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# MULTI-STATE RELIABILITY THEORY

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**Abstract**

One inherent weakness of traditional reliability theory (see eqr 340) is that the system and the components are always described just as functioning or failed. The first attempts to replace this by a theory for multistate systems of multistate components were made in the late 1970s. By the mid 1980s the basic theory in this area was established. The objective of this article is to introduce concepts of multistate systems and present some upper and lower bounds for the availabilities and unavailabilities, to any level, in a fixed time interval. A series of applications of multistate reliability theory have been suggested during the last years. For example, the theory enables one to consider applications in electrical power generation systems, where the system state is the amount of power generated by the system or in offshore gas pipeline networks where the system state is the amount of gas delivered.

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## 1 Introduction

One inherent weakness of traditional reliability theory (see eqr 340) is that the system and the components are always described just as functioning or failed. The first attempts to replace this by a theory for multistate systems of multistate components were made in the late 1970s in [1], [2] and [3]. This was followed up by independent work in [4], [5] and [6] giving proper definitions of a multistate monotone system and of multistate coherent systems and also of minimal path and cut vectors. Furthermore, in [7] upper and lower bounds for the availabilities and unavailabilities, to any level, in a fixed time interval were obtained for multistate monotone systems based on corresponding information on the multistate components. These were assumed to be maintained and interdependent. Such bounds are of great interest when trying to predict the performance process of the system, noting that exact expressions are obtainable just for trivial systems. Hence by the mid 1980s the basic multistate reliability theory was established. A review of the early development in this area is given in [8]. Very recently probabilistic modelling of partial monitoring of components with applications to preventive system maintenance has been extended in [9] to multistate monotone systems of multistate components.

The theory was applied in [10] to an offshore electrical power generation system for two nearby oilrigs, where the amounts of power that may possibly be supplied to the two oilrigs are considered as system states. This application is also used to illustrate the theory in [9]. In [11] the theory was applied to the Norwegian offshore gas pipeline network in the North Sea, as of the end of the 1980s, transporting gas to Emden in Germany. The system state depends on the amount of gas actually delivered, but also to some extent on the amount of gas compressed mainly by the compressor component closest to Emden. Recently the first book [12] on multistate system reliability analysis and optimization appeared. The book also contains many examples of application of reliability assessment and optimization methods to real engineering problems.

## 2 Basic concepts and basic bounds

Let $S = \{0, 1, \ldots, M\}$ be the set of states of the system; the $M + 1$ states representing successive levels of performance ranging from the perfect functioning level $M$ down to the complete failure level 0. Furthermore, let $C = \{1, \ldots, n\}$ be the set of components and $S_i$ $(i = 1, \ldots, n)$ the set of states of the $i$th component. We claim $\{0, M\} \subseteq S_i \subseteq S$. Hence, the states 0 and $M$ are chosen to represent the endpoints of a performance scale that might be used for both the system and its components. Let $x_i$ $(i = 1, \ldots, n)$ denote the state or performance level of the $i$th component and $\boldsymbol{x} = (x_1, \ldots, x_n)$. It is assumed that the state, $\phi$, of the system is given by the structure function $\phi = \phi(\boldsymbol{x})$. For the following type of multistate systems a series of results can be derived.

**Definition 1** A system is *a multistate monotone system(MMS)* iff its structure $\phi$ satisfies:

(i) $\phi(\boldsymbol{x})$ is non-decreasing in each argument

(ii) $\phi(\mathbf{0}) = 0$ and $\phi(\boldsymbol{M}) = M \quad (\mathbf{0} = (0, \ldots, 0),\ \boldsymbol{M} = (M, \ldots, M))$.

The first assumption roughly says that improving one of the components cannot harm the system, whereas the second says that if all components are in the complete failure (perfect functioning) state, then the system is in the complete failure (perfect functioning) state.

We now impose some further restrictions on the structure function $\phi$. The following notation is needed:

$$(\cdot_i, \boldsymbol{x}) = (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n),$$

$$S_{i,j}^0 = S_i \cap \{0, \ldots, j-1\} \qquad \text{and} \qquad S_{i,j}^1 = S_i \cap \{j, \ldots, M\}.$$

**Definition 2** Consider an MMS with structure function $\phi$ satisfying

(i) $\min\limits_{1 \le i \le n} x_i \le \phi(\boldsymbol{x}) \le \max\limits_{1 \le i \le n} x_i$ .

If in addition $\forall i \in \{1, \ldots, n\}$, $\forall j \in \{1, \ldots, M\}$, $\exists (\cdot_i, \boldsymbol{x})$ such that

(ii) $\phi(k_i, \boldsymbol{x}) \ge j$, $\phi(\ell_i, \boldsymbol{x}) < j$, $\forall k \in S_{i,j}^1$, $\forall \ell \in S_{i,j}^0$, we have a *multistate strongly coherent system (MSCS)*,

(iii) $\phi(k_i, \boldsymbol{x}) > \phi(\ell_i, \boldsymbol{x})$ $\forall k \in S_{i,j}^1$, $\forall \ell \in S_{i,j}^0$, we have a *multistate coherent system (MCS)*,

(iv) $\phi(M_i, \boldsymbol{x}) > \phi(0_i, \boldsymbol{x})$, we have a *multistate weakly coherent system (MWCS)*.

When $M = 1$, all reduce to the established binary coherent system(BCS) (see eqr 340). The structure function $\min\limits_{1 \le i \le n} x_i \big( \max\limits_{1 \le i \le n} x_i \big)$ is often denoted the multistate series (parallel) structure.

Now choose $j \in \{1, \ldots, M\}$ and let the states $S_{i,j}^0, (S_{i,j}^1)$ correspond to the failure (functioning) state for the $i$th component if a binary approach is used. Condition (ii) above means that for all components $i$ and any level $j$, there shall exist a combination of the states of the other components, $(\cdot_i, \boldsymbol{x})$, such that if the $i$th component is in the binary failure (functioning) state, the system itself is in the corresponding binary failure (functioning) state. Loosely speaking, modifying [6], condition (ii) says that every level of each component is relevant to the same level of the system, condition (iii) says that every level of each component is relevant to the system, whereas condition (iv) simply says that every component is relevant to the system.

For a BCS one can prove the following practically very useful principle: Redundancy at the component level is superior to redundancy at the system level except for a parallel system where it makes no difference. Assuming $S_i = S$ $(i = 1, \dots, n)$ this is also true for an MCS, but not for an MWCS.

We now mention a special type of an MSCS. Define the indicators $(j = 1, \dots, M)$ $I_j(x_i) = 1(0)$ if $x_i \geq j (x_i < j)$, and the indicator vector $\boldsymbol{I}_j(\boldsymbol{x}) = (I_j(x_1), \dots, I_j(x_n))$.

**Definition 3** An MSCS is said to be a *binary type multistate strongly coherent system (BTMSCS)* iff there exist binary coherent structures $\phi_j$, $j = 1, \dots, M$ such that its structure function $\phi$ satisfies $\phi(\boldsymbol{x}) \geq j \Leftrightarrow \phi_j(\boldsymbol{I}_j, \boldsymbol{x})) = 1$ for all $j \in \{1, \dots, M\}$ and all $\boldsymbol{x}$.

Choose again $j \in \{1, \dots, M\}$ and let the states $S^0_{i,j}(S^1_{i,j})$ correspond to the failure (functioning) state for the $i$th component if a binary approach is applied. By the definition above $\phi_j$ will from the binary states of the components uniquely determine the corresponding binary state of the system.

In what follows $\boldsymbol{y} < \boldsymbol{x}$ means $y_i \leq x_i$ for $i = 1, \dots, n$, and $y_i < x_i$ for some $i$.

**Definition 4** Let $\phi$ be the structure function of an MMS and let $j \in \{1, \dots, M\}$. A vector $\boldsymbol{x}$ is said to be a *minimal path (cut) vector to level* $j$ iff $\phi(\boldsymbol{x}) \geq j$ and $\phi(\boldsymbol{y}) < j$ for all $\boldsymbol{y} < \boldsymbol{x}$ ($\phi(\boldsymbol{x}) < j$ and $\phi(\boldsymbol{y}) \geq j$ for all $\boldsymbol{y} > \boldsymbol{x}$).

**Definition 5** The *performance process of the* $i$*th component* $(i = 1, \dots, n)$ is a stochastic process $\{X_i(t), t \in [0, \infty)\}$, where for each fixed $t \in [0, \infty)$ $X_i(t)$ is a random variable which takes values in $S_i$. The *joint performance process for the components* $\{\boldsymbol{X}(t), t \in [0,\infty)\} = \{(X_1(t),\ldots,X_n(t)), t \in [0,\infty)\}$ is the corresponding vector stochastic process. The *performance process of an MMS* with structure function $\phi$ is a stochastic process $\{\phi(\boldsymbol{X}(t)), t \in [0,\infty)\}$, where for each fixed $t \in [0,\infty), \phi(\boldsymbol{X}(t))$ is a random variable which takes values in $S$.

**Definition 6** The performance processes $\{X_i(t), t \in [0,\infty)\}$, $i = 1,\ldots,n$ are *independent* in the time interval $I$ iff, for any integer $m$ and $\{t_1,\ldots,t_m\} \subset I$ the random vectors $\{X_1(t_1),\ldots,X_1(t_m)\},\ldots,\{X_n(t_1),\ldots,X_n(t_m)\}$ are independent.

**Definition 7** Let $j \in \{1,\ldots,M\}$. The *availability*, $h_\phi^{j(I)}$ and the *unavailability*, $g_\phi^{j(I)}$ to level $j$ in the time interval $I$ for an *MMS* with structure function $\phi$ are given by

$$h_\phi^{j(I)} = P[\phi(\boldsymbol{X}(s)) \geq j \ \ \forall s \in I], \qquad g_\phi^{j(I)} = P[\phi(\boldsymbol{X}(s)) < j \ \ \forall s \in I].$$

Note that $h_\phi^{j(I)} + g_\phi^{j(I)} \leq 1$, with equality for the case $I = [t,t]$.

As an example of the bounds for $h_\phi^{j(I)}$ and $g_\phi^{j(I)}$ given in [7], we give the following theorem by first introducing the $n \times M$ matrices

$$\boldsymbol{P}_\phi^{(I)} = \{p_i^{j(I)}\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,M}} \qquad \boldsymbol{Q}_\phi^{(I)} = \{q_i^{j(I)}\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,M}}.$$

Note that according to [13] we don't need to assume that each of the performance processes of the components is associated in $I$.

**Theorem 1** Let $(C,\phi)$ be an MMS with the marginal performance processes of its components being independent in $I$. Furthermore, for $j \in \{1,\ldots,M\}$ let $\boldsymbol{y}_k^j = (y_{1k}^j,\ldots,y_{nk}^j)$,

$k = 1, \ldots, n^j$ ($\boldsymbol{z}_k^j = (z_{1k}^j, \ldots, z_{nk}^j)$, $k = 1, \ldots, m^j$) be its minimal path (cut) vectors to level $j$. Define

$$\ell_\phi^{j'}(\boldsymbol{P}_\phi^{(I)}) = \max_{1\le k\le n^j} \prod_{i=1}^{n} p_i^{y_{ik}^j(I)} \qquad \bar{\ell}_\phi^{j'}(\boldsymbol{Q}_\phi^{(I)}) = \max_{1\le k\le m^j} \prod_{i=1}^{n} q_i^{z_{ik}^j+1(I)}$$

$$\ell_\phi^{j^*}(\boldsymbol{P}_\phi^{(I)}) = \prod_{k=1}^{m^j} \coprod_{i=1}^{n} p_i^{z_{ik}^j+1(I)} \qquad \bar{\ell}_\phi^{j^*}(\boldsymbol{Q}_\phi^{(I)}) = \prod_{k=1}^{n^j} \coprod_{i=1}^{n} q_i^{y_{ik}^j(I)}$$

$$B_\phi^j(\boldsymbol{P}_\phi^{(I)}) = \max_{j\le k\le M} \left\{ \max[\ell_\phi^{k'}(\boldsymbol{P}_\phi^{(I)}), \ell_\phi^{k^*}(\boldsymbol{P}_\phi^{(I)})] \right\}$$

$$\bar{B}_\phi^j(\boldsymbol{Q}_\phi^{(I)}) = \max_{1\le k\le j} \left\{ \max[\bar{\ell}_\phi^{k'}(\boldsymbol{Q}_\phi^{(I)}), \bar{\ell}_\phi^{k^*}(\boldsymbol{Q}_\phi^{(I)})] \right\}$$

Then

$$B_\phi^j(\boldsymbol{P}_\phi^{(I)}) \le h_\phi^{j(I)} \le \inf_{t\in I} \left[1 - \bar{B}_\phi^j(\boldsymbol{Q}_\phi^{([t,t])})\right] \le 1 - \bar{B}_\phi^j(\boldsymbol{Q}_\phi^{(I)})$$

$$\bar{B}_\phi^j(\boldsymbol{Q}_\phi^{(I)}) \le g_\phi^{j(I)} \le \inf_{t\in I} \left[1 - B_\phi^j(\boldsymbol{P}_\phi^{([t,t])})\right] \le 1 - B_\phi^j(\boldsymbol{P}_\phi^{(I)}).$$

Here $\coprod_{i=1}^n a_i \stackrel{\text{def}}{=} 1 - \prod_{i=1}^n (1 - a_i)$. By specializing $M = 1$ and $I = [t, t]$ the bounds reduce to the familiar ones from binary theory as given in [14].

## 3 An offshore electrical power generation system

The purpose of the offshore electrical power generation system considered in [9] and [10], depicted in Figure 1, is to supply two nearby oilrigs with electrical power. Both oilrigs have their own main generation, represented by equivalent generators $A_1$ and $A_3$ each having a capacity of 50 MW. In addition oilrig 1 has a standby generator $A_2$ that is switched into the network in case of outage of $A_1$ or $A_3$. $A_2$ also has a capacity of 50 MW. The control unit, $U$, continuously supervises the supply from each of the generators with automatic control of the switches. If for instance the supply from $A_3$ to oilrig 2 is not sufficient, whereas the supply from $A_1$ to oilrig 1 is sufficient, $U$ can activate $A_2$ to supply oilrig 2 with electrical power through the standby subsea cables $L$.

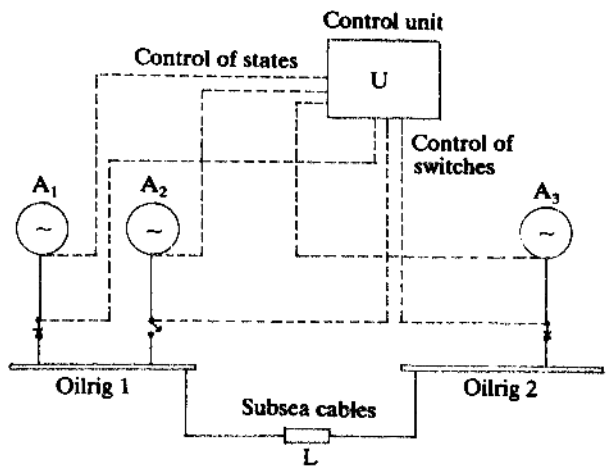


Figure 1. Outline of an offshore electrical power generation system

The components to be considered here are $A_1, A_2, A_3, U$ and $L$. We let the perfect functioning level $M$ equal 4 and let the set of states of all components be $\{0, 2, 4\}$. For $A_1, A_2$ and $A_3$ these states are interpreted as

0: The generator cannot supply any power;

2: The generator can supply maximum 25 MW;

4: The generator can supply maximum 50 MW.

Note that as an approximation we have chosen to describe supply capacity of the generators using a discrete scale of three points. The supply capacity is not a measure of the actual amount of power delivered at a fixed point of time.

The control unit $U$ has the states

0: $U$ will by mistake switch the main generators $A_1$ and $A_3$ off without switching $A_2$ on;

2: $U$ will not switch $A_2$ on when needed;

4: $U$ is functioning perfectly.

The subsea cables $L$ are actually assumed to be constructed as double cables transferring half of the power through each simple cable. This leads to the following states of $L$

0: No power is transferred;

2: 50% of the power is transferred;

4: 100% of the power is transferred.

Let us now for simplicity assume that the mechanism that distributes the power from $A_2$ to platform 1 or 2 is working perfectly, transferring excess power from $A_2$ to platform 2 if platform 1 is ensured a delivery corresponding to state 4. Now let $\phi(A_1, A_2, A_3, U, L) =$ the amount of power that can be supplied to platform 2. In addition to the states taken by $A_1, A_2, A_3$, $\phi$ can also take the following states

1: The amount of power that can be supplied is maximum $12 \cdot 5$ MW;

3: The amount of power that can be supplied is maximum $37 \cdot 5$ MW.

Number the components $A_1, A_2, A_3, U, L$ successively 1, 2, 3, 4, 5. Then a little thought leads to

$$\phi(\boldsymbol{x}) = I(x_4 > 0)\min(x_3 + \max(x_1 + x_2 I(x_4 = 4) - 4, 0)x_5/4, 4),$$

noting that $\max(x_1 + x_2 I(x_4 = 4) - 4, 0)$ is just the excess power from $A_2$ which one tries to transfer to platform 2. This is obviously a multistate monotone system.

# 4 Related articles

eqr 113, eqr 126, eqr 340, eqr 343, eqr 346, eqr 347, eqr 349

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