal{E}$ to get the familiar form

$$\mathbf{E}(\mathbf{r}, t) = \int_{\text{all } k\text{-space}} d^3 k \, \mathcal{E}(\omega) \sum_{l=1}^{2} \left[a_l(\mathbf{k}) \mathbf{e}_l(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \text{c.c.} \right], \tag{A.8}$$

$$\mathbf{B}(\mathbf{r}, t) = \int_{\text{all } k\text{-space}} d^3 k \, \mathcal{E}(\omega) \sum_{l=1}^{2} \left[a_l(\mathbf{k}) \hat{\mathbf{k}} \times \mathbf{e}_l(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \text{c.c.} \right] \tag{A.9}$$

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[1]Noting that $\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$.