Dept. of Math. University of Oslo
Statistical Research Report No. 5
ISSN 0806–3842 October 2007

Copula goodness-of-fit testing: an overview and power comparison

Daniel Berg
University of Oslo and Norwegian Computing Center

Abstract

Several copula goodness-of-fit approaches are examined, three of which are proposed in this paper. Results are presented from an extensive Monte Carlo study, where we examine the effect of dimension, sample size and strength of dependence on the nominal level and power of the different approaches. While no approach is always the best, some stand out and conclusions and recommendations are made. A novel study of p-value variation due to permuation order, for approaches based on Rosenblatt's transformation is also carried out. Results show significant variation due to permutation order for some of the approaches based on this transform. However, when approaching rejection regions, the additional variation is negligible. Finally, motivated by the permutation study, new versions of some goodness-of-fit approaches are proposed and examined. The new versions consider all permutation orders of the variables and we see some power improvement over the approaches that consider one permutation order only.

Key words: Copula, Cramér-von Mises statistic, empirical copula, goodness-of-fit, parametric bootstrap, pseudo-observations, Rosenblatt's transform.

1 Introduction

A copula contains all the information about the dependency structure of a random vector. Due to the representation theorem of Sklar (1959), every distribution function H can be written as $H(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}$, where $F_1, \ldots, F_d$ are the marginal distributions and $C : [0,1]^d \to [0,1]$ is the copula. This enables the modelling of marginal distributions and the dependence structure in separate steps. This feature in particular has motivated successful applications in areas such as survival analysis, hydrology, actuarial science and finance. For exhaustive and general introductions to copulae, the reader is referred to Joe (1997) and Nelsen (1999), and for introductions oriented to financial applications, Malevergne and Sornette (2006) and Cherubini et al. (2004). While the evaluation of univariate distributions is well documented, the study of goodness-of-fit (GoF) tests for copulas emerged only recently as a challenging inferential problem.

Let C be the underlying d-variate copula of a population. Suppose one wants to test the composite GoF hypothesis

$$\mathcal{H}_0 : C \in \mathcal{C} = \{C_\theta; \theta \in \Theta\} \quad \text{vs.} \quad \mathcal{H}_1 : C \notin \mathcal{C} = \{C_\theta; \theta \in \Theta\}, \tag{1}$$

where Θ is the parameter space. Lately, several contributions have been made to test this hypothesis, e.g. Genest and Rivest (1993), Shih (1998), Breymann et al. (2003), Malevergne and Sornette (2003), Scaillet (2005), Genest and Rémillard (2008), Fermanian (2005), Panchenko (2005), Genest et al. (2006a), Berg and Bakken (2007), Dobrić and Schmid (2007), Quessy et al. (2007), Genest et al. (2008), among others. However, the field is still in its infancy and general guidelines and recommendations are sparse.

For univariate distributions, the GoF assessment can be performed using e.g. the well-known Anderson-Darling statistic (Anderson and Darling, 1954), or less quantitatively using a QQ-plot. In the multivariate domain there are fewer alternatives. A simple way to build GoF approaches for multivariate random variables is to consider multi-dimensional chi-square approaches, as in for example Dobrić and Schmid (2005). The problem with this approach, as with all binned approaches based on gridding the probability space, is that they will not be feasible for high dimensional problems due to the curse of dimensionality. Another issue with binned approaches is that the grouping of the data is not trivial. Grouping too coarsely destroys valuable information and the ability to contrast distributions becomes very limited. On the other hand, too small groups leads to a highly irregular empirical cumulative distribution function (cdf) due to the limited amount of data. For these reasons, multivariate binned approaches are not considered in this study. Multivariate kernel density estimation (KDE) approaches such as the ones proposed by Fermanian (2005) and Scaillet (2005) are also excluded from this study as they will simply be too computationally exhaustive for high dimensional problems. The author believes GoF to be most useful for high-dimensional problems since copulae are then harder to conceptualize. Moreover, the consequences of poor model choice is often much greater in higher dimensional problems, e.g. risk assessments for high dimensional financial portfolios.

The class of dimension reduction approaches is a more promising alternative. Dimension reduction approaches reduce the multivariate problem to a univariate problem, and then apply some univariate test, leading to numerically efficient approaches even for high dimensional problems. These approaches primarily differ in the way the dimension reduction is carried out. For the univariate test it is common to apply standard univariate statistics such as Kolmogorov- or Cramér-von Mises type statistics. Examples include Breymann et al. (2003), Malevergne and Sornette (2003), Genest et al. (2006a), Berg and Bakken (2007), Quessy et al. (2007) and Genest and Rémillard (2008) among others.

This paper is organized as follows. In Section 2 some preliminaries are presented. Section 3 gives an overview of the nine GoF approaches considered, including three new ones. In Section 4 results are presented from an extensive Monte Carlo study where we examine the effect of dimension, sample size and strength of dependence on the nominal level and power of the approaches. Several null- and alternative hypothesis copulae are considered. Further, this section also presents results from a novel numerical study of the effect of permutation order for approaches based on Rosenblatt's transform. New versions of some of the approaches based on this transform are proposed and examined. These new versions utilize all permutation orders of the data in an attempt to extract more information, and hence increase the power. Finally, Section 5 discusses and recommends. In addition, detailed testing procedures, leading to proper p-value estimates for all approaches, are given in the appendix.

2 Preliminaries

For copula GoF testing one is interested in the fit of the copula alone. Typically, one does not wish to introduce any distributional assumptions for the margins. Instead the testing is carried out using rank data. Suppose we have a random d-variate vector $\mathbf{X}$. The inference is then based on the so-called pseudo-vector $\mathbf{Z}$:

$$\mathbf{Z}_j = (Z_{j1}, \ldots, Z_{jd}) = \left(\frac{R_{j1}}{n+1}, \ldots, \frac{R_{jd}}{n+1} \right), \tag{2}$$

where R_{ji} is the rank of X_{ji} amongst $(X_{1i}, \ldots, X_{ni})$. This transformation of each margin through their normalized ranks is often denoted the empirical marginal transformation. Given the independent samples $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ can be considered to be samples from the underlying copula C. However, due to the rank transformation, $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ are no longer independent samples. In addition, since we are testing a hypothesized parametric copula model, as summarized by (1), parameter estimation error will influence the limiting distribution of any GoF approach. The practical consequence is the need for parametric bootstrap procedures to obtain reliable p-value estimates. This is treated in more detail in Secion 3.10.

2.1 Rosenblatt's transformation

The Rosenblatt transformation, proposed by Rosenblatt (1952), transforms a set of dependent variables into a set of independent $U[0,1]$ variables, given the multivariate distribution. The transformation is a universally applicable way of creating a set of i.i.d. $U[0,1]$ variables from any set of dependent variables with known distribution. Given a test for multivariate, independent uniformity, the transformation can be used to test the fit of any assumed model.

Definition 2.1 (Rosenblatt's transformation)
Let $\mathbf{Z} = (Z_1, \ldots, Z_d)$ *denote a random vector with marginal distributions* $F_i(z_i) = P(Z_i \leq z_i)$ *and conditional distributions* $F_{i|1\ldots i-1}(Z_i \leq z_i | Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1})$ *for* $i = 1, \ldots, d$*. Rosenblatt's*

transformation of $\mathbf{Z}$ *is defined as* $\mathcal{R}(\mathbf{Z}) = (\mathcal{R}_1(Z_1), \ldots, \mathcal{R}_d(Z_d))$ *where*

$$\begin{aligned}
\mathcal{R}_1(Z_1) &= P(Z_1 \leq z_1) = F_1(z_1), \\
\mathcal{R}_2(Z_2) &= P(Z_2 \leq z_2 | X_1 = z_1) = F_{2|1}(z_2|z_1), \\
&\vdots \\
\mathcal{R}_d(Z_d) &= P(Z_d \leq x_d | Z_1 = z_1, \ldots, Z_{d-1} = z_{d-1}) = F_{d|1\ldots d-1}(z_d|z_1, \ldots, z_{d-1}).
\end{aligned}$$

The random vector $\mathbf{V} = (V_1, \ldots, V_d)$*, where* $V_i = \mathcal{R}_i(Z_i)$*, is now i.i.d.* $U[0,1]^d$.

A recent application of this transformation is multivariate GoF tests. The Rosenblatt transformation is applied to a data set, assuming a multivariate null hypothesis distribution, and then a test of multivariate independence is carried out on the resulting transformed data set. The null hypothesis is typically a parametric copula family. The parameters of this copula family needs to be estimated before performing the transformation.

One advantage with Rosenblatt's transformation in a GoF setting is that the null- and alternative hypotheses are the same, regardless of the distribution before the transformation. Hong and Li (2005) report Monte Carlo evidence of multivariate tests using transformed variables outperforming tests using the original random variables. Chen et al. (2004) believe that a similar conclusion also applies to GoF tests for copulae. Another advantage is that computationally intensive double bootstrap procedures can be avoided for some approaches.

A disadvantage with tests based on Rosenblatt's transformation is the lack of invariance with respect to the permutation of the variables since there are $d!$ possible permutations. However, as long as the permutation is decided randomly, the results will not be influenced in any particular direction. The practical implications of this disadvantage is studied in Section 4.2.

2.2 Parameter estimation

Testing the hypothesis in (1) involves the estimation of the copula parameters $\boldsymbol{\theta}$ by some consistent estimator $\widehat{\boldsymbol{\theta}}$. There are mainly two ways of estimating these parameters; the fully parametric method or a semi-parametric method. The fully parametric method, termed the inference functions for margins (IFM) method (Joe, 1997), relies on the assumption of parametric, univariate margins. First, the parameters of the margins are estimated and then each parametric margin is plugged into the copula likelihood which is then maximized with respect to the copula parameters. Since we treat the margins as nuisance parameters we rather proceed with the pseudo-vector $\mathbf{Z}$ and the semi-parametric method. This method is denoted the pseudo-likelihood (Demarta and McNeil, 2005) or the canonical maximum likelihood (CML) (Romano, 2002) method, and is described in Genest et al. (1995) and in Shih and Louis (1995) in the presence of censorship. Having obtained the pseudo-vector $\mathbf{Z}$ as described in (2), the copula parameters can be estimated using either maximum likelihood (ML) or using the well-known relation to Kendall's tau.

For the elliptical copulae in higher dimensions the pairwise sample Kendall's tau's are inverted. This gives the correlation- and scale matrix for the Gaussian and Student copulae, respectively. For the Student copula one must also estimate the degree-of-freedom. We follow Mashal and Zeevi (2002) and Demarta and McNeil (2005), who propose a two-stage approach in which the scale matrix is first estimated by inversion of Kendall's tau, and then the pseudo-likelihood function is maximized with respect to the degree-of-freedom ν, using the estimate of the scale matrix. For the Archimedean copulae the parameter is estimated by inversion of Kendall's tau. For dimension $d > 2$ we estimate the parameter as the average of the $d(d-1)/2$ pairs of Kendall's tau's.

3 Copula goodness-of-fit approaches

The following nine copula GoF approaches are examined:

$\mathcal{A}_1$: Based on Rosenblatt's transformation, proposed by Berg and Bakken (2007). This approach includes, as special cases, the approaches proposed by Malevergne and Sornette (2003), Breymann et al. (2003), and the second approach in Chen et al. (2004).

$\mathcal{A}_2$: Based on the the empirical copula and the copula distribution function, proposed by Genest and Rémillard (2008).

$\mathcal{A}_3$: Based on approach $\mathcal{A}_2$ and the Rosenblatt transformation, proposed by Genest et al. (2008).

$\mathcal{A}_4$: Based on the empirical copula and the cdf of the copula function, proposed by Savu and Trede (2004) and Genest et al. (2006a).

$\mathcal{A}_5$: Based on Spearman's dependence function, proposed by Quessy et al. (2007).

$\mathcal{A}_6$: A new approache that extends Shih's test (Shih, 1998) for the bivariate Clayton model to arbitrary dimension.

$\mathcal{A}_7$: Based on the inner product between two vectors as a measure of their distance, proposed by Panchenko (2005).

$\mathcal{A}_8$: A new approach based on approach $\mathcal{A}_7$ and the Rosenblatt transformation.

$\mathcal{A}_9$: A new approach based on averages of the approaches above.

Approaches $\mathcal{A}_1$-$\mathcal{A}_5$ are all dimension reduction approaches, while $\mathcal{A}_6$ is a moment-based approach and $\mathcal{A}_7$-$\mathcal{A}_8$ are denoted full multivariate approaches. For all the dimension reduction approaches only the Cramér-von Mises statistic is considered for the unviariate test.

3.1 Approach $\mathcal{A}_1$

Berg and Bakken (2007) propose a generalization of the approches proposed by Breymann et al. (2003) and Malevergne and Sornette (2003). The approach is based on Rosenblatt's transformation applied to the pseudo-vector $\mathbf{Z}$ from (2), assuming a null hypothesis copula $C_{\widehat{\theta}}$. The d-variate vector $\mathbf{V}$, resulting from the transformation, is i.i.d. $U[0,1]^d$ under the null hypothesis.[1] Berg and Bakken (2007) also propose a second Rosenblatt transformation, applied to $\mathbf{V}$ but this term will not be considered here.

The dimension reduction of approach $\mathcal{A}_1$ is based on $\mathbf{V}$:

$$W_1 = \sum_{i=1}^{d} \Gamma\{V_i; \boldsymbol{\alpha}\}, \tag{3}$$

where Γ is any weight function used to weight the information in $\mathbf{V}$ and $\boldsymbol{\alpha}$ is the set of weight parameters. Any weight function may be used, depending on the use and the region of $\mathbf{V}$ one wishes to emphasize. Consider for example the special case $\Gamma\{V_i; \boldsymbol{\alpha}\} = \Phi^{-1}(V_i)^2$ which corresponds to the approach proposed by Breymann et al. (2003). If the null hypothesis is the Gaussian copula this is also equivalent with the approach proposed by Malevergne and Sornette (2003). Both of the latter studies apply the Anderson-Darling (Anderson and Darling, 1954) statistic. Berg and Bakken (2007) show that the Anderson-Darling statistic with $\Gamma\{V_i; \boldsymbol{\alpha}\} = |V_i - 0.5|$ performs particularly well for testing the Gaussian null hypothesis. Hence, when performing the numerical studies in Section 4.1 the following two special cases of approach $\mathcal{A}_1$ are considered:

$$\mathcal{A}_1^{(i)} : \Gamma\{V_i; \boldsymbol{\alpha}\} = \Phi^{-1}(V_i)^2 \quad \text{and} \quad \mathcal{A}_1^{(ii)} : \Gamma\{V_i; \boldsymbol{\alpha}\} = |V_i - 0.5|.$$

For approach $\mathcal{A}_1^{(i)}$ it is easy to see that the distribution of W_1 is a χ^2_d distribution[1]. However, for approach $\mathcal{A}_1^{(ii)}$, and in general, the distribution of W_1 is not known and one must turn to a double bootstrap procedure to approximate the cdf F_1 under the null hypothesis. The test observator S_1 of approach $\mathcal{A}_1$ is defined as the cdf of $F_1(W_1)$:

$$S_1(w) = P\{F_1(W_1) \le w\}, \qquad w \in [0,1].$$

Under the null hypothesis, all V_i are i.i.d. $U[0,1]$, hence $S_1(w) = w$. Suppose we have the random samples $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ from $\mathbf{V}$. Then the empirical version of the test observator can be computed as

$$\widehat{S}_1(w) = \frac{1}{n+1} \sum_{j=1}^{n} I\{F_1(W_1) \le w\}. \tag{4}$$

[1]Since we are working with rank data this is only close to, but not exactly true. This issue is discussed in Section 3.10. Until then it is assumed that this holds.

This paper only considers the Cramér-von Mises statistic, a version of which becomes (shown in Appendix B):

$$
\begin{aligned}
\widehat{T}_1 &= n \int_0^1 \{\widehat{S}_1(w) - S_1(w)\}^2 \, \mathrm{d}S_1(w) \\
&= \frac{n}{3} + \frac{n}{n+1} \sum_{j=1}^{n} \widehat{S}_1 \left(\frac{j}{n+1} \right)^2 - \frac{n}{(n+1)^2} \sum_{j=1}^{n} (2j+1) \widehat{S}_1 \left(\frac{j}{n+1} \right).
\end{aligned} \tag{5}
$$

3.2 Approach $\mathcal{A}_2$

Genest and Rémillard (2008) propose to use the copula distribution function for the dimension reduction. The approach is based on the empirical copula process, introduced by Deheuvels (1979):

$$
\widehat{C}(\mathbf{u}) = \frac{1}{n+1} \sum_{j=1}^{n} I\left\{ Z_{j1} \le u_1, \ldots, Z_{jd} \le u_d \right\}. \tag{6}
$$

where $\mathbf{Z}_j$ is given by (2) and $\mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d$. The empirical copula is the observed frequency of $P(Z_1 < u_1, \ldots, Z_d < u_d)$. Suppose we have the random samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ from $\mathbf{Z}$. The idea is then to compare $\widehat{C}(\mathbf{z})$ with an estimation $C_{\widehat{\theta}}(\mathbf{z})$ of C_θ. This is a very natural approach for copula GoF testing considering that most univariate GoF tests are based on a distance between an empirical- and null hypothesis distribution function. Genest et al. (2008) state that, given that it is entirely non-parametric, $\widehat{C}$ is the most objective benchmark for testing the copula GoF. We expect this approach to be very powerful since there are so few transformations of the data. A Cramér-von Mises statistic for approach $\mathcal{A}_2$ becomes (Genest et al., 2008):

$$
\widehat{T}_2 = n \int_{[0,1]^d} \left\{ \widehat{C}(\mathbf{z}) - C_{\widehat{\theta}}(\mathbf{z}) \right\}^2 \, \mathrm{d}\widehat{C}(\mathbf{z}) = \sum_{j=1}^{n} \left\{ \widehat{C}(\mathbf{z}_j) - C_{\widehat{\theta}}(\mathbf{z}_j) \right\}^2. \tag{7}
$$

3.3 Approach $\mathcal{A}_3$

Genest et al. (2008) propose to apply approach $\mathcal{A}_2$ to the vector $\mathbf{V}$ resulting from applying the Rosenblatt transform to $\mathbf{Z}$. Suppose we have the random samples $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ from $\mathbf{V}$. The idea is then to compare $\widehat{C}$ with the independence copula $C_\perp$. A Cramér-von Mises statistic for approach $\mathcal{A}_3$ becomes (Genest et al., 2008):

$$
\widehat{T}_3 = n \int_{[0,1]^d} \left\{ \widehat{C}(\mathbf{v}) - C_\perp(\mathbf{v}) \right\}^2 \, \mathrm{d}\widehat{C}(\mathbf{v}) = \sum_{j=1}^{n} \left\{ \widehat{C}(\mathbf{v}_j) - C_\perp(\mathbf{v}_j) \right\}^2. \tag{8}
$$

3.4 Approach $\mathcal{A}_4$

Savu and Trede (2004) and Genest et al. (2006a) propose to use Kendall's dependence function $K(w) = P(C(\mathbf{Z}) \le w)$ as a GoF approach. The test observator S_4 of approach $\mathcal{A}_4$ becomes

$$
S_4(w) = P\{C(\mathbf{Z}\} \le w\}, \qquad w \in [0,1],
$$

where $\mathbf{Z}$ is the pseudo-vector from (2). Under the null hypothesis, $S_4(w) = S_{4,\widehat{\theta}}(w)$ which is copula specific. Suppose we have the random samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ from $\mathbf{Z}$. The empirical version of test observator S_4 then equals

$$
\widehat{S}_4(w) = \frac{1}{n+1} \sum_{j=1}^{n} I\{\widehat{C}(\mathbf{z}_j) \le w\}. \tag{9}
$$

A Cramér-von Mises statistic for approach $\mathcal{A}_4$ is given by:

$$
\widehat{T}_4 = n \int_0^1 \{\widehat{S}_4(w) - S_{4,\widehat{\theta}}(w)\}^2 \, \mathrm{d}\widehat{S}_4(w) = \sum_{j=1}^{n} \left\{ \widehat{S}_4 \left(\frac{j}{n+1} \right) - S_{4,\widehat{\theta}} \left(\frac{j}{n+1} \right) \right\}^2. \tag{10}
$$

3.5 Approach $\mathcal{A}_5$

Quessy et al. (2007) propose a GoF approach for bivariate copulae based on Spearman's dependence function $L_2(w) = P(Z_1 Z_2 \leq w)$. Notice that $L_2(w) = P(C_\perp(Z_1, Z_2) \leq w)$. A natural extension to arbitrary dimension d is then $L_d(w) = P(C_\perp(\mathbf{Z}) \leq w)$ and the test observator S_5 of approach $\mathcal{A}_5$ becomes

$$S_5(w) = P\{C_\perp(\mathbf{Z}) \leq w\}, \quad w \in [0, 1],$$

where $\mathbf{Z}$ is the pseudo-vector from (2). Under the null hypothesis, $S_5(w) = S_{5,\widehat{\theta}}(w)$, which is copula specific. Suppose we have the random samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ from $\mathbf{Z}$. The empirical version of test observator S_5 then equals

$$\widehat{S}_5(w) = \frac{1}{n+1} \sum_{j=1}^{n} I\{C_\perp(\mathbf{z}_j) \leq w\}. \tag{11}$$

A Cramér-von Mises statistic for approach A_5 is given by:

$$\widehat{T}_5 = n \int_0^1 \{\widehat{S}_5(w) - S_{5,\widehat{\theta}}(w)\}^2 \, \mathrm{d}\widehat{S}_5(w) = \sum_{j=1}^{n} \left\{ \widehat{S}_5 \left(\frac{j}{n+1} \right) - S_{5,\widehat{\theta}} \left(\frac{j}{n+1} \right) \right\}^2. \tag{12}$$

3.6 Approach $\mathcal{A}_6$

Shih (1998) propose a moment-based GoF test for the bivariate gamma frailty model, also known as Clayton's copula. Shih (1998) considered unweighted and weighted estimators of the dependency parameter θ via Kendall's tau and a weighted rank-based estimator, namely

$$\widehat{\theta}_\tau = \frac{2\widehat{\tau}}{1 - \widehat{\tau}} \quad \text{and} \quad \widehat{\theta}_W = \frac{\sum_{i<j} \Delta_{ij}/W_{ij}}{\sum_{i<j} (1 - \Delta_{ij})/W_{ij}}, \tag{13}$$

where $\widehat{\tau} = -1 + 4 \sum_{i<j} \Delta_{ij}/\{n(n-1)\}$, $\Delta_{ij} = I\{(Z_{i1} - Z_{j1})(Z_{i2} - Z_{j2}) > 0\}$ and $W_{ij} = \sum_{k=1}^{n} I\{Z_{k1} \leq \max(Z_{i1}, Z_{j1}), Z_{k2} \leq \max(Z_{i2}, Z_{j2})\}$. Since $\widehat{\theta}_\tau$ and $\widehat{\theta}_W$ are both unbiased estimators of θ under the null hypothesis that $C = C_\theta$ for some $\theta \geq 0$, Shih (1998) propose the GoF statistic

$$\widehat{T}_{Shih} = \sqrt{n}\{\widehat{\theta}_\tau - \widehat{\theta}_W\}.$$

Shih (1998) shows that this statistic is asymptotically normal under the null hypothesis. Unfortunately, the variance provided by Shih (1998) was found to be wrong by Genest et al. (2006b), where a corrected formula is provided.

One way of extending this approach to arbitrary dimension d is comparing each pairwise element of $\widehat{\theta}_\tau$ and $\widehat{\theta}_W$. The resulting vector of $d(d-1)/2$ statistics will tend, asymptotically, to a $d(d-1)/2$ dimensional normal vector with a non-trivial covariance matrix. The normalized version of the vector, i.e. the inverted square root of the covariance matrix multiplied with the vector of statistics, will be asymptotically standard normal and hence the sum of squares will now be chi-squared with $d(d-1)/2$ degrees of freedom. The covariance matrix of the vector of statistics remains to be computed and is deferred to future research. For now we simply compute the non-normalized sum of squares and perform a parametric bootstrap to estimate the p-value, as for all other approaches.

The test statistic for approach $\mathcal{A}_6$ then becomes:

$$\widehat{T}_6 = \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \left\{ \widehat{\theta}_{\tau,ij} - \widehat{\theta}_{W,ij} \right\}^2. \tag{14}$$

$\widehat{\theta}_W$, and hence approach $\mathcal{A}_6$, is constructed specifically for testing the Clayton copula and will not be considered for testing any other copula model.

3.7 Approach $\mathcal{A}_7$

Approaches $\mathcal{A}_1$-$\mathcal{A}_5$ are all two-stage dimension reduction approaches. First the problem is reduced to a univariate problem, second a univariate test statistic is applied. In contrast, the approach proposed by Panchenko (2005) tests the entire data set in one step. The approach is based on the inner product of

$\mathbf{Z}$ and $\mathbf{Z}_{\widehat{\theta}}$, where $\mathbf{Z}$ is the pseudo-vector from (2) and $\mathbf{Z}_{\widehat{\theta}}$ is the null hypothesis vector with $\widehat{\theta}$ being a consistent estimator of the copula parameter. The inner product can be used as a measure of the distance between two vectors. Now define the squared distance Q between the two vectors as

$$Q = \left\langle \mathbf{Z} - \mathbf{Z}_{\widehat{\theta}} \, |\kappa_d| \, \mathbf{Z} - \mathbf{Z}_{\widehat{\theta}} \right\rangle .$$

Here κ_d is a positive definite symmetric kernel such as the Gaussian kernel:

$$\kappa_d(\mathbf{Z}, \mathbf{Z}') = \exp\left\{ -\|\mathbf{Z} - \mathbf{Z}'\|^2 / (2dh^2) \right\},$$

with $\| \cdot \|$ denoting the Euclidean norm in $\mathbb{R}^d$ and $h > 0$ being a bandwidth. Q will be zero if and only if $\mathbf{Z} = \mathbf{Z}_{\widehat{\theta}}$. Suppose we have the random samples $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ from $\mathbf{Z}$. Now generate the random samples $(\mathbf{z}_1^*, \ldots, \mathbf{z}_n^*)$ from the null hypothesis vector $\mathbf{Z}_{\widehat{\theta}}$. Following the properties of an inner product, Q can be decomposed as $Q = Q_{11} - 2Q_{12} + Q_{22}$. Each term of this decomposition is estimated using V-statistics (see Denker and Keller (1983) for an introduction to U- and V-statistics) and the test statistic for approach $\mathcal{A}_7$ is given by:

$$\widehat{T}_7 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{z}_i, \mathbf{z}_j) - \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{z}_i, \mathbf{z}_j^*) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{z}_i^*, \mathbf{z}_j^*). \tag{15}$$

3.8 Approach $\mathcal{A}_8$

Along the lines of approach $\mathcal{A}_3$ we propose a version of approach $\mathcal{A}_7$ based on $\mathbf{V}$, the vector resulting from the Rosenblatt transformation applied to $\mathbf{Z}$. Suppose we have the random samples $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ from $\mathbf{V}$. Now generate the random samples $(\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*)$ from the independence copula. The statistic for approach $\mathcal{A}_8$ is simply

$$\widehat{T}_8 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{v}_i, \mathbf{v}_j) - \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{v}_i, \mathbf{v}_j^*) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\mathbf{v}_i^*, \mathbf{v}_j^*). \tag{16}$$

3.9 Approach $\mathcal{A}_9$

Finally, we propose to use averages of the approaches already introduced, as new approaches. Such averages will capture several aspects of the data and its potential deviation from the null hypothesis. Surely one can find optimal weights for a weighted average and the average should be taken over standardized variables, i.e. all approaches should be scaled appropriately. However, due to the computational load, this approach is included here in its most simple form as an interesting supplement and a hint of further research. Two averages are considered, first the average of all approaches and second the average of the three approaches based on the empirical copula, i.e. $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$. The corresponding statistics are defined as

$$\widehat{T}_9^{(i)} = \frac{1}{9} \left\{ \widehat{T}_1^{(i)} + \widehat{T}_1^{(ii)} + \sum_{j=2}^{8} \widehat{T}_j \right\} \quad \text{and} \quad \widehat{T}_9^{(ii)} = \frac{1}{3} \left\{ \widehat{T}_2 + \widehat{T}_3 + \widehat{T}_4 \right\}. \tag{17}$$

3.10 Testing procedures

In Section 3.1 it was assumed that $\mathbf{V}$, resulting from applying Rosenblatt's transformation to $\mathbf{Z}$, is i.i.d. $U[0,1]^d$. The non-parametric margins introduce dependence in $\mathbf{V}$. Hence, it is only close to, but not exactly independent. This applies to all approaches considered here. In addition, we have small sample estimation error from the estimation of the null hypothesis copula parameter. To cope with these issues and obtain a proper estimate of the p-value of a statistic, one turns to parametric bootstrap procedures. The parametric bootstrap procedure used in Genest et al. (2006a) is adopted, the validity of which is established in Genest and Rémillard (2008). Dobrić and Schmid (2007) and Berg and Bakken (2007) propose a very similar procedure in their modification of the original procedure used in Breymann et al. (2003). The asymptotic validity of the bootstrap procedure has only been proved for the approaches $\mathcal{A}_2$ and $\mathcal{A}_4$. However, the results in Dobrić and Schmid (2007) and Berg and Bakken (2007) strongly indicates that the procedure is valid also for approach $\mathcal{A}_1$. This is further discussed in view of the results in Section 4.1 and in Section 5. The test procedure for approach $\mathcal{A}_7$, originally proposed in Panchenko

(2005), gave us too low nominal levels (i.e. the rejection rate when the null hypothesis is true is lower than the prescribed size). However, a small fix, in line with the procedure of Genest and Rémillard (2008), solved this issue. The test procedures for all approaches are detailed in Appendix C. In many cases one must resort to a double parametric bootstrap to compute a statistic. This means that there are two bootstrap parameters that needs to be chosen, the sample size N_b for the double bootstrap step and the number of replications K for the estimation of p-values. In this paper the number of replications K is chosen to equal 1000, while the double bootstrap sample size N_b is chosen to equal 10000 for approach $\mathcal{A}_1$, and 2500 in dimension $d = \{2, 4\}$ and 5000 in dimension $d = 8$ for approaches $\mathcal{A}_2$, $\mathcal{A}_4$ and $\mathcal{A}_5$. See Appendix C for details.

4 Numerical experiments

4.1 Size and power simulations

A large Monte Carlo study is performed to assess the properties of the approaches for various dimensions, sample sizes, levels of dependence and alternative dependence structures. The nominal levels and the power against fixed alternatives are of particular interest. The simulations are carried out according to the following factors:

- $\mathcal{H}_0$ copula (5 choices: Gaussian, Student, Clayton, Gumbel, Frank),
- $\mathcal{H}_1$ copula (5 choices: Gaussian, Student ($\nu = 6$), Clayton, Gumbel, Frank),
- Kendall's tau (2 choices: $\tau = \{0.2, 0.4\}$),
- Dimension (3 choices: $d = \{2, 4, 8\}$),
- Sample size (2 choices: $n = \{100, 500\}$).

Due to extreme computational load, the Student copula is only considered as null hypothesis in the bivariate case. In each of the remaining 260 cases, a sample of dimension d and size n is drawn from the $\mathcal{H}_1$ copula with dependence parameter corresponding to τ. The statistics of the various GoF approaches are then computed under the null hypothesis $\mathcal{H}_0$ and p-values are estimated. This entire procedure is repeated $10,000$ times in order to estimate the nominal level and power for each approach under consideration.

Since we apply a parametric bootstrap procedure in the estimation of p-values, critical values are obtained by simulating from the null hypothesis, and hence all reported powers are so-called size-adjusted powers and approaches can be compared appropriately (see e.g. Hendry (2006) and Florax et al. (2006) for size-adjustment suggestions).

The critical values of each statistic under the true null hypothesis were tabulated for each dimension and sample size considered and for many levels of dependence. For the power simulations we used table look-up with linear interpolation to ensure comparison with the appropriate critical value. Despite the tabulation this computationally exhaustive experiment would not have been feasible without access to the *Titan* computer grid at the University of Oslo, a cluster of $1,750$ computing cores, 6.5 TB memory, 350 TB local disk and 12.5 Tflops.

4.1.1 Testing the Gaussian hypothesis

Let us first consider testing the Gaussian hypothesis under several fixed alternatives. Table 1 shows the results from our simulations.

Notice that the nominal levels of all approaches match the prescribed size of 5% well. Note next that the power generally (but not always) increases with level of dependence, as expected since two copulae differs more and more as we move away from independence where all copulae are equivalent to the independence copula. Also note that the power increases with sample size, as it should for the approaches to be consistent. The power generally (but not always) also increases with dimension. This is also expected since it is natural to believe that the difference between two distributions increases with dimension, see for example Chen et al. (2004) who show that the Kullbach-Leibler Information Criterion (a measure of distance between two copulae) between the Gaussian- and Student copulae increases with dimension. Also, one can imagine that there is more for a GoF approach to work with the higher dimension is.

Next, we note that no approach is always the best, they all have special cases where they perform well and cases where they perform poor. For example, approaches $\mathcal{A}_1$ and $\mathcal{A}_3$ perform particularly well for testing against heavy tails, i.e. the Student copula alternative. $\mathcal{A}_1^{(i)}$ performs extremely well for high dimensions and large sample sizes while $\mathcal{A}_3$ performs very well for the bivariate case and for small sample sizes in higher dimensions. When Clayton and Gumbel are the alternatives, two of the approaches based on the empirical copula, $\mathcal{A}_2$ and $\mathcal{A}_4$, perform very well. In addition, in particular for Gumbel alternatives in higher dimensions, approach $\mathcal{A}_5$ performs very well. And finally, as expected, approach $\mathcal{A}_9^{(ii)}$, the average of $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ perform very well for Clayton and Gumbel alternatives. For the Frank alternative, approach $\mathcal{A}_3$ performs particularly well for the bivariate case, but then, surprisingly, extremely poor for higher dimensions while approaches $\mathcal{A}_4$ and $\mathcal{A}_5$ perform quite well for all dimensions. This shows us the danger of concluding for higher dimensions based on bivariate power results. We also note from the table that approaches $\mathcal{A}_7$ and $\mathcal{A}_8$ are generally quite poor, they almost never perform among the best. However, at the same time they are usually not among the worst. Finally we see that the average approaches perform quite well in most cases, sometimes being the most powerful ones.

One aspect of the power comparison that is lost when only looking at the best approach (bold in the tables), or when ranking the approaches, is that an approach can be almost as good as the best approach in all cases, but not necessarily the very best. For example when the alternative is the Gumbel copula for $d = 4$, $n = 500$ and $\tau = 0.40$, approach $\mathcal{A}_9^{(ii)}$ will be ranked 1 with a power of 99.8 while approach $\mathcal{A}_5$ will be ranked number 5 when its power is 98.1. This small difference in power may not even be statistically significant and purely due to Monte Carlo variation. Hence, in addition to the table we also examine a boxplot of the differences in power, from the best performing approach. This is depicted in Figure 1. From this figure we see that although approaches $\mathcal{A}_2$ and $\mathcal{A}_4$ are the best performing approaches in addition to the average approaches, the power in some very few cases is remarkably low compared to the best in those specific cases. All cases of poor performance of these approaches are for the Student alternative. Hence, for testing the Gaussian copula one should apply more than one approach, e.g $\mathcal{A}_2$ and $\mathcal{A}_3$ and in higher dimensions with large sample sizes also approach $\mathcal{A}_1^{(i)}$. The average approaches is an attempt of applying several approaches simultaneously and we see that they have very stable and good performance. However, also for these approaches there are cases, although very few, of very poor performance compared to the very best approach.

For approach $\mathcal{A}_1$, Berg and Bakken (2007) report results where the weight function $\Gamma\{V_i; \boldsymbol{\alpha}\} = |V_i - 0.5|$ outperformed $\Gamma\{V_i; \boldsymbol{\alpha}\} = \Phi^{-1}(V_i)^2$, in particular for small sample sizes. These results are not confirmed in this paper where the conclusion is the opposite in almost all cases. However, in this paper the Cramér–von Mises statistic was applied while Berg and Bakken (2007) considered the Anderson–Darling statistic. Since the Anderson–Darling statistic emphasizes the tails of the distribution, when mixed with the extreme weight on the corners and edges of the unit hypercube from $\Phi^{-1}(V_i)$ it may be too extreme for small sample sizes. When using the Cramér–von Mises statistic this is apparently not the case.

4.1.2 Testing the Student hypothesis

Next, we consider testing the Student copula hypothesis, for the bivariate case only. Table 2 and Figure 2 show the results. Again we note that the nominal levels match the prescribed size well. The powers against the Gaussian copula are also very close to the nominal levels which makes sense since the Student copula approaches the Gaussian as the degrees of freedom increases. As for testing the Gaussian hypothesis, approaches $\mathcal{A}_2$, $\mathcal{A}_4$, and particularly $\mathcal{A}_9^{(ii)}$, perform very well. Approaches $\mathcal{A}_1$, $\mathcal{A}_7$ and $\mathcal{A}_8$ all perform rather poorly. While approach $\mathcal{A}_1$ performed very well for Student alternatives when testing the Gaussian copula, this is of course not the case when testing the Student copula since this is now the null hypothesis and nominal levels should, and do indeed, match the prescribed size of 5%.

4.1.3 Testing the Clayton hypothesis

Table 3 shows the results of testing the Clayton hypothesis and Figure 3 shows the power differences. The nominal levels match the prescribed size well. Again notice the very good performance of approaches $\mathcal{A}_2$ and $\mathcal{A}_4$. $\mathcal{A}_6$ does however outperform all other approaches. $\mathcal{A}_9^{(i)}$ also perform very well, but is highly dominated by $\mathcal{A}_6$ and does not provide additional knowledge in this case. Approach $\mathcal{A}_6$, the multivariate version of Shih's statistic, is constructed specifically for testing the Clayton copula. With this in mind,

Table 1: Percentage of rejections (at 5% significance level) of the Gaussian copula by approaches $\mathcal{A}_1$-$\mathcal{A}_9$.

d	n	τ	*True copula*	$\mathcal{A}_1^{(i)}$	$\mathcal{A}_1^{(ii)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(i)}$	$\mathcal{A}_9^{(ii)}$
2	100	0.2	Gaussian	*5.3*	*5.0*	*5.0*	*4.6*	*5.4*	*5.7*	–	*4.7*	*5.2*	*5.0*	*5.1*
			Student-t	0.9	4.2	7.0	**8.8**	6.1	5.3	–	5.6	6.0	3.3	6.4
			Clayton	2.6	5.0	19.7	19.6	19.9	15.6	–	7.1	6.9	10.6	**24.0**
			Gumbel	1.9	4.6	10.7	3.6	**11.6**	8.4	–	6.2	5.9	4.9	9.7
			Frank	3.4	3.2	6.0	**7.4**	6.0	6.2	–	5.4	5.5	3.4	6.1
		0.4	Gaussian	*5.2*	*5.0*	*4.7*	*5.4*	*4.8*	*4.7*	–	*5.0*	*4.7*	*5.0*	*4.9*
			Student-t	1.3	2.4	5.9	**11.6**	4.8	3.9	–	5.3	5.8	2.3	6.4
			Clayton	1.1	2.5	57.4	59.6	49.7	33.7	–	14.9	15.8	22.2	**63.9**
			Gumbel	1.3	2.6	**19.1**	5.0	18.5	8.2	–	7.0	7.9	4.1	16.2
			Frank	0.8	1.2	10.6	11.6	10.1	8.9	–	6.1	6.3	1.5	**11.8**
	500	0.2	Gaussian	*4.7*	*4.9*	*5.2*	*4.8*	*5.2*	*5.1*	–	*5.1*	*4.9*	*4.9*	*5.0*
			Student-t	19.5	16.9	10.0	16.9	8.4	8.5	–	10.3	9.8	**21.4**	10.0
			Clayton	2.0	5.8	72.5	71.3	71.9	57.2	–	23.8	20.3	56.5	**79.5**
			Gumbel	2.5	6.9	33.2	8.5	33.9	25.8	–	12.3	11.1	21.2	**34.3**
			Frank	2.2	2.9	11.4	**21.9**	11.1	9.9	–	7.6	8.1	5.8	14.5
		0.4	Gaussian	*5.0*	*5.0*	*4.6*	*5.4*	*4.9*	*4.8*	–	*4.9*	*5.5*	*5.1*	*4.8*
			Student-t	23.8	12.5	8.2	**30.5**	6.6	6.9	–	10.1	12.6	20.6	12.0
			Clayton	6.8	4.3	99.8	**100**	99.6	96.2	–	78.1	84.3	99.0	99.9
			Gumbel	8.8	6.0	**65.3**	18.9	62.9	39.8	–	26.4	32.4	42.3	**65.3**
			Frank	15.1	12.2	36.9	**60.7**	33.4	26.4	–	17.0	20.6	36.9	52.1
4	100	0.2	Gaussian	*4.8*	*5.0*	*4.6*	*4.8*	*4.8*	*5.3*	–	*5.6*	*5.0*	*5.0*	*4.9*
			Student-t	5.1	6.5	8.9	**15.4**	8.5	7.0	–	6.7	6.6	7.5	9.7
			Clayton	1.1	5.0	45.6	30.5	52.5	19.2	–	9.4	7.0	20.2	**55.9**
			Gumbel	1.2	3.1	12.8	0.7	42.5	**56.4**	–	13.9	8.8	13.2	34.9
			Frank	2.0	1.4	1.8	3.0	12.2	**19.6**	–	7.5	6.8	2.0	8.4
		0.4	Gaussian	*4.5*	*4.8*	*5.2*	*5.4*	*5.1*	*5.1*	–	*4.9*	*5.3*	*4.9*	*5.3*
			Student-t	9.2	3.7	8.6	**24.4**	6.1	5.3	–	6.9	7.1	7.5	8.1
			Clayton	1.1	1.8	**90.8**	80.4	84.0	45.6	–	27.9	18.3	48.8	90.1
			Gumbel	1.5	1.7	41.0	3.6	**52.0**	48.7	–	25.8	15.4	17.1	50.1
			Frank	1.6	2.2	10.1	7.3	**23.6**	20.6	–	12.6	8.3	5.6	21.2
	500	0.2	Gaussian	*5.8*	*5.3*	*5.3*	*5.0*	*4.8*	*4.9*	–	*5.0*	*5.5*	*4.9*	*4.7*
			Student-t	**98.5**	71.8	16.5	47.1	11.2	12.6	–	13.6	15.0	96.5	15.7
			Clayton	4.3	7.7	99.0	94.4	98.0	88.4	–	39.3	22.2	94.6	**99.2**
			Gumbel	8.0	5.9	84.2	48.0	97.7	**98.5**	–	70.3	34.7	92.3	98.0
			Frank	3.6	6.6	25.4	5.0	64.3	**66.2**	–	20.3	17.2	39.1	63.8
		0.4	Gaussian	*4.7*	*4.7*	*4.8*	*4.9*	*4.7*	*4.8*	–	*5.1*	*5.0*	*4.4*	*4.6*
			Student-t	**98.1**	67.5	11.6	72.1	8.0	8.8	–	16.4	18.7	94.0	13.8
			Clayton	44.3	13.2	**100**	**100**	**100**	99.9	–	97.2	91.2	**100**	**100**
			Gumbel	63.2	34.7	98.9	70.1	99.6	98.1	–	95.5	77.4	99.4	**99.8**
			Frank	79.3	74.2	73.2	19.5	88.6	74.5	–	61.2	40.7	**97.4**	90.6
8	100	0.2	Gaussian	*5.0*	*5.2*	*5.9*	*4.7*	*5.8*	*5.2*	–	*5.3*	*5.2*	*5.4*	*5.7*
			Student-t	**40.4**	16.4	9.8	15.0	12.3	7.7	–	7.9	6.9	35.9	12.4
			Clayton	0.7	4.1	48.7	24.3	**66.0**	1.2	–	11.8	6.6	19.5	65.5
			Gumbel	0.6	1.7	22.0	2.3	61.5	**98.3**	–	56.9	13.8	14.0	56.1
			Frank	0.4	0.6	3.8	1.3	7.3	**56.0**	–	14.4	7.2	0.6	4.7
		0.4	Gaussian	*5.1*	*5.2*	*5.0*	*4.6*	*5.3*	*5.7*	–	*5.5*	*5.1*	*5.3*	*5.1*
			Student-t	**51.7**	16.1	8.3	17.6	7.4	6.1	–	8.0	8.5	39.2	7.8
			Clayton	1.6	2.4	**96.6**	49.2	93.3	28.1	–	40.4	19.9	59.9	95.0
			Gumbel	16.2	10.1	70.5	2.7	78.4	**92.8**	–	67.9	28.1	52.7	78.6
			Frank	4.8	8.3	19.6	2.9	**28.7**	23.9	–	26.7	7.5	14.6	25.7
	500	0.2	Gaussian	*5.5*	*4.8*	*4.4*	*5.1*	*4.8*	*5.4*	–	*5.2*	*5.1*	*4.6*	*4.8*
			Student-t	**100**	99.9	23.7	56.4	19.1	11.8	–	21.7	20.9	**100**	21.3
			Clayton	11.8	12.9	**100**	74.3	99.7	84.8	–	50.5	13.6	97.2	99.9
			Gumbel	30.0	13.4	**100**	71.7	**100**	**100**	–	**100**	63.0	99.9	**100**
			Frank	22.9	38.3	99.8	10.5	98.4	**99.9**	–	69.6	19.4	90.7	99.8
		0.4	Gaussian	*4.9*	*5.4*	*4.9*	*5.2*	*5.4*	*5.1*	–	*4.7*	*5.9*	*5.1*	*5.2*
			Student-t	**100**	99.8	16.9	71.5	12.2	10.6	–	21.4	32.0	**100**	13.7
			Clayton	78.0	52.6	**100**	99.8	**100**	**100**	–	99.2	81.5	**100**	**100**
			Gumbel	**100**	98.7	**100**	33.9	**100**	**100**	–	**100**	94.7	**100**	**100**
			Frank	99.5	99.5	**100**	1.9	99.8	95.6	–	97.3	37.7	**100**	**100**

Note: The Student copula alternative hypothesis with degree-of-freedom $\nu = 6$. Numbers in *italic* are nominal levels and should correspond to the prescribed size of 5%. Numbers in **bold** indicates the best performing approach. All powers are size-adjusted.

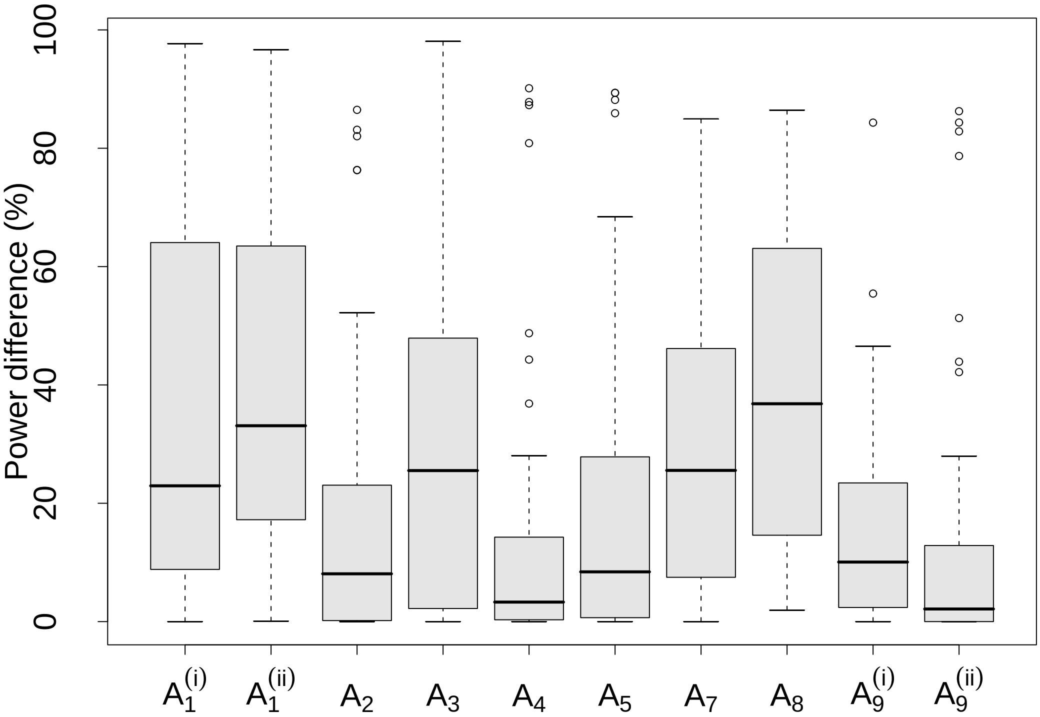


Figure 1: Distribution of power difference from the very best approach for testing the Gaussian copula.

Table 2: Percentage of rejections (at 5% significance level) of the bivariate Student copula by approaches $\mathcal{A}_1$-$\mathcal{A}_9$.

d	n	τ	*True copula*	$\mathcal{A}_1^{(i)}$	$\mathcal{A}_1^{(ii)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(i)}$	$\mathcal{A}_9^{(ii)}$
2	100	0.2	Gaussian	**5.7**	5.4	4.9	4.0	5.0	5.2	–	5.6	5.3	5.6	4.8
			Student-t	*4.4*	*4.6*	*4.8*	*4.1*	*5.1*	*4.8*	–	*5.1*	*5.0*	*4.6*	*4.8*
			Clayton	4.8	5.3	19.2	11.0	20.1	17.2	–	7.3	6.8	15.4	**21.3**
			Gumbel	4.7	5.1	9.2	4.9	**10.5**	7.0	–	5.9	5.8	7.6	10.1
			Frank	4.9	5.4	6.0	4.4	6.6	**7.1**	–	5.8	5.7	6.5	6.6
		0.4	Gaussian	4.7	5.4	4.9	4.0	5.2	5.4	–	**5.7**	4.9	5.2	4.8
			Student-t	*4.1*	*4.5*	*4.2*	*4.4*	*4.8*	*5.1*	–	*4.9*	*4.9*	*4.4*	*4.4*
			Clayton	4.2	4.9	55.0	31.7	53.3	41.1	–	15.4	14.8	39.9	**57.3**
			Gumbel	4.4	5.0	17.2	6.1	**18.7**	9.1	–	7.2	7.4	10.5	17.5
			Frank	2.9	3.4	11.8	5.3	**12.5**	10.5	–	7.5	6.3	6.9	11.6
	500	0.2	Gaussian	5.8	5.8	5.1	5.1	5.0	5.6	–	5.8	5.5	**6.0**	5.3
			Student-t	*5.1*	*5.1*	*4.5*	*4.5*	*4.5*	*5.3*	–	*5.1*	*5.2*	*4.8*	*4.6*
			Clayton	5.6	4.8	69.9	60.4	72.4	61.3	–	22.0	19.9	65.7	**77.5**
			Gumbel	5.2	5.3	28.6	18.6	30.0	19.7	–	11.0	10.0	23.5	**33.2**
			Frank	5.2	6.3	12.3	8.3	12.7	12.6	–	7.4	7.8	11.6	**13.4**
		0.4	Gaussian	**5.6**	5.2	4.5	5.3	5.0	5.5	–	5.2	4.9	5.4	5.0
			Student-t	*4.9*	*4.6*	*5.3*	*4.4*	*4.5*	*4.8*	–	*4.7*	*5.0*	*4.7*	*4.6*
			Clayton	6.4	7.0	99.8	99.6	99.6	97.7	–	74.6	78.4	99.5	**99.9**
			Gumbel	4.5	5.1	61.7	40.0	61.2	34.1	–	22.4	24.1	49.2	**68.3**
			Frank	11.6	5.9	41.2	15.4	40.4	31.7	–	17.2	14.2	36.0	**44.8**

Note: Student copula alternative hypothesis with degree-of-freedom $\nu = 6$. Numbers in *italic* are nominal levels and should correspond to the prescribed size of 5%. Numbers in **bold** indicates the best performing approach. All powers are size-adjusted.

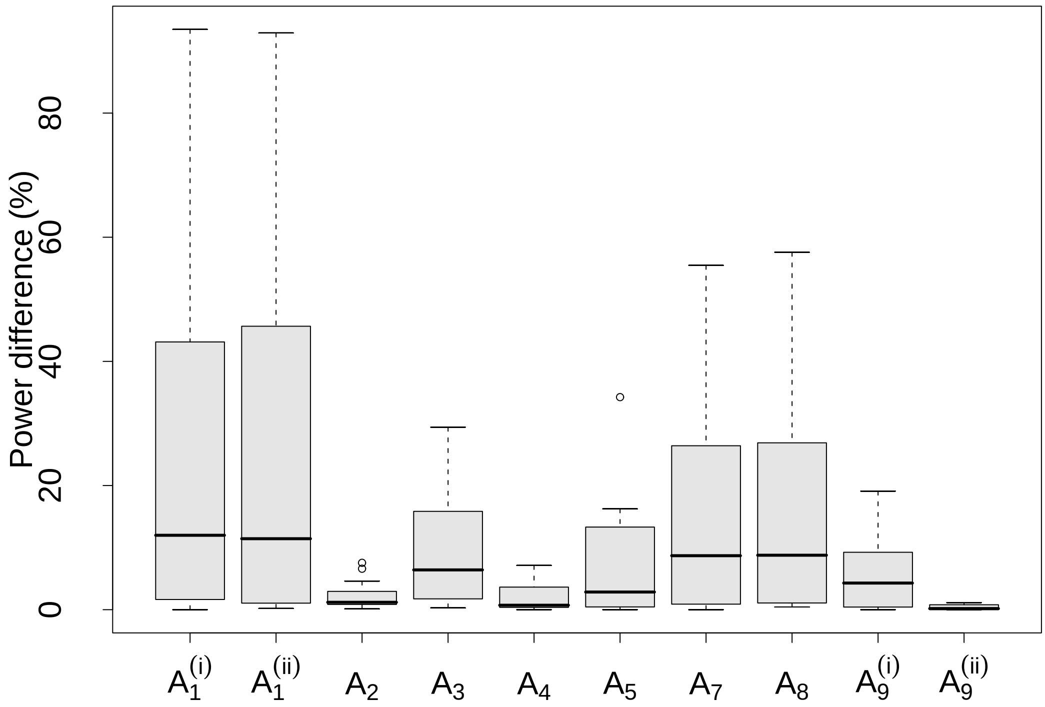


Figure 2: Distribution of power difference from the very best approach for testing the bivariate Student copula.

the performance of approaches $\mathcal{A}_2$ and $\mathcal{A}_4$ is quite impressive. While approach $\mathcal{A}_3$ performed very well for testing the Gaussian copula it performs very poor for testing the Clayton copula, with terrible performance in some cases. Finally, note that the powers are in general higher than that for testing the Gaussian hypothesis, i.e. it is simpler to detect deviations from the Clayton copula than from the Gaussian copula.

4.1.4 Testing the Gumbel hypothesis

We now test the Gumbel hypothesis. The results are shown in Table 4 and the power differences in Figure 4. Notice that the nominal levels match the prescribed size well. Note also, again, the very good performance of approaches $\mathcal{A}_2$ and $\mathcal{A}_4$. Finally, approach $\mathcal{A}_9^{(ii)}$ perform very well. This is not surprising since it is the average of $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$.

4.1.5 Testing the Frank hypothesis

Finally, we test the Frank hypothesis. The results are shown in Table 5 and the power differences in Figure 5. The nominal levels match the prescribed size well. Note again the very good performance of approach $\mathcal{A}_2$. Approaches $\mathcal{A}_4$ and $\mathcal{A}_9^{(ii)}$ also perform very well.

4.2 Effect of permutation order for Rosenblatt's transform

Approaches $\mathcal{A}_1$, $\mathcal{A}_3$ and $\mathcal{A}_8$ are all based on Rosenblatt's transform and a consecutive test of independence. The lack of invariance to the order of permutation may pose a problem to these approaches in the sense that the statistic for a given data set may prove very different depending on the permutation order. This is an undesirable feature of a statistical testing procedure. However, the practical consequence of this permutation invariance has not yet been investigated.

Table 6 shows the effect of permutation order on the estimated p-value for the three approaches based on Rosenblatt's transformation. The reported values are means and standard deviations of the estimated p-values (over $d!$ permutations). The study is restricted to dimension $d = 5$ for which there

Table 3: Percentage of rejections (at 5% significance level) of the Clayton copula by approaches $\mathcal{A}_1$-$\mathcal{A}_9$.

d	n	τ	*True copula*	$\mathcal{A}_1^{(i)}$	$\mathcal{A}_1^{(ii)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(i)}$	$\mathcal{A}_9^{(ii)}$
2	100	0.2	Gaussian	7.5	7.3	21.3	6.6	**23.2**	14.5	20.9	7.3	6.9	20.8	22.4
			Student-t	8.0	8.5	23.8	8.4	**24.1**	16.3	15.9	7.5	7.0	21.0	23.7
			Clayton	*4.9*	*5.1*	*5.0*	*5.2*	*5.0*	*5.2*	*4.5*	*5.2*	*5.2*	*5.0*	*5.1*
			Gumbel	6.2	9.4	46.7	13.0	**47.3**	32.3	40.4	12.4	11.1	41.2	47.1
			Frank	7.0	6.9	24.6	6.4	27.1	16.3	**30.3**	8.6	7.4	25.1	25.8
		0.4	Gaussian	24.0	26.7	58.9	26.4	58.2	33.7	62.1	16.6	15.3	**66.5**	60.6
			Student-t	13.4	19.0	**60.6**	16.0	58.4	35.1	53.6	15.4	13.7	58.2	57.3
			Clayton	*4.4*	*4.8*	*4.8*	*5.4*	*4.9*	*4.9*	*4.8*	*4.7*	*4.8*	*4.6*	*4.8*
			Gumbel	29.7	38.9	91.6	41.2	90.6	70.1	90.2	34.9	31.7	**92.0**	90.2
			Frank	24.1	19.2	64.8	24.2	66.2	35.6	**84.3**	19.3	16.5	77.0	65.6
	500	0.2	Gaussian	20.6	13.3	78.7	44.8	70.2	52.9	**85.9**	24.0	20.5	68.5	75.3
			Student-t	26.9	23.3	**82.1**	33.4	73.7	64.8	68.5	26.1	22.2	76.1	77.6
			Clayton	*5.2*	*5.1*	*5.0*	*4.8*	*5.1*	*5.4*	*5.1*	*5.3*	*4.5*	*4.8*	*5.2*
			Gumbel	12.6	23.2	**99.2**	84.9	97.9	94.0	99.0	60.1	52.0	97.2	98.6
			Frank	18.8	9.0	86.6	42.9	82.2	63.4	**97.6**	30.4	22.7	78.3	84.8
		0.4	Gaussian	94.8	85.6	**100**	99.5	99.7	95.5	**100**	77.7	82.3	99.9	99.9
			Student-t	65.3	71.4	**99.9**	89.7	99.6	97.3	99.8	74.7	74.9	99.8	99.8
			Clayton	*5.3*	*5.1*	*5.0*	*5.2*	*4.7*	*4.8*	*4.9*	*4.7*	*4.4*	*5.0*	*4.7*
			Gumbel	98.4	97.8	**100**	**100**	**100**	**100**	**100**	99.4	99.5	**100**	**100**
			Frank	97.8	69.9	**100**	99.4	99.9	96.7	**100**	84.6	86.8	**100**	**100**
4	100	0.2	Gaussian	10.8	10.6	37.4	3.2	38.5	39.1	**49.8**	10.6	6.5	49.2	37.9
			Student-t	27.1	21.3	48.4	17.8	37.7	42.2	37.7	10.1	7.3	**57.2**	42.5
			Clayton	*4.7*	*5.1*	*5.3*	*5.6*	*5.2*	*5.1*	*4.6*	*6.3*	*4.7*	*5.0*	*5.2*
			Gumbel	8.8	12.0	64.4	3.0	91.1	**94.1**	81.5	31.9	14.0	88.4	88.6
			Frank	7.7	6.5	36.0	1.4	**74.7**	68.9	73.0	15.1	7.2	72.8	68.8
		0.4	Gaussian	78.3	65.7	89.8	3.0	83.0	73.9	91.6	31.0	16.7	**95.2**	84.3
			Student-t	53.9	45.7	**92.9**	6.1	82.6	76.0	86.2	29.9	15.8	92.2	85.6
			Clayton	*5.2*	*4.7*	*5.6*	*5.5*	*5.2*	*5.1*	*4.5*	*5.3*	*4.9*	*5.1*	*5.3*
			Gumbel	79.1	62.1	99.3	4.9	99.8	99.8	99.8	80.8	40.1	**99.9**	99.8
			Frank	68.7	37.9	91.4	3.2	97.0	84.8	**99.6**	52.4	15.1	99.3	96.3
	500	0.2	Gaussian	89.6	38.1	99.4	18.1	97.0	91.2	**99.9**	38.8	23.0	99.4	98.0
			Student-t	93.7	76.9	99.9	89.7	95.8	94.5	97.9	44.1	30.8	**100**	98.7
			Clayton	*4.8*	*4.7*	*5.2*	*5.6*	*5.6*	*4.7*	*5.0*	*4.8*	*5.3*	*5.1*	*5.6*
			Gumbel	71.1	37.8	**100**	80.3	**100**	**100**	**100**	97.8	83.4	**100**	**100**
			Frank	82.6	11.8	99.8	14.5	**100**	99.9	**100**	67.9	24.8	**100**	**100**
		0.4	Gaussian	**100**	**100**	**100**	99.7	**100**	99.9	**100**	97.4	95.5	**100**	**100**
			Student-t	**100**	99.8	**100**	80.0	**100**	**100**	**100**	96.9	90.1	**100**	**100**
			Clayton	*4.9*	*5.2*	*5.3*	*5.7*	*5.6*	*5.2*	*5.6*	*4.8*	*5.5*	*5.1*	*5.4*
			Gumbel	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
			Frank	**100**	99.0	**100**	99.9	**100**	**100**	**100**	**100**	93.6	**100**	**100**
8	100	0.2	Gaussian	14.3	12.6	29.9	9.9	21.4	53.5	**82.6**	8.1	6.6	74.2	22.3
			Student-t	57.8	61.0	44.3	40.9	20.2	54.3	65.9	9.3	8.6	**85.5**	24.4
			Clayton	*5.5*	*5.0*	*5.2*	*5.5*	*5.6*	*5.4*	*4.3*	*4.7*	*5.2*	*5.1*	*5.5*
			Gumbel	7.6	10.5	63.2	52.6	91.9	**100**	98.0	68.7	26.5	97.0	90.8
			Frank	3.2	6.0	16.6	4.2	74.8	96.5	**96.7**	20.4	6.3	93.4	68.9
		0.4	Gaussian	97.5	91.7	96.9	2.5	87.1	89.0	98.2	34.8	10.9	**99.1**	90.2
			Student-t	86.3	80.5	**98.4**	29.5	86.1	89.4	96.0	32.4	10.7	97.7	91.4
			Clayton	*5.7*	*5.4*	*4.8*	*5.1*	*4.7*	*4.8*	*4.6*	*5.3*	*5.0*	*4.7*	*4.7*
			Gumbel	93.0	82.2	99.8	19.9	**100**	**100**	**100**	97.3	43.4	**100**	**100**
			Frank	85.2	62.8	93.7	0.6	99.6	97.7	**100**	76.5	8.1	**100**	99.6
	500	0.2	Gaussian	**100**	71.6	**100**	24.9	98.9	97.4	**100**	41.8	17.0	**100**	99.5
			Student-t	**100**	**100**	**100**	99.3	96.7	98.1	**100**	50.8	32.0	**100**	99.3
			Clayton	*5.3*	*4.8*	*5.0*	*4.8*	*4.9*	*5.3*	*4.6*	*5.3*	*5.4*	*5.4*	*4.7*
			Gumbel	98.3	40.7	**100**	96.6	**100**	**100**	**100**	**100**	96.8	**100**	**100**
			Frank	99.9	11.0	**100**	3.7	**100**	**100**	**100**	92.8	15.5	**100**	**100**
		0.4	Gaussian	**100**	**100**	**100**	96.1	**100**	**100**	**100**	98.7	84.4	**100**	**100**
			Student-t	**100**	**100**	**100**	93.2	**100**	**100**	**100**	98.7	78.1	**100**	**100**
			Clayton	*4.5*	*4.8*	*4.8*	*4.9*	*4.9*	*5.2*	*5.1*	*5.5*	*4.9*	*4.8*	*4.8*
			Gumbel	**100**	**100**	**100**	88.5	**100**	**100**	**100**	**100**	**100**	**100**	**100**
			Frank	**100**	**100**	**100**	69.5	**100**	**100**	**100**	**100**	76.0	**100**	**100**

Note: The Student copula alternative hypothesis with degree-of-freedom $\nu = 6$. Numbers in *italic* are nominal levels and should correspond to the prescribed size of 5%. Numbers in **bold** indicates the best performing approach. All powers are size-adjusted.

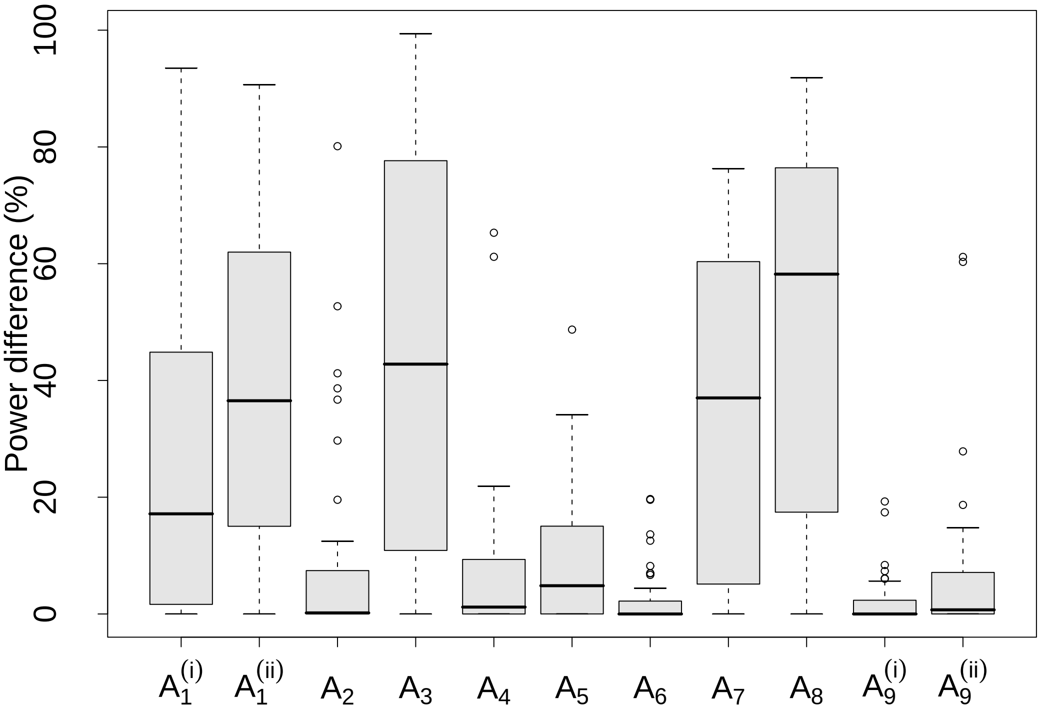


Figure 3: Distribution of power difference from the very best approach for testing the Clayton copula.

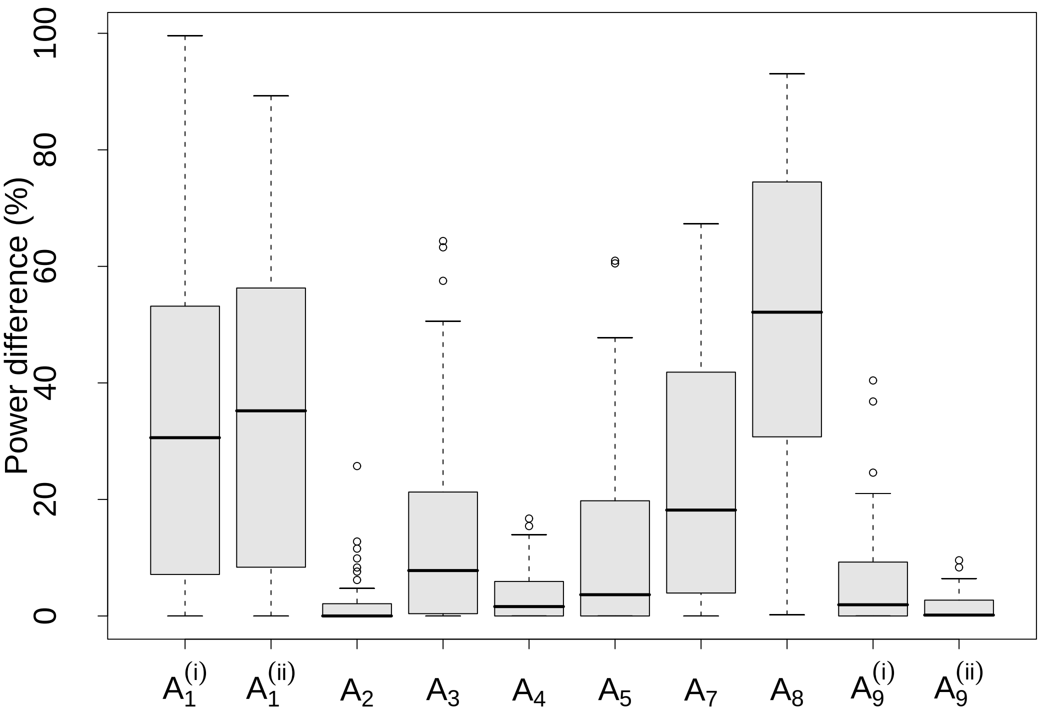


Figure 4: Distribution of power difference from the very best approach for testing the Gumbel copula.

Table 4: Percentage of rejections (at 5% significance level) of the Gumbel copula by approaches $\mathcal{A}_1$-$\mathcal{A}_9$.

d	n	τ	*True copula*	$\mathcal{A}_1^{(i)}$	$\mathcal{A}_1^{(ii)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(i)}$	$\mathcal{A}_9^{(ii)}$
2	100	0.2	Gaussian	7.7	6.6	9.9	7.3	9.6	9.6	–	6.4	6.6	**10.2**	9.8
			Student-t	7.1	6.2	**11.2**	9.8	9.0	7.6	–	5.9	6.2	8.8	10.4
			Clayton	5.9	6.5	45.8	31.1	44.0	35.1	–	12.3	10.8	33.1	**47.5**
			Gumbel	*5.3*	*5.1*	*5.1*	*4.9*	*5.1*	*5.1*	–	*5.1*	*5.3*	*5.1*	*4.9*
			Frank	6.7	5.2	12.1	8.0	11.3	**13.3**	–	7.4	6.8	10.4	11.7
		0.4	Gaussian	11.4	11.2	17.5	8.9	16.4	13.7	–	8.1	7.2	**19.1**	17.6
			Student-t	5.8	6.2	**20.2**	15.2	16.1	11.3	–	7.5	6.7	13.9	19.7
			Clayton	8.1	14.0	**92.6**	75.4	89.8	75.3	–	34.7	31.4	83.4	**92.6**
			Gumbel	*4.8*	*4.6*	*4.8*	*5.1*	*4.9*	*4.7*	–	*4.7*	*5.2*	*4.8*	*5.0*
			Frank	8.1	7.1	**28.7**	9.4	24.8	24.3	–	10.3	9.0	20.9	25.7
	500	0.2	Gaussian	19.9	9.8	**37.0**	23.9	29.2	26.9	–	11.7	10.2	31.4	33.1
			Student-t	16.6	11.6	**39.1**	33.7	25.2	17.3	–	11.8	10.2	27.7	30.8
			Clayton	8.4	10.3	**99.6**	98.5	98.5	95.9	–	57.5	51.5	97.1	99.3
			Gumbel	*4.7*	*4.6*	*5.1*	*4.8*	*4.6*	*5.1*	–	*5.0*	*4.6*	*4.6*	*4.6*
			Frank	16.0	7.4	**53.9**	30.7	38.5	42.6	–	16.2	12.7	37.1	44.3
		0.4	Gaussian	49.9	32.4	**74.1**	38.4	61.6	46.8	–	25.4	28.9	73.8	67.7
			Student-t	9.0	10.8	**74.1**	56.7	57.3	36.0	–	20.9	21.1	53.0	68.4
			Clayton	43.6	57.8	**100**	**100**	**100**	**100**	–	99.3	99.6	**100**	**100**
			Gumbel	*5.4*	*4.9*	*5.2*	*5.5*	*5.0*	*5.0*	–	*4.8*	*5.2*	*5.0*	*4.9*
			Frank	45.3	13.8	**95.5**	47.8	85.1	82.2	–	44.4	42.1	86.2	89.2
4	100	0.2	Gaussian	6.8	13.0	54.7	43.4	51.1	24.0	–	14.9	7.5	41.6	**57.3**
			Student-t	24.9	24.8	56.8	55.7	52.8	21.1	–	13.0	8.8	58.7	**60.1**
			Clayton	3.4	15.1	89.6	85.4	97.1	82.2	–	29.9	10.1	90.6	**97.2**
			Gumbel	*5.0*	*4.9*	*5.0*	*4.5*	*5.0*	*5.3*	–	*5.0*	*5.6*	*4.8*	*5.0*
			Frank	4.6	5.4	22.2	13.1	29.2	**30.6**	–	12.6	5.5	18.6	30.0
		0.4	Gaussian	29.7	36.6	66.7	44.0	59.9	33.7	–	28.8	9.2	**70.5**	65.0
			Student-t	15.1	22.0	68.0	66.1	60.7	30.2	–	26.2	9.9	60.0	**68.9**
			Clayton	26.8	29.9	99.9	99.1	**100**	98.8	–	82.4	32.8	99.8	**100**
			Gumbel	*5.0*	*5.0*	*5.0*	*5.2*	*5.1*	*5.1*	–	*5.0*	*5.4*	*5.5*	*5.0*
			Frank	17.8	9.0	51.4	12.5	54.3	**56.1**	–	26.2	7.3	46.5	53.7
	500	0.2	Gaussian	75.9	59.1	**99.4**	98.5	98.3	96.0	–	68.4	19.5	**99.4**	99.2
			Student-t	92.0	88.5	99.1	99.7	97.7	94.5	–	67.4	27.3	**100**	99.2
			Clayton	34.2	64.9	**100**	**100**	**100**	**100**	–	98.1	53.3	**100**	**100**
			Gumbel	*4.7*	*4.8*	*4.8*	*4.6*	*4.7*	*5.0*	–	*4.7*	*4.2*	*4.6*	*4.7*
			Frank	47.7	10.0	86.6	47.5	92.7	**98.1**	–	58.0	9.8	93.2	94.0
		0.4	Gaussian	99.9	98.2	**100**	99.7	99.6	97.6	–	95.9	54.8	**100**	99.9
			Student-t	86.1	91.3	**100**	**100**	99.6	97.1	–	93.9	60.2	**100**	**100**
			Clayton	**100**	95.7	**100**	**100**	**100**	**100**	–	**100**	99.8	**100**	**100**
			Gumbel	*4.7*	*5.1*	*4.9*	*5.3*	*5.1*	*4.8*	–	*4.6*	*5.1*	*4.8*	*5.2*
			Frank	99.4	31.8	99.9	58.9	99.8	**100**	–	93.0	23.7	**100**	99.9
8	100	0.2	Gaussian	1.0	30.0	89.8	73.2	87.1	29.9	–	37.6	6.7	50.0	**90.4**
			Student-t	52.3	70.3	89.4	76.6	86.2	30.9	–	36.1	8.3	**91.9**	89.9
			Clayton	0.2	29.9	93.6	95.4	**99.8**	81.2	–	53.3	8.6	89.3	99.7
			Gumbel	*5.4*	*5.1*	*4.1*	*4.8*	*4.9*	*4.8*	–	*4.6*	*5.1*	*5.1*	*4.8*
			Frank	0.3	4.3	14.6	10.3	**40.4**	19.4	–	28.4	5.5	3.6	36.8
		0.4	Gaussian	36.8	68.2	**98.1**	72.3	90.2	50.3	–	70.1	6.8	93.7	93.7
			Student-t	45.3	65.7	**97.8**	83.8	90.8	51.8	–	65.0	11.7	94.1	94.6
			Clayton	38.5	45.9	**100**	99.6	**100**	99.9	–	98.2	42.0	**100**	**100**
			Gumbel	*5.2*	*5.1*	*5.3*	*5.1*	*5.3*	*5.4*	–	*5.0*	*5.5*	*5.2*	*5.4*
			Frank	16.0	8.7	54.3	9.6	**67.1**	63.5	–	53.4	4.9	42.5	66.2
	500	0.2	Gaussian	99.9	99.1	**100**	**100**	**100**	**100**	–	99.2	14.8	**100**	**100**
			Student-t	**100**	**100**	**100**	**100**	**100**	**100**	–	98.9	31.7	**100**	**100**
			Clayton	79.4	98.9	**100**	**100**	**100**	**100**	–	**100**	33.0	**100**	**100**
			Gumbel	*5.1*	*4.9*	*4.1*	*4.8*	*5.1*	*5.2*	–	*4.3*	*4.8*	*5.2*	*5.0*
			Frank	78.6	18.6	90.1	36.7	99.9	**100**	–	93.7	7.0	99.2	99.9
		0.4	Gaussian	**100**	**100**	**100**	**100**	**100**	**100**	–	**100**	37.5	**100**	**100**
			Student-t	**100**	**100**	**100**	**100**	**100**	**100**	–	**100**	67.5	**100**	**100**
			Clayton	**100**	99.9	**100**	**100**	**100**	**100**	–	**100**	99.7	**100**	**100**
			Gumbel	*5.3*	*4.9*	*5.1*	*5.3*	*5.2*	*5.4*	–	*4.9*	*5.0*	*5.2*	*5.1*
			Frank	**100**	48.8	**100**	35.6	**100**	**100**	–	99.8	9.5	**100**	**100**

Note: Student copula alternative hypothesis with degree-of-freedom $\nu = 6$. Numbers in *italic* are nominal levels and should correspond to the prescribed size of 5%. Numbers in **bold** indicates the best performing approach. All powers are size-adjusted.

Table 5: Percentage of rejections (at 5% significance level) of the Frank copula by approaches $\mathcal{A}_1$-$\mathcal{A}_9$.

d	n	τ	*True copula*	$\mathcal{A}_1^{(i)}$	$\mathcal{A}_1^{(ii)}$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9^{(i)}$	$\mathcal{A}_9^{(ii)}$
2	100	0.2	Gaussian	5.8	5.5	6.0	**7.5**	6.9	6.6	–	4.9	5.1	6.2	7.4
			Student-t	10.6	8.4	8.8	9.9	8.9	7.9	–	6.0	5.7	**11.9**	10.1
			Clayton	5.1	5.3	24.4	21.3	26.2	18.5	–	7.9	7.4	17.4	**29.4**
			Gumbel	5.2	6.0	13.5	8.8	14.2	11.4	–	6.3	6.3	10.0	**14.9**
			Frank	*5.8*	*5.6*	*5.5*	*7.3*	*5.6*	*5.4*	–	*5.4*	*4.8*	*5.7*	*5.9*
		0.4	Gaussian	12.2	9.1	9.4	9.2	9.5	6.8	–	5.6	6.5	**13.1**	10.7
			Student-t	8.2	6.4	13.7	10.4	13.3	9.4	–	6.2	7.1	12.0	**14.7**
			Clayton	6.8	5.2	65.4	47.5	62.4	34.6	–	15.9	16.9	46.6	**68.2**
			Gumbel	6.5	6.0	**29.1**	9.6	26.0	15.7	–	8.4	9.1	18.0	26.6
			Frank	*5.9*	*4.8*	*4.9*	*6.3*	*5.2*	*4.7*	–	*4.1*	*5.1*	*5.3*	*5.3*
	500	0.2	Gaussian	7.6	6.7	11.2	**15.3**	10.3	10.3	–	6.7	7.3	10.3	11.8
			Student-t	47.8	26.9	28.0	20.5	26.5	25.2	–	12.4	13.4	**48.0**	29.2
			Clayton	7.6	7.1	87.7	81.0	84.2	66.4	–	27.5	27.5	74.3	**87.8**
			Gumbel	11.4	10.3	**55.6**	31.9	44.5	41.8	–	15.1	15.9	41.1	49.2
			Frank	*5.5*	*4.9*	*4.5*	*7.2*	*5.4*	*5.1*	–	*4.6*	*5.4*	*4.9*	*5.5*
		0.4	Gaussian	30.3	23.1	42.5	35.1	32.7	23.2	–	14.0	14.9	**47.5**	42.2
			Student-t	20.9	14.5	**68.5**	28.6	57.1	46.2	–	22.3	21.5	58.9	63.8
			Clayton	11.9	9.5	**100**	99.9	**100**	97.6	–	83.9	85.2	99.9	**100**
			Gumbel	9.9	12.2	**95.2**	47.5	85.8	77.3	–	41.7	41.2	81.2	89.9
			Frank	*6.0*	*4.8*	*4.2*	*6.4*	*4.7*	*4.0*	–	*4.6*	*5.0*	*4.9*	*5.0*
4	100	0.2	Gaussian	4.8	9.3	27.6	27.0	24.8	10.3	–	6.9	6.9	18.2	**29.8**
			Student-t	44.0	25.9	40.0	41.1	36.8	20.3	–	8.2	7.7	**59.2**	44.5
			Clayton	6.5	8.5	68.0	75.0	87.1	41.9	–	13.2	8.5	71.9	**88.4**
			Gumbel	10.2	5.3	19.6	3.9	33.8	**50.5**	–	11.2	7.2	27.3	31.1
			Frank	*5.5*	*5.3*	*4.5*	*4.9*	*4.8*	*4.7*	–	*5.2*	*5.1*	*5.2*	*4.8*
		0.4	Gaussian	14.1	29.4	30.1	33.1	31.3	18.4	–	10.8	7.6	**43.9**	37.3
			Student-t	18.5	16.7	47.4	53.0	43.3	29.2	–	13.0	9.3	49.8	**53.6**
			Clayton	4.5	9.8	95.5	97.5	98.0	62.1	–	47.1	19.4	93.8	**98.8**
			Gumbel	9.7	5.1	58.0	7.2	54.7	**65.3**	–	21.3	9.1	44.0	56.6
			Frank	*5.6*	*4.8*	*5.4*	*5.4*	*5.3*	*5.7*	–	*5.2*	*4.6*	*5.4*	*5.5*
	500	0.2	Gaussian	13.4	38.1	**86.1**	79.1	66.0	57.7	–	19.8	15.9	77.3	76.2
			Student-t	99.0	90.2	97.4	95.7	88.3	88.7	–	34.3	27.9	**99.9**	95.2
			Clayton	11.2	31.1	**100**	**100**	**100**	99.7	–	66.7	37.3	**100**	**100**
			Gumbel	26.6	7.8	84.7	22.0	91.9	**97.5**	–	56.8	25.5	91.2	92.5
			Frank	*5.6*	*5.4*	*5.1*	*4.9*	*4.4*	*5.6*	–	*4.9*	*5.0*	*5.8*	*4.5*
		0.4	Gaussian	78.9	93.7	98.3	95.3	90.9	74.2	–	58.9	40.3	**99.9**	95.7
			Student-t	72.0	78.8	99.9	99.6	98.6	95.8	–	72.2	52.2	**100**	99.6
			Clayton	8.0	36.9	**100**	**100**	**100**	**100**	–	99.9	96.5	**100**	**100**
			Gumbel	35.0	6.9	**99.9**	51.9	99.7	**99.9**	–	91.5	54.4	99.7	99.8
			Frank	*4.9*	*5.1*	*5.3*	*6.0*	*5.0*	*5.1*	–	*5.7*	*4.8*	*5.0*	*5.3*
8	100	0.2	Gaussian	1.0	20.5	**81.2**	68.2	60.8	12.5	–	11.2	6.3	26.9	72.6
			Student-t	75.6	68.9	84.6	73.1	69.2	27.1	–	12.6	7.9	**94.3**	79.5
			Clayton	2.6	15.5	83.6	94.6	**97.7**	36.5	–	22.7	8.6	79.5	97.4
			Gumbel	20.3	5.0	35.7	22.2	63.2	**87.7**	–	39.8	7.8	43.7	60.4
			Frank	*4.5*	*5.1*	*4.7*	*5.2*	*4.8*	*4.8*	–	*5.5*	*5.1*	*4.9*	*4.8*
		0.4	Gaussian	11.7	62.0	**93.6**	81.4	60.1	24.2	–	25.7	8.2	78.1	73.4
			Student-t	47.8	55.9	**95.2**	91.3	74.1	38.4	–	28.3	10.8	90.9	86.2
			Clayton	1.3	18.1	98.7	99.8	**99.9**	69.4	–	81.0	39.4	98.5	**99.9**
			Gumbel	26.5	7.9	72.8	29.5	74.7	**93.7**	–	50.3	11.0	67.6	77.0
			Frank	*5.0*	*4.8*	*4.6*	*5.2*	*5.1*	*5.5*	–	*4.7*	*4.4*	*4.9*	*5.0*
	500	0.2	Gaussian	47.7	94.1	**100**	**100**	99.8	99.0	–	66.6	15.1	**100**	**100**
			Student-t	**100**	**100**	**100**	**100**	**100**	**100**	–	77.4	32.3	**100**	**100**
			Clayton	6.3	82.8	**100**	**100**	**100**	**100**	–	93.7	35.8	**100**	**100**
			Gumbel	71.4	6.0	95.9	74.3	**100**	**100**	–	98.5	34.1	98.9	**100**
			Frank	*4.5*	*4.8*	*4.3*	*5.1*	*5.2*	*5.3*	–	*5.6*	*5.3*	*5.5*	*5.1*
		0.4	Gaussian	**100**	**100**	**100**	**100**	99.9	93.1	–	97.6	37.9	**100**	**100**
			Student-t	**100**	**100**	**100**	**100**	**100**	99.7	–	98.6	61.5	**100**	**100**
			Clayton	8.3	83.7	**100**	**100**	**100**	**100**	–	**100**	99.6	**100**	**100**
			Gumbel	93.3	16.3	**100**	95.1	**100**	**100**	–	99.9	62.5	**100**	**100**
			Frank	*5.0*	*4.6*	*4.7*	*4.9*	*4.6*	*4.2*	–	*5.3*	*4.7*	*4.4*	*4.6*

Note: Student copula alternative hypothesis with degree-of-freedom $\nu = 6$. Numbers in *italic* are nominal levels and should correspond to the prescribed size of 5%. Numbers in **bold** indicates the best performing approach. All powers are size-adjusted.

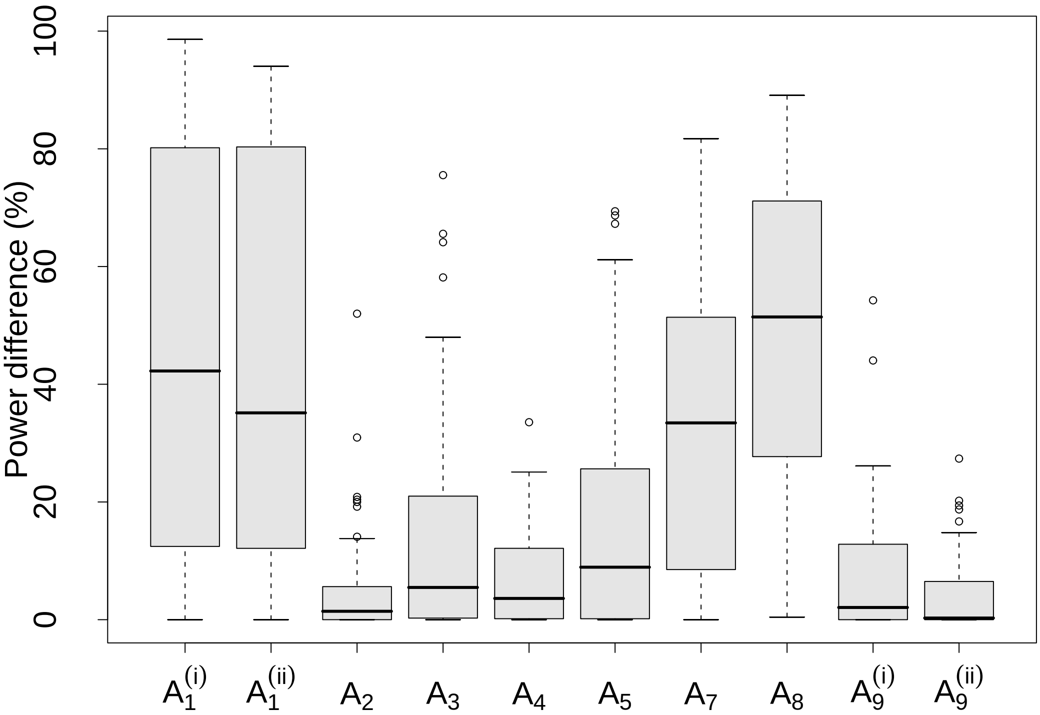


Figure 5: Distribution of power difference from the very best approach for testing the Frank copula.

are $d! = 120$ different permutations, sample size $n = 100$ and dependence $\tau = 0.5$. All reported values are averaged over 1000 independent simulations. For some of the approaches there are two sources of variation; permutation order and double bootstrap procedure. In order to see the effect of permutation order only, we report the same p-value variation results when the permutation is kept fixed, see Table 7.

From the two tables one can see that the permutation order adds no variance for approach $\mathcal{A}_1^{(i)}$ when the null hypothesis is the Gaussian copula. This permutation invariance of approach $\mathcal{A}_1^{(i)}$ under the Gaussian null hypothesis is proved in Appendix A. However, when using a different weight function or when the null hypothesis is different from the Gaussian copula, variation is added due to the permutation order. Note that in- or close to rejection regions, the variation due to permutation order is as great as in other regions, relative to the mean. However, the practical effect will not be so important as the conclusion will most probably be the same, regardless of permutation order. We see the same for the other approaches. When the null- and alternative hypotheses are the same we see that the average of the mean p-values are approximately 0.5 as they should be. We also see that the variation in these cases are quite large, typically around 0.25 for approaches $\mathcal{A}_1^{(ii)}$, $\mathcal{A}_3$ and $\mathcal{A}_8$. For approach $\mathcal{A}_1^{(i)}$ we see that the variation is in general lower than for the other approaches. Also note that for approach $\mathcal{A}_8$ the permutation order adds almost no variation in any case as the estimated p-value will vary heavily even when keeping the permutation order fixed. This is due to the construction of the approach. Random samples from the null hypothesis copula are drawn in every computation of the statistic, hence inducing large variation, at least when we are far from rejection regions.

To further illustrate, we look at so-called mixing tests. Two copulae are mixed in the following way:

$$C^{\text{Mix}} = (1-\beta)C_1 + \beta C_2,$$

where $\beta \in [0,1]$ is the mixing parameter. We consider the case where C_1 is the Clayton copula while C_2 is the Gumbel copula. So when $\beta = 0$, the mixed copula is equivalent with the Clayton copula, while when $\beta = 1$ it is equivalent with the Gumbel copula. We draw $n = 500$ random samples from the $d = 5$ dimensional mixed copula with dependences $\tau_1 = \tau_2 = 0.4$. We then estimate the p-value under a Clayton null hypothesis for all values of β, using approaches $\mathcal{A}_1^{(i)}$, $\mathcal{A}_1^{(ii)}$, $\mathcal{A}_3$ and $\mathcal{A}_8$, i.e. all approaches based on Rosenblatt's transformation. The p-value is estimated for each of the $d!$ permutations of the variables and the 95% confidence interval over the $d!$ permutations is computed. This is repeated 1000

Table 6: Estimated mean p-value (mean of $d!$ permutations) for approaches based on Rosenblatt's transformation. In parentheses the standard deviation over all permutations is given. All quoted values are averaged over 100 simulations.

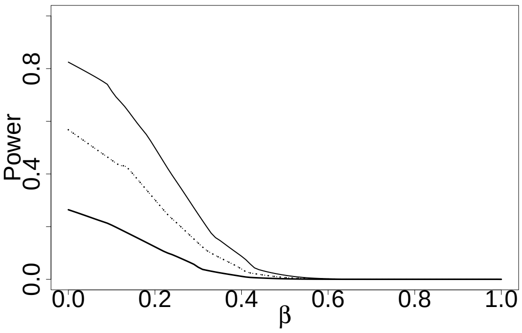
$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{A}_1^{(i)}$	$\mathcal{A}_1^{(ii)}$	$\mathcal{A}_3$	$\mathcal{A}_8$
Gaussian	Gaussian	0.514 (0.000)	0.520 (0.263)	0.513 (0.287)	0.510 (0.290)
	Clayton	0.501 (0.000)	0.480 (0.239)	0.021 (0.038)	0.205 (0.201)
	Gumbel	0.479 (0.000)	0.460 (0.237)	0.549 (0.294)	0.294 (0.247)
	Frank	0.415 (0.000)	0.419 (0.232)	0.535 (0.311)	0.428 (0.287)
Clayton	Gaussian	0.003 (0.002)	0.008 (0.015)	0.312 (0.187)	0.248 (0.237)
	Clayton	0.520 (0.159)	0.535 (0.263)	0.519 (0.269)	0.501 (0.283)
	Gumbel	0.002 (0.002)	0.016 (0.024)	0.370 (0.222)	0.103 (0.139)
	Frank	0.008 (0.004)	0.040 (0.051)	0.424 (0.226)	0.265 (0.242)
Gumbel	Gaussian	0.082 (0.027)	0.095 (0.118)	0.109 (0.100)	0.390 (0.279)
	Clayton	0.035 (0.012)	0.214 (0.181)	0.000 (0.001)	0.101 (0.129)
	Gumbel	0.533 (0.110)	0.533 (0.270)	0.528 (0.264)	0.506 (0.287)
	Frank	0.113 (0.034)	0.340 (0.239)	0.417 (0.246)	0.463 (0.286)
Frank	Gaussian	0.242 (0.102)	0.129 (0.152)	0.104 (0.086)	0.380 (0.274)
	Clayton	0.536 (0.153)	0.400 (0.248)	0.000 (0.001)	0.173 (0.184)
	Gumbel	0.396 (0.135)	0.492 (0.265)	0.325 (0.227)	0.365 (0.267)
	Frank	0.509 (0.151)	0.508 (0.272)	0.506 (0.245)	0.486 (0.281)

Note: Applied to $n = 100$ samples of $d = 5$ dimensional copulae with dependence parameter $\tau = 0.5$.

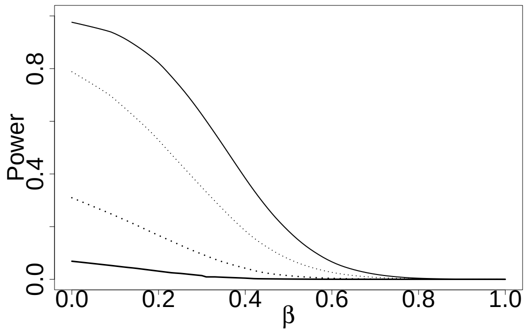
Table 7: Estimated mean p-value (mean of $d!$ separate estimations based on the same data set) for approaches based on Rosenblatt's transformation. In parentheses the standard deviation over all permutations is given. All quoted values are averaged over 100 simulations.

$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{A}_1^{(i)}$	$\mathcal{A}_1^{(ii)}$	$\mathcal{A}_3$	$\mathcal{A}_8$
Gaussian	Gaussian	0.514 (0.000)	0.530 (0.057)	0.523 (0.000)	0.510 (0.284)
	Clayton	0.501 (0.000)	0.483 (0.056)	0.021 (0.000)	0.205 (0.194)
	Gumbel	0.479 (0.000)	0.458 (0.052)	0.559 (0.000)	0.294 (0.239)
	Frank	0.415 (0.000)	0.416 (0.048)	0.551 (0.000)	0.432 (0.282)
Clayton	Gaussian	0.002 (0.000)	0.008 (0.003)	0.318 (0.000)	0.250 (0.216)
	Clayton	0.517 (0.000)	0.535 (0.056)	0.524 (0.000)	0.501 (0.275)
	Gumbel	0.002 (0.000)	0.013 (0.003)	0.382 (0.000)	0.105 (0.125)
	Frank	0.008 (0.000)	0.038 (0.007)	0.436 (0.000)	0.262 (0.218)
Gumbel	Gaussian	0.080 (0.000)	0.089 (0.023)	0.104 (0.000)	0.390 (0.268)
	Clayton	0.036 (0.000)	0.205 (0.036)	0.000 (0.000)	0.100 (0.123)
	Gumbel	0.527 (0.000)	0.531 (0.061)	0.532 (0.000)	0.508 (0.281)
	Frank	0.112 (0.000)	0.342 (0.050)	0.421 (0.000)	0.461 (0.278)
Frank	Gaussian	0.240 (0.000)	0.129 (0.031)	0.109 (0.000)	0.381 (0.263)
	Clayton	0.541 (0.000)	0.395 (0.055)	0.000 (0.000)	0.170 (0.174)
	Gumbel	0.391 (0.000)	0.489 (0.059)	0.320 (0.000)	0.366 (0.257)
	Frank	0.502 (0.000)	0.510 (0.063)	0.501 (0.000)	0.485 (0.274)

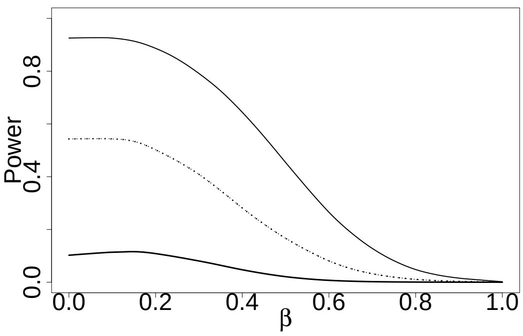
Note: Applied to $n = 100$ samples of $d = 5$ dimensional copulae with dependence parameter $\tau = 0.5$.



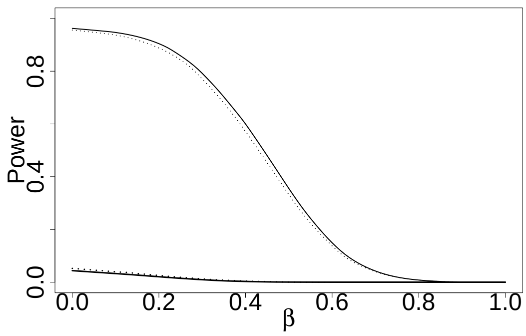
(a) Approach $\mathcal{A}_1^{(i)}$



(b) Approach $\mathcal{A}_1^{(ii)}$



(c) Approach $\mathcal{A}_3$



(d) Approach $\mathcal{A}_8$

Figure 6: P-value variation due to permutation order for approaches based on Rosenblatt's transformation. Average 95% confidence intervals over 1000 separate mixed copula simulations. The null hypothesis is the Clayton copula and the alternative hypothesis is the Gumbel copula. The solid line represents the variation over $d!$ permutation orders while the dotted line represents the variation when permutation is kept fixed.

times and Figure 6 shows the resulting confidence intervals, averaged over the 1000 repetitions. Included in the figure are also the corresponding confidence intervals when the permutation order is kept fixed. This way we can see the additional p-value variation solely due to permutation order. We see that for approach $\mathcal{A}_1^{(i)}$ the additional variation is substantial when the null hypothesis is true. However, as we move towards rejection, the additional variation becomes negligible in the sense that the conclusion will be the same no matter which permutation order is chosen. Again we note that the additional variation due to permutation order is smaller for approach $\mathcal{A}_1^{(i)}$ than for the other approaches based on Rosenblatt's transformation. Note also, that for approach $\mathcal{A}_1^{(ii)}$ there is p-value variation even when the permutation is kept fixed. This is due to the double parametric bootstrap step concerned with the approximation of F_1 in (3.1). This is also the case for approach $\mathcal{A}_8$ where we see only marginal additional variation due to permutation order.

Finally, we examine whether the utilization of all permutations may give us additional power. The idea is that by computing a statistic for each permutation of the data, more information is extracted from the data and we may achieve higher power. This is investigated for approaches $\mathcal{A}_1^{(i)}$ and $\mathcal{A}_3$ in the case $d = 4$, $n = 100$, $\tau = 0.4$ for $\mathcal{H}_0$ and $\mathcal{H}_1$ both being one of the Gaussian-, Clayton-, Gumbel- or Frank copulae. We simply compute the average of the statistics over the $d!$ permutations. Table 8 shows the results, along with corresponding results (permutation fixed) from Tables 1, 3, 4 and 5. We see that averaging over all $d!$ permutations adds some power, e.g. for $\mathcal{A}_3$ for $\mathcal{H}_0$ =Gaussian, $\mathcal{H}_1$ =Clayton where the power increases from 81% to 95%. Hence, this might be a fruitful idea to pursuit in future research. Perhaps one can find clever ways of averaging only over some few of the $d!$ permutations, and still gain most of the power increase.

Table 8: Percentage of rejections (at 5% significance level) by approaches $\mathcal{A}_1^{(i)}$ and $\mathcal{A}_3$ when computing the average of $\widehat{T}_1^{(i)}$ and $\widehat{T}_3$ over all $d!$ permutations. These are denoted by $\mathcal{A}_{1,d!}^{(i)}$ and $\mathcal{A}_{3,d!}$ and they are compared to corresponding rejection rates for the original approaches $\mathcal{A}_1^{(i)}$ and $\mathcal{A}_3$, that only consider one, fixed permutation.

$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{A}_1^{(i)}$	$\mathcal{A}_{1,d!}^{(i)}$	$\mathcal{A}_3$	$\mathcal{A}_{3,d!}$
Gaussian	Gaussian	*4.7*	*5.2*	*5.1*	*4.9*
	Clayton	1.0	1.0	80.8	94.9
	Gumbel	1.5	1.7	3.6	2.2
	Frank	1.6	1.8	7.3	6.6
Clayton	Gaussian	78.3	83.4	3.0	4.7
	Clayton	*5.2*	*5.7*	*5.5*	*5.1*
	Gumbel	79.1	83.2	4.9	6.1
	Frank	68.7	74.5	3.2	3.4
Gumbel	Gaussian	29.7	30.0	44.0	62.9
	Clayton	26.8	26.0	99.1	99.9
	Gumbel	*5.0*	*5.3*	*5.2*	*5.2*
	Frank	17.8	18.4	12.5	18.3
Frank	Gaussian	14.1	14.6	33.1	48.4
	Clayton	4.5	4.0	97.5	99.5
	Gumbel	9.7	8.9	7.2	6.8
	Frank	*5.6*	*5.1*	*5.4*	*5.3*

Note: Dimension $d = 4$, sample size $n = 100$ and dependence $\tau = 0.4$. Numbers in *italic* are nominal levels and should correspond to the prescribed size of 5%. All powers are size-adjusted.

5 Discussion and recommendations

An overview of six copula GoF approaches was given, along with the proposal of three new approaches. A large Monte Carlo study was presented, examining the nominal levels and the power against some fixed alternatives under several combinations of problem dimension, sample size and dependence.

Results show, in general, increasing power with dimension, sample size and dependence, as expected. Further, the results show that approach $\mathcal{A}_2$, the approach based on a distance between the empirical- and null hypothesis copula distribution functions, is in general the best approach, with approach $\mathcal{A}_4$ as a strong runner up. However, in some cases, e.g. when testing the Gaussian hypothesis against heavy tails, approach $\mathcal{A}_2$ does not perform so well. In this case, however, the otherwise poor approach $\mathcal{A}_1$ performs very well for high dimensions and large sample sizes. Hence, in general, approach $\mathcal{A}_2$ is recommended. However, one should consider supplmenting it with approaches $\mathcal{A}_4$ and $\mathcal{A}_1^{(i)}$, in particular if no strong a priori opinions exist as to which distribution we are testing for and what kind of deviations to detect. Average approaches merge the qualities of all the approaches included in the averaging and provides more stable power performance than the individual approaches. However, the topic of averaging different approaches was included as a hint of further research and needs more work. Finally, to decide which approaches to consider, a preliminary test of ellipticity (see e.g. Huffera and Park (2007)) can be helpful.

When doing model evaluation, it is recommended to also examine various diagnostic tests such as GoF plots, e.g. plotting $S_4(w)$ with simulated null hypothesis confidence bands as done in Genest et al. (2006a). This may give valuable information on the fit of a copula. However, there is still a need for intuitive and informative diagnostic plots. Ideally such a plot should show, in some way and in case of rejection by the formal tests, which variable (i.e. which dimension) and/or which samples causes the rejection. Is it actually a deviation in the dependence structure between the variables or is the rejection due to some extreme samples? More research is needed on this topic.

Next, results were reported on the variation of the p-value estimates due to permutation order for approaches based on Rosenblatt's transformation. In general, one does not want a statistical testing procedure to give different values when running it several times on the same data set. However, for some of the approaches based on Rosenblatt's transformation, the estimated p-value will be different depending on which permuation order that is chosen for the variables. This effect decreases as the p-value estimates approach critical levels. Hence, the author does not believe that the permutation effect is something to worry about. On the contrary, the permutational invariance may actually be useful, as seen when averaging over all permutations increases the power of some of the approaches. Also, as long as the permutation order is chosen in a random fashion, the results are not influenced in any particular direction.

The results concerning the permutation of variables also point in direction of important future research. The variation of p-value estimates also depends on the bootstrap parameters M and N_b. These parameters are usually, in a rather arbitrary way, set to what is believed to be large values. This is also the case in this paper. However, there has been no study of the effect that these choices may have on the power, and even more importantly the nominal levels of an approach. Originally, in the power studies of Section 4.1, a double bootstrap parameter $N_b = 2500$ was chosen for all combinations of dimension, sample size, dependence and alternative copula. However, for dimension $d = 8$ we observed some peculiar results, e.g. decreasing power as sample size increased. These peculiarities vanished when increasing N_b to 5000 for dimension $d = 8$. Choosing appropriately large values for these parameters and thus achieving proper nominal levels is crucial for any study and/or application of these GoF approaches. Hence, a study of the effects of these parameters and required minimum values would be highly interesting.

The computational aspect also deserves some attention. An important quality of approaches based on Rosenblatt's transform is computational efficiency. Approaches $\mathcal{A}_2$, $\mathcal{A}_4$ and $\mathcal{A}_5$ need computationally intensive double parametric bootstrap procedures to estimate p-values in some cases (e.g. for the elliptical copulae, in particular for higher dimensions and large sample sizes). Approaches based on Rosenblatt's transformation does not, in general, need this double bootstrap step, since after Rosenblatt's transformation, the null hypothesis is always independence.

Finally, a word of warning. As emphasized in Genest et al. (2008), the asymptotics of several of the procedures presented here are not known. Hence, one cannot know for sure whether a bootstrap procedure will converge in every case. However, all the results so far on the performance of the proposed approaches and bootstrap procedures are comforting and strongly indicate the validity of the test procedures. Keep in mind though, the original approach and test procedure proposed by Breymann et al. (2003), which showed terrible performance in the study of Dobrić and Schmid (2007). This shows how wrong it can all go if our test procedure is not valid. Approaches $\mathcal{A}_2$ and $\mathcal{A}_4$, that turned out to be the best in our study, both have known asymptotics and the bootstrap procedures for these approaches are well established from Quessy (2005), Genest et al. (2006a) and Genest and Rémillard (2008). Hence, for the time being, these are the recommended ones to use for copula goodness-of-fit testing.

Acknowledgements

The author acknowledges the support and guidance of colleagues at the Norwegian Computing Center, in particular Assistant Research Director Kjersti Aas and Chief Research Scientist Xeni Kristine Dimakos. I would especially like to express my gratitude to Kjersti Aas for introducing me to the field of copulae and copula goodness-of-fit testing and for her continuous drive and interest. Credit is also due to my friend and former colleague Henrik Bakken for fruitful discussions and collaboration in the early phase of this and related work. Finally, I would like to thank colleagues and participants at various workshops and conferences for valuable comments, in particular Professor Nils Lid Hjort at the University of Oslo, Professor Jean-François Quessy at Université du Québec à Trois-Rivières, Professor Mark Salmon at Warwick Business School and Professor Alexander McNeil at Heriott-Watt University.

References

Anderson, T. W. and D. A. Darling (1954). A test of goodness of fit. *Journal of the American Statistical Association 49*, 765–769.

Berg, D. and H. Bakken (2007). A copula goodness-of-fit approach based on the conditional probability integral transform. Working paper, University of Oslo and Norwegian Computing Center.

Breymann, W., A. Dias, and P. Embrechts (2003). Dependence structures for multivariate high-frequency data in finance. *Quantitative Finance 1*, 1–14.

Chen, X., Y. Fan, and A. Patton (2004). Simple tests for models of dependence between multiple financial time series, with applications to U.S. equity returns and exchange rates. Financial Markets Group, London School of Economics, Discussion Paper 483. Revised July 2004.

Cherubini, U., E. Luciano, and W. Vecchiato (2004). *Copula Methods in Finance*. Wiley.

Deheuvels, P. (1979). La fonction de dépendance empirique et ses propriétés: Un test non paramétrique d'indépendence. *Acad. Royal Bel., Bull. Class. Sci.,* 5^{e} *série 65*, 274–292.

Demarta, S. and A. J. McNeil (2005). The t copula and related copulas. *International Statistical Review 73*(1), 111–129.

Denker, M. and G. Keller (1983). On u-statistics and v. mises' statistics for weakly dependent processes. *Zeitschrift fuer Wahrscheinlichkeitstheorie und Verwandte Gebiete 64*, 505–522.

Dobrić, J. and F. Schmid (2005). Testing goodness-of-fit for parametric families of copulas - application to financial data. *Communications in Statistics: Simulation and Computation 34*(4), 1053–1068.

Dobrić, J. and F. Schmid (2007). A goodness of fit test for copulas based on rosenblatt's transformation. *Comput. Stat. Data Anal. 51*(9), 4633–4642.

Fermanian, J. (2005). Goodness of fit tests for copulas. *Journal of Multivariate Analysis 95*, 119–152.

Florax, R. J. G. M., H. Folmer, and S. J. Rey (2006). A comment on specification searches in spatial econometrics: The relevance of hendry's methodology: A reply. *Regional Science and Urban Economics 36*, 300–308.

Genest, C., K. Ghoudi, and L. Rivest (1995). A semi-parametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika 82*, 543–552.

Genest, C., J.-F. Quessy, and B. Rémillard (2006a). Goodness-of-fit procedures for copula models based on the probability integral transform. *Scandinavian Journal of Statistics 33*, 337–366.

Genest, C., J.-F. Quessy, and B. Rémillard (2006b). On the joint asymptotic behavior of two rank-based estimators of the association parameter in the gamma frailty model. *Stat. Prob. Lett. 76*, 10–18.

Genest, C. and B. Rémillard (2008). Validity of the parametric bootstrap for goodness-of-fit testing in semiparametric models. *Ann. Henri Poincaré 44*. In press.

Genest, C., B. Rémillard, and D. Beaudoin (2008). Omnibus goodness-of-fit tests for copulas: A review and a power study. *Insurance: Mathematics and Economics 42*. In press.

Genest, C. and L.-P. Rivest (1993). Statistical inference procedures for bivariate archimedean copulas. *Journal of the American Statistical Association*, 1034–1043.

Hendry, D. F. (2006). Specification searches in spatial econometrics: The relevance of hendry's methodology. *Regional Science and Urban Economics 36*, 309–312.

Hong, Y. and H. Li (2005). Nonparametric Specification Testing for Continuous-Time Models with Application to Spot Interest Rates. *Rev. Financial Stud. 18*, 37–84.

Huffera, F. W. and C. Park (2007). A test for elliptical symmetry. *Journal of Multivariate Analysis 98*(2), 256–281.

Joe, H. (1997). *Multivariate Models and Dependence Concepts.* London: Chapman & Hall.

Malevergne, Y. and D. Sornette (2003). Testing the gaussian copula hypothesis for financial assets dependence. *Quantitative Finance 3*, 231–250.

Malevergne, Y. and D. Sornette (2006). *Extreme Financial Risks: From Dependence to Risk Management.* Springer-Verlag Berlin Heidelberg.

Mashal, R. and A. Zeevi (2002). Beyond correlation: Extreme co-movements between financial assets. Technical report, Columbia University.

Nelsen, R. B. (1999). *An Introduction to Copulas.* New York: Springer Verlag.

Panchenko, V. (2005). Goodness-of-fit test for copulas. *Physica A 355(1)*, 176–182.

Quessy, J.-F. (2005). *Théorie et application des copules : tests d'adéquation, tests d'indépendance et bornes pour la valeur-à-risque.* Ph. D. thesis, Université Laval.

Quessy, J.-F., M. Mesfioui, and M.-H. Toupin (2007). A goodness-of-fit test based on Spearmans dependence function. Working paper, Université du Québec à Trois-Rivières.

Romano, C. (2002). Calibrating and simulating copula functions: An application to the Italian stock market. Working paper n. 12, CIDEM, Universit'a degli Studi di Roma "La Sapienza".

Rosenblatt, M. (1952). Remarks on a multivariate transformation. *The Annals of Mathematical Statistics 23*, 470–472.

Savu, C. and M. Trede (2004). Goodness-of-fit tests for parametric families of archimedean copulas. CAWM discussion paper, No. 6.

Scaillet, O. (2005, May). Kernel based goodness-of-fit tests for copulas with fixed smoothing parameters. FAME Research Paper Series rp145, International Center for Financial Asset Management and Engineering. Available at http://ideas.repec.org/p/fam/rpseri/rp145.html.

Shih, J. H. (1998). A goodness-of-fit test for association in a bivariate survival model. *Biometrika 85*, 189–200.

Shih, J. H. and T. A. Louis (1995). Inferences on the association parameter in copula models for bivariate survival data. *Biometrics 51*, 1384–1399.

Sklar, A. (1959). Fonctions dé repartition á n dimensions et leurs marges. *Publ. Inst. Stat. Univ. Paris 8*, 299–231.

A Proof of permutation invariance of $\mathcal{A}_1^{(i)}$ under Gaussian copula null hypothesis

To prove that approach $\mathcal{A}_1^{(i)}$ is permutatoin invariant under the Gaussian copula null hypothesis, let us first look at how Rosenblatt's transformation is carried out. For the Gaussian copula null hypothesis, this transformation is easily computed using the Cholesky decomposition of the covariance matrix. Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be a d-dimensional vector, where $\boldsymbol{\mu} = \mathbb{E}(\mathbf{X})$ and $\boldsymbol{\Sigma}$ is the $d \times d$ positive definite covariance matrix.

Since $\boldsymbol{\Sigma}$ is positive definite it can be written as $\boldsymbol{\Sigma} = \mathbf{A}^T\mathbf{A}$, where $\mathbf{A}$ is a lower triangular matrix and $\mathbf{A}^T$ denotes its transpose. Next, it is well known that $\mathbf{X}$ can be expressed as $\mathbf{X} = \boldsymbol{\mu} + \mathbf{A}^T\mathbf{Y}$ where $\mathbf{Y} \sim \mathcal{N}(0, \boldsymbol{I})$ and $\boldsymbol{I}$ is the d-dimensional identity matrix. I.e. $\mathbf{Y}$ is a vector of d i.i.d. standard normally distributed variables. Solving for $\mathbf{Y}$ gives $\mathbf{Y} = (\mathbf{A}^T)^{-1}(\mathbf{X} - \boldsymbol{\mu})$. We now see that the vector $\mathbf{V} = \Phi(\mathbf{Y})$ is i.i.d. $U(0,1)^d$ under the Gaussian null hypothesis.

For approach $\mathcal{A}_1^{(i)}$ we now need to compute $W_1 = \sum_{i=1}^d \Phi^{-1}(V_i)^2 = \sum_{i=1}^d Y_i^2 = \mathbf{Y}^T\mathbf{Y}$. We now proceed with the bivariate setting for simplicity but the proof can easily be extended to arbitrary dimension d. Consider the Cholesky decomposition of the covariance matrix $\boldsymbol{\Sigma} = \mathbf{A}^T\mathbf{A}$ in detail:

$$\boldsymbol{\Sigma}^1 = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}^2 & a_{12}a_{22} \\ a_{12}a_{22} & a_{22}^2 \end{pmatrix},$$

where the superscript 1 in $\boldsymbol{\Sigma}^1$ denotes permutation order 1. We see now that $a_{11} = \sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}/\sigma_2$, $a_{12} = \sigma_{12}/\sigma_2$ and $a_{22} = \sigma_2$. Next, we see that

$$(\mathbf{A}^T)^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}a_{22}} \\ 0 & \frac{1}{a_{22}} \end{pmatrix}$$

and that

$$\mathbf{Y} = (\mathbf{A}^T)^{-1}(\mathbf{X} - \boldsymbol{\mu}) = \begin{pmatrix} \frac{1}{a_{11}}(X_1 - \mu_1) - \frac{a_{12}}{a_{11}a_{22}}(X_2 - \mu_2) \\ \frac{1}{a_{22}}(X_2 - \mu_2) \end{pmatrix}.$$

Now to compute $W_1^1 = \mathbf{Y}^T\mathbf{Y}$, superscript 1 denoting permutation order 1, we get

$$\begin{aligned} W_1^1 &= \frac{(X_1 - \mu_1)^2}{a_{11}^2} + \frac{a_{12}^2}{a_{11}^2a_{22}^2}(X_2 - \mu_2)^2 - \frac{2a_{12}}{a_{11}^2a_{22}}(X_1 - \mu_1)(X_2 - \mu_2) + \frac{(X_2 - \mu_2)^2}{a_{22}^2} \\ &= \frac{(X_1 - \mu_1)^2\sigma_2^2 + (X_2 - \mu_2)^2\sigma_1^2 - 2(X_1 - \mu_1)(X_2 - \mu_2)\sigma_{12}}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \end{aligned}$$

by inserting σ's for the a's.

By doing the same exercise with permutation order 2 we first get

$$\boldsymbol{\Sigma}^2 = \begin{pmatrix} \sigma_2^2 & \sigma_{12} \\ \sigma_{12} & \sigma_1^2 \end{pmatrix}$$

and $a_{11} = \sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}/\sigma_1$, $a_{12} = \sigma_{12}/\sigma_1$ and $a_{22} = \sigma_1$. Next, in the same manner as above, it is easily shown that

$$W_1^2 = \frac{(X_2 - \mu_2)^2\sigma_1^2 + (X_1 - \mu_1)^2\sigma_2^2 - 2(X_1 - \mu_1)(X_2 - \mu_2)\sigma_{12}}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} = W_1^1.$$

Hence we have shown that approach $\mathcal{A}_1^{(i)}$ is permutation invariant under the Gaussian copula null hypothesis. This is not so for other weight functions or other null hypothesis copulae. The invariance stems from the use of Φ^{-1} which cancels out with the Φ in $\mathbf{V} = \Phi(\mathbf{Y})$ and the squaring $\Phi(V_i)^2$.

B Derivation of a Cramér-von Mises statistic

Consider the Cramér–von Mises (CvM) statistic

$$T = n\int_0^1 \{\widehat{F}(w) - F(w)\}^2 \mathrm{d}F(w),$$

where $\widehat{F}(w) = \frac{1}{n+1}\sum_{j=1}^n I(X_j \le t)$ is the empirical distribution function, equivalent to the normalized ranks used in the construction of the pseudo-vector $\mathbf{Z}$ in (2). Given a random sample $(x_1, \ldots, x_n)$, the empirical version $\widehat{T}$ of the CvM statistic can be derived as follows.

$$\begin{aligned}\widehat{T} =& n\int_0^1 \{\widehat{F}(w) - F(w)\}^2 \mathrm{d}F(w)\\ =& n\int_0^1 \widehat{F}(w)^2 \mathrm{d}F(w) - 2n\int_0^1 \widehat{F}(w)F(w)\mathrm{d}F(w) + n\int_0^1 F(w)^2\mathrm{d}F(w).\end{aligned}$$

Since $\widehat{F}(w)$ is constant and equal to $\widehat{F}(j/(n+1))$ between $j/(n+1)$ and $(j+1)/(n+1)$ for $j = 1, \ldots, n$, the first two integrals can be split into n smaller integrals:

$$\begin{aligned}\widehat{T} =& n\sum_{j=1}^n \int_{j/(n+1)}^{(j+1)/(n+1)} \widehat{F}\left(\frac{j}{n+1}\right)^2 \mathrm{d}F(w)\\ &-2n\sum_{j=1}^n \int_{j/(n+1)}^{(j+1)/(n+1)} \widehat{F}\left(\frac{j}{n+1}\right) F(w)\mathrm{d}F(w) + \frac{n}{3}\left[F(w)^3\right]_0^1\\ =& \frac{n}{3} + n\sum_{j=1}^n \widehat{F}\left(\frac{j}{n+1}\right)^2 \left\{F\left(\frac{j+1}{n+1}\right) - F\left(\frac{j}{n+1}\right)\right\}\\ &-n\sum_{j=1}^n \widehat{F}\left(\frac{j}{n+1}\right)\left\{F\left(\frac{j+1}{n+1}\right)^2 - F\left(\frac{j}{n+1}\right)^2\right\}.\end{aligned}$$

For approach $\mathcal{A}_1$ the test observator $S_1(w)$ is $U[0,1]$ under the null hypothesis. Hence $F(w) = w$ and we easily see that $\widehat{T}$ reduces to

$$\widehat{T}' = \frac{n}{3} + \frac{n}{n+1}\sum_{j=1}^n \widehat{F}\left(\frac{j}{n+1}\right)^2 - \frac{n}{(n+1)^2}\sum_{j=1}^n (2j+1)\widehat{F}\left(\frac{j}{n+1}\right).$$

C Test procedures

Suppose we have observed the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. The following parametric bootstrap procedures lead to proper p-value estimates for a parametric null hypothesis copula.

C.1 Approach $\mathcal{A}_1$ (Berg and Bakken, 2007)

(1) Extract the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) Compute $(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathbf{R}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ assuming the parametric null hypothesis copula $C_{\widehat{\theta}}$. Here $\mathbf{R}(\mathbf{z}_1, \ldots, \mathbf{z}_n) = (\mathcal{R}(z_{11}, \ldots, z_{1d}), \ldots, \mathcal{R}(z_{n1}, \ldots, z_{nd}))$ where $\mathcal{R}(z_{11}, \ldots, z_{1d}) = (\mathcal{R}_1(z_{11}), \ldots, \mathcal{R}_d(z_{1d}))$ denotes Rosenblatt's transformation as presented in Definition 2.1.

(4) Compute $(\mathbf{h}_1, \ldots, \mathbf{h}_n) = \mathbf{R}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

(5) Compute W_1 according to (3), using weight functions Γ_V and Γ_H on $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $(\mathbf{h}_1, \ldots, \mathbf{h}_n)$ respectively.

(6) If W_1 follows a known distribution under the null hypothesis, compute $F_1(W_1)$ accordingly and jump to step (8).
If not, approximate F_1 as follows. For some large integer N_b, repeat the following steps for every $l \in \{1, \ldots, N_b\}$:

(i) Generate a random sample $\mathbf{v}_l^* = (v_{1,l}^*, \dots, v_{d,l}^*)$ from the null hypothesis copula, namely an i.i.d. $U[0,1]^d$ vector.

(ii) Compute $\mathbf{h}_l^* = (h_{1,l}^*, \dots, h_{d,l}^*) = \mathcal{R}(v_{1,l}^*, \dots, v_{d,l}^*)$.

(iii) Compute $W_{1,l}^*$ according to (3) using the same weight functions Γ_V and Γ_H as in step (5) but now on $(v_{1,l}^*, \dots, v_{d,l}^*)$ and $(h_{1,l}^*, \dots, h_{d,l}^*)$ respectively.

(7) Compute $F_1(W_1) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{W_{1,l}^* > W_1\}$.

(8) Compute $\widehat{T}_1$ according to (4) and (5).

(9) For some large integer K, repeat the following steps for every $k \in \{1, \dots, K\}$:

(a) Generate a random sample $(\mathbf{x}_{1,k}^0, \dots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \dots, \mathbf{z}_{n,k}^0)$ according to (2).

(b) Estimate the parameters θ^0 with $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \dots, \mathbf{z}_{n,k}^0)$.

(c) Compute $(\mathbf{v}_{1,k}^0, \dots, \mathbf{v}_{n,k}^0) = \mathbf{R}(\mathbf{z}_{1,k}^0, \dots, \mathbf{z}_{n,k}^0)$ assuming the parametric null hypothesis copula $C_{\widehat{\theta}_k^0}$.

(d) Compute $(\mathbf{h}_{1,k}^0, \dots, \mathbf{h}_{n,k}^0) = \mathbf{R}(\mathbf{v}_{1,k}^0, \dots, \mathbf{v}_{n,k}^0)$.

(d) Compute $W_{1,k}^0$ according to (3), using the same weight functions Γ_V and Γ_H as in step (5), now on $(\mathbf{v}_{1,k}^0, \dots, \mathbf{v}_{n,k}^0)$ and $(\mathbf{h}_{1,k}^0, \dots, \mathbf{h}_{n,k}^0)$ respectively.

(e) If $W_{1,k}^0$ follows a known distribution under the null hypothesis, compute $F_1(W_{1,k}^0)$ accordingly and jump to step (g).
If not, approximate F_1 as follows. For some large integer N_b, repeat the following steps for every $l \in \{1, \dots, N_b\}$:

(i) Generate a random sample $\mathbf{v}_{l,k}^{0*} = (v_{1,l,k}^{0*}, \dots, v_{d,l,k}^{0*})$ from the null copula, an i.i.d. $U[0,1]^d$ vector.

(ii) Compute $\mathbf{h}_{l,k}^{0*} = (h_{1,l,k}^{0*}, \dots, h_{d,l,k}^{0*}) = \mathcal{R}(v_{1,l,k}^{0*}, \dots, v_{d,l,k}^{0*})$.

(iii) Compute $W_{1,l,k}^{0*}$ according to (3) using the same weight functions Γ_V and Γ_H as in step (5) but now on $(v_{1,l,k}^{0*}, \dots, v_{d,l,k}^{0*})$ and $(h_{1,l,k}^{0*}, \dots, h_{d,l,k}^{0*})$ respectively.

(f) Compute $F_1(W_1^0) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{W_{1,l,k}^{0*} > W_{1,k}^0\}$.

(g) Compute $\widehat{T}_{1,k}^0$ according to (4) and (5).

(10) An approximate p-value for approach $\mathcal{A}_1$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^{K} I\{\widehat{T}_{1,k}^0 > \widehat{T}_1\}$.

C.2 Approach $\mathcal{A}_2$ (Genest et al., 2008)

(1) Extract the pseudo-observations $(\mathbf{z}_1, \dots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \dots, \mathbf{z}_n)$.

(3) Compute $\widehat{C}(\mathbf{z})$ according to (6).

(4) If there is an analytical expression for C_θ, compute the estimated statistic $\widehat{T}_2$ by plugging $\widehat{C}(\mathbf{z})$ and $C_{\widehat{\theta}}(\mathbf{z})$ into (7). Jump to step (5).
If there is no analytical expression for C_θ then choose $N_b \geq n$ and carry out the following steps:

(i) Generate a random sample $(\mathbf{x}_1^*, \dots, \mathbf{x}_{N_b}^*)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_1^*, \dots, \mathbf{z}_{N_b}^*)$ according to (2).

(ii) Approximate $C_{\widehat{\theta}}$ by $C_{\widehat{\theta}}^*(\mathbf{u}) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{\mathbf{z}_l^* \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$.

(iii) Approximate the CvM statistic in (7) by $\widehat{T}_2 = \sum_{j=1}^{n} \left\{ \widehat{C}(\mathbf{z}_j) - C_{\widehat{\theta}}^*(\mathbf{z}_j) \right\}^2$.

(5) For some large integer K, repeat the following steps for every $k \in \{1, \dots, K\}$:

(a) Generate a random sample $(\mathbf{x}_{1,k}^0, \dots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \dots, \mathbf{z}_{n,k}^0)$ according to (2).

(b) Estimate the parameters θ^0 with a consistent estimator $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \dots, \mathbf{z}_{n,k}^0)$.

(c) Let $\widehat{C}_k^0(\mathbf{u}) = \frac{1}{n+1} \sum_{j=1}^{n} I\{\mathbf{z}_{j,k}^0 \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$.

(d) If there is an analytical expression for C_θ, let $\widehat{T}_{2,k}^0 = \sum_{j=1}^{n} \left\{ \widehat{C}_k^0(\mathbf{z}_{j,k}^0) - C_{\widehat{\theta}_k^0}(\mathbf{z}_{j,k}^0) \right\}^2$ and jump to step (6).
If there is no analytical expression for C_θ then choose $N_b \geq n$ and proceed as follows:

(i) Generate a random sample $(\mathbf{x}_{1,k}^{0*}, \ldots, \mathbf{x}_{N_b,k}^{0*})$ from the null hypothesis copula $C_{\widehat{\theta}_k^0}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^{0*}, \ldots, \mathbf{z}_{N_b,k}^{0*})$ according to (2).

(ii) Approximate $C_{\widehat{\theta}_k^0}$ by $C_{\widehat{\theta}_k^0}^{0*}(\mathbf{u}) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{\mathbf{z}_{l,k}^{0*} \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$,

(iii) Approximate the CvM statistic in (7) by $\widehat{T}_{2,k}^{*} = \sum_{j=1}^{n} \left\{ \widehat{C}_k^0(\mathbf{z}_{j,k}^0) - C_{\widehat{\theta}_k^0}^{0*}(\mathbf{z}_{j,k}^0) \right\}^2$.

(6) An approximate p-value for approach $\mathcal{A}_2$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^{K} I\{\widehat{T}_{2,k}^0 > \widehat{T}_2\}$.

C.3 Approach $\mathcal{A}_3$ (Genest et al., 2008)

(1) Extract the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) Compute $(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathbf{R}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ assuming the parametric null hypothesis copula $C_{\widehat{\theta}}$.

(3) Compute $\widehat{C}(\mathbf{v})$ according to (6).

(4) Compute $\widehat{T}_3$ according to (8).

(5) For some large integer K, repeat the following steps for every $k \in \{1, \ldots, K\}$:

(a) Generate a random sample $(\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ according to (2).

(b) Estimate the parameters θ^0 with a consistent estimator $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$.

(c) Compute $(\mathbf{v}_{1,k}^0, \ldots, \mathbf{v}_{n,k}^0) = \mathbf{R}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$.

(d) Let $\widehat{C}_k^0(\mathbf{u}) = \frac{1}{n+1} \sum_{j=1}^{n} I\{\mathbf{v}_{j,k}^0 \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$.

(e) Compute $\widehat{T}_{3,k}^0 = \sum_{j=1}^{n} \left\{ \widehat{C}_k^0(\mathbf{v}_{j,k}^0) - C_{\perp}(\mathbf{v}_{j,k}^0) \right\}^2$.

(6) An approximate p-value for approach $\mathcal{A}_3$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^{K} I\{\widehat{T}_{3,k}^0 > \widehat{T}_3\}$.

C.4 Approach $\mathcal{A}_4$ (Genest et al., 2008)

(1) Extract the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) Compute $\widehat{C}(\mathbf{z})$ according to (6).

(4) If there is an analytical expression for $S_{4,\theta}$, compute the statistic $\widehat{T}_4$ according to (9) and (10). Jump to step (5).
If there is no analytical expression for $S_{4,\theta}$ then choose $N_b \geq n$ and proceed as follows:

(i) Generate a random sample $(\mathbf{x}_1^*, \ldots, \mathbf{x}_{N_b}^*)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_1^*, \ldots, \mathbf{z}_{N_b}^*)$ according to (2).

(ii) Approximate $S_{4,\widehat{\theta}}$ by $\widehat{S}_4^*(w) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{\widehat{C}^*(\mathbf{z}_l^*) \leq w\}$, where
$\widehat{C}^*(\mathbf{u}) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{\mathbf{z}_l^* \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$.

(iii) Approximate the CvM statistic in (10) by
$\widehat{T}_4 = \frac{n}{N_b} \sum_{l=1}^{N_b} \left\{ \widehat{S}_4\left(\widehat{C}^*(\mathbf{z}_l^*)\right) - \widehat{S}_4^*\left(\widehat{C}^*(\mathbf{z}_l^*)\right) \right\}$.

(5) For some large integer K, repeat the following steps for every $k \in \{1, \ldots, K\}$:

(a) Generate a random sample $(\mathbf{x}_{1,k}^0, \ldots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ according to (2).

(b) Estimate the parameters θ^0 with a consistent estimator $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$.

(c) Let $\widehat{S}_{4,k}^0(w) = \frac{1}{n+1} \sum_{j=1}^{n} I\{\widehat{C}_k^0(\mathbf{z}_{j,k}^0) \leq w\}$, where $\widehat{C}_k^0(\mathbf{u}) = \frac{1}{n+1} \sum_{j=1}^{n} I\{\mathbf{z}_{j,k}^0 \leq \mathbf{u}\}$.

(d) If there is an analytical expression for $S_{4,\theta}$, compute the statistic $\widehat{T}_{4,k}^0$ by using $\widehat{S}_{4,k}^0$ and $S_{4,\widehat{\theta}_k^0}$ in (10). Jump to step (6).
If there is no analytical expression for $S_{4,\theta}$ then choose $N_b \geq n$ and proceed as follows:

(i) Generate a random sample $(\mathbf{x}_{1,k}^{0*}, \ldots, \mathbf{x}_{N_b,k}^{0*})$ from the null hypothesis copula $C_{\widehat{\theta}_k^0}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^{0*}, \ldots, \mathbf{z}_{N_b,k}^{0*})$ according to (2).

(ii) Approximate $S_{4,\widehat{\theta}_k^0}$ by $\widehat{S}_{4,k}^{0*}(w) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{\widehat{C}_k^{0*}(\mathbf{z}_{l,k}^{0*}) \leq w\}$, where $\widehat{C}_k^{0*}(\mathbf{u}) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{\mathbf{z}_{l,k}^{0*} \leq \mathbf{u}\}, \quad \mathbf{u} \in [0,1]^d$.

(iii) Approximate the CvM statistic in (10) by $\widehat{T}_{4,k}^0 = \frac{n}{N_b} \sum_{l=1}^{N_b} \left\{\widehat{S}_{4,k}^0 \left(\widehat{C}_k^{0*}(\mathbf{z}_{l,k}^{0*})\right) - \widehat{S}_{4,k}^{0*} \left(\widehat{C}_k^{0*}(\mathbf{z}_{l,k}^{0*})\right)\right\}$.

(6) An approximate p-value for approach $\mathcal{A}_4$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^{K} I\{\widehat{T}_{4,k}^0 > \widehat{T}_4\}$.

C.5 Approach $\mathcal{A}_5$

(1) Extract the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) If there is an analytical expression for $S_{5,\theta}$, compute the statistic $\widehat{T}_5$ according to (11) and (12). Jump to step (4).
If there is no analytical expression for $S_{5,\theta}$ then choose $N_b \geq n$ and proceed as follows:

  (i) Generate a random sample $(\mathbf{x}_1^*, \ldots, \mathbf{x}_{N_b}^*)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_1^*, \ldots, \mathbf{z}_{N_b}^*)$ according to (2).

  (ii) Approximate $S_{5,\widehat{\theta}}$ by $\widehat{S}_5^*(w) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{C_\perp(\mathbf{z}_l^*) \leq w\}$.

  (iii) Approximate the CvM statistic in (12) by $\widehat{T}_5 = \frac{n}{N_b} \sum_{l=1}^{N_b} \left\{\widehat{S}_5\left(C_\perp(\mathbf{z}_l^*)\right) - \widehat{S}_5^*\left(C_\perp(\mathbf{z}_l^*)\right)\right\}$.

(4) For some large integer K, repeat the following steps for every $k \in \{1, \ldots, K\}$:

  (a) Generate a random sample $(\mathbf{x}_{1,k}^0, \ldots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ according to (2).

  (b) Estimate the parameters θ^0 with a consistent estimator $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$.

  (c) Let $\widehat{S}_{5,k}^0(w) = \frac{1}{n+1} \sum_{j=1}^{n} I\{C_\perp(\mathbf{z}_{j,k}^0) \leq w\}$.

  (d) If there is an analytical expression for $S_{5,\theta}$, compute the statistic $\widehat{T}_{5,k}^0$ by using $\widehat{S}_{5,k}^0$ and $S_{5,\widehat{\theta}_k^0}$ in (12). Jump to step (5).
  If there is no analytical expression for $S_{5,\theta}$ then choose $N_b \geq n$ and proceed as follows:

    (i) Generate a random sample $(\mathbf{x}_{1,k}^{0*}, \ldots, \mathbf{x}_{N_b,k}^{0*})$ from the null hypothesis copula $C_{\widehat{\theta}_k^0}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^{0*}, \ldots, \mathbf{z}_{N_b,k}^{0*})$ according to (2).

    (ii) Approximate $S_{5,\widehat{\theta}_k^0}$ by $\widehat{S}_{5,k}^{0*}(w) = \frac{1}{N_b+1} \sum_{l=1}^{N_b} I\{C_\perp(\mathbf{z}_{l,k}^{0*}) \leq w\}$.

    (iii) Approximate the CvM statistic in (12) by $\widehat{T}_{5,k}^0 = \frac{n}{N_b} \sum_{l=1}^{N_b} \left\{\widehat{S}_{5,k}^0\left(C_\perp(\mathbf{z}_{l,k}^{0*})\right) - \widehat{S}_{5,k}^{0*}\left(C_\perp(\mathbf{z}_{l,k}^{0*})\right)\right\}$.

  (5) An approximate p-value for approach $\mathcal{A}_5$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^{K} I\{\widehat{T}_{5,k}^0 > \widehat{T}_5\}$.

C.6 Approach $\mathcal{A}_6$

(1) Extract the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) Estimate the parameters $\widehat{\theta}_\tau$ and $\widehat{\theta}_W$ according to (13).

(4) Compute $\widehat{T}_6$ according to (14).

(5) For some large integer K, repeat the following steps for every $k \in \{1, \ldots, K\}$:

  (a) Generate a random sample $(\mathbf{x}_{1,k}^0, \ldots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ according to (2).

  (b) Estimate the parameters $\widehat{\theta}_{\tau,k}^0$ and $\widehat{\theta}_{W,k}^0$ according to (13).

  (c) Compute $\widehat{T}_{6,k}^0$ according to (14) using $\widehat{\theta}_{\tau,k}^0$ and $\widehat{\theta}_{W,k}^0$.

(6) An approximate p-value for approach $\mathcal{A}_6$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^{K} I\{\widehat{T}_{6,k}^0 > \widehat{T}_6\}$.

C.7 Approach $\mathcal{A}_7$ (Panchenko (2005) – corrected)

(1) Extract the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) Generate a random sample $(\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_1^*, \ldots, \mathbf{z}_n^*)$ according to (2).

(4) Compute $\widehat{T}_7$ according to (15) using $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ and $(\mathbf{z}_1^*, \ldots, \mathbf{z}_n^*)$.

(5) For some large integer K, repeat the following steps for each $k \in \{1, \ldots, K\}$:

   (a) Generate a random sample $(\mathbf{x}_{1,k}^0, \ldots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ according to (2).

   (b) Estimate the parameters θ^0 with a consistent estimator $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$.

   (c) Generate a random sample $(\mathbf{x}_{1,k}^{0*}, \ldots, \mathbf{x}_{n,k}^{0*})$ from the null hypothesis copula $C_{\widehat{\theta}_k^0}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^{0*}, \ldots, \mathbf{z}_{n,k}^{0*})$ according to (2).

   (d) Compute $\widehat{T}_{7,k}^0$ according to (15) using $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ and $(\mathbf{z}_{1,k}^{0*}, \ldots, \mathbf{z}_{n,k}^{0*})$.

(6) An approximate p-value for approach $\mathcal{A}_7$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^K I\{\widehat{T}_{7,k}^0 > \widehat{T}_7\}$.

C.8 Approach $\mathcal{A}_8$

(1) Extract the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) Compute $(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathbf{R}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ assuming the parametric null hypothesis copula $C_{\widehat{\theta}}$.

(4) Generate a random sample $(\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*)$ from the independence copula.

(5) Compute $\widehat{T}_8$ according to (16) using $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $(\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*)$.

(6) For some large integer K, repeat the following steps for each $k \in \{1, \ldots, K\}$:

   (a) Generate a random sample $(\mathbf{x}_{1,k}^0, \ldots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ according to (2).

   (b) Estimate the parameters θ^0 with a consistent estimator $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$.

   (c) Compute $(\mathbf{v}_{1,k}^0, \ldots, \mathbf{v}_{n,k}^0) = \mathbf{R}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ assuming the parametric null hypothesis copula $C_{\widehat{\theta}_k^0}$.

   (d) Generate a random sample $(\mathbf{v}_{1,k}^{0*}, \ldots, \mathbf{v}_{n,k}^{0*})$ from the independence copula.

   (e) Compute $\widehat{T}_{8,k}^0$ according to (16) using $(\mathbf{v}_{1,k}^0, \ldots, \mathbf{v}_{n,k}^0)$ and $(\mathbf{v}_{1,k}^{0*}, \ldots, \mathbf{v}_{n,k}^{0*})$.

(7) An approximate p-value for approach $\mathcal{A}_8$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^K I\{\widehat{T}_{8,k}^0 > \widehat{T}_8\}$.

C.9 Approach $\mathcal{A}_9$

(1) Extract the pseudo-observations $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ by converting the sample data $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ into normalized ranks according to (2).

(2) Estimate the parameters θ with a consistent estimator $\widehat{\theta} = \widehat{\mathcal{V}}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$.

(3) Compute $\widehat{T}_1^{(i)}$, $\widehat{T}_1^{(ii)}$, $\widehat{T}_2$-$\widehat{T}_8$ by carrying out the appropriate steps of test procedures C.1-C.8 using $(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ and $\widehat{\theta}$.

(4) Compute $\widehat{T}_9$ according to (17).

(5) For some large integer K, repeat the following steps for every $k \in \{1, \ldots, K\}$:

   (a) Generate a random sample $(\mathbf{x}_{1,k}^0, \ldots, \mathbf{x}_{n,k}^0)$ from the null hypothesis copula $C_{\widehat{\theta}}$ and compute the associated pseudo-samples $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ according to (2).

   (b) Estimate the parameters θ^0 with a consistent estimator $\widehat{\theta}_k^0 = \widehat{\mathcal{V}}(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$.

   (c) Compute $\widehat{T}_{1,k}^{0,(i)}$, $\widehat{T}_{1,k}^{0,(ii)}$, $\widehat{T}_{2,k}^0$-$\widehat{T}_{8,k}^0$ by carrying out the appropriate steps of test procedures C.1-C.8 using $(\mathbf{z}_{1,k}^0, \ldots, \mathbf{z}_{n,k}^0)$ and $\widehat{\theta}_k^0$.

   (d) Compute $\widehat{T}_{9,k}^0$ according to (17) using $\widehat{T}_{1,k}^{0,(i)}$, $\widehat{T}_{1,k}^{0,(ii)}$, $\widehat{T}_{2,k}^0$-$\widehat{T}_{8,k}^0$.

(6) An approximate p-value for approach $\mathcal{A}_9$ is then given by $\widehat{p} = \frac{1}{K+1} \sum_{k=1}^K I\{\widehat{T}_{9,k}^0 > \widehat{T}_9\}$.