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C*-algebras with quantum group symmetry

Noncommutative boundaries and equivariant subproduct systems

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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo, under the supervision of professor Sergey Neshveyev and associate professor Makoto Yamashita. This work was partially supported by the Norwegian Research Council through grant 300837 “Quantum Symmetry”.

The thesis is a collection of four papers, presented in chronological order of writing. The papers are preceded by an introductory chapter that relates them to each other and provides background information and motivation for the work.

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First and foremost I must thank my main supervisor Sergey Neshveyev. A big part of this thesis has been developed through our weekly meetings, often in parallel with good laughs and non-mathematical discussions. Without his guidance and encouragement I would not have gotten very far.

I am also grateful to my co-supervisor Makoto Yamashita, who is always glad to help.

The last four years have been an interesting and enjoyable journey. Much of the credit for this goes to my colleagues at the seventh floor. Special mention goes to my two office-mates Floris and Lucas, my "personal trainer" Ulrik and to Gaute, with whom it is possible to discuss sports. Thanks also to my fellow dog-lovers in the administration, for our many chats and all the chocolate.

I would like to thank my family and friends, not only for being a great family and great friends, but also for supporting me in pursuing a PhD. Mamma deserves a special thanks for the many months we stayed at her house during the pandemic.

Finally I would like to thank my wife, Ranveig, whose support is too extensive to describe here. I will instead end with a mantra of hers, which for some reason is very motivating: *Du kan alltids slutte*.

• **Erik Habbestad**

Oslo, June 2023

Summary

English

This thesis consists of four papers, which are all concerned with compact quantum groups and their actions on C^* -algebras. The results shed light on both the C^* -algebras which are studied, and on the quantum groups that act on them.

In the first paper we construct a Furstenberg–Hamana boundary $\partial_{FH}D(G)$ of the Drinfeld double $D(G)$ of a compact quantum group G , and compare it to the Poisson boundary. For a natural class the Poisson boundaries of the discrete dual \hat{G} coincides with Poisson boundaries of $D(G)$. We show that when G has a weakly amenable quantum dimension function there is an *absolute* Poisson boundary $\partial_{\Pi}\hat{G}$, which extends the notion of an absolute Poisson boundary in the case where there exists an ergodic probability measure on $\text{Irr } G$. Moreover, in this case

$$\partial_{FH}D(G) \cong \partial_{\Pi}\hat{G}.$$

This allows us to compute $\partial_{FH}D(G)$ in some cases, by relating it to known Poisson boundaries.

The construction of $\partial_{FH}D(G)$ can be carried out on the level of the representation category $\text{Rep } G$. We generalize the construction to any rigid C^* -tensor category (without the assumption of a fibre functor). When the rigid C^* -tensor category has weakly amenable quantum dimension function and simple unit, we prove that the Furstenberg–Hamana boundary coincides with the categorical Poisson boundary constructed by Neshveyev and Yamashita.

In papers II and III we consider subproduct systems associated to Temperley–Lieb polynomials. Any such polynomial P is stabilized up to a scalar by a compact quantum group \tilde{O}_P^+ monoidally equivalent to $U_q(2)$, for appropriate $q > 0$. As a consequence, the associated C^* -algebras \mathcal{T}_P and \mathcal{O}_P admit actions by \tilde{O}_P^+ . We show that there is an \tilde{O}_P^+ -equivariant $*$ -isomorphism

$$\mathcal{O}_P \rtimes \mathbb{Z} \cong B(U_q(2), \tilde{O}_P^+),$$

where the latter C^* -algebra is the linking algebra implementing the aforementioned monoidal equivalence. For a subclass of Temperley–Lieb polynomials it is enough to consider a free orthogonal quantum group O_P^+ , in which case we have

$$\mathcal{O}_P \cong B(SU_q(2), O_P^+).$$

On our way we show that equivariant subproduct systems behave well with respect to monoidal equivalence (II), and find some sufficient conditions for when an algebraic bi-Galois object admits a completion to a linking algebra (III).

We show that \mathcal{T}_P is $KK^{\bar{O}_P^+}$ -equivalent to the complex numbers, and also compute the K -theory of \mathcal{O}_P . These K -theoretic results extend results by Arici–Kaad and De Commer–Yamashita. Some of the key tools in our arguments are KK -theoretic results established by Voigt. In particular we use (and slightly extend) the Baum–Connes conjecture for the dual of $SU_q(2)$.

For any two-dimensional Temperley–Lieb polynomial, we have a $*$ -isomorphism

$$\mathcal{O}(\mathcal{H}_P) \cong C(SU_q(2)), \quad q \in \mathbb{C}^*,$$

where the latter C^* -algebra is the functions on braided $SU_q(2)$, first defined by Kasprzak–Meyer–Roy–Woronowicz. This is connected to the fact that, in a braided sense, $SU_q(2)$ stabilizes the polynomial $P = X_1X_2 - \bar{q}X_2X_1$.

Our initial motivation for paper IV was to find, for each Temperley–Lieb polynomial P , a braided compact quantum group stabilizing P . In the paper we show that this is indeed possible. Moreover, on the way to this result we found a natural setting for producing what we call *braided matrix quantum groups*. We thus produce braided versions of the most well-known examples of compact matrix quantum groups, and show that many of the braided compact quantum groups considered in the literature are covered by this procedure. In particular we recover the braided $SU_q(2)$ mentioned above and the \mathbb{T} -braided free orthogonal quantum groups defined by Meyer–Roy.

The construction itself is quite general. We use Majid’s transmutation procedure on Hopf-algebra maps $H \rightarrow \mathbb{C}[T]$ where T is a compact abelian group to obtain a braided Hopf $*$ -algebra H_β : For any H -cocentral subgroup $T_0 \subset T$, H_β can be viewed as a Hopf $*$ -algebra object in the category of right $\mathbb{C}[T/T_0]$ -comodules with braiding β .

A general result in the paper is that the bosonization $\mathbb{C}[T/T_0] \# H_\beta$ is always a 2-cocycle twist of the Hopf $*$ -algebra of coinvariants $(\mathbb{C}[T/T_0] \otimes H)^{\Delta(T_0)}$. This, of course, has immediate consequences for representation theory.

Norwegian

Denne avhandlinga består av fire artiklar, som alle omhandlar kompakte kvantegrupper og deira verknadar på C^* -algebraer. Resultata belyser eigenskapar ved dei relevante C^* -algebraene, men fortel oss òg noko om om kvantegruppene sjølve.

I den første artikkelen studerer me Furstenberg–Hamana randa $\partial_{FH}D(G)$ til Drinfeld-dobblinga $D(G)$ av ei kompakt kvantegruppe G , og samanliknar ho med Poissonrendene. Me viser at ein naturleg klasse av Poissonrendene til \hat{G} samanfaller med Poissonrendene til $D(G)$. Vidare viser me at når G har ein svakt amenabel dimensjonsfunksjon, så finst det ei absolutt Poissonrand $\partial_{\Pi}\hat{G}$, som naturleg utvidar omgrepet i tilfellet kor det finst eit ergodisk mål på $\text{Irr } G$. I dette tilfellet har me

$$\partial_{FH}D(G) \cong \partial_{\Pi}\hat{G}.$$

Dette gjer det mogleg å finne randa $\partial_{FH}D(G)$, ved å relatere ho til kjente Poissonrender.

Konstruksjonen av $\partial_{FH}D(G)$ kan òg utførast for representasjonskategorien $\text{Rep } G$, og generaliserast til ein kvar rigid C^* -tensorkategori (uten ein fiberfunctor). Spesielt viser me at for ein rigid C^* -tensorkategori med svakt amenabel dimensjonsfunksjon og enkel eining, så samsvarar Furstenberg–Hamana randen med den kategoriske Poissonranden konstruert av Neshveyev og Yamashita.

I artikkel II og III ser me på Temperley–Lieb polynom og naturleg tilhøyrande delproduktsystem. Eit kvart slikt polynom P blir stabilisert, modulo ein skalar, av ei kompakt kvantegruppe \tilde{O}_P^+ , som, for passande $q > 0$, er monoidalt ekvivalent til $U_q(2)$. Som ein konsekvens av dette er dei tilhøyrande C^* -algebraene \mathcal{T}_P og \mathcal{O}_P utstyrt med ein \tilde{O}_P^+ -verknad. Me viser at det finst ein \tilde{O}_P^+ -ekvivalent $*$ -isomorfi

$$\mathcal{O}_P \rtimes \mathbb{Z} \cong B(U_q(2), \tilde{O}_P^+),$$

kor C^* -algebraen til høgre er linkalgebraen som implementerer den monoidale ekvivalensen nemnt ovanfor. For somme Temperley–Lieb polynom P kan me i staden for bruke ei fri ortogonal kvantegruppe O_P^+ , og me har følgande isomorfi:

$$\mathcal{O}_P \cong B(SU_q(2), O_P^+).$$

På vegen viser me blant anna at ekvivalente delproduktsystem er "kompatible" med monoidal ekvivalens av kvantegrupper (II), og finn tilstrekkelege vilkår for når eit algebraisk bi-Galois objekt har ein C^* -komplettering (III).

Vidare viser me at \mathcal{T}_P er $KK^{\tilde{O}_P^+}$ -ekvivalent til \mathbb{C} , og me finn K -gruppene til \mathcal{O}_P . Desse resultatata utvidar funn gjort av Arici–Kaad og De Commer–Yamashita. Ein kritisk del av argumentasjonen baserer seg på resultat av Voigt. Spesielt treng me (ei utviding av) Baum–Connes konjekturen for dualen til $SU_q(2)$.

For alle todimensjonal Temperley–Lieb polynom P har me ein $*$ -isomorfi

$$\mathcal{O}(\mathcal{H}_P) \cong C(SU_q(2)), \quad q \in \mathbb{C}^*,$$

kor C^* -algebraen til høgre er algebraen av kontinuerlege funksjonar på ei fletta kvantegruppa $SU_q(2)$, først definert av Kasprzak–Meyer–Roy–Woronowicz. Dette kan ses i samanheng med at $SU_q(2)$ stabiliserer polynomet $P = X_1X_2 - \bar{q}X_2X_1$, i "fletta forstand".

Dette forklarar den opphavelige motivasjonen for arbeidet bak den fjerde artikkelen. Gitt eit Temperley–Lieb polynom P , kan me finne ei fletta kompakt kvantegruppe som stabiliserer P ? I artikkelen viser me at det faktisk er mogleg, og på vegen dit vart det naturleg å finne ein generell metode for å produsere fletta kvantegrupper over kompakte abelske grupper. Me finn så fletta versjoner av dei mest kjende kompakte matrisekvantegruppene, og viser at fleire fletta kvantegrupper i litteraturen blir dekkja av desse eksempla. Eit eksempel er den fletta kvantegruppa $SU_q(2)$, og eit anna er dei \mathbb{T} -fletta frie ortogonale kvantegruppene definert av Meyer og Roy.

Konstruksjonen i seg sjølv er relativt generell. Me bruker Majids *transmutasjon* på ein morfi mellom Hopfalgebraer $H \rightarrow \mathbb{C}[T]$, kor T er ei kompakt abelsk

Summary

gruppe: For einkvan H -kosentral undergruppe $T_0 \subset T$ får me eit Hopf $*$ -algebra objekt H_β i kategorien av $\mathbb{C}[T/T_0]$ -komodular utstyrt med ei fletting β .

Eit generelt resultat i artikkelen er at bosoniseringa $\mathbb{C}[T/T_0] \# H_\beta$ alltid er ein 2-kosykeldeformasjon av Hopf $*$ -algebraen $(\mathbb{C}[T/T_0] \otimes H)^{\Delta(T_0)}$ bestående av koinvariantar under verknaden av den naturlege diagonale undergruppa $\Delta(T_0) \subset T \times T$. Dette har umiddelbare representasjonsteoretiske konsekvensar.

List of Papers

Paper I

Erik Habbestad, Lucas Hataishi and Sergey Neshveyev “Noncommutative Poisson boundaries and Furstenberg–Hamana boundaries of Drinfeld doubles”. In *Journal de Mathématiques Pures et Appliquées*, March 2022, volume 159, issue 2, pp. 313–347. DOI: 10.1016/j.matpur.2021.12.006.

Paper II

Erik Habbestad and Sergey Neshveyev “Subproduct systems with quantum group symmetry”. To appear in *Journal of Noncommutative Geometry*.

Paper III

Erik Habbestad and Sergey Neshveyev “Subproduct systems with quantum group symmetry. II”. Preprint, arXiv: 2212.08512

Paper IV

Erik Habbestad and Sergey Neshveyev “Cocycle twisting of semidirect products and transmutation”. Preprint, arXiv: 2304.00494

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Chapter 1

Introduction

This thesis is concerned with compact quantum groups and their actions on certain C^* -algebras. On one hand we consider Poisson and Furstenberg–Hamana boundaries, which are canonically attached to the quantum group in question. On the other hand we consider C^* -algebras associated to certain subproduct systems, a construction which, at least a priori, does not rely on quantum groups at all. Nevertheless, also these C^* -algebras turn out to be closely related to the representation theory of some of their quantum symmetry groups, and in some cases even braided quantum groups appear naturally.

The introduction starts from the Hopf $*$ -algebra of regular functions on a compact group, which serves as a model for understanding the theory of compact quantum groups. Then we briefly discuss quantum group actions of various kinds, leading us to the more specialized topics of noncommutative boundaries and equivariant subproduct systems. Along the way we highlight some of the relevant results in the mathematical literature, hopefully preparing the reader for the summaries of the papers.

1.1 Compact quantum groups

1.1.1 Motivation and basic theory

Because of Gelfand duality [GN43], the category of unital C^* -algebras can be seen as an enlargement of the category of compact Hausdorff spaces. From this point of view a *compact quantum group* should be a unital C^* -algebra A with some extra structure. In a subcategory of C^* -algebras consisting of "continuous functions on compact quantum groups" the commutative C^* -algebras should correspond to genuine groups, and for it to be interesting, it should of course contain many noncommutative examples.

As a first step towards understanding the theory of compact quantum groups, one must realize that genuine compact groups are intimately connected to their representation theory. This is nicely captured by Peter–Weyl theory [PW27], or, for the more categorically inclined, the classic Tannaka–Krein duality [Tan39; Kre49]. A nice way to connect these two theories is, as noted by Hochschild already in the 60's [Hoc65], the language of *Hopf algebras*. This observation is essential also in the quantum setting, so it will serve as our starting point.

Let G be a compact topological group with unit element e . Recall that if $\pi : G \rightarrow B(H_\pi)$ is a finite dimensional unitary representation on a Hilbert space H_π then a *matrix coefficient* for π is a complex valued function on G of the form

$$\pi_{\xi, \zeta}(s) = \langle \pi(s)\xi, \zeta \rangle, \quad \xi, \zeta \in H_\pi,$$

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where $\langle \cdot, \cdot \rangle$ is the inner product on H_π . The space of *regular functions* on G is the vector space

$$\mathbb{C}[G] = \text{span}_{\mathbb{C}}\{\pi_{\xi, \zeta} \mid \pi \text{ is finite dimensional}\}.$$

Because we can form tensor products of representations and every representation has a contragredient representation, $\mathbb{C}[G]$ becomes a $*$ -algebra under pointwise multiplication and complex conjugation. In fact $\mathbb{C}[G]$ is a classic example of a *Hopf $*$ -algebra*. Namely, by taking duals of the structure maps on G we get a *coproduct* Δ (dual product), *antipode* S (dual inverse) and *counit* ε (dual unit) on $\mathbb{C}[G]$:

$$\begin{aligned} \Delta : \mathbb{C}[G] &\rightarrow \mathbb{C}[G] \otimes \mathbb{C}[G], & S : \mathbb{C}[G] &\rightarrow \mathbb{C}[G], & \varepsilon : \mathbb{C}[G] &\rightarrow \mathbb{C}, \\ \Delta(f)(s, t) &= f(st), & S(f)(s) &= f(s^{-1}), & \varepsilon(f) &= f(e). \end{aligned}$$

To see that this is well-defined, let π be a unitary representation and write π_{ij} for the matrix coefficients of π with respect to an orthonormal basis $(e_i)_i$ in H_π . Then, because π is a group homomorphism,

$$\Delta(\pi_{ij}) = \sum_k \pi_{ik} \otimes \pi_{kj}, \quad S(\pi_{ij}) = \pi_{ji}^*, \quad \varepsilon(\pi_{ij}) = \delta_{ij}. \quad (1.1)$$

It is a simple exercise to verify that $\mathbb{C}[G]$ is indeed a Hopf $*$ -algebra.

Integration against the (normalized) Haar measure defines an important linear functional on $\mathbb{C}[G]$. It is characterized by being the unique faithful state h on $\mathbb{C}[G]$ (so $h(1) = 1$ and $h(a^*a) > 0$ for all $a \neq 0$) which is invariant, meaning that

$$(h \otimes \iota)\Delta = h(\cdot)1 = (\iota \otimes h)\Delta.$$

On the other hand, if $(\mathcal{A}, \Delta, S, \varepsilon)$ is a commutative Hopf $*$ -algebra, then the set

$$\mathbb{G}^\circ(\mathcal{A}) = \{\phi : \mathcal{A} \rightarrow \mathbb{C} \mid \phi \text{ is a nonzero } * \text{-homomorphism}\},$$

becomes a group with unit element ε , product

$$(\phi * \psi) = (\phi \otimes \psi) \circ \Delta, \quad \phi, \psi \in \mathbb{G}^\circ(\mathcal{A}),$$

and inversion map

$$\phi^{-1} = \phi \circ S, \quad \phi \in \mathbb{G}^\circ(\mathcal{A}).$$

If \mathcal{A} has a faithful invariant state h we can define a norm on \mathcal{A} by $\|a\| = h(a^*a)^{1/2}$, and $\mathbb{G}^\circ(\mathcal{A})$ becomes a compact group when equipped with the weak- $*$ topology. We arrive at a Hopf algebraic formulation of the Tannaka–Krein theorem:

Theorem 1.1.1 ([Hoc65]). *The assignments*

$$\mathcal{A} \rightarrow \mathbb{G}^\circ(\mathcal{A}), \quad G \mapsto \mathbb{C}[G],$$

define an equivalence between the category of reduced¹ commutative Hopf $$ -algebras with a faithful invariant state and the category of compact groups. That is*

$$\mathcal{A} \cong \mathbb{C}[\mathbb{G}^\circ(\mathcal{A})], \quad \mathbb{G}^\circ(\mathbb{C}[G]) \cong G.$$

¹Here *reduced* means that $\mathbb{G}^\circ(\mathcal{A})$ separates the points of \mathcal{A} .

In view of Gelfand duality, the reader might wonder why we didn't phrase theorem 1.1.1 in terms of some class of "Hopf- C^* -algebras", by extending the maps Δ , S , ε to $C(G)$. This is indeed possible, and, more generally, Takesaki proved a similar theorem for locally compact groups in terms of commutative *Hopf-von Neumann algebras* [Tak69], building on the work by Kac, Tatsuuma and Ernest [Kac63; Tat67; Ern67]. However, it turns out that requiring such structure on the level of operator algebras is too strong for the quantum setting, as it would exclude many of the interesting examples. Indeed, in many cases the appropriate notion of an antipode is not a bounded map. Theorem 1.1.1 on the other hand, will in some sense extend to compact quantum groups, where a Hopf $*$ -algebra of matrix coefficients will still be present.

In the 80's Woronowicz came up with the definition of *compact matrix pseudogroups* [Wor87a], which generalized $C(G)$ for a compact Lie group G . This definition is inspired by the following fact: If G is a compact Lie group, then it has a faithful finite dimensional unitary representation π , and the finite set $\{\pi_{ij}\}$ generates $\mathbb{C}[G]$ as a $*$ -algebra. Together with the formulas (1.1) this determines the Hopf $*$ -algebra $\mathbb{C}[G]$. We return to this in example 1.1.4 below.

A definition covering all compact groups and compact matrix pseudogroups was found a few years later. It is phrased on the level of C^* -algebras, where by the symbol \otimes we mean the minimal tensor product.

Definition 1.1.2 ([Wor98]). A compact quantum group is a pair (A, Δ) , where A is a unital C^* -algebra and $\Delta : A \rightarrow A \otimes A$ is a unital $*$ -homomorphism such that

- (i) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$, and
- (ii) $\text{span}\{(a \otimes 1)\Delta(b) \mid a, b \in A\}$ and $\text{span}\{(1 \otimes a)\Delta(b) \mid a, b \in A\}$ are dense in $A \otimes A$.

When A is commutative (i) implies that the (compact) spectrum is a semigroup, while (ii) implies that it has cancellation. From this one can show that the spectrum is in fact a group, implying that the definition covers compact groups. In general we will write $A = C(G)$ also for noncommutative A , and refer to G as the compact quantum group.

As for compact groups there is always a Haar state on $C(G)$, and much of the Peter-Weyl theory holds also for compact quantum groups. We briefly sketch some of the main ingredients, leading us again to a Hopf $*$ -algebra of regular functions.

A (unitary) representation U on a finite dimensional Hilbert space H_U is an invertible (unitary) element $U \in B(H_U) \otimes C(G)$ such that

$$(\iota \otimes \Delta)(U) = U_{12}U_{13}, \quad (1.2)$$

where the notation indicates in which way we embed $B(H_U) \otimes C(G)$ in $B(H_U) \otimes C(G) \otimes C(G)$. A matrix coefficient for U is an element in $C(G)$ of the form

$$u_{\xi, \zeta} = (\langle \cdot, \zeta \rangle \otimes \iota)(U), \quad \xi, \zeta \in H_U.$$

1. Introduction

However, not everything generalizes directly. Most notably the contragredient representation U^c of a unitary representation U is not automatically unitary. This complicates the theory, but luckily there is always a canonical equivalent representation \bar{U} , called the *conjugate* representation. The $*$ -algebra of regular functions $\mathbb{C}[G]$ is thus defined in the same way as for compact groups, and using the new Peter–Weyl theory it can be shown that $\mathbb{C}[G]$ is dense in $C(G)$. Like before $\mathbb{C}[G]$ becomes a Hopf $*$ -algebra with the restricted coproduct. Precisely, the structure is given by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad S(u_{ij}) = u_{ji}^*, \quad \varepsilon(u_{ij}) = \delta_{ij},$$

where $U = (u_{ij})_{i,j}$ is any finite dimensional unitary representation written in terms of an orthonormal basis in H_U , compare to (1.1). By definition the (unitary) representations of G coincides with the (unitary) comodules of $\mathbb{C}[G]$: If U is a finite dimensional representation then

$$H_U \rightarrow H_U \otimes \mathbb{C}[G], \quad \xi \mapsto U(\xi \otimes 1),$$

defines a right $\mathbb{C}[G]$ -comodule. We are ready to state a characterization which extends theorem 1.1.1:

Theorem 1.1.3 ([DK94]). *Let \mathcal{A} be a Hopf $*$ -algebra. Then the following are equivalent:*

- (i) $\mathcal{A} = \mathbb{C}[G]$ for a compact quantum group G .
- (ii) \mathcal{A} is spanned by the matrix coefficients of its finite dimensional unitary comodules.
- (iii) \mathcal{A} has a faithful invariant state.

We remark that $\mathbb{C}[G]$ may have several C^* -completions $C(G)$, which sits in a sequence of surjective $*$ -homomorphisms

$$C_u(G) \rightarrow C(G) \rightarrow C_r(G),$$

where the first and last C^* -algebras are called the *universal* and *reduced* forms respectively. We will not dwell on this here, but mention that if the map $C_u(G) \rightarrow C_r(G)$ is an isomorphism, we say that G is *coamenable*. This terminology comes from the fact that if Γ is a discrete group, then $\mathbb{C}[G] := \mathbb{C}\Gamma$ defines a compact quantum group G which is coamenable if and only if Γ is amenable.

It is time for some examples.

Example 1.1.4. This is rather a class of examples coinciding with the notion of compact matrix pseudogroups mentioned above, [Wor87a], [DK94]. A *compact matrix quantum group* is a Hopf $*$ -algebra $(\mathbb{C}[G], \Delta, \varepsilon, S)$ generated by elements u_{ij} , $1 \leq i, j \leq m$, such that the matrices

$$(u_{ij})_{ij}, (u_{ij}^*)_{ij} \in M_m(\mathbb{C}[G])$$

are unitarizable and

$$\Delta(u_{ij}) = u_{ik} \otimes u_{kj}, \varepsilon(u_{ij}) = \delta_{ij}.$$

One way to see that this indeed defines a compact quantum group is to note that $(u_{ij})_{i,j}$ and $(u_{ij}^*)_{i,j}$ define $\mathbb{C}[G]$ -comodules and use theorem 1.1.3. \diamond

As a more concrete example we consider the famous quantum group $SU_q(2)$, an example which came about even before the formal definitions discussed above, [Wor87b].

Example 1.1.5. Fix $q > 0$. By definition $\mathbb{C}[SU_q(2)]$ is the universal $*$ -algebra generated by elements α and γ subject to the relations

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha, & \alpha\gamma^* &= q\gamma^*\alpha, & \gamma^*\gamma &= \gamma\gamma^*, \\ \alpha^*\alpha + \gamma^*\gamma &= 1, & \alpha\alpha^* + q^2\gamma\gamma^* &= 1. \end{aligned}$$

The coproduct is given by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

$SU_q(2)$ is a compact quantum matrix group, where the generating (fundamental) representation is

$$U = (u_{ij})_{i,j} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

When $q = 1$ we recover the continuous functions on ordinary $SU(2)$ by identifying the u_{ij} with the matrix coefficients π_{ij} corresponding to the canonical representation $\pi : SU(2) \rightarrow B(\mathbb{C}^2)$ and the standard orthonormal basis in \mathbb{C}^2 . That is

$$\pi_{ij}(a) = a_{ij}, \quad a = (a_{ij})_{i,j} \in SU(2).$$

We mention that, independently from Woronowicz, $\mathbb{C}[SU_q(2)]$ was studied by Vaksman and Soibelman [VS88], in the setting of q -deformed Lie groups. If G is a simply connected semisimple compact Lie group, there is a systematic way to deform the universal enveloping algebra $U(\mathfrak{g})$ of its complexified Lie algebra \mathfrak{g} due to Jimbo and Drinfeld [Jim85; Dri86]. The resulting deformed object $U_q(\mathfrak{g})$ is a Hopf $*$ -algebra in a canonical way, and the q -deformation G_q is obtained as a certain $*$ -subalgebra $\mathbb{C}[G_q] \subset U_q(\mathfrak{g})^*$. These are always compact matrix quantum groups. \diamond

Example 1.1.6. Let $m \geq 2$ be a natural number, $F \in GL_m(\mathbb{C})$ and assume that $F\bar{F} = \pm 1$. Let $\mathbb{C}[O_F^\pm]$ denote the universal $*$ -algebra generated by elements u_{ij} , $1 \leq i, j \leq m$, subject to the relations

$$U = (u_{ij})_{i,j} \text{ is unitary and } U = F\bar{U}F^{-1},$$

where $\bar{U} = (u_{ij}^*)_{i,j}$. Define a coproduct on $\mathbb{C}[O_F^\pm]$ by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

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Then O_F^+ is a compact (matrix) quantum group, called a free orthogonal quantum group [Wan95; VW96]. The condition on F makes sure that the fundamental representation U is irreducible, and the defining relations in $\mathbb{C}[O_F^+]$ says that F is an intertwiner making the contragredient representation equivalent to the unitary representation U .

If F is the identity matrix I_m , then $O_m^+ := O_{I_m}^+$ contains the orthogonal group $O(m, \mathbb{R})$ as a *closed quantum subgroup*: There is a surjective $*$ -homomorphism $\theta : \mathbb{C}[O_m^+] \rightarrow \mathbb{C}[O(m, \mathbb{R})]$ such that $\Delta\theta = (\theta \otimes \theta)\Delta$. To see this, just notice that in $\mathbb{C}[O(m, \mathbb{R})]$ we have the same relations as in $\mathbb{C}[O_m^+]$ plus commutativity. \diamond

1.1.2 The discrete dual

To any compact quantum group G one can define the *discrete dual quantum group* \hat{G} . The structure on \hat{G} is obtained by dualizing the Hopf $*$ -algebra structure on $\mathbb{C}[G]$. This gives a $*$ -algebra $\mathbb{C}[G]^*$, which is not quite a Hopf algebra, because the image of the coproduct becomes a map

$$\hat{\Delta} : \mathbb{C}[G]^* \rightarrow (\mathbb{C}[G] \otimes \mathbb{C}[G])^* \neq \mathbb{C}[G]^* \otimes \mathbb{C}[G]^*.$$

Formally speaking it is rather a *multiplier Hopf algebra*, [Van94]. Nevertheless, $\mathbb{C}[G]^*$ is a quite concrete object and in particular we have the $*$ -isomorphism

$$\mathbb{C}[G]^* \cong \prod_{U \in \text{Irr } G} B(H_U), \quad \omega \mapsto \{(\iota \otimes \omega)(U)\}_U,$$

where $\text{Irr } G$ is a maximal family of pairwise inequivalent irreducible unitary representations of G . Its $*$ -subalgebra of *compactly supported functions* is by definition

$$c_c(\hat{G}) = \{h(\cdot a) \mid a \in \mathbb{C}[G]\} \cong \bigoplus_U B(H_U),$$

where h is the Haar state on G . Very often, however, we need to work with the operator algebras

$$c_o(\hat{G}) = c_0 - \bigoplus_U B(H_U), \quad \ell^\infty(\hat{G}) = \ell^\infty - \bigoplus_U B(H_U).$$

These algebras turn \hat{G} into a *locally compact quantum group*, which is a class of quantum groups also covering compact quantum groups, [KV00]. In this thesis the only non-compact and non-discrete locally compact quantum group we will encounter is the *Drinfeld double* $D(G)$. However, we will very rarely need to work with $D(G)$ directly, as its actions can be described in terms of G . These actions are considered in section 1.1.4 of this introduction, and the connection to $D(G)$ is briefly considered in section I.1.3 of paper I.

1.1.3 Categorical duality

Let G be a compact quantum group, and $\text{Rep } G$ be its category of finite dimensional unitary representations. Let also Hilb_f denote the category of

finite dimensional Hilbert spaces. These are both examples of *rigid C^* -tensor categories*. Roughly speaking, this means that we can form tensor products, the morphism spaces are Banach spaces, there is a $*$ -operation on morphisms and every object has a dual (rigidity). There is a canonical forgetful functor

$$F_G : \text{Rep } G \rightarrow \text{Hilb}_f, \quad F_G(U) = H_U,$$

respecting the tensor structure (it is *monoidal*). The following theorem is usually called *Woronowicz's Tannaka–Krein duality*.

Theorem 1.1.7 ([Wor88]). *Let \mathcal{C} be a rigid C^* -tensor category with a unitary fibre functor $F : \mathcal{C} \rightarrow \text{Hilb}_f$. Then there is a compact quantum group G and a unitary monoidal equivalence $E : \mathcal{C} \rightarrow \text{Rep } G$ such that $F_G \circ E$ is unitarily monoidally isomorphic to F .*

We remark that it is not enough for two compact quantum groups to have monoidally equivalent representation categories for them to be isomorphic. Quantum groups G_1 and G_2 with an equivalence $\text{Rep } G_1 \cong \text{Rep } G_2$ as C^* -tensor categories are called *monoidally equivalent*.

Concretely $\mathbb{C}[G]$ is recovered from F_G through the fact we have $*$ -algebra isomorphisms

$$\text{End}(F_G) \cong \prod_{U \in \text{Irr } G} B(H_U) \cong \mathbb{C}[G]^*,$$

where $\text{End}(F_G)$ denotes the $*$ -algebra of natural transformations $F_G \rightarrow F_G$. This extends to an isomorphism of multiplier Hopf $*$ -algebras, where the structure on $\text{End}(F_G)$ is based on the tensor structure and rigidity of $\text{Rep } G$.

Let us briefly mention how this is connected to the Tannaka–Krein duality in the beginning of this introduction. When G is a compact group we can recover G as the group of monoidal unitary natural transformations $\text{Aut}^{\otimes}(F_G) \subset \text{End}(F_G)$. Indeed, the isomorphism $\text{End}(F_G) \cong \mathbb{C}[G]^*$ restricts to an isomorphism

$$\text{Aut}^{\otimes}(F_G) \cong \mathbb{G}^{\circ}(\mathbb{C}[G]) \cong G.$$

We can also realize G directly as the closed subgroup

$$\{(\pi(s))_{\pi} \mid s \in G\} \subset \prod_{\pi \in \text{Irr } G} U(H_{\pi}),$$

where $U(H_{\pi}) \subset B(H_{\pi})$ denotes the group of unitary operators on H_{π} .

1.1.4 Quantum group actions

Let G be a compact quantum group. A right action by G on a C^* -algebra B is a $*$ -homomorphism $\alpha : B \rightarrow B \otimes C(G)$ such that $(\iota \otimes \Delta)\alpha = (\alpha \otimes \iota)\alpha$ and

$$\overline{\text{span}\{(1 \otimes x)\alpha(b) \mid x \in C(G), b \in B\}}^{\|\cdot\|} = B \otimes C(G).$$

We refer to B as a (right) G - C^* -algebra, and left actions are defined analogously. It is not too hard to see that this definition, which is due to Podleś [Pod95],

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covers actions by genuine compact groups, see e.g. [DC17]. The fixed point algebra B^G is by definition

$$B^G = \{b \in B \mid \alpha(b) = b \otimes 1\},$$

and we have the canonical conditional expectation

$$E : (\iota \otimes h)\alpha : B \rightarrow B^G \tag{1.3}$$

where h denotes the Haar state on G . As opposed to the case of compact groups E is not always faithful, but when it is we call the action *reduced*.

Example 1.1.8. Let G be a compact quantum group and H a closed quantum subgroup with corresponding map $\theta : C(G) \rightarrow C(H)$. Then H acts on G by

$$\alpha = (\iota \otimes \theta)\Delta : C(G) \rightarrow C(G) \otimes C(H).$$

We write $C(G/H) = C(G)^H$. It is readily verified that the coproduct Δ on $C(G)$ satisfies

$$\Delta(C(G/H)) \subset C(G) \otimes C(G/H)$$

so G acts on $C(G/H)$ from the left. The quantum space G/H is called a quantum homogeneous space. Let us also mention that in this case $C(G/H)^G = \mathbb{C}1$, which is the quantum group analogue of transitivity. In general we refer to actions with trivial fixed point algebras as *ergodic*. \diamond

An extremely useful class of G - C^* -algebras are the *linking algebras*, because they are closely connected to monoidal equivalence. A linking algebra for two compact quantum groups G_1 and G_2 is a unital C^* -algebra $B(G_1, G_2)$ together with commuting *free* (see e.g. III) ergodic actions

$$C(G_1) \otimes B(G_1, G_2) \leftarrow B(G_1, G_2) \rightarrow B(G_1, G_2) \otimes C(G_2).$$

The existence of a linking algebra is equivalent to G_1 and G_2 being monoidally equivalent. This was shown by Bichon, De Rijdt and Vaes [BDV06], building on the algebraic counterparts of Hopf bi-Galois extensions [Ul87; Sch96].

The proof of the above result relies on the fact that any G - C^* -algebra B can be decomposed into the so-called *spectral subspaces*. To any finite dimensional unitary representation U of G , the associated spectral subspace K_U is

$$K_U := \{T\xi \mid \xi \in H_U \text{ and } T : H_U \rightarrow B \text{ is equivariant}\} \subset B,$$

and the direct sum

$$\mathcal{B} = \bigoplus_{U \in \text{Irr } G} K_U \subset B$$

is dense in B [Pod95]. The $*$ -algebra \mathcal{B} is called the *algebraic core* of B , and it is a $\mathbb{C}[G]$ -comodule $*$ -algebra under the restricted action. Spectral decomposition is useful for many purposes, and is often the main tool when studying quantum group actions. For instance it played a key role in establishing a categorical duality for actions, which we will return to in a moment.

First we will consider a special type of G - C^* -algebras. Assume that we are given a left G - C^* -algebra (B, α) which is also a left $\mathbb{C}[G]$ -module with action $\triangleright : \mathbb{C}[G] \otimes B \rightarrow B$. Then B is called a *Yetter–Drinfeld G - C^* -algebra* if

$$x \triangleright b^* = (S(x)^* \triangleright b)^*$$

and

$$\alpha(x \triangleright b) = x_{(1)} b_{(1)} S(x_{(3)}) \otimes (x_{(2)} \triangleright b_{(2)}),$$

where we use the Sweedler notation, so $\alpha(b) = b_{(1)} \otimes b_{(2)}$ and $\Delta(x) = x_{(1)} \otimes x_{(2)}$. Yetter–Drinfeld G - C^* -algebras can equivalently be seen as $D(G)$ - C^* -algebras, where $D(G)$ denotes the Drinfeld double mentioned earlier [NV10].

Example 1.1.9. There is a canonical left action on $B(H_U)$, $U \in \text{Irr } G$, given by

$$B(H_U) \ni T \mapsto U_{21}(1 \otimes T)U_{21}^*,$$

Combining these actions for all U in $\text{Irr } G$ induces a normal map

$$\ell^\infty(\hat{G}) \rightarrow L^\infty(G) \bar{\otimes} \ell^\infty(\hat{G}),$$

where $L^\infty(G) = C(G)''$. This is an example of a *von Neumann algebraic action*, which is defined similarly to C^* -algebraic actions. The norm-closure of the algebraic core of $\ell^\infty(\hat{G})$, defined in the same way as for C^* -algebras, is a G - C^* -algebra $\mathcal{R}(\ell^\infty(\hat{G}))$. On the other hand $\mathcal{R}(\ell^\infty(\hat{G}))$ is also a right $\mathbb{C}[G]$ -module under the action

$$a \triangleright x = (\iota \otimes x) \hat{\Delta}(a),$$

where $\hat{\Delta}$ denotes the coproduct on $\ell^\infty(\hat{G})$ and we use that $\mathcal{R}(\ell^\infty(\hat{G})) \subset \mathbb{C}[G]^*$. One can check that $\mathcal{R}(\ell^\infty(\hat{G}))$ is a Yetter–Drinfeld G - C^* -algebra, see e.g. [NV10]. This will be heavily used in paper I. \diamond

1.1.5 Duality for actions

As promised, we turn to some categorical considerations. Using the spectral decomposition mentioned above, Pinzari and Roberts [PR08], showed that an ergodic action can be described by the *spectral-functor*

$$\text{Rep } G \rightarrow \text{Hilb}_f, \quad U \mapsto K_U,$$

where K_U is equipped with a natural inner product induced from the unique G -invariant state given by (1.3). De Commer and Yamashita gave a description of this in terms of *module-categories* [DY13], which is similar to earlier work by Ostrik in the algebraic setting [Ost03]. Soon after, a full categorical characterization of (possibly also non-ergodic) actions was found by Neshveyev [Nes14].

Let us state one of these results somewhat explicitly. If B is a unital G - C^* -algebra, we can assign to B the singly generated² (by B) $(\text{Rep } G)$ -module category \mathcal{D}_B consisting of finitely generated G -equivariant Hilbert B -modules.

²*Singly generated* means that any object in \mathcal{D}_B is a subobject of some product $X \otimes B$.

Theorem 1.1.10 ([Nes14], [DY13]). *The assignment $B \mapsto (\mathcal{D}_B, B)$ induces a bijection between isomorphism classes of unital reduced G - C^* -algebras and unitary equivalence classes of singly generated $(\text{Rep } G)$ -module C^* -categories (\mathcal{M}, M) .*

This leads naturally to questions of which categories correspond to various types of G - C^* -algebras. One result in this direction, due to Neshveyev and Yamashita, is a characterization of *braided commutative* Yetter–Drinfeld G - C^* -algebras. In this case \mathcal{D}_B can be turned into a C^* -tensor category and the assignment $U \mapsto H_U \otimes B$ becomes a tensor functor $\text{Rep } G \rightarrow \mathcal{D}_B$. This functor is *dominant*, in the sense that the image of $\text{Rep } G$ generates \mathcal{D}_B . This describes the characterization: Any C^* -tensor category \mathcal{C} with a dominant unitary tensor functor $\text{Rep } G \rightarrow \mathcal{C}$ corresponds to a braided commutative Yetter–Drinfeld G - C^* -algebra [NY14]. We rely on this correspondence in paper I when discussing categorical boundaries.

1.2 Noncommutative boundaries

Let G be a locally compact group, and fix a probability measure μ on G which is absolutely continuous with respect to the Haar measure. We say that a bounded measurable function f on G is μ -*harmonic* if

$$f(s) = \int_G f(st) \, d\mu(t), \quad s \in G. \quad (1.4)$$

We write $H^\infty(G, \mu)$ for the set of μ -harmonic functions. In the 60's Furstenberg proved that such functions may alternatively be expressed in terms of a *Poisson formula* depending on functions on a so-called *Poisson boundary*, [Fur63; Fur73]. More precisely, the Poisson boundary is a G -measure space (Π, ν) , depending on μ , such that

$$H^\infty(G, \mu) \cong L^\infty(\Pi, \nu), \quad f \mapsto \hat{f},$$

and the corresponding Poisson formula is then

$$f(s) = \int_\Pi \hat{f}(s.x) \, d\nu(x).$$

In nice cases the Poisson boundaries are related to a certain universal compact G -space $\partial_F G$, today called the *Furstenberg boundary* of G . In particular, if G is a semisimple Lie group with finite center, then the Furstenberg boundary is a homogeneous space and coincides with the Poisson boundary corresponding to a natural choice of measure on G [Fur63].

On the other hand, in the late 70's, Hamana studied the seemingly unrelated topic of *injective envelopes* [Ham78]. He writes in a remark that the Furstenberg boundary $\partial_F \Gamma$ of a discrete group Γ is the spectrum of the Γ -injective envelope $I_\Gamma(\mathbb{C})$ of \mathbb{C} . Let us mention that this C^* -algebra can concretely be realized as an operator subspace

$$C(\partial_F \Gamma) = I_\Gamma(\mathbb{C}) \subset \ell^\infty(\Gamma).$$

This was rediscovered by Kalantar and Kennedy in [KK17], who showed, using Hamana’s theory, that properties of the Furstenberg boundary has implications for, among several things, C^* -simplicity and amenability.

Because Hamana’s theory is phrased in terms of unital C^* -algebras and unital completely positive maps between them, it lends itself nicely to the quantum setting. This observation allowed Kalantar and co-authors to extend the notion of Furstenberg boundaries to discrete quantum groups in [Kal+22].

As noted in example 1.1.9 the C^* -algebra $\mathcal{R}(\ell^\infty(\hat{G}))$ is a Yetter-Drinfeld G - C^* -algebra. In view of the results in [Kal+22], it is then a natural question whether there is a $D(G)$ -injective envelope inside $\mathcal{R}(\ell^\infty(\hat{G}))$, an object which we call a *Furstenberg–Hamana boundary* of $D(G)$. We answer this question in the affirmative in paper I, and denote the resulting C^* -algebra by $C(\partial_{FH}D(G))$.

On the other hand, the Poisson boundary $H^\infty(\hat{G}, \phi)$ of a discrete quantum group was introduced by Izumi, [Izu02]. For a fixed normal state ϕ on $\ell^\infty(\hat{G})$ the space $H^\infty(\hat{G}, \phi)$ consists of the fixed points of the *Poisson integral*

$$P_\phi : \ell^\infty(\hat{G}) \rightarrow \ell^\infty(\hat{G}), \quad P_\phi(x) = (\iota \otimes \phi)\hat{\Delta}(x),$$

which is the quantum analogue of (1.4). For coamenable quantum groups a classification of (ergodic) Poisson boundaries was established in 2007 [Tom07]: If $H^\infty(\hat{G}, \phi)^G = \mathbb{C}1$, then

$$H^\infty(\hat{G}, \phi) \cong L^\infty(G/K)$$

where K is the maximal quantum subgroup of Kac type, see [Sol05]. Using linking algebras, this result was later generalized to compact quantum groups which are monoidally equivalent to coamenable ones [DV10].

Now, the Poisson boundary of a discrete quantum group \hat{G} also inherits the structure of a Yetter–Drinfeld G -von Neumann algebra from $\ell^\infty(\hat{G})$. Thus, an important part of paper I, is a comparison between the Furstenberg–Hamana boundary of $D(G)$ and the Poisson boundary of \hat{G} . In particular we show that when $H^\infty(\hat{G}, \phi)^G = \mathbb{C}1$ we have

$$C(\partial_{FH}D(G)) \cong \mathcal{R}(H^\infty(\hat{G}, \phi)).$$

Hence, by Tomatsu’s result above, the Furstenberg–Hamana boundary for coamenable G is a quantum homogeneous space.

On the categorical side a Poisson boundary for weakly amenable C^* -tensor categories \mathcal{C} was defined by Neshveyev and Yamashita [NY17]; it is a unitary tensor functor $\Pi : \mathcal{C} \rightarrow \mathcal{P}$ with a certain universal property. When the category is $\text{Rep } G$ for a compact quantum group G it corresponds to the (*absolute*) Poisson boundary of \hat{G} under the correspondence described in section 1.1.5.

In paper I we construct a categorical Furstenberg–Hamana boundary, which similar to the Poisson boundary behaves well under the categorical duality. When the C^* -tensor category in question has a weakly amenable quantum dimension function and simple unit the two boundaries coincide.

1.3 Equivariant subproduct systems

In the 30's Fock proposed a physical model for understanding a quantum system of bosons [Foc32]. The starting point was a Hilbert space H representing the possible states of a single boson. Then, taking into account properties of bosons, a model for an n -particle state is a vector in the n 'th symmetric power $\text{Sym}^n(H)$. Fock modelled the procedure of adding (and removing) particles by introducing so-called *creation* (and *annihilation*) operators on the large Hilbert space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \text{Sym}^n(H).$$

It has later been realized that many C^* -algebras can be represented by creation operators on similar Hilbert spaces, or even Hilbert C^* -modules. A formal, and very general, framework for this is the notion of *subproduct systems* of C^* -correspondences [SS09; BM10]. The class of C^* -algebras coming from subproduct systems includes the celebrated Toeplitz and Cuntz–Pimsner algebras (associated to *product systems*) [Pim97], and hence gives rise to an abundance of interesting C^* -algebras. Below we discuss another subclass which is more relevant to us.

Following Shalit and Solel a *standard subproduct system* is a collection $\mathcal{H} = \{H_n\}_{n \in \mathbb{Z}_+}$ of Hilbert spaces such that

$$H_0 = \mathbb{C}, \quad \dim(H_1) = m < \infty, \quad H_{m+n} \subset H_m \otimes H_n.$$

The associated *Fock space* is the Hilbert space direct sum

$$\mathcal{F}(\mathcal{H}) = \bigoplus_n H_n.$$

The corresponding Toeplitz algebra $\mathcal{T}(\mathcal{H})$ is the unital C^* -subalgebra of $B(\mathcal{F}(\mathcal{H}))$ generated by the operators $S_\xi \in B(\mathcal{F}(\mathcal{H}))$, $\xi \in H_1$, given by

$$S_\xi(\zeta) = f_{n+1}(\xi \otimes \zeta), \quad \zeta \in H_n,$$

where f_n is the projection $H_1^{\otimes n} \rightarrow H_n$. As it turns out the algebra of compact operators on $\mathcal{F}(\mathcal{H})$ sits inside $\mathcal{T}(\mathcal{H})$, so we can also consider the *Cuntz–Pimsner algebra*

$$\mathcal{O}(\mathcal{H}) = \mathcal{T}(\mathcal{H})/K(\mathcal{F}(\mathcal{H})).$$

If G is a compact quantum group with representations $V_n \in B(H_n) \otimes C(G)$ such that $V_n \subset V_1^{\otimes n}$ is a subrepresentation, we say that the subproduct system is *G-equivariant*. There is then a canonical right action on $B(\mathcal{F}(\mathcal{H}))$, given by conjugation by $\bigoplus_n V_n$, which in turn induces actions on the C^* -algebras $\mathcal{T}(\mathcal{H})$ and $\mathcal{O}(\mathcal{H})$. Below is a simple example of this, inspiring paper II and III.

Example 1.3.1. Let H be a 2-dimensional Hilbert space. We consider the *symmetric subproduct system* $\mathbb{S}\mathbb{P}_2 = \{\text{Sym}^n(H)\}_n$ consisting of symmetric powers of H . Let us fix an orthonormal basis $\{\xi_1, \xi_2\}$ in H , and write

$$D = \xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1.$$

Let $U(2) = U(H)$ be the group of unitary transformations. Then, for any $u \in U(2)$

$$(u \otimes u)(D) = \det(u)(D),$$

or in other words, the projection $e : H \otimes H \rightarrow \mathbb{C}D$ is an intertwiner onto the determinant representation. Thus, because $\text{Sym}^n(H)$ can be identified with the subspace

$$H_n = \sum_{i=0}^{n-2} H^{\otimes i} \otimes (\mathbb{C}D)^\perp \otimes H^{\otimes(n-i+2)} \subset H^{\otimes n}, \quad (1.5)$$

we have recovered the well known fact that the symmetric powers are (irreducible) $U(2)$ -subrepresentations of $H^{\otimes n}$. That is, \mathbb{SSP}_2 is a $U(2)$ -equivariant subproduct system. \diamond

In the 90's, studying algebras of d -contractions, Arveson proved in [Arv98] that for any $d \geq 2$,

$$\mathcal{O}(\mathbb{SSP}_d) \cong C(S^d).$$

A key part of the argument is that $U(d)$ acts transitively on the d -sphere, or equivalently the C^* -algebra $C(S^d)$ does not have any $U(d)$ -equivariant quotients. In our terminology, he took advantage of the fact that the subproduct system was $U(d)$ -equivariant.

Recently, Arici and Kaad initiated a study of a certain family of $SU(2)$ -equivariant subproduct systems [AK21]. The subproduct system \mathbb{SSP}_2 is a member of this family, but they generalize this example in a different direction than Arveson. Their starting point is an $(m+1)$ -dimensional Hilbert space and a vector $P \in H \otimes H$ which is fixed under the representation $\pi_{m/2} \otimes \pi_{m/2}$ of $SU(2)$. The corresponding subproduct system is determined by the expression (1.5), with P instead of D . Their main result is that the Toeplitz algebras are equivalent to the complex numbers in the $SU(2)$ -equivariant KK -category.

In paper II and III, inspired by the work of Arici and Kaad, we study a larger class of subproduct systems \mathcal{H}_P , parametrized by what we call *Temperly–Lieb vectors* $P \in H \otimes H$, $\dim H \geq 2$. Any such vector possesses a quantum symmetry by a compact quantum group \tilde{O}_P^\pm defined by Mrozinski [Mro14]. We take advantage of this to determine the C^* -algebras associated to \mathcal{H}_P , and also compute equivariant KK -theory.

In the two-dimensional case \mathcal{H}_P is generated by a vector of the form

$$P = \xi_1 \otimes \xi_2 - \bar{q} \xi_2 \otimes \xi_1, \quad (1.6)$$

for $q \in \mathbb{C}^*$, in which case the relevant quantum group is $U_q(2)$ [ZZ05]. For the corresponding Cuntz–Pimsner algebra we have an isomorphism

$$\mathcal{O}(\mathcal{H}_P) \cong C(SU_q(2)),$$

where, when q is complex, $SU_q(2)$ denotes the *braided compact quantum group* constructed in [Kas+16a]. When $q = 1$ we are in the situation of example 1.3.1, recovering Arveson's result in the 2-dimensional case. In general the C^* -algebras corresponding to \mathcal{H}_P are closely connected to linking algebras.

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Let us end this section by mentioning a link to paper I. Consider the *gauge action* σ by the circle \mathbb{T} on $\mathcal{T}(\mathcal{H}_P)$. It is the action induced by the unitaries $u_z \in B(\mathcal{F}(\mathcal{H}_P))$, $z \in \mathbb{T}$, given by $u_z \xi = z^n \xi$ for $\xi \in H_n$. Then

$$C(\partial_{FH} D(\tilde{\mathcal{O}}_P^+)) \cong \mathcal{O}(\mathcal{H}_P)^\sigma$$

where the latter C^* -algebra denotes the fixed point algebra by the induced gauge action on $\mathcal{O}(\mathcal{H}_P)$. This result is not stated explicitly in any of the papers, but it is a straightforward consequence of the relation between both objects and linking algebras. See in particular example I.3.19.

1.4 Braided quantum groups

As mentioned in the previous section, the Temperley–Lieb subproduct systems are in some cases related to braided quantum groups. Below we explain how this led us to paper IV, but first we briefly explain what we mean by a braided quantum group.

Let G and H be a compact quantum groups. Suppose we are given Hopf $*$ -algebra maps

$$\pi : \mathbb{C}[G] \rightarrow \mathbb{C}[H], \quad i : \mathbb{C}[H] \rightarrow \mathbb{C}[G], \quad \pi \circ i = \text{Id}_{\mathbb{C}[H]}.$$

We say that $\mathbb{C}[G]$ is a *Hopf algebra with projection*, [Rad85]. In this case the quantum homogeneous space G/H admits a structure of a *braided compact quantum group*. That is, the $*$ -algebra $\mathbb{C}[G/H]$ has a canonical structure of a Hopf $*$ -algebra object in the category of Yetter–Drinfeld modules for $\mathbb{C}[H]$. We can think of G as a semidirect product $G = H \ltimes (G/H)$.

Now, as we saw above, for the 2-dimensional Temperley–Lieb subproduct systems the Cuntz–Pimmser algebras are isomorphic to $C(SU_q(2))$ for appropriate complex $q \neq 0$, where $SU_q(2)$ denotes a braided quantum group. These braided quantum groups are defined in [Kas+16b], and fit into the definition above because

$$U_q(2) = \mathbb{T} \ltimes SU_q(2).$$

Moreover, with the proper interpretation, we can say that the vector in equation (1.6) is stabilized by $SU_q(2)$ even for complex q , [Kas+16b]. Our motivation for the work leading to paper IV was to extend this to the higher dimensional case: For a certain subclass the Temperley–Lieb vectors are stabilized by appropriate free orthogonal quantum groups. Thus, similar to the 2-dimensional case, one can try to find braided versions of the free orthogonal quantum groups, one for each Temperley–Lieb vector P . In paper IV we show that this is indeed possible, although we do not need this in paper II or III.

Along the way we realized that the resulting braided quantum groups are related to Majid’s *transmutation* procedure [Maj93], and it became natural to define *braided matrix quantum groups* over abelian groups in greater generality. This in turn enabled us to unify many of the recent examples of braided compact quantum groups, [Ans+22; BJR22; Kas+16b; MR22; BS19].

1.5 Summary of Papers

Paper I is a collaboration with fellow PhD candidate Lucas Hataishi and the author's main supervisor Sergey Neshveyev. Papers II, III and IV are collaborations with Sergey Neshveyev.

1.5.1 Summary of paper I

This paper is concerned with the Poisson boundary and Furstenberg–Hamana boundary of the Drinfeld double $D(G)$ of a compact quantum group G . In practice we work with Yetter–Drinfeld G - C^* -algebras, while $D(G)$ itself is mainly in the background as a conceptual tool.

The first part of the paper consists of constructing the Furstenberg–Hamana boundary $\partial_{FH}D(G)$. To do this we must first find a proper notion of $D(G)$ -equivariance in a category where there are both G - C^* -algebras and G -von Neumann algebras, and prove that some of the standard tools work in this setting. For instance, we show that $\mathcal{R}(\ell^\infty(\hat{G}))$ is $D(G)$ -injective (in the sense of operator spaces) among $D(G)$ - C^* -algebras, recall example 1.1.9. This allows us to construct $C(\partial_{FH}D(G))$ as an operator subspace of $\mathcal{R}(\ell^\infty(\hat{G}))$ following the procedure by Hamana, [Ham78; KK17; Kal+22].

Next, we explain that a natural class of Poisson boundaries of $D(G)$ coincides with Poisson boundaries of \hat{G} , leading us to compare $C(\partial_{FH}D(G))$ to $H^\infty(\hat{G}, \phi)$ where ϕ is G -invariant. There is always a surjective $D(G)$ -equivariant ucp map

$$\mathcal{R}(H^\infty(\hat{G}, \phi)) \rightarrow C(\partial_{FH}D(G)),$$

which is an isomorphism exactly when the Poisson boundary is a $D(G)$ -boundary. Then, by slightly extending the results in [NY17], we show that an *absolute* Poisson boundary $\partial_\Pi \hat{G}$ exists when G has a weakly amenable quantum dimension function. In this case we always have an isomorphism $\partial_\Pi \hat{G} \cong \partial_{FH}D(G)$. This allows us to describe the Furstenberg–Hamana boundary in many cases where the absolute Poisson boundary has been computed. (To be precise, these Poisson boundaries were computed also under the extra assumption that $\text{Irr } G$ is countable, so the results in the paper also removes this assumption.)

In the final section we use the duality for braided commutative Yetter–Drinfeld G - C^* -algebras to define Furstenberg–Hamana boundaries for C^* -tensor categories. This requires the notion of \mathcal{C} -linear transformations, which were already defined in [JP17] under a different name, and most of the work is to show that these maps indeed recover exactly the $D(G)$ -equivariant ucp maps under the categorical duality. After this we show that the categorical Furstenberg–Hamana boundary has similar properties to the quantum group counterpart. In particular it coincides with the categorical Poisson boundary when the C^* -tensor category has simple unit and weakly amenable quantum dimension function.

1.5.2 Summary of paper II

In this paper we study subproduct systems corresponding to what we call *Temperley–Lieb polynomials*.

Definition 1.5.1. A noncommutative quadratic polynomial $P = \sum_{ij=1}^m a_{ij} X_i X_j$ is called Temperley–Lieb if $A\bar{A}$ is unitary, where $A = (a_{ij})_{i,j}$.

The name comes from the fact that the projection $e : \mathbb{C}^m \otimes \mathbb{C}^m \rightarrow \mathbb{C}P$ generates representations of Temperley–Lieb algebras. This implies that the subproduct system \mathcal{H}_P corresponding to P is determined by the famous Jones–Wenzl projections [Jon83; Wen87], which in turn allows us to quite quickly find several relations in the Toeplitz algebra $\mathcal{T}(\mathcal{H}_P)$ already in section 1.

In section 2 we consider general equivariant subproduct systems, and how they behave under monoidal equivalence. More precisely, we show that if G and \tilde{G} are monoidally equivalent with linking algebra $B(G, \tilde{G})$, then the cotensor product $(-) \boxtimes B(G, \tilde{G})$ transforms a G -subproduct system $\mathcal{H} = \{H_n\}_n$ into a \tilde{G} -subproduct system $\tilde{\mathcal{H}} = \{H_n \boxtimes B(G, \tilde{G})\}_n$. Under this correspondence

$$\mathcal{T}(\mathcal{H}) \boxtimes B(G, \tilde{G}) \cong \mathcal{T}(\tilde{\mathcal{H}}), \quad \mathcal{O}(\mathcal{H})_r \boxtimes B(G, \tilde{G}) \cong \mathcal{O}(\tilde{\mathcal{H}})_r$$

where the subscript r means that we are considering the reduced actions.

For the rest of the paper a standing assumption is that $A\bar{A} = \pm 1$. The reasoning is that then

$$(U \otimes U)(P \otimes 1) = P \otimes 1$$

where U is the fundamental representation of O_A^+ . This implies that \mathcal{H}_P is O_A^+ -equivariant, see example 1.3.1.

In section 3 we use the relations found in section 1 and the fact that $C(SU_q(2))$ does not have any G -equivariant quotients to conclude that

$$\mathcal{O}(\mathcal{H}_Q) \cong C(SU_q(2)), \quad Q = X_1 X_2 - q X_2 X_1.$$

The results from section 2 implies that, for any Temperley–Lieb polynomial P ,

$$\mathcal{O}(\mathcal{H}_P) \cong C(SU_q(2)) \boxtimes B(SU_q(2), O_A^+) \cong B(SU_q(2), O_A^+),$$

but it is actually not too hard to see this directly, using the relations found in [BDV06]. From this we also conclude that the relations we found in section 1 determines $\mathcal{T}(\mathcal{H}_P)$.

In section 4 we consider the gauge invariant part $\mathcal{T}(\mathcal{H}_P)^\sigma$ of $\mathcal{T}(\mathcal{H}_P)$. We show, again relying on the Jones–Wenzl formulas, that $\mathcal{T}(\mathcal{H}_P)^\sigma$ coincides with the so-called end-compactification $C(\overline{\mathbb{F}O_A})$ studied by Vaes and Vergnioux [VV07]. Here $\mathbb{F}O_A$ denotes the dual discrete quantum group of O_A^+ .

In the fifth and final section we prove that the inclusion $\mathbb{C} \rightarrow \mathcal{T}(\mathcal{H}_P)$ induces an O_A^+ -equivariant equivalence in KK -theory. Again, due to a result by Voigt [Voi11], we can use monoidal equivalence and focus on the case of $SU_q(2)$. Then, using an argument relying on the fact that the discrete dual of $SU_q(2)$ satisfies the Baum–Connes conjecture [Voi11], we are able to reduce the proof to showing that

the natural inclusion $C^*(SU_q(2)) \rightarrow \mathcal{T}(\mathcal{H}_P) \rtimes SU_q(2)$ induces an isomorphism of K_0 -groups. This is a quite concrete task, especially because we know that $\mathcal{O}(\mathcal{H}_P) = C(SU_q(2))$ from section 3.

1.5.3 Summary of paper III

This paper is a natural continuation of paper II. We prove similar results, but now for all Temperley–Lieb polynomials (we remove the assumption that $A\bar{A} = \pm 1$). To do this we must consider a class of larger quantum groups, denoted by $\tilde{\mathcal{O}}_A^+$, defined by Mrozinski [Mro14]. Similarly to in the previous paper, the subproduct system \mathcal{H}_P is $\tilde{\mathcal{O}}_A^+$ -equivariant, and $\tilde{\mathcal{O}}_A^+$ is monoidally equivalent to a fairly tractable quantum group, namely $U_q(2)$.

In the first section there is a discussion on the precise relations between algebraic bi-Galois objects and the C^* -algebraic notion of a linking algebra, building on an unpublished note by Bichon [Bic99]. The main take away from this section is that if a bi-Galois object has an invariant $*$ -structure, then there is always a (possibly different) $*$ -structure on it admitting a C^* -completion. We believe this discussion is interesting in itself, but it also allows us to conclude that the (algebraic) bi-Galois object $\mathcal{B}(U_q(2), \tilde{\mathcal{O}}_A^+)$, provided to us by Mrozinski, admits a C^* -completion turning it into a linking algebra.

In the second section we prove that there is an isomorphism

$$\mathcal{O}(\mathcal{H}_P) \rtimes \mathbb{Z} \cong B(U_q(2), \tilde{\mathcal{O}}_A^+)$$

where the crossed product is induced from the unitary $A\bar{A} \in B(H_1)$. Again, this allows us to conclude that the relations found in section 1 of the previous paper determine $\mathcal{T}(\mathcal{H}_P)$.

In the third section we prove that the inclusion $\mathbb{C} \rightarrow \mathcal{T}(\mathcal{H}_P)$ induces an $\tilde{\mathcal{O}}_A^+$ -equivariant equivalence in KK -theory. The procedure and proof is similar to that in the previous paper. However, to use similar arguments, we must first upgrade the Baum–Connes conjecture for the dual of $SU_q(2)$ to a statement about $U_q(2)$. We do this, in an appendix, by adapting Voigt’s arguments in [Voi11]. The rest of the proof is similar to the proof in paper II, but slightly more involved.

In the final section we compute the K -groups of $\mathcal{O}(\mathcal{H}_P)$. This is done by adapting an argument by Arici and Kaad, in [AK21]. The main part of the argument consists of finding a concrete inverse to the KK -equivalence induced by the inclusion $\mathbb{C} \rightarrow \mathcal{T}(\mathcal{H}_P)$. Again, a key role is played by the Jones–Wenzl formulas.

1.5.4 Summary of paper IV

In this paper we apply Majid’s transmutation theory [Maj93] to Hopf $*$ -algebra maps $\pi : H \rightarrow \mathbb{C}[T]$ where T is a compact abelian group; for each bicharacter β on \hat{T} there is a canonical Hopf algebra object H_β in the category $\mathcal{M}^{\mathbb{C}[T]}$ of $\mathbb{C}[T]$ -comodules equipped with the braiding β .

One of the main results is that for any H -cocentral subgroup $T_0 \subset T$, the bosonisation $\mathbb{C}[T/T_0] \# H_\beta$ is isomorphic to a 2-cocycle twist of $\mathbb{C}[T/T_0] \rtimes H$. Here the latter Hopf $*$ -algebra is $\mathbb{C}[T/T_0] \otimes H$ as an algebra, but equipped with the smash coproduct.

We proceed to discuss applications to compact quantum groups, and in particular define *braided matrix quantum groups* over triples (T, β, Z) where T is an abelian group, β is a bicharacter on \hat{T} and Z is a unitary corepresentation matrix for T . The triple indicates that the corresponding braided Hopf $*$ -algebra lives in $(\mathcal{M}^{\mathbb{C}[T]}, \beta)$ and that its *fundamental representation* has underlying T -representation Z .

The final part of the paper consists of computing examples. In particular we show that for any matrix $A \in \mathrm{GL}_n(\mathbb{C})$ such that $A\bar{A}$ is unitary, there is a corresponding braided free orthogonal quantum group. This is connected to paper II and III because it means that for any Temperley–Lieb polynomial there is a braided quantum group with a representation stabilizing the polynomial.

We show that a number of braided quantum groups already defined in the literature fit into this framework [Ans+22; BJR22; Kas+16b; MR22; BS19]. They are obtained from compact matrix quantum groups by transmuting and (possibly) passing to quotients by cocentral subgroups.

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Papers

Noncommutative Poisson boundaries and Furstenberg–Hamana boundaries of Drinfeld doubles

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Abstract

We clarify the relation between noncommutative Poisson boundaries and Furstenberg–Hamana boundaries of quantum groups. Specifically, given a compact quantum group G , we show that in many cases where the Poisson boundary of the dual discrete quantum group \hat{G} has been computed, the underlying topological boundary either coincides with the Furstenberg–Hamana boundary of the Drinfeld double $D(G)$ of G or is a quotient of it. This includes the q -deformations of compact Lie groups, free orthogonal and free unitary quantum groups, quantum automorphism groups of finite dimensional C^* -algebras. In particular, the boundary of $D(G_q)$ for the q -deformation of a compact connected semisimple Lie group G is G_q/T (for $q \neq 1$), in agreement with the classical results of Furstenberg and Moore on the Furstenberg boundary of $G_{\mathbb{C}}$.

We show also that the construction of the Furstenberg–Hamana boundary of $D(G)$ respects monoidal equivalence and, in fact, can be carried out entirely at the level of the representation category of G . This leads to a notion of the Furstenberg–Hamana boundary of a rigid C^* -tensor category.

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Introduction

In his celebrated work on the Poisson formula for semisimple Lie groups [Fur63] Furstenberg attached two boundaries to every locally compact group G , which are now called the Poisson and Furstenberg boundaries of G . For real semisimple Lie groups with finite centers he showed that the two boundaries coincide and give rise to an integral representation of the bounded harmonic functions on G . As has been observed since then both constructions can be phrased in operator algebraic terms, at least when G is discrete, paving the way to their generalizations to the noncommutative setting.

For the Furstenberg boundary $\partial_F G$, it was noticed by Hamana [Ham78] that $C(\partial_F G)$ coincides with the injective envelope of the G -algebra \mathbb{C} . In fact, Hamana’s motivation was quite different from that of Furstenberg and the connection between the two works was mentioned only in passing. This connection had not attracted any attention until a few years ago, when it was rediscovered by Kalantar and Kennedy in their work on C^* -simple groups [KK17]. Since then Hamana’s construction of injective envelopes has been used to develop a Furstenberg boundary-type theory in several different contexts - for unitary representations of discrete groups [BK19], étale groupoids [Bor19], discrete quantum groups [Kal+22].

On the other hand, a construction of the Poisson boundary in the operator algebraic framework was given by Izumi [Izu02]. He initiated the study of Poisson boundaries of discrete quantum groups and computed the boundary of the dual of $SU_q(2)$. The answer - the standard Podleś quantum sphere S_q^2 - turned out to be a quantization of the Poisson boundary of the complexification $SL_2(\mathbb{C})$ of $SU(2)$. The same phenomenon was then demonstrated for $SU_q(n)$ [INT06], at which point it became clear that this is not a coincidence and requires an explanation [Nes05]. A computation of the Poisson boundaries of the duals of

all q -deformed compact semisimple Lie groups was done soon afterwards by Tomatsu [Tom07].

A satisfactory conceptual explanation of the above phenomenon is actually not difficult to find. Given a compact semisimple Lie group G , the Poisson boundary of \hat{G}_q carries an action of the Drinfeld double $D(G_q)$. It can then be shown (see Proposition I.3.1) that the Poisson boundary of $D(G_q)$ is isomorphic, as a noncommutative $D(G_q)$ -space, to that of \hat{G}_q . Since $D(G_q)$ can be viewed as a quantization of $G_{\mathbb{C}}$, it is therefore not so surprising that the Poisson boundary of \hat{G}_q is a quantization of that of $G_{\mathbb{C}}$. In hindsight, even several steps in the two computations are related. For example, an important property both in the classical [Fur63] and quantum [Hay00] (see also [INT06]) cases is that the Poisson boundary is a homogeneous space of G and G_q , resp.

It is then natural to ask what the Furstenberg boundary of $D(G_q)$, and possibly of some other quantum groups, is. In this paper we answer this question using already known properties of noncommutative Poisson boundaries. It is curious that this direction - from Poisson to Furstenberg boundaries - is opposite to the one in Furstenberg's work, but we leave a detailed comparison of the two approaches to another occasion.

In more detail, the contents of the paper is as follows. After a short preliminary Section I.1, in Section I.2 we define the Furstenberg–Hamana boundary $C(\partial_{\text{FH}}D(G))$ of the Drinfeld double $D(G)$ of a compact quantum group G as the $D(G)$ -injective envelope of \mathbb{C} . The existence and uniqueness of this object (Theorem I.2.4) are proved similarly to Hamana's work and subsequent papers. It is worth stressing though that we make use of special properties of the Drinfeld doubles and do not attempt to work with arbitrary locally compact quantum groups (cf. [Ham11]). Our setting is essentially a G -equivariant version of the case of discrete quantum groups studied recently in [Kal+22].

In Section I.3 we show that basic properties of the Poisson and Furstenberg–Hamana boundaries quickly imply that if the action of G on the Poisson boundary of \hat{G} is ergodic, then this boundary coincides with the Furstenberg–Hamana boundary of $D(G)$. (To be more precise, a noncommutative Poisson boundary is in general only a measure-theoretic object, but under the ergodicity assumption there is no question what the underlying topological structure is.) The quantum groups G with this property are precisely the ones with countable isomorphism classes of irreducible representations and weakly amenable quantum dimension functions. The corresponding Poisson boundaries have been shown to be universal objects measuring how far these dimension functions are from amenable ones [NY17]. Using this property we define a Poisson-type boundary $\partial_{\Pi}\hat{G}$ of \hat{G} for any compact quantum group G with weakly amenable quantum dimension function (Theorem I.3.7), and then show that we still have $\partial_{\text{FH}}D(G) = \partial_{\Pi}\hat{G}$ (Theorem I.3.10). We draw some consequences of this equality and illustrate it with several examples. Going beyond the weakly amenable case, in Section I.3.3 we consider a free unitary quantum group U_F^+ and show that the topological boundary of its dual defined in [VV10] is a quotient of $\partial_{\text{FH}}D(U_F^+)$ (Theorem I.3.21).

One of the consequences of the construction of the $D(G)$ -algebra $C(\partial_{\text{FH}}D(G))$ is that it is braided-commutative (Corollary I.2.8), which by results of [NY14] implies that there is a C^* -tensor category associated with it. It is natural to expect that this category depends only on the representation category of G , similarly to the cases of noncommutative Poisson and Martin boundaries [DV10; NY14; Jor17]. In Section I.4 we show that this is indeed the case and define Furstenberg–Hamana boundaries of arbitrary rigid C^* -tensor categories with simple units. We explain, without going into too many details, how a number of results from Section I.3 generalize to this setting. But the main new technical point of Section I.4, which is of independent interest, is a categorical description of G - and $D(G)$ -equivariant completely positive, completely bounded and completely isometric maps (Propositions I.4.3 and I.4.5). Particular cases of this description have appeared in [NY17] to define categorical analogues of invariant means and in the work of Popa and Vaes [PV15], which developed one of the equivalent approaches [PV15; NY16b; GJ16] to defining a maximal C^* -norm on the fusion algebra of a rigid C^* -tensor category. Furthermore, categorical analogues of completely positive maps have been already proposed in full generality in [JP17], so the definitions we give are natural and essentially known. The crux of the matter is to show that they indeed reflect properties of equivariant maps under a Tannaka–Krein duality for quantum group actions.

I.1 Preliminaries

I.1.1 Compact quantum groups and their duals

We follow the conventions of [NT13], but we recall some of the main concepts for the reader’s convenience. Let G be a compact quantum group and $(\mathbb{C}[G], \Delta)$ be the Hopf $*$ -algebra of regular functions on G . Throughout the entire paper we will work only with the reduced form of G , so $C(G)$ denotes the closure of $\mathbb{C}[G]$ in the GNS-representation defined by the Haar state h .

A finite dimensional unitary representation of G is a unitary element U of $B(H_U) \otimes C(G)$ (or equivalently, of $B(H_U) \otimes \mathbb{C}[G]$), where H_U is a finite dimensional Hilbert space, such that

$$(\iota \otimes \Delta)(U) = U_{12}U_{13}.$$

The tensor product of two representations U and V is defined by $U_{13}V_{23}$ and denoted by $U \oplus V$ or simply by $U \otimes V$, when there is no danger of confusing it with the tensor product of operators. The C^* -tensor category of finite dimensional unitary representations of G is denoted by $\text{Rep } G$. The unit $\mathbb{1}$ of $\text{Rep } G$ is the trivial representation $\mathbb{1} \in C(G)$. We denote by $\text{Irr}(G)$ the set of equivalence classes of irreducible unitary representations of G . For every $s \in \text{Irr}(G)$ we fix a representative U_s and write H_s for H_{U_s} .

The dual space $\mathcal{U}(G) = \mathbb{C}[G]^*$ has the structure of a $*$ -algebra, defined by duality from the Hopf $*$ -algebra $(\mathbb{C}[G], \Delta)$. We also define $\mathcal{U}(G^n) = (\mathbb{C}[G]^{\otimes n})^*$. Then the dual of the product map $\mathbb{C}[G] \otimes \mathbb{C}[G] \rightarrow \mathbb{C}[G]$ is a $*$ -homomorphism

$\hat{\Delta}: \mathcal{U}(G) \rightarrow \mathcal{U}(G \times G)$. Every finite dimensional unitary representation U of G defines a $*$ -representation π_U of $\mathcal{U}(G)$ on H_U by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. The representations $\pi_s = \pi_{U_s}$, $s \in \text{Irr}(G)$, allow us to identify $\mathcal{U}(G)$ with $\prod_{s \in \text{Irr}(G)} B(H_s)$. Then the dual discrete quantum group \hat{G} , in the von Neumann algebra setting, is described by the von Neumann algebra

$$\ell^\infty(\hat{G}) = \ell^\infty\text{-}\bigoplus_{s \in \text{Irr}(G)} B(H_s) \subset \mathcal{U}(G)$$

with comultiplication $\hat{\Delta}|_{\ell^\infty(\hat{G})}$. We also define

$$c_0(\hat{G}) = c_0\text{-}\bigoplus_{s \in \text{Irr}(G)} B(H_s), \quad c_c(\hat{G}) = \bigoplus_{s \in \text{Irr}(G)} B(H_s).$$

The fundamental unitary of G is defined by

$$W = \bigoplus_{s \in \text{Irr}(G)} (U_s)_{21} \in M(C(G) \otimes c_0(\hat{G})).$$

We have the following identities, which reflect the duality between G and \hat{G} and are equivalent to the pentagon relation for W in the regular representations of G and \hat{G} :

$$(\Delta \otimes \iota)(W) = W_{13}W_{23}, \quad (\iota \otimes \hat{\Delta})(W) = W_{12}W_{13}. \quad (\text{I.1})$$

The Woronowicz character $f_1 \in \mathcal{U}(G)$ is denoted by ρ . Then, given a finite dimensional unitary representation U of G , the conjugate unitary representation is defined by

$$\bar{U} = (j(\rho_U)^{1/2} \otimes 1)(j \otimes \iota)(U^*)(j(\rho_U)^{-1/2} \otimes 1) \in B(\bar{H}_U) \otimes \mathbb{C}[G],$$

where $\rho_U = \pi_U(\rho)$ and j denotes the canonical $*$ -anti-isomorphism $B(H_U) \cong B(\bar{H}_U)$ defined by $j(T)\bar{\xi} = \overline{T^* \xi}$. We have morphisms $R_U: \mathbb{1} \rightarrow \bar{U} \otimes U$ and $\bar{R}_U: \mathbb{1} \rightarrow U \otimes \bar{U}$ defined by

$$R_U(1) = \sum_i \bar{\xi}_i \otimes \rho_U^{-1/2} \xi_i \quad \text{and} \quad \bar{R}_U(1) = \sum_i \rho_U^{1/2} \xi_i \otimes \bar{\xi}_i, \quad (\text{I.2})$$

where $\{\xi_i\}_i$ is any orthonormal basis in H_U . They solve the conjugate equations for U and \bar{U} , meaning that

$$(R_U^* \otimes \iota)(\iota \otimes \bar{R}_U) = \iota_{\bar{U}} \quad \text{and} \quad (\bar{R}_U^* \otimes \iota)(\iota \otimes R_U) = \iota_U.$$

The quantum dimension of U is

$$\dim_q U = \text{Tr}(\rho_U^{\pm 1}) = \|R_U\|^2 = \|\bar{R}_U\|^2.$$

I.1.2 Quantum group actions

A (continuous) left action of a compact quantum group G on a C^* -algebra A is an injective nondegenerate $*$ -homomorphism $\alpha: A \rightarrow C(G) \otimes A$ such that $(\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$ and such that the space $(C(G) \otimes 1)\alpha(A)$ is norm dense in $C(G) \otimes A$ (the *Podleś condition*). Given such an action, we also say that A is a G - C^* -algebra.

An element $a \in A$ is called *regular*, if $\alpha(a)$ lies in the algebraic tensor product $\mathbb{C}[G] \otimes_{\text{alg}} A$. The injectivity of α implies that $(\varepsilon \otimes \iota)\alpha(a) = a$ for regular a , where ε is the counit of $(\mathbb{C}[G], \Delta)$. It follows that the set of regular elements forms a $*$ -subalgebra $\mathcal{A} \subset A$ and α defines a coaction of $(\mathbb{C}[G], \Delta)$ on \mathcal{A} . We have a right $c_c(\hat{G})$ -module structure on A defined by

$$\blacktriangleleft: A \otimes c_c(\hat{G}) \rightarrow A, \quad a \blacktriangleleft \omega = (\omega \otimes \iota)\alpha(a).$$

Here we use that every element $\omega \in c_c(\hat{G}) \subset \mathbb{C}[G]^*$ has the form $\omega = h(\cdot x)$ for some $x \in \mathbb{C}[G]$ and therefore extends to a bounded linear functional on $C(G)$, and even to a normal linear functional on $L^\infty(G) = \pi_h(C(G))''$. We have $\mathcal{A} = A \blacktriangleleft c_c(\hat{G})$. The Podleś condition implies then that \mathcal{A} is dense in A . The converse is also true, see, e.g., [NT04, Corollary 1.4].

A left action of G on a von Neumann algebra N is an injective normal unital $*$ -homomorphism $\alpha: N \rightarrow L^\infty(G) \otimes N$ such that $(\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$. We then say that N is a G -von Neumann algebra. We can define in the same way as above the subalgebra $\mathcal{N} \subset N$ of regular elements. As a consequence of the Takesaki duality (see [Vae01]), the algebra \mathcal{N} is dense in N in the ultrastrong operator topology. Denote by $\mathcal{R}(N)$ the norm closure of \mathcal{N} in N . Then the restriction of α to $\mathcal{R}(N)$ defines an action of G on $\mathcal{R}(N)$. Thus, we get a functor

$$\mathcal{R}: (G\text{-von Neumann algebras}) \rightarrow (G\text{-}C^*\text{-algebras}).$$

Given two G - C^* -algebras A_1 and A_2 , we say that a bounded linear map $T: A_1 \rightarrow A_2$ is G -equivariant, if it is a $c_c(\hat{G})$ -module map. We use the same definition for G -von Neumann algebras. Furthermore, we can and will consider the situations when one of the algebras is a G - C^* -algebra and the other is a G -von Neumann algebra, the equivariance understood this way still makes sense.

A right action of the dual discrete quantum group \hat{G} on a C^* -algebra A is an injective nondegenerate $*$ -homomorphism $\beta: A \rightarrow M(A \otimes c_0(\hat{G}))$ such that $(\iota \otimes \hat{\Delta})\beta = (\beta \otimes \iota)\beta$. We then say that A is a \hat{G} - C^* -algebra. The Podleś condition – density of $(1 \otimes c_0(\hat{G}))\beta(A)$ in $A \otimes c_0(\hat{G})$ – is automatically satisfied in this case. Indeed, since $(\iota \otimes \hat{\varepsilon})\beta(a) = a$ for all $a \in A$ by the injectivity of β , this follows, e.g., from [NT04, Corollary 1.4].

Given a right action β of \hat{G} , by duality we get a left $\mathbb{C}[G]$ -module structure on A defined by

$$\triangleright: \mathbb{C}[G] \otimes A \rightarrow A, \quad x \triangleright a = (\iota \otimes x)\beta(a). \quad (\text{I.3})$$

Then A becomes a $\mathbb{C}[G]$ -module algebra and

$$x \triangleright a^* = (S(x)^* \triangleright a)^* \quad \text{for all } x \in \mathbb{C}[G], a \in A. \quad (\text{I.4})$$

Conversely, if a C^* -algebra A is a $\mathbb{C}[G]$ -module algebra and condition (I.4) is satisfied, then there is a $*$ -homomorphism $\beta: A \rightarrow \prod_{s \in \text{Irr}(G)} A \otimes B(H_s)$ uniquely determined by (I.3), where $\iota \otimes x$ is well-defined, as every $x \in \mathbb{C}[G] \subset \mathcal{U}(G)^*$ factors through a finite direct sum of the algebras $B(H_s)$. We then have

$$(\iota \otimes \hat{\Delta})\beta = (\beta \otimes \iota)\beta: A \rightarrow \prod_{s,t \in \text{Irr}(G)} A \otimes B(H_s) \otimes B(H_t),$$

$(\iota \otimes \hat{\varepsilon})\beta = \iota$ and $(1 \otimes c_c(\hat{G}))\beta(A) = A \otimes_{\text{alg}} c_c(\hat{G})$ (see, e.g., the proof of [NT04, Proposition 1.3]). As the image of β is automatically contained in

$$\ell^\infty\text{-} \bigoplus_{s \in \text{Irr}(G)} A \otimes B(H_s) \subset M(A \otimes c_0(\hat{G})),$$

it follows that β defines an action of \hat{G} on A . Therefore the actions of discrete quantum groups admit a purely algebraic description.

A \hat{G} -von Neumann algebra is a \hat{G} - C^* -algebra N such that N is a von Neumann algebra and the action map $N \rightarrow M(N \otimes c_0(\hat{G})) = N \bar{\otimes} \ell^\infty(\hat{G})$ is normal. The last condition is equivalent to saying that $\mathbb{C}[G]$ acts on N by normal operators; it doesn't seem to be known whether this condition is really necessary.

Given two \hat{G} - C^* -algebras A_1 and A_2 , we say that a linear map $A_1 \rightarrow A_2$ is \hat{G} -equivariant if it is a $\mathbb{C}[G]$ -module map.

I.1.3 Yetter–Drinfeld algebras

Assume G is a compact quantum group, A is a C^* -algebra and we have a left action $\alpha: A \rightarrow M(C(G) \otimes A)$ of G and a right action $\beta: A \rightarrow M(A \otimes c_0(\hat{G}))$ of the dual discrete quantum group \hat{G} . The Yetter–Drinfeld condition is commutativity of the following diagram:

$$\begin{array}{ccc} M(C(G) \otimes A) & \xleftarrow{\alpha} & A & \xrightarrow{\beta} & M(A \otimes c_0(\hat{G})) \\ \iota \otimes \beta \downarrow & & & & \downarrow \alpha \otimes \iota \\ M(C(G) \otimes A \otimes c_0(\hat{G})) & \xrightarrow{\text{Ad } W_{13}} & & & M(C(G) \otimes A \otimes c_0(\hat{G})) \end{array}$$

If it is satisfied, we say that A is a Yetter–Drinfeld G - C^* -algebra. (We could equally well say that it is a Yetter–Drinfeld \hat{G} - C^* -algebra.) In a similar way we can define Yetter–Drinfeld G -von Neumann algebras.

Let $\mathcal{A} \subset A$ be the subalgebra of regular elements (with respect to the G -action). Then the Yetter–Drinfeld condition is equivalent to

$$\alpha(x \triangleright a) = x_{(1)} a_{(1)} S(x_{(3)}) \otimes (x_{(2)} \triangleright a_{(2)}) \tag{I.5}$$

for all $x \in \mathbb{C}[G]$ and $a \in \mathcal{A}$, where we use Sweedler's sumless notation, so we write $\Delta(x) = x_{(1)} \otimes x_{(2)}$ and $\alpha(a) = a_{(1)} \otimes a_{(2)}$. This implies that if A is a Yetter–Drinfeld G - C^* - or G -von Neumann algebra, then $\mathcal{A} \subset A$ is a $\mathbb{C}[G]$ -submodule.

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As a consequence, if N is a Yetter–Drinfeld G -von Neumann algebra, then $\mathcal{R}(N)$ is a Yetter–Drinfeld G - C^* -algebra.

The most important examples of Yetter–Drinfeld G - C^* -algebras for us are $C(G)$, with the G -action defined by Δ and the \hat{G} -action defined by

$$C(G) \ni x \mapsto W(x \otimes 1)W^*,$$

and $c_0(\hat{G})$, with the \hat{G} -action given by $\hat{\Delta}$ and the G -action defined by

$$c_0(\hat{G}) \ni \omega \mapsto W^*(1 \otimes \omega)W.$$

In the same way we get the Yetter–Drinfeld G -von Neumann algebras $L^\infty(G)$ and $\ell^\infty(\hat{G})$.

To have the structure of a Yetter–Drinfeld G -algebra on a C^* -algebra A is the same as having a left action of the Drinfeld double $D(G)$ of G on A [NV10, Proposition 3.2]. Let us briefly recall this correspondence; it will be used only in Section I.3.1, as most of the time it is more convenient to use the Yetter–Drinfeld condition directly.

The reduced C^* -algebra of continuous functions on $D(G)$ is

$$C(D(G)) = C(G) \otimes c_0(\hat{G})$$

while the coproduct on it is defined by

$$\Delta_{D(G)}(a \otimes \omega) = W_{32}\Delta(a)_{13}\hat{\Delta}(\omega)_{42}W_{32}^*. \quad (\text{I.6})$$

To recognize that this is exactly the same definition as in [NV10], note that the dual of $(C(G), \Delta)$ within the general theory of locally compact quantum groups is $(c_0(\hat{G}), \hat{\Delta}^{\text{op}})$ rather than $(c_0(\hat{G}), \hat{\Delta})$.

Now, for any Yetter–Drinfeld G - C^* -algebra A , the corresponding left action

$$\gamma: A \rightarrow M(C(D(G)) \otimes A)$$

of $D(G)$ is defined by $\gamma = ((\iota \otimes \beta)\alpha(\cdot))_{132}$. For example, this way $\Delta_{D(G)}$ corresponds to

$$\alpha = \Delta \otimes \iota, \quad \beta = W_{13}(\iota \otimes \hat{\Delta})(\cdot)W_{13}^*, \quad (\text{I.7})$$

which are the actions of the quantum subgroups G and \hat{G}^{op} of $D(G)$ by left translations.

Given two Yetter–Drinfeld G - C^* - or G -von Neumann algebras A_1 and A_2 , we say that a bounded linear map $A_1 \rightarrow A_2$ is $D(G)$ -equivariant if it is both G - and \hat{G} -equivariant.

I.2 Furstenberg–Hamana boundaries of Drinfeld doubles

I.2.1 Minimal idempotents in convex contraction semigroups

As discussed in the introduction, Hamana’s construction of an injective envelope has been adapted to several different contexts. We ourselves will need two versions of it. The following proposition is an attempt to capture the essence of Hamana’s arguments in one general statement.

Proposition I.2.1. *Assume X is a subspace of a dual Banach space Y^* and S is a convex semigroup of contractive linear maps $X \rightarrow X$ such that if we consider S as a set of maps $X \rightarrow Y^*$, then S is closed in the topology of pointwise weak* convergence. Then there is an idempotent $\phi_0 \in S$ such that*

$$\phi_0\psi\phi_0 = \phi_0 \quad \text{for all } \psi \in S. \tag{I.8}$$

Proof. Define a pre-order on S by

$$\phi \prec \psi \quad \text{iff} \quad \|\phi(x)\| \leq \|\psi(x)\| \quad \text{for all } x \in X.$$

Since by assumption S is compact in the topology of pointwise weak* convergence, every decreasing chain in S with respect to this pre-order has a lower bound. By Zorn’s lemma we conclude that S has a minimal element ϕ_0 .

Now, assume $\psi \in S$ is such that $\psi(\phi_0(X)) \subset \phi_0(X)$. We claim that $\psi|_{\phi_0(X)}$ is the identity map. Consider a cluster point $\Psi \in S$ of the sequence $\{\frac{1}{n} \sum_{k=1}^n \psi^k\}_{n=1}^\infty$ in S . Then $\Psi\psi = \Psi$. As ϕ_0 is minimal and $\Psi\phi_0 \prec \phi_0$, the map $\Psi|_{\phi_0(X)}$ must be isometric. Hence, for every $x \in \phi_0(X)$, we have

$$\|\psi(x) - x\| = \|\Psi(\psi(x) - x)\| = \|\Psi\psi(x) - \Psi(x)\| = 0,$$

proving our claim.

Applying the claim to $\psi = \phi_0$, we conclude that ϕ_0 is an idempotent. Applying it to $\phi_0\psi$, we get that $\phi_0\psi\phi_0 = \phi_0$ for all $\psi \in S$. ■

We will call any such ϕ_0 a *minimal idempotent* in S .

Remark I.2.2. This agrees with minimality with respect to the standard order on the idempotents defined by

$$\phi_1 \leq \phi_2 \quad \text{iff} \quad \phi_2\phi_1 = \phi_1\phi_2 = \phi_1,$$

cf. [Pau11, Theorem 2.9]. Indeed, if an idempotent ϕ_0 satisfies (I.8), then it is immediate that it is minimal with respect to \leq . Conversely, assume ϕ_1 is a minimal idempotent with respect to \leq . Let ϕ_0 be an idempotent satisfying (I.8). Replacing ϕ_0 by $\phi_0\phi_1$, we may assume that $\phi_0\phi_1 = \phi_0$. Then the idempotent $\phi_1\phi_0$ satisfies $\phi_1\phi_0 \leq \phi_1$, hence $\phi_1\phi_0 = \phi_1$. Then, for every $\psi \in S$, we have

$$\phi_1\psi\phi_1 = \phi_1\phi_0\psi\phi_1\phi_0 = \phi_1\phi_0 = \phi_1.$$

For general compact semigroups S , however, minimality with respect to \leq does not imply (I.8), with any nontrivial compact group giving an example.

I.2.2 Furstenberg–Hamana boundary

Let G be a compact quantum group. Recall that a unital G - C^* -algebra A is called G -injective if, given unital G - C^* -algebras B and C , a completely isometric

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G -equivariant ucp map $\phi: B \rightarrow C$ and a G -equivariant ucp map $\psi: B \rightarrow A$, there is a G -equivariant ucp map $\tilde{\psi}: C \rightarrow A$ making the diagram

$$\begin{array}{ccc} & C & \\ \phi \uparrow & \swarrow \tilde{\psi} & \\ B & \xrightarrow{\psi} & A \end{array}$$

commutative. In a similar way one defines $D(G)$ -injectivity, that is, injectivity for Yetter–Drinfeld G - C^* -algebras.

Definition I.2.3. We say that a unital Yetter–Drinfeld G - C^* -algebra A is a $D(G)$ -boundary, or that the action of $D(G)$ on A is a *boundary action*, if for every unital Yetter–Drinfeld G - C^* -algebra B and every $D(G)$ -equivariant ucp map $\psi: A \rightarrow B$, the map ψ is automatically completely isometric. A $D(G)$ -boundary A is called a *Furstenberg–Hamana boundary* of $D(G)$, if it is in addition a $D(G)$ -injective C^* -algebra.

The term *boundary action* is suggested in [Kal+22]. Adapting Hamana’s terminology, a $D(G)$ -boundary is also called a $D(G)$ -essential extension of the trivial Yetter–Drinfeld G - C^* -algebra \mathbb{C} , and a Furstenberg–Hamana boundary is called a $D(G)$ -injective envelope of \mathbb{C} .

Theorem I.2.4. *For any compact quantum group G , a Furstenberg–Hamana boundary of $D(G)$ exists and is unique up to isomorphism.*

We denote the Furstenberg–Hamana boundary of $D(G)$ by $C(\partial_{\text{FH}}D(G))$.

The proof of the theorem follows by now standard lines [Ham78; Ham79; Pau11; Kal+22], we mainly have to take care of G -equivariance.

We will need the following construction. Given a \hat{G} - C^* -algebra A , with the action of \hat{G} given by $\beta: A \rightarrow M(A \otimes c_0(\hat{G}))$, and a state ϕ on A , we have a \hat{G} -equivariant cp map

$$\mathcal{P}_\phi = (\phi \otimes \iota)\beta: A \rightarrow \ell^\infty(\hat{G}),$$

called a *Poisson integral*. Such maps are often also called Izumi’s Poisson integrals, since their usage in noncommutative probability was pioneered by Izumi in [Izu02]. It has been observed in a number of cases that if we also have an action of G on A , then the maps \mathcal{P}_ϕ are G -equivariant as well [Izu02, Lemmas 2.2(3), 3.8(2)]. Let us prove a general result of this sort.

Assume N is a G -von Neumann algebra, with the action of G given by $\alpha: N \rightarrow L^\infty(G) \bar{\otimes} N$. Consider the von Neumann algebra $N \bar{\otimes} \ell^\infty(\hat{G})$. We have a right action of \hat{G} on it given by $\iota \otimes \hat{\Delta}$. Define a left action of G by

$$\alpha_W: N \bar{\otimes} \ell^\infty(\hat{G}) \rightarrow L^\infty(G) \bar{\otimes} N \bar{\otimes} \ell^\infty(\hat{G}), \quad \alpha_W(x) = W_{13}^*(\alpha \otimes \iota)(x)W_{13}.$$

Using (I.1) it is easy to check that this way $N \bar{\otimes} \ell^\infty(\hat{G})$ becomes a Yetter–Drinfeld G -von Neumann algebra.

Now, given a \hat{G} - C^* -algebra A as above and a completely bounded (cb) linear map $\phi: A \rightarrow N$, define

$$\mathcal{P}_\phi: A \rightarrow N \bar{\otimes} \ell^\infty(\hat{G}) \quad \text{by} \quad \mathcal{P}_\phi(a) = (\phi \otimes \iota)\beta(a).$$

To make sense of this definition, note that

$$\beta(A) \subset \ell^\infty\text{-} \bigoplus_{s \in \text{Irr}(G)} A \otimes B(H_s)$$

and

$$N \bar{\otimes} \ell^\infty(\hat{G}) = \ell^\infty\text{-} \bigoplus_{s \in \text{Irr}(G)} N \otimes B(H_s),$$

so by $\phi \otimes \iota$ we simply mean a collection of maps $A \otimes B(H_s) \rightarrow N \otimes B(H_s)$.

Proposition I.2.5. *For any Yetter–Drinfeld G - C^* -algebra A and G -von Neumann algebra N , the map $\phi \mapsto \mathcal{P}_\phi$ defines a one-to-one correspondence between the G -equivariant cb maps $\phi: A \rightarrow N$ and the $D(G)$ -equivariant cb maps $\mathcal{P}: A \rightarrow N \bar{\otimes} \ell^\infty(\hat{G})$, with the inverse given by $\mathcal{P} \mapsto (\iota \otimes \hat{\varepsilon})\mathcal{P}$.*

Obviously, the map \mathcal{P}_ϕ is ucp if and only if ϕ is ucp, so we also get a correspondence between the ucp maps.

Proof. It is easy to see that the map $\phi \mapsto \mathcal{P}_\phi$ defines a one-to-one correspondence between the cb maps $\phi: A \rightarrow N$ and the \hat{G} -equivariant cb maps $\mathcal{P}: A \rightarrow N \bar{\otimes} \ell^\infty(\hat{G})$, with the inverse given by $\mathcal{P} \mapsto (\iota \otimes \hat{\varepsilon})\mathcal{P}$. Therefore we only need to show that \mathcal{P}_ϕ is G -equivariant if and only if ϕ is G -equivariant. The “only if” direction is immediate, as the map $\iota \otimes \hat{\varepsilon}: N \bar{\otimes} \ell^\infty(\hat{G}) \rightarrow N$ is G -equivariant.

Assume now that ϕ is G -equivariant. Denote by α_A the G -action on A and by α the G -action on N . For an element y of $\prod_s A \otimes B(H_s)$ or $\prod_s N \otimes B(H_s)$, denote by y_s the component of y in $A \otimes B(H_s)$, resp., $N \otimes B(H_s)$. Then the Yetter–Drinfeld condition for A can be written as

$$((\iota \otimes \beta)\alpha_A(a))_s = (U_s)_{31}^*(\alpha_A \otimes \iota)(\beta(a)_s)(U_s)_{31}.$$

If $a \in A$ is regular, then the above expressions live in $\mathbb{C}[G] \otimes_{\text{alg}} A \otimes B(H_s)$. This justifies the following computations for such a and all $s \in \text{Irr}(G)$:

$$\begin{aligned} ((\iota \otimes \mathcal{P}_\phi)\alpha_A(a))_s &= (\iota \otimes \phi \otimes \iota)((\iota \otimes \beta)\alpha_A(a))_s \\ &= (\iota \otimes \phi \otimes \iota)((U_s)_{31}^*(\alpha_A \otimes \iota)(\beta(a)_s)(U_s)_{31}) \\ &= (U_s)_{31}^*((\iota \otimes \phi)\alpha_A \otimes \iota)(\beta(a)_s)(U_s)_{31} \\ &= (U_s)_{31}^*(\alpha \otimes \iota)(\phi \otimes \iota)(\beta(a)_s)(U_s)_{31} \\ &= \alpha_W(\mathcal{P}_\phi(a))_s, \end{aligned}$$

where we used the Yetter–Drinfeld condition in the second equality and the equivariance of ϕ in the fourth equality. This implies G -equivariance of \mathcal{P}_ϕ . ■

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For the proof of Theorem I.2.4 we will need only the following corollary. Consider the Yetter–Drinfeld G -von Neumann algebra $\ell^\infty(\hat{G})$ as in Section I.1.3, or in other words, as defined above for $N = \mathbb{C}$. Then $\mathcal{R}(\ell^\infty(\hat{G}))$ is a Yetter–Drinfeld G - C^* -algebra.

Corollary I.2.6. *The C^* -algebra $\mathcal{R}(\ell^\infty(\hat{G}))$ is $D(G)$ -injective.*

Proof. Take a unital Yetter–Drinfeld G - C^* -algebra B . Since every G -equivariant bounded linear map $B \rightarrow \ell^\infty(\hat{G})$ has image in $\mathcal{R}(\ell^\infty(\hat{G}))$, by Proposition I.2.5 we conclude that the map $\phi \mapsto \mathcal{P}_\phi$ defines a one-to-one correspondence between the G -invariant states on B and the $D(G)$ -equivariant ucp maps $B \rightarrow \mathcal{R}(\ell^\infty(\hat{G}))$. This implies the corollary, since every G -invariant state on B can be extended to a G -invariant state on any given Yetter–Drinfeld G - C^* -algebra C containing B as an operator subsystem (see Lemma I.2.10 for a stronger result). ■

Proof of Theorem I.2.4. Consider the convex semigroup S of $D(G)$ -equivariant ucp maps

$$\mathcal{R}(\ell^\infty(\hat{G})) \rightarrow \mathcal{R}(\ell^\infty(\hat{G})).$$

We can equally well consider all $D(G)$ -equivariant ucp maps $\mathcal{R}(\ell^\infty(\hat{G})) \rightarrow \ell^\infty(\hat{G})$, since every such map has image in $\mathcal{R}(\ell^\infty(\hat{G}))$ by the G -equivariance. Since $\mathbb{C}[G]$ and $c_c(\hat{G})$ act on $\ell^\infty(\hat{G})$ by normal operators, this implies that S is closed in the topology of pointwise ultraweak operator convergence. Hence, by Proposition I.2.1, there is a minimal idempotent ϕ_0 in S . Consider the C^* -algebra $A = \phi_0(\mathcal{R}(\ell^\infty(\hat{G})))$ with the Choi–Effros product $a \cdot b = \phi_0(ab)$. We claim that A equipped with the actions of G and \hat{G} obtained by restriction from those on $\mathcal{R}(\ell^\infty(\hat{G}))$ is a Furstenberg–Hamana boundary of $D(G)$.

Step 1. A is a G - C^ -algebra.* Denote by α the G -action on $\mathcal{R}(\ell^\infty(\hat{G}))$ and by $\mathcal{N} \subset \mathcal{R}(\ell^\infty(\hat{G}))$ the subalgebra of regular elements. The G -equivariance of ϕ_0 can be written as $(\iota \otimes \phi_0)\alpha = \alpha\phi_0$. Note also that the C^* -algebra $C(G) \otimes A$ can be viewed as an operator subsystem of $C(G) \otimes \mathcal{R}(\ell^\infty(\hat{G}))$ equipped with the product $x \cdot y = (\iota \otimes \phi_0)(xy)$. Hence $\alpha_A := \alpha|_A$ is a well-defined ucp map $A \rightarrow C(G) \otimes A$ and, for all $a, b \in A$, we have

$$\alpha(a \cdot b) = \alpha(\phi_0(ab)) = (\iota \otimes \phi_0)\alpha(ab) = (\iota \otimes \phi_0)(\alpha(a)\alpha(b)) = \alpha(a) \cdot \alpha(b),$$

so that α_A is a homomorphism.

For any $a \in \mathcal{R}(\ell^\infty(\hat{G}))$ and $x \in C(G)$, we have

$$(x \otimes 1) \cdot \alpha(\phi_0(a)) = (x \otimes 1)\alpha(\phi_0(a)) = (\iota \otimes \phi_0)((x \otimes 1)\alpha(a)).$$

This implies that the Podleś condition for α_A is satisfied. This follows also from the density of the subalgebra $\mathcal{A} = A \cap \mathcal{N} = \phi_0(\mathcal{N})$ of regular elements.

Step 2. A is a \hat{G} - C^ -algebra.* By the \hat{G} -equivariance of ϕ_0 , the space A is a $\mathbb{C}[G]$ -submodule of $\mathcal{R}(\ell^\infty(\hat{G}))$. Since condition (I.4) is obviously satisfied for A , in order to define an action of \hat{G} on A we only need to check that the module structure respects the product, that is,

$$x \triangleright (a \cdot b) = (x_{(1)} \triangleright a) \cdot (x_{(2)} \triangleright b)$$

for all $x \in \mathbb{C}[G]$ and $a, b \in A$. But this follows immediately by applying ϕ_0 to the same identity for the original product on $\mathcal{R}(\ell^\infty(\hat{G}))$.

Step 3. The Yetter–Drinfeld condition is satisfied. As condition (I.5) holds for the elements of \mathcal{N} , it obviously holds for the elements of $\mathcal{A} \subset \mathcal{N}$.

Step 4. A is $D(G)$ -injective. Since $A = \phi_0(\mathcal{R}(\ell^\infty(\hat{G})))$ and ϕ_0 is an idempotent, this follows from Corollary I.2.6.

Step 5. A is a $D(G)$ -boundary. Assume $\psi: A \rightarrow B$ is a $D(G)$ -equivariant ucp map for some unital Yetter–Drinfeld G - C^* -algebra B . By the $D(G)$ -injectivity of A , there is a $D(G)$ -equivariant ucp map $\phi: B \rightarrow A$. Then $\phi\psi\phi_0$ is an element of the semigroup S . By the minimality of ϕ_0 we get that $\phi\psi\phi_0 = \phi_0\phi\psi\phi_0 = \phi_0$, that is, $\phi\psi$ is the identity map on A . Hence ψ is completely isometric.

It remains to prove the uniqueness up to isomorphism. This is a standard argument, which we recall for the reader's convenience, cf. [Ham79, Theorem 4.1]. Assume A_1 and A_2 are two Furstenberg–Hamana boundaries of $D(G)$. Then by the injectivity of A_2 there is an equivariant ucp map $\gamma: A_1 \rightarrow A_2$. It must be completely isometric, as A_1 is a boundary. Hence, by the injectivity of A_1 , there is an equivariant ucp map $\tilde{\gamma}: A_2 \rightarrow A_1$ such that $\tilde{\gamma}\gamma = \iota$. But then $\gamma\tilde{\gamma}: A_2 \rightarrow A_2$ is an idempotent, which must be completely isometric, as A_2 is a boundary. It follows that $\gamma\tilde{\gamma} = \iota$. Therefore γ is a complete order isomorphism of A_1 onto A_2 , hence it is an isomorphism of C^* -algebras. ■

The following rigidity property of the Furstenberg–Hamana boundary can be deduced from the definition (see [Ham79, Lemma 3.7]), but it is immediate from the construction.

Corollary I.2.7. *The only $D(G)$ -equivariant ucp map $C(\partial_{\text{FH}}D(G)) \rightarrow C(\partial_{\text{FH}}D(G))$ is the identity map.*

Proof. Using the notation from the proof of Theorem I.2.4, if $\psi: A \rightarrow A$ is a $D(G)$ -equivariant ucp map, then $\psi\phi_0 = \phi_0\psi\phi_0 = \phi_0$ by the minimality of ϕ_0 , that is, ψ is the identity map. ■

The proof of the theorem implies one more nonobvious property of $C(\partial_{\text{FH}}D(G))$. Recall that a Yetter–Drinfeld G - C^* -algebra A is called *braided-commutative*, if

$$ab = b_{(2)}(S^{-1}(b_{(1)}) \triangleright a) \tag{I.9}$$

for all $a, b \in \mathcal{A}$. Both $C(G)$ and $c_0(\hat{G})$ are braided-commutative.

Corollary I.2.8. *The Yetter–Drinfeld G - C^* -algebra $C(\partial_{\text{FH}}D(G))$ is braided-commutative.*

Proof. Using the notation from the proof of Theorem I.2.4, identity (I.9) holds for all $a, b \in \mathcal{A} = A \cap \mathcal{N}$ for the original product on \mathcal{N} . By applying ϕ_0 we conclude that it also holds for the Choi–Effros product. ■

I.2.3 Boundary actions

In this subsection we follow closely the last part of [Kal+22, Section 4]. Our goal is to prove the following result.

Theorem I.2.9. *Let G be a compact quantum group. Then up to isomorphism the $D(G)$ -boundaries are precisely the unital Yetter–Drinfeld G - C^* -subalgebras of $C(\partial_{\text{FH}}D(G))$. Furthermore, given two such subalgebras $A_1, A_2 \subset C(\partial_{\text{FH}}D(G))$, the embedding map $A_1 \hookrightarrow A_2$ is the only $D(G)$ -equivariant ucp map $A_1 \rightarrow A_2$ if $A_1 \subset A_2$, and there are no such maps if $A_1 \not\subset A_2$.*

Therefore there is a one-to-one correspondence between the isomorphism classes of $D(G)$ -boundaries and the unital Yetter–Drinfeld G - C^* -subalgebras of $C(\partial_{\text{FH}}D(G))$.

For the proof of the theorem we need the following equivariant version of Arveson’s extension theorem.

Lemma I.2.10. *Given a unitary representation $U \in M(K(H) \otimes C(G))$ of G on a Hilbert space H , consider the G -von Neumann algebra $B(H)$, with the G -action*

$$\alpha_U: B(H) \rightarrow L^\infty(G) \bar{\otimes} B(H) \quad \text{given by} \quad \alpha_U(T) = U_{21}^*(1 \otimes T)U_{21}.$$

Then $\mathcal{R}(B(H))$ is a G -injective C^ -algebra.*

Proof. Assume B and C are unital G - C^* -algebras, with the actions of G denoted by α_B and α_C , $\phi: B \rightarrow C$ is a completely isometric G -equivariant ucp map, and $\psi: B \rightarrow B(H)$ is a G -equivariant ucp map. By Arveson’s extension theorem, there is a ucp map $\eta: C \rightarrow B(H)$ such that $\eta\phi = \psi$. Define

$$\tilde{\psi}: C \rightarrow B(H) \quad \text{by} \quad \tilde{\psi}(c) = (h \otimes \iota)(U_{21}(\iota \otimes \eta)\alpha_C(c)U_{21}^*).$$

A simple computation shows that $\tilde{\psi}$ is G -equivariant and $\tilde{\psi}\phi = \psi$. ■

If we combine this lemma with Proposition I.2.5 for $N = B(H)$, then we can conclude that the C^* -algebra $\mathcal{R}(B(H) \bar{\otimes} \ell^\infty(\hat{G}))$ is $D(G)$ -injective. This generalizes Corollary I.2.6.

Proof of Theorem I.2.9. Let A be a $D(G)$ -boundary. By the injectivity of $C(\partial_{\text{FH}}D(G))$ we get a $D(G)$ -equivariant ucp map $\phi: A \rightarrow C(\partial_{\text{FH}}D(G))$, which then must be completely isometric. We claim that it is a homomorphism.

By considering, e.g., a representation of the crossed product $G \rtimes A$, we can find a unitary representation U of G on a Hilbert space H and a representation $\pi: A \rightarrow B(H)$ that is G -equivariant with respect to the action $T \mapsto U_{21}^*(1 \otimes T)U_{21}$ of G on $B(H)$. By Lemma I.2.10, there is a G -equivariant ucp map $\tilde{\pi}: C(\partial_{\text{FH}}D(G)) \rightarrow B(H)$ such that $\tilde{\pi}\phi = \pi$. Consider also any $D(G)$ -equivariant ucp map $\psi: \mathcal{R}(B(H) \bar{\otimes} \ell^\infty(\hat{G})) \rightarrow C(\partial_{\text{FH}}D(G))$. By Proposition I.2.5

we then get the following commutative diagram of $D(G)$ -equivariant ucp maps:

$$\begin{array}{ccccc}
 C(\partial_{\text{FH}}D(G)) & \xrightarrow{\mathcal{P}_{\bar{\pi}}} & \mathcal{R}(B(H) \bar{\otimes} \ell^\infty(\hat{G})) & \xrightarrow{\psi} & C(\partial_{\text{FH}}D(G)) \\
 \uparrow \phi & & \nearrow \mathcal{P}_\pi & & \\
 A & & & &
 \end{array}$$

The map \mathcal{P}_π is a homomorphism, since π is. By Corollary I.2.7, the composition $\psi\mathcal{P}_{\bar{\pi}}$ is the identity map, hence $E = \mathcal{P}_{\bar{\pi}}\psi$ is an idempotent with image $\mathcal{P}_{\bar{\pi}}(C(\partial_{\text{FH}}D(G)))$. It follows that $\mathcal{P}_{\bar{\pi}}$ defines a C^* -algebra isomorphism of $C(\partial_{\text{FH}}D(G))$ onto $\mathcal{P}_{\bar{\pi}}(C(\partial_{\text{FH}}D(G)))$ equipped with the Choi–Effros product $a \cdot b = E(ab)$. Since the Choi–Effros product coincides with the original product on $\mathcal{P}_\pi(A) \subset \mathcal{P}_{\bar{\pi}}(C(\partial_{\text{FH}}D(G)))$, we conclude that ϕ is indeed a homomorphism.

Next, let A be a unital Yetter–Drinfeld G - C^* -subalgebra of $C(\partial_{\text{FH}}D(G))$ and $\phi: A \rightarrow B$ be a $D(G)$ -equivariant ucp map for some unital Yetter–Drinfeld G - C^* -algebra B . Take any $D(G)$ -equivariant ucp map $\psi: B \rightarrow C(\partial_{\text{FH}}D(G))$. The map $\psi\phi: A \rightarrow C(\partial_{\text{FH}}D(G))$ extends to a $D(G)$ -equivariant ucp map $C(\partial_{\text{FH}}D(G)) \rightarrow C(\partial_{\text{FH}}D(G))$, which then must be the identity map. This shows that ϕ is completely isometric, proving that A is a $D(G)$ -boundary. The argument shows also that in the case $B = C(\partial_{\text{FH}}D(G))$ and $\psi = \iota$, the map $\phi: A \rightarrow C(\partial_{\text{FH}}D(G))$ must be the embedding map. This implies the last part of the theorem. \blacksquare

I.3 Comparison with Poisson boundaries

I.3.1 Noncommutative Poisson boundaries

Given a von Neumann algebra M , by a Markov operator on M one means a normal ucp map $P: M \rightarrow M$. The corresponding Poisson boundary [Izu02] is defined as the space

$$H^\infty(M, P) = \{x \in M \mid P(x) = x\}$$

of P -harmonic elements. This space is ultraweakly operator closed in M and it is the image of a ucp projection $M \rightarrow H^\infty(M, P)$ obtained as a cluster point of $\{\frac{1}{n} \sum_{k=1}^n P^k\}_{n=1}^\infty$. As a consequence it has a unique structure of a von Neumann algebra such that the embedding $H^\infty(M, P) \rightarrow M$ is a normal completely isometric ucp map.

Let G be a compact quantum group. For a normal state ϕ on $\ell^\infty(\hat{G})$, consider the convolution operator

$$P_\phi = (\phi \otimes \iota)\hat{\Delta}: \ell^\infty(\hat{G}) \rightarrow \ell^\infty(\hat{G}).$$

We will be mainly interested in the left G -invariant normal states ϕ . Every such state is determined by its restriction to $\ell^\infty(\hat{G})^G = Z(\ell^\infty(\hat{G})) \cong \ell^\infty(\text{Irr}(G))$, that is, by a probability measure μ on $\text{Irr}(G)$. Explicitly, the G -invariant normal

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state ϕ_μ corresponding to μ is given by $\sum_{s \in \text{Irr}(G)} \mu(s) \phi_s$, where ϕ_s is the state on $B(H_s)$ defined by

$$\phi_s = (\dim_q U_s)^{-1} \text{Tr}(\cdot \pi_s(\rho)^{-1}).$$

We will write P_s and P_μ instead of P_{ϕ_s} and P_{ϕ_μ} . We will also use the lighter notation $H^\infty(\hat{G}, \mu)$ for $H^\infty(\ell^\infty(\hat{G}), P_{\phi_\mu})$.

The Markov operators P_μ are G -equivariant by Proposition I.2.5 (or by [Izu02, Lemma 2.2]). In particular, they leave $\ell^\infty(\text{Irr}(G))$ invariant and therefore define random walks on $\text{Irr}(G)$. In fact, the states ϕ_μ are the only ones with this property: if P_ϕ , for a normal state ϕ , leaves $\ell^\infty(\text{Irr}(G))$ invariant, then $\phi = \phi_\mu$ for some μ , see [NT04, Proposition 2.1]. The operators P_μ are right \hat{G} -equivariant as well, and as a result $H^\infty(\hat{G}, \mu)$ is a Yetter–Drinfeld G -von Neumann algebra.

Random walks can also be considered on $D(G)$, but as the following result shows, for a natural class of states the corresponding Poisson boundary is described entirely in terms of \hat{G} .

For a probability measure μ on $\text{Irr}(G)$, consider the normal state $h \otimes \phi_\mu$ on $L^\infty(D(G)) = L^\infty(G) \bar{\otimes} \ell^\infty(\hat{G})$ and the Markov operator $(\iota \otimes (h \otimes \phi_\mu)) \Delta_{D(G)}$ on $L^\infty(D(G))$. We denote by $H^\infty(D(G), \mu)$ the corresponding Poisson boundary.

Proposition I.3.1. *For any compact quantum group G and any probability measure μ on $\text{Irr}(G)$, we have an isomorphism*

$$H^\infty(\hat{G}, \mu) \cong H^\infty(D(G), \mu), \quad \omega \mapsto W^*(1 \otimes \omega)W,$$

of Yetter–Drinfeld G -von Neumann algebras.

Proof. Denote by Q the Markov operator $(\iota \otimes (h \otimes \phi_\mu)) \Delta_{D(G)}$ on $L^\infty(D(G))$. By the definition (I.6) of $\Delta_{D(G)}$ we have

$$Q(a \otimes \omega) = (\iota \otimes h \otimes \iota)(W_{23}(\Delta(a) \otimes P_\mu(\omega))W_{23}^*).$$

Using first that $W_{23} = W_{13}^*(\Delta \otimes \iota)(W)$ by (I.1) and then the invariance of the Haar state h , we see that this equals

$$W^*(\iota \otimes h \otimes \iota)(\Delta \otimes \iota)(W(a \otimes P_\mu(\omega))W^*)W = W^*(h(\cdot)1 \otimes \iota)(W(a \otimes P_\mu(\omega))W^*)W.$$

It follows that

$$H^\infty(D(G), \mu) \subset \text{Im } Q \subset W^*(1 \otimes \ell^\infty(\hat{G}))W.$$

Next, consider an element of the form $x = W^*(1 \otimes \omega)W$, $\omega \in \ell^\infty(\hat{G})$. Then the above computation gives

$$Q(x) = W^*(h(\cdot)1 \otimes \iota)(W(\iota \otimes P_\mu)(W^*(1 \otimes \omega)W)W^*)W.$$

As $(\iota \otimes P_\mu)(W^*(1 \otimes \omega)W) = W^*(1 \otimes P_\mu(\omega))W$ by the G -equivariance of P_μ , we thus get

$$Q(x) = W^*(1 \otimes P_\mu(\omega))W,$$

so that x is Q -harmonic if and only if ω in P_μ -harmonic. This proves the isomorphism $H^\infty(\hat{G}, \mu) \cong H^\infty(D(G), \mu)$ stated in the proposition. The claim that this isomorphism respects the actions of G and \hat{G} is easy to check using (I.1), if we recall that the actions on $H^\infty(D(G), \mu) \subset L^\infty(D(G))$ are given by (I.7). ■

Remark I.3.2. In the Kac case, and only in this case, the states of the form $h \otimes \phi_\mu$ are exactly the normal states on $L^\infty(D(G))$ that are invariant under the actions of the quantum subgroup G of $D(G)$ by left and right translations.

I.3.2 Quantum groups with weakly amenable dimension functions

Recall that given a right action $\beta: A \rightarrow M(A \otimes c_0(\hat{G}))$ of \hat{G} and a normal state ϕ on $\ell^\infty(\hat{G})$, a state ω on A is called ϕ -stationary if $\omega * \phi := (\omega \otimes \phi)\beta = \omega$, or in other words, if $\phi \mathcal{P}_\omega = \omega$ [Fur63; Kal+22]. It is easy to see that the Poisson integral $\mathcal{P}_\omega: A \rightarrow \ell^\infty(\hat{G})$ has image in $H^\infty(\ell^\infty(\hat{G}), P_\phi)$ if and only if ω is ϕ -stationary.

Proposition I.3.3. *Assume G is a compact quantum group and μ is a probability measure on $\text{Irr}(G)$. Then there is a $D(G)$ -equivariant ucp map $\psi: \mathcal{R}(H^\infty(\hat{G}, \mu)) \rightarrow C(\partial_{\text{FH}}D(G))$. Any such map ψ is surjective and the following conditions are equivalent:*

- (1) ψ is an isomorphism;
- (2) the only $D(G)$ -equivariant ucp map $\mathcal{R}(H^\infty(\hat{G}, \mu)) \rightarrow \mathcal{R}(H^\infty(\hat{G}, \mu))$ is the identity map;
- (3) the state $\hat{\varepsilon}|_{\mathcal{R}(H^\infty(\hat{G}, \mu))}$ is the only G -invariant ϕ_μ -stationary state on $\mathcal{R}(H^\infty(\hat{G}, \mu))$;
- (4) $\mathcal{R}(H^\infty(\hat{G}, \mu))$ is a $D(G)$ -boundary.

Proof. A $D(G)$ -equivariant ucp map $\psi: \mathcal{R}(H^\infty(\hat{G}, \mu)) \rightarrow C(\partial_{\text{FH}}D(G))$ exists by the $D(G)$ -injectivity of $C(\partial_{\text{FH}}D(G))$. On the other hand, $\mathcal{R}(H^\infty(\hat{G}, \mu))$ is $D(G)$ -injective as well, being the image of an equivariant ucp projection $\mathcal{R}(\ell^\infty(\hat{G})) \rightarrow \mathcal{R}(H^\infty(\hat{G}, \mu))$. Hence there is an equivariant ucp map $\eta: C(\partial_{\text{FH}}D(G)) \rightarrow \mathcal{R}(H^\infty(\hat{G}, \mu))$. Then $\psi\eta = \iota$ by Corollary I.2.7, which shows that ψ is surjective. Let us show equivalence of (1)–(4).

The implication (1) \Rightarrow (2) follows again from Corollary I.2.7.

Assuming (2), let ω be a G -invariant ϕ_μ -stationary state. Then $\mathcal{P}_\omega: \mathcal{R}(H^\infty(\hat{G}, \mu)) \rightarrow \ell^\infty(\hat{G})$ has image in $H^\infty(\hat{G}, \mu)$, hence in $\mathcal{R}(H^\infty(\hat{G}, \mu))$ by the G -equivariance. But then by assumption it must be the identity map. Hence $\hat{\varepsilon} = \hat{\varepsilon} \mathcal{P}_\omega = \omega$ on $\mathcal{R}(H^\infty(\hat{G}, \mu))$. Note that the state $\hat{\varepsilon}|_{\mathcal{R}(H^\infty(\hat{G}, \mu))}$ is indeed ϕ_μ -stationary, since $\hat{\varepsilon} = \phi_\mu$ on $H^\infty(\hat{G}, \mu)$. Thus, (2) \Rightarrow (3).

Next, assume that (3) holds and we are given a $D(G)$ -equivariant ucp map $\psi: \mathcal{R}(H^\infty(\hat{G}, \mu)) \rightarrow A$ for a unital Yetter–Drinfeld G - C^* -algebra A . As $\mathcal{R}(H^\infty(\hat{G}, \mu))$ is $D(G)$ -injective, there is also an equivariant ucp $\eta: A \rightarrow$

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$\mathcal{R}(H^\infty(\hat{G}, \mu))$. Consider the state $\omega = \hat{\varepsilon}\eta\psi$. Then, by Proposition I.2.5, we have $\eta\psi = \mathcal{P}_\omega$. But since $\hat{\varepsilon}|_{\mathcal{R}(H^\infty(\hat{G}, \mu))}$ is ϕ_μ -stationary, the state ω is ϕ_μ -stationary as well, hence $\omega = \hat{\varepsilon}|_{\mathcal{R}(H^\infty(\hat{G}, \mu))}$ and therefore \mathcal{P}_ω is the identity map. Thus, $\eta\psi = \iota$, which implies that ψ is a complete isometry. This shows that (3) \Rightarrow (4).

The implication (4) \Rightarrow (1) follows, e.g., from the surjectivity of ψ and Theorem I.2.9. \blacksquare

Remark I.3.4. The proof of the implication (3) \Rightarrow (4) shows that, for any G and μ , if $B \subset \mathcal{R}(H^\infty(\hat{G}, \mu))$ is a unital Yetter–Drinfeld G - C^* -subalgebra such that $\hat{\varepsilon}|_B$ is the only G -invariant ϕ_μ -stationary state on B , then B is a $D(G)$ -boundary, cf. [Kal+22, Theorem 4.19]. \diamond

The simplest case when condition (3) in Proposition I.3.3 is satisfied is when G acts ergodically on $H^\infty(\hat{G}, \mu)$ and therefore $\mathcal{R}(H^\infty(\hat{G}, \mu))$ has a unique G -invariant state. This is probably the closest to the original setup and motivation of Furstenberg, see Corollary to Theorem 3.1 in [Fur63]. It is very much possible that Proposition I.3.3 cannot be applied in any other case: if $H^\infty(\hat{G}, \mu)^G$ is nontrivial, the C^* -algebra $\mathcal{R}(H^\infty(\hat{G}, \mu))$ might be too large to be the Furstenberg–Hamana boundary.

The condition $H^\infty(\hat{G}, \mu)^G = \mathbb{C}1$ is equivalent to triviality of the Poisson boundary of the classical random walk on $\text{Irr}(G)$ defined by $P_\mu|_{\ell^\infty(\text{Irr}(G))}$. A probability measure μ satisfying this condition is called *ergodic*. As a byproduct of the isomorphism $C(\partial_{\text{FH}}D(G)) \cong \mathcal{R}(H^\infty(\hat{G}, \mu))$, we conclude that up to isomorphism the Yetter–Drinfeld G -von Neumann algebra $H^\infty(\hat{G}, \mu)$ is independent of an ergodic measure μ . In fact, a stronger result is known and is easy to prove. Define

$$H^\infty(\hat{G}) = \{x \in \ell^\infty(\hat{G}) \mid P_s(x) = x \text{ for all } s \in \text{Irr}(G)\}.$$

The elements of $H^\infty(\hat{G})$ are called *absolutely harmonic* [BNY18].

Lemma I.3.5. *For any ergodic probability measure μ on $\text{Irr}(G)$, we have $H^\infty(\hat{G}, \mu) = H^\infty(\hat{G})$.*

Proof. Take a probability measure ν on $\text{Irr}(G)$. Since the action of G on $H^\infty(\hat{G}, \mu)$ is ergodic, the state $\hat{\varepsilon}$ is the only normal G -invariant state on $H^\infty(\hat{G}, \mu)$. Hence $\phi_\nu = \hat{\varepsilon}$ on $H^\infty(\hat{G}, \mu)$, and therefore $P_\nu|_{H^\infty(\hat{G}, \mu)}$ is the identity map. It follows that $H^\infty(\hat{G}, \mu) \subset H^\infty(\hat{G})$. The opposite inclusion is obvious. \blacksquare

Recall that the quantum dimension function $U \mapsto \dim_q U$ on $\text{Rep } G$ is said to be *weakly amenable*, if there is a state on $\ell^\infty(\text{Irr}(G)) = Z(\ell^\infty(\hat{G}))$ that is invariant under P_s for all $s \in \text{Irr}(G)$. By a version of the Furstenberg–Kaimanovich–Vershik–Rosenblatt theorem [HY00, Theorem 2.5], the quantum groups admitting ergodic measures are exactly the ones that have weakly amenable quantum dimension functions and countable $\text{Irr}(G)$. The corresponding Poisson boundaries have been shown to have a universal property [NY14; NY17]. We want to show next that this property implies that $H^\infty(\hat{G})$ leads to a model of

the Furstenberg–Hamana boundary of $D(G)$ without the countability assumption on $\text{Irr}(G)$. We need some preparation to formulate the result.

Recall (see [NY17, Section 2] and references there) that a dimension function d on $\text{Rep } G$ is called *amenable*, if

$$d(U) = \|\Gamma_U\|$$

for all representations $U \in \text{Rep } G$, where $\Gamma_U \in B(\ell^2(\text{Irr}(G)))$ is the matrix defined by

$$\Gamma_U = (\dim \text{Mor}(U_s, U \otimes U_t))_{s,t}.$$

Since the inequality $\|\Gamma_U\| \leq d(U)$ holds for any dimension function, an amenable dimension function is unique if it exists, and when it exists, it is the smallest dimension function on $\text{Rep } G$. Coamenability of G is equivalent to amenability of the classical dimension function $U \mapsto \dim H_U$ on $\text{Rep } G$. In particular, the quantum dimension function is amenable if and only if G is coamenable and of Kac type.

Now, consider a unital braided-commutative Yetter–Drinfeld C^* -algebra A , with coactions

$$\alpha: A \rightarrow C(G) \otimes A \quad \text{and} \quad \beta: A \rightarrow M(A \otimes c_0(\hat{G})).$$

Then by [NY14] we can associate to A a C^* -tensor category \mathcal{C}_A together a unitary tensor functor $\text{Rep } G \rightarrow \mathcal{C}_A$. A convenient way of doing this is to start with $\text{Rep } G$, enlarge the morphism spaces and then consider the idempotent completion of the new category. Namely, the new morphism spaces are defined by

$$\mathcal{C}_A(U, V) = \{T \in A \otimes B(H_U, H_V) : V_{31}^*(\alpha \otimes \iota)(T)U_{31} = 1 \otimes T\}.$$

We will write $\mathcal{C}_A(U)$ for $\mathcal{C}_A(U, U) = \text{End}_{\mathcal{C}_A}(U)$. The tensor products of morphisms are described by the following rules:

$$T \otimes \iota_Y = T \otimes 1 \in A \otimes B(H_U, H_V) \otimes B(H_Y), \quad \iota_Y \otimes T = (\beta_Y \otimes \iota)(T) \quad (\text{I.10})$$

for all $Y \in \text{Rep } G$, where $\beta_Y = (\iota \otimes \pi_Y)\beta: A \rightarrow A \otimes B(H_Y)$. The morphisms T in the category $\text{Rep } G$ are viewed as morphisms in \mathcal{C}_A via the map $T \mapsto 1 \otimes T$.

If the action of G on A is ergodic, then \mathcal{C}_A becomes a rigid C^* -tensor category with simple unit, so it has a well-defined intrinsic dimension function. Standard solutions of the conjugate equations in \mathcal{C}_A can be expressed in terms of the solutions (I.2), which might not be standard in \mathcal{C}_A . Namely, cf. [NY17, Lemma 4.7], for every $U \in \text{Rep } G$ there is a unique positive invertible element $a_U \in \mathcal{C}_A(U) = (A \otimes B(H_U))^G$ such that

$$(\iota \otimes a_U^{1/2})R_U, \quad (a_U^{-1/2} \otimes \iota)\bar{R}_U$$

form a standard solution of the conjugation equation for U and \bar{U} in \mathcal{C}_A . In other words, $a_U > 0$ is characterized by the properties that the scalars

$$R_U^*(\iota \otimes a_U)R_U \quad \text{and} \quad \bar{R}_U^*(a_U^{-1} \otimes \iota)\bar{R}_U$$

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in $\mathcal{C}_A(\mathbb{1}) = A^G = \mathbb{C}1$ are equal and their product is minimal possible. The dimension $d^{\mathcal{C}_A}(U)$ of $U \in \text{Rep } G$ in \mathcal{C}_A is therefore given by

$$d^{\mathcal{C}_A}(U)1_A = (\dim_q U)(\iota \otimes \psi_U)(a_U^{-1}),$$

where

$$\psi_U = (\dim_q U)^{-1} \text{Tr}(\cdot \rho_U).$$

Example I.3.6. Assume K is a closed quantum subgroup of G , so that we have a surjective homomorphism $\pi: \mathbb{C}[G] \rightarrow \mathbb{C}[K]$ of Hopf $*$ -algebras. Consider the Yetter–Drinfeld G - C^* -subalgebra $A = C(G/K)$ of $C(G)$. In this case \mathcal{C}_A is monoidally equivalent to $\text{Rep } K$, see [NY14, Section 3.1], so the corresponding dimension function on $\text{Rep } G$ is defined by the quantum dimension function for K . If ρ^G and ρ^K denote the Woronowicz characters for G and K , then the elements a_U are given by

$$a_U = U_{21}(1 \otimes \rho_U^G (\rho_U^K)^{-1})U_{21}^* \quad \text{for all } U \in \text{Rep } G.$$

Note (see [INT06, Lemma 2.7]) that $\rho_U^G = \rho_U^K T = T \rho_U^K$ for a positive operator $T \in \text{End}_K(H_U)$, since every right K -invariant state on $B(H_U)$ has the form $\text{Tr}(\cdot T \rho_U^K)$ for a positive $T \in \text{End}_K(H_U)$. In particular, the operator $\rho_U^G (\rho_U^K)^{-1} = (\rho_U^K)^{-1} \rho_U^G$ is indeed positive and π intertwines the scaling groups of G and K . \diamond

We are now ready to reformulate, and slightly extend, the main results of [NY17] in the quantum group setting.

Theorem I.3.7. *Assume G is a compact quantum group with weakly amenable quantum dimension function. Then there is a unique up to isomorphism unital braided-commutative Yetter–Drinfeld G - C^* -algebra A with $A^G = \mathbb{C}1$ such that the following conditions are satisfied:*

- (i) *the dimension function on $\text{Rep } G$ defined by \mathcal{C}_A is amenable, that is,*

$$\|\Gamma_U\|1_A = (\dim_q U)(\iota \otimes \psi_U)(a_U^{-1}) \quad \text{for all } U \in \text{Rep } G;$$

- (ii) *if B is another unital braided-commutative Yetter–Drinfeld G - C^* -algebra such that $B^G = \mathbb{C}1$ and B defines the amenable dimension function on $\text{Rep } G$, then there is a unique unital $*$ -homomorphism $A \rightarrow B$ of Yetter–Drinfeld G - C^* -algebras.*

Furthermore, if ϕ is the unique G -invariant state on A , then the Poisson integral $\mathcal{P}_\phi: A \rightarrow \ell^\infty(\hat{G})$ is completely isometric and extends to a normal completely isometric map of $\pi_\phi(A)''$ onto $H^\infty(\hat{G})$.

We denote the Yetter–Drinfeld G - C^* -algebra A given by Theorem I.3.7 by $C(\partial_\Pi \hat{G})$ and call $\partial_\Pi \hat{G}$ the *Poisson boundary* of \hat{G} .

Proof. When $\text{Irr}(G)$ is countable, this is essentially Theorems 4.1 and 5.1 in [NY17] combined with the correspondence between braided-commutative Yetter–Drinfeld C^* -algebras and tensor functors [NY14]. The only part that requires an additional explanation is (ii), namely, that a homomorphism $A \rightarrow B$ is truly unique, not just unique up to an automorphism of B . But this follows from the proof of [NY17, Theorem 4.1]. Namely, denote by $b_U \in (B \otimes B(H_U))^G$ the elements defined similarly to a_U . Then any $D(G)$ -equivariant unital $*$ -homomorphism $\eta: A \rightarrow B$ must satisfy

$$(\eta \otimes \iota)(a_U) = b_U \quad \text{for all } U \in \text{Rep } G. \quad (\text{I.11})$$

On the other hand, by the proof of [NY17, Theorem 4.1], the morphism spaces $\mathcal{C}_A(U, V)$ are generated by the morphisms in $\text{Rep } G$, the morphisms a_U and their tensor products. It follows that η is completely determined by (I.11).

To deal with the general case, we will first construct a net of quotients G_i of G with countable $\text{Irr}(G_i)$ and weakly amenable quantum dimension functions.

Since there is an invariant state on $\ell^\infty(\text{Irr}(G))$, a standard argument (see, e.g., [Gre69, §2.4]) shows that there is a net $(m_j)_j$ of normal states such that $m_j P_s|_{\ell^\infty(\text{Irr}(G))} - m_j \rightarrow 0$ in norm for all $s \in \text{Irr}(G)$. The states m_j , viewed as elements of $\ell^1(\text{Irr}(G))$, have at most countable supports $I_j \subset \text{Irr}(G)$.

Take any countable set $X_0 \subset \text{Irr}(G)$. Then we can find a sequence $\{j_n\}_{n=1}^\infty$ such that $m_{j_n} P_s|_{\ell^\infty(\text{Irr}(G))} - m_{j_n} \rightarrow 0$ for all $s \in X_0$. Let $X_1 \subset \text{Irr}(G)$ consist of all $s \in \text{Irr}(G)$ such that U_s is a subrepresentation of a tensor product of representations U_t and their conjugates for $t \in X_0 \cup (\cup_{n=1}^\infty I_{j_n})$. The set X_1 is countable. Repeat the same procedure with X_0 replaced by X_1 , and so on. We thus get a sequence of countable sets $X_0 \subset X_1 \subset \dots$. Let G_0 be the quotient of G with $\text{Irr}(G_0) = \cup_{n \geq 0} X_n$. In other words, $\mathbb{C}[G_0] \subset \mathbb{C}[G]$ is spanned by the matrix coefficients of U_s for all $s \in \cup_{n \geq 0} X_n$. By construction, we can find a sequence $\{j'_n\}_{n=1}^\infty$ such that $I_{j'_n} \subset \text{Irr}(G_0)$ and $m_{j'_n} P_s|_{\ell^\infty(\text{Irr}(G))} - m_{j'_n} \rightarrow 0$ for all $s \in \text{Irr}(G_0)$. This implies that if we identify $\ell^\infty(\text{Irr}(G_0))$ with a direct summand of $\ell^\infty(\text{Irr}(G))$ and view the states $m_{j'_n}$ as states on $\ell^\infty(\text{Irr}(G_0))$, then a weak* cluster point of the sequence $\{m_{j'_n}\}_{n=1}^\infty$ is an invariant mean on $\ell^\infty(\text{Irr}(G_0))$. Therefore G_0 has weakly amenable quantum dimension function.

We apply the above procedure to every countable subset X_0 of $\text{Irr}(G)$ and this way get a collection of quotients G_i of G with countable $\text{Irr}(G_i)$ and weakly amenable quantum dimension functions. Define a partial order on this collection by the inclusions $\text{Irr}(G_i) \subset \text{Irr}(G_k)$. This gives us the required net of quotients of G .

For every k , consider the Yetter–Drinfeld G_k - C^* -algebra A_k given by the theorem in the countable case. Note that if $U \in \text{Rep } G_k$ is viewed as a representation of G , then the matrix $(\dim \text{Mor}(U_s, U \otimes U_t))_{s,t \in \text{Irr}(G_k)}$ is only a corner of Γ_U , but its norm equals that of Γ_U , see the proof of [HI98, Proposition 4.8]. Therefore

$$d^{\mathcal{C}A_k}(U) = \|\Gamma_U\| \quad \text{for all } U \in \text{Rep } G_k. \quad (\text{I.12})$$

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Assume now that $\text{Irr}(G_i) \subset \text{Irr}(G_k)$ for some i and k . Consider the C^* -algebra $A_{ik} \subset A_k$ that is the closure of the span of the spectral subspaces of A_k corresponding to $s \in \text{Irr}(G_i)$. Then A_{ik} has the structure of a braided-commutative Yetter–Drinfeld G_i - C^* -algebra. For every $U \in \text{Rep } G_i$, we have

$$(A_k \otimes B(H_U))^{G_k} = (A_{ik} \otimes B(H_U))^{G_i},$$

where on the left hand side we view U as a representation of G_k . This and (I.12) imply that A_{ik} defines the amenable dimension function on $\text{Rep } G_i$. It follows that there exists a unique unital $*$ -homomorphism $A_i \rightarrow A_{ik}$ of Yetter–Drinfeld G_i - C^* -algebras. We thus get an inductive system of C^* -algebras A_i . Let A be the limit of this inductive system.

It follows almost immediately by the construction that A has the structure of a braided-commutative Yetter–Drinfeld G - C^* -algebra and satisfies properties (i) and (ii). It remains to prove the claim about the Poisson integral.

Consider the von Neumann algebra $M = \pi_\phi(A)''$. The state ϕ is faithful on A , as $\phi(\cdot)1 = (h \otimes \iota)\alpha$. Hence we can view A as a subalgebra of M . We continue to denote by ϕ the normal state $(\cdot \xi_\phi, \xi_\phi)$ on M . The quantum group G acts ergodically on M and ϕ is the unique normal G -invariant state on M , hence ϕ is faithful on M for the same reason as that $\phi|_A$ is faithful.

For every index i , let $M_i = \pi_\phi(A_i)'' \subset M$. By the faithfulness of ϕ on M , we have $M_i \cong \pi_{\phi_i}(A_i)''$, where $\phi_i = \phi|_{A_i}$. Since the theorem is true for quantum groups with countable isomorphism classes of irreducible representations, we already know that \mathcal{P}_{ϕ_i} extends to an isomorphism of von Neumann algebras $M_i \cong H^\infty(\hat{G}_i) = H^\infty(\hat{G}_i, \mu_i)$, where μ_i is any ergodic measure on $\text{Irr}(G_i)$.

Denote by π_i the map $\ell^\infty(\hat{G}) \rightarrow \ell^\infty(\hat{G}_i)$ dual to the embedding $\mathbb{C}[G_i] \rightarrow \mathbb{C}[G]$. If $\text{Irr}(G_i) \subset \text{Irr}(G_k)$, define in a similar way $\pi_{ki}: \ell^\infty(\hat{G}_k) \rightarrow \ell^\infty(\hat{G}_i)$. We then have the following commutative diagram:

$$\begin{array}{ccc} M_k & \xrightarrow{\mathcal{P}_{\phi_k}} & H^\infty(\hat{G}_k) \\ \uparrow & & \downarrow \pi_{ki} \\ M_i & \xrightarrow{\mathcal{P}_{\phi_i}} & H^\infty(\hat{G}_i), \end{array}$$

where $M_i \rightarrow M_k$ is the embedding map. Since the horizontal maps are isomorphisms, it follows that $\pi_{ki}(H^\infty(\hat{G}_k)) = H^\infty(\hat{G}_i)$ and we have a conditional expectation $E_{ki}: M_k \rightarrow M_i$ such that $\mathcal{P}_{\phi_i} E_{ki} = \pi_{ki} \mathcal{P}_{\phi_k}$. This conditional expectation is ϕ_k -preserving. The existence of these conditional expectations for all k large enough implies that there is a (necessarily unique) ϕ -preserving conditional expectation $E_i: M \rightarrow M_i$. We remark that the existence of E_i follows also from the description of the modular group of ϕ given in [BDV06, Proposition 2.10].

Now, take $x \in M$. Let $y_i = \mathcal{P}_{\phi_i}(E_i(x)) \in H^\infty(\hat{G}_i)$. As $\pi_{ki}(y_k) = y_i$ for $\text{Irr}(G_i) \subset \text{Irr}(G_k)$, there is a unique element $y \in \ell^\infty(\hat{G})$ such that $y_i = \pi_i(y)$ for all i . As the homomorphisms π_i respect the comultiplications, it is easy to see

that we must have $y \in H^\infty(\hat{G})$. We define $\mathcal{P}(x) = y$. It is then straightforward to check that $\mathcal{P}: M \rightarrow H^\infty(\hat{G})$ is a normal ucp map that agrees with \mathcal{P}_ϕ on A . As $E_i(x) \rightarrow x$ in the ultrastrong operator topology, we have $\|x_i\| \nearrow \|x\|$. It is also clear that $\|y_i\| \nearrow \|y\|$. It follows that $\|x\| = \|y\|$, so that \mathcal{P} is an isometric map. The same argument applies to the matrix algebras over M and $\ell^\infty(\hat{G})$, hence \mathcal{P} is completely isometric.

What is left to prove is that $\mathcal{P}(M) = H^\infty(\hat{G})$. Take $y \in H^\infty(\hat{G})$. For every i there is a unique $x_i \in M_i$ such that $\pi_i(y) = \mathcal{P}_{\phi_i}(x_i)$. Then $\|x_i\| = \|\pi_i(y)\| \leq \|y\|$. We also have $E_{ki}(x_k) = x_i$ if $\text{Irr}(G_i) \subset \text{Irr}(G_k)$. It follows that the net $(x_i)_i$ converges in the ultrastrong operator topology to a unique element $x \in M$ such that $E_i(x) = x_i$ for all i . Then $\mathcal{P}(x) = y$. ■

The theorem implies that $H^\infty(\hat{G})$ has a von Neumann algebra structure. Not surprisingly, it is given by a Choi–Effros product by the following elaboration on the proof.

Corollary I.3.8. *There is a $D(G)$ -equivariant ucp projection $\ell^\infty(\hat{G}) \rightarrow H^\infty(\hat{G})$.*

Proof. We will use the net of quotients G_i of G constructed in the proof of Theorem I.3.7. For every i , choose a $D(G_i)$ -equivariant ucp projection $e_i: \ell^\infty(\hat{G}_i) \rightarrow H^\infty(\hat{G}_i)$. Consider the maps $e_i\pi_i: \ell^\infty(\hat{G}) \rightarrow H^\infty(\hat{G}_i)$. Identifying $\ell^\infty(\hat{G}_i)$ with a direct summand of $\ell^\infty(\hat{G})$, we can view $e_i\pi_i$ as a cp map on $\ell^\infty(\hat{G})$. This map is G - and \hat{G}_i -equivariant. It follows that any cluster point $e: \ell^\infty(\hat{G}) \rightarrow \ell^\infty(\hat{G})$ of the net $(e_i\pi_i)_i$ is a $D(G)$ -equivariant cp map.

We need to show that e is a projection onto $H^\infty(\hat{G})$. Take $x \in \ell^\infty(\hat{G})$. If $\text{Irr}(G_i) \subset \text{Irr}(G_k)$, then $\pi_i(e_k\pi_k(x)) \in H^\infty(\hat{G}_i)$, hence by passing to the limit we get that $\pi_i(e(x)) \in H^\infty(\hat{G}_i)$ for all i . As we already used in the proof of Theorem I.3.7, this implies that $e(x) \in H^\infty(\hat{G})$. Furthermore, if we start with $x \in H^\infty(\hat{G})$, then $\pi_i(e_k\pi_k(x)) = \pi_i(x)$, hence $\pi_i(e(x)) = \pi_i(x)$ for all i and therefore $e(x) = x$. ■

Corollary I.3.9. *If ϕ is the unique G -invariant state on $C(\partial_\Pi\hat{G})$, then \mathcal{P}_ϕ is a $D(G)$ -equivariant isomorphism of $C(\partial_\Pi\hat{G})$ onto $\mathcal{R}(H^\infty(\hat{G}))$.*

Proof. Since G acts ergodically on $H^\infty(\hat{G})$, all spectral subspaces of $H^\infty(\hat{G})$ are finite dimensional and hence $\mathcal{R}(H^\infty(\hat{G}))$ is the unique ultrastrongly operator dense G - C^* -subalgebra of $H^\infty(\hat{G})$. It follows that $\mathcal{P}_\phi(C(\partial_\Pi\hat{G})) = \mathcal{R}(H^\infty(\hat{G}))$. ■

We now return to the Furstenberg–Hamana boundaries.

Theorem I.3.10. *For any compact quantum group G with weakly amenable quantum dimension function, we have an isomorphism*

$$C(\partial_{\text{FH}}D(G)) \cong C(\partial_\Pi\hat{G})$$

of Yetter–Drinfeld G - C^ -algebras.*

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Proof. When $\text{Irr}(G)$ is countable, then $C(\partial_{\Pi}\hat{G}) \cong \mathcal{R}(H^{\infty}(\hat{G}, \mu))$ for any ergodic probability measure μ and the theorem follows from Proposition I.3.3. In the general case the arguments are similar.

The C^* -algebra $C(\partial_{\Pi}\hat{G}) \cong \mathcal{R}(H^{\infty}(\hat{G}))$ is $D(G)$ -injective by Corollary I.3.8. To show that it is a $D(G)$ -boundary, assume B is a unital Yetter–Drinfeld G - C^* -algebra and $\psi: C(\partial_{\Pi}\hat{G}) \rightarrow B$ is a $D(G)$ -equivariant ucp map. Take a G -invariant state ω on B . Then $\omega\psi = \phi$, the unique G -invariant state on $C(\partial_{\Pi}\hat{G})$. Hence $\mathcal{P}_{\omega}\psi = \mathcal{P}_{\phi}$ is completely isometric by Theorem I.3.7. It follows that ψ is completely isometric as well. ■

Corollary I.3.11. *For any compact quantum group G , we have $C(\partial_{\text{FH}}D(G))^G = \mathbb{C}1$ if and only if the quantum dimension function of G is weakly amenable.*

Proof. The “if” direction follows immediately from Theorem I.3.10, as $C(\partial_{\Pi}\hat{G})^G = \mathbb{C}1$.

Conversely, assume $C(\partial_{\text{FH}}D(G))^G = \mathbb{C}1$. Take any $D(G)$ -equivariant ucp map

$$\phi: \mathcal{R}(\ell^{\infty}(\hat{G})) \rightarrow C(\partial_{\text{FH}}D(G)).$$

Then it maps $\ell^{\infty}(\text{Irr}(G)) = \mathcal{R}(\ell^{\infty}(\hat{G}))^G$ into scalars, hence it defines an invariant mean on $\ell^{\infty}(\text{Irr}(G))$. ■

Corollary I.3.12. *For any compact quantum group G , the Furstenberg–Hamana boundary of $D(G)$ is trivial if and only if G is coamenable and of Kac type.*

Proof. If G is coamenable and of Kac type, then $\partial_{\text{FH}}D(G)$ is trivial by Theorem I.3.10, as then the quantum dimension function of G is amenable and therefore $\partial_{\Pi}\hat{G}$ is trivial.

Conversely, assume $\partial_{\text{FH}}D(G)$ is trivial. Then, by the previous corollary, the quantum dimension function is weakly amenable. By Theorem I.3.10 the triviality of $\partial_{\text{FH}}D(G)$ is then equivalent to triviality of $\partial_{\Pi}\hat{G}$. But this means that the quantum dimension function of G is amenable, that is, G is coamenable and of Kac type. ■

Remark I.3.13. The “if” direction can be proved in a more elementary way as follows. We have to show that \mathbb{C} is a $D(G)$ -injective C^* -algebra. For this it suffices to show that the $D(G)$ -injective C^* -algebra $\mathcal{R}(\ell^{\infty}(\hat{G}))$ has a $D(G)$ -invariant state. To construct such a state, we can start with any right \hat{G} -invariant mean on $\ell^{\infty}(\hat{G})$, compose it with the left G -invariant conditional expectation $\ell^{\infty}(\hat{G}) \rightarrow \ell^{\infty}(\text{Irr}(G))$ and then restrict the composition to $\mathcal{R}(\ell^{\infty}(\hat{G}))$, see the proof of [Vae05, Lemma 7.1].

For the “only if” direction we can also argue as follows. If $\partial_{\text{FH}}D(G)$ is trivial, then G has weakly amenable quantum dimension function and $H^{\infty}(\hat{G}) = \mathbb{C}1$. Corollary I.3.8 implies then that G is coamenable, while [Izu02, Corollary 3.9] shows that G must be of Kac type. But if $\text{Irr}(G)$ is uncountable, this argument still relies on Theorem I.3.7. It would be interesting to find a more direct proof of such a basic property. ◇

For the time being it seems the only examples of compact quantum groups with weakly amenable quantum dimension functions are the quantum groups that are monoidally equivalent to coamenable compact quantum groups. For such quantum groups the noncommutative space $\partial_{\text{FH}}D(G) = \partial_{\Pi}\hat{G}$ has a more explicit description.

Let us start with the coamenable case. Recall from [Sol05] (see also [NY16a, Section 2.3]) that every compact quantum group G has the largest closed quantum subgroup K of Kac type, namely, $\mathbb{C}[K]$ is the quotient of $\mathbb{C}[G]$ by the ideal generated by the elements $a - S^2(a)$, $a \in \mathbb{C}[G]$.

Proposition I.3.14. *Assume G is a coamenable compact quantum group and K is its maximal quantum subgroup of Kac type. Then we have the following isomorphisms of Yetter–Drinfeld G - C^* -algebras:*

$$C(\partial_{\text{FH}}D(G)) \cong C(\partial_{\Pi}\hat{G}) \cong C(G/K).$$

Proof. The first isomorphism holds by Theorem I.3.10. If $\text{Irr}(G)$ is countable, then the second isomorphism is just a reformulation of [NY16a, Theorem 3.1]; together with the last part of Theorem I.3.7 it recovers the description of $H^\infty(\hat{G}, \mu)$ for ergodic μ given by Tomatsu [Tom07]. The general case is similar and can be dealt with as follows.

It is clear from Example I.3.6 that the dimension function on $\text{Rep } G$ defined by $C(G/K)$ coincides with the classical dimension function $U \mapsto \dim H_U$, so it is amenable by assumption. By the universality of $C(\partial_{\Pi}\hat{G})$ we can therefore identify $C(\partial_{\Pi}\hat{G})$ with a Yetter–Drinfeld G - C^* -subalgebra of $C(G/K)$. But by [NY14, Theorem 3.1] (which removes an extra assumption in [Tom07, Theorem 3.18]) any such C^* -subalgebra has the form $C(G/H)$ for some intermediate closed quantum subgroup $K \subset H \subset G$. Since $C(G/H)$ defines the classical dimension function on $\text{Rep } G$, H must be of Kac type. Hence $H = K$ by the maximality of K . ■

Remark I.3.15. Proposition I.3.14 covers all cases where $\partial_{\text{FH}}D(G)$ has the form G/H , that is, if G is a compact quantum group such that $C(\partial_{\text{FH}}D(G)) \cong C(G/H)$ for a closed quantum subgroup H of G , then G is coamenable and H is its maximal quantum subgroup K of Kac type. Indeed (cf. [NY14, Proposition 4.3]), if $\partial_{\text{FH}}D(G) \cong G/H$, then $C(\partial_{\text{FH}}D(G))^G = \mathbb{C}1$, so G has weakly amenable quantum dimension function. Hence $C(G/H) \cong C(\partial_{\Pi}G)$, which implies that H is of Kac type and the classical dimension function on $\text{Rep } G$ is amenable, so that G is coamenable. As $C(G/H) \cong C(\partial_{\Pi}\hat{G})$ has no nontrivial $D(G)$ -equivariant endomorphisms, we must have $H = K$.

Example I.3.16. Let G be a compact connected semisimple Lie group and consider its q -deformation G_q , $q > 0$. Then by [Tom07, Lemma 4.10], for $q \neq 1$, the maximal quantum subgroup of G_q of Kac type is the nondeformed maximal torus $T \subset G_q$. Therefore $\partial_{\text{FH}}D(G_q) = G_q/T$ (for $q \neq 1$). Since $D(G_q)$ should be thought of as a quantization of the complexification $G_{\mathbb{C}}$ of G , this agrees with [Fur63; Moo64] showing that the Furstenberg boundary of $G_{\mathbb{C}}$ is $G_{\mathbb{C}}/P = G/T$,

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where P is a minimal parabolic subgroup of $G_{\mathbb{C}}$. Note also that $\partial_{\text{FH}}D(G)$ is trivial, as G is coamenable and of Kac type. \diamond

If we have a compact quantum group G that is monoidally equivalent to a coamenable one, G_0 , then, under the correspondence between braided-commutative Yetter–Drinfeld C^* -algebras and tensor functors [NY14], the algebras $C(\partial_{\Pi}\hat{G})$ and $C(\partial_{\Pi}\hat{G}_0)$ correspond to the same functor from $\text{Rep } G \sim \text{Rep } G_0$, since the defining properties (i) and (ii) in Theorem I.3.7 can be formulated at the categorical level (see also Section I.4.2 below). Therefore by the results of [NY14, Section 3.2] we get the following generalization of Proposition I.3.14.

Proposition I.3.17. *Assume G is a compact quantum group that is monoidally equivalent to a coamenable compact quantum group G_0 , and $B(G, G_0)$ is a G - G_0 -Galois object defining such an equivalence. Let K_0 be the maximal quantum subgroup of G_0 of Kac type. Then we have the following isomorphisms of Yetter–Drinfeld G - C^* -algebras:*

$$C(\partial_{\text{FH}}D(G)) \cong C(\partial_{\Pi}\hat{G}) \cong B(G, G_0)^{K_0}.$$

When $\text{Irr}(G)$ is countable, the second isomorphism is basically a reformulation of a result of De Rijdt and Vander Venet [DV10, Theorem 9.3].

We remind (see [NY14, Section 3.2]) that the action of \hat{G} on $B(G, G_0)^{K_0}$ is given by the Miyashita–Ulbrich action. Namely, for the subalgebra of regular elements $\mathcal{B} \subset B(G, G_0)$ with respect to the action $\alpha: B(G, G_0) \rightarrow C(G) \otimes B(G, G_0)$ of G we have a bijective Galois map

$$\Gamma: \mathcal{B} \otimes_{\text{alg}} \mathcal{B} \rightarrow \mathbb{C}[G] \otimes_{\text{alg}} \mathcal{B}, \quad a \otimes b \mapsto \alpha(a)(1 \otimes b).$$

Then

$$x \triangleright a = \Gamma^{-1}(x \otimes 1)_1 a \Gamma^{-1}(x \otimes 1)_2 \quad \text{for } x \in \mathbb{C}[G], a \in B(G, G_0),$$

where we again use the sumless notation $\Gamma^{-1}(x \otimes 1) = \Gamma^{-1}(x \otimes 1)_1 \otimes \Gamma^{-1}(x \otimes 1)_2$. The action of \hat{G} can also be defined by a multiplicative unitary type formula, see [VV08, Eq. (5.2)].

Corollary I.3.18. *Under the assumptions of Proposition I.3.17, all $D(G)$ -boundaries up to isomorphism are the Yetter–Drinfeld G - C^* -algebras $B(G, G_0)^{H_0}$ for the intermediate closed quantum subgroups $K_0 \subset H_0 \subset G_0$.*

Proof. By Theorem I.2.9 all $D(G)$ -boundaries up to isomorphism are the unital Yetter–Drinfeld G - C^* -subalgebras of $C(\partial_{\text{FH}}D(G))$. Hence the corollary follows from Proposition I.3.17 and [NY14, Theorem 3.2]. \blacksquare

Example I.3.19. For $N \geq 2$ and $Q \in \text{Mat}_N(\mathbb{C})$ such that $Q\bar{Q} = \pm 1$, consider the free orthogonal quantum group O_Q^+ . It is monoidally equivalent to $\text{SU}_q(2)$, where $q \in [-1, 1] \setminus \{0\}$ is determined by

$$\text{Tr}(Q^*Q) = |q + q^{-1}|, \quad \text{sign}(Q\bar{Q}) = -\text{sign}(q).$$

An O_Q^+ - $SU_q(2)$ -Galois object $B(O_Q^+, SU_q(2))$ defining such an equivalence is given in [BDV06, Theorem 5.5] (with $F_1 = Q$ and $F_2 = \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}$). If $q \neq \pm 1$, then $\mathbb{T} \subset SU_q(2)$ is the maximal quantum subgroup of Kac type, hence

$$C(\partial_{\text{FH}}D(O_Q^+)) \cong B(O_Q^+, SU_q(2))^{\mathbb{T}}.$$

Furthermore, by [Pod95, Theorem 2.1] the only intermediate closed quantum subgroups $\mathbb{T} \subset H \subset SU_q(2)$ are $H = \mathbb{T}$ and $H = SU_q(2)$. Hence $B(O_Q^+, SU_q(2))^{\mathbb{T}}$ is the only nontrivial $D(O_Q^+)$ -boundary. We remark that, thanks to [VV08, Theorem 6.1] and [VV07, Theorem 5.8], $B(O_Q^+, SU_q(2))^{\mathbb{T}}$ has an alternative description as the space of ends of the quantum Cayley graph of $\mathbb{F}O_Q := \widehat{Q_Q^+}$. Note also that by [Kal+22, Corollary 7.3], if $N \geq 3$, then $B(O_Q^+, SU_q(2))^{\mathbb{T}}$ is an $\mathbb{F}O_Q$ -boundary.

If $q = \pm 1$, then $O_Q^+ \cong SU_{\pm 1}(2)$ is coamenable and of Kac type, hence the Furstenberg–Hamana boundary of $D(O_Q^+)$ is trivial.

Example 1.3.20. Let C be a finite dimensional C^* -algebra of dimension ≥ 4 and ω be a faithful state on C such that for the multiplication map $m: C \otimes C \rightarrow C$ we have $mm^* = \delta^2 \iota$ for some $\delta \geq (\dim C)^{1/2}$, where m^* is the adjoint of m with respect to the scalar product on C defined by ω . Then it follows from [DV10, Theorem 4.7] that the quantum automorphism group $\text{QAut}(C, \omega)$ of (C, ω) is monoidally equivalent to

$$\text{SO}_{q^2}(3) \cong \text{SU}_q(2)/\{\pm 1\} \cong \text{QAut}(\text{Mat}_2(\mathbb{C}), (q + q^{-1})^{-1} \text{Tr}(\cdot \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix})),$$

where $q \in (0, 1]$ is determined by

$$\delta = q + q^{-1}$$

and the first isomorphism is from [Dij94, Corollary 2.3]. Let $B(\text{QAut}(C, \omega), \text{SU}_q(2)/\{\pm 1\})$ be the bi-Galois object as described in [DV10, Theorem 4.7]. If $q \neq 1$, then it follows from [Pod95, Theorem 3.5] that the torus $\mathbb{T}/\{\pm 1\} \cong \mathbb{T}$ is the maximal quantum subgroup of Kac type in $\text{SU}_q(2)/\{\pm 1\}$ and there are no intermediate quantum subgroups $\mathbb{T} \subset H \subset \text{SU}_q(2)/\{\pm 1\}$ apart from $H = \mathbb{T}$ and $H = \text{SU}_q(2)/\{\pm 1\}$. Therefore

$$C(\partial_{\text{FH}}D(\text{QAut}(C, \omega))) \cong B(\text{QAut}(C, \omega), \text{SU}_q(2)/\{\pm 1\})^{\mathbb{T}}$$

and, moreover, $B(\text{QAut}(C, \omega), \text{SU}_q(2)/\{\pm 1\})^{\mathbb{T}}$ is the only nontrivial $D(\text{QAut}(C, \omega))$ -boundary.

If $q = 1$, then either C is \mathbb{C}^4 and ω is given by the uniform probability distribution, or C is $\text{Mat}_2(\mathbb{C})$ and ω is the normalized trace. Therefore $\text{QAut}(C, \omega)$ is either the quantum permutation group S_4^+ or the group $\text{SO}(3)$. In either case we get a coamenable quantum group of Kac type, so the Furstenberg–Hamana boundary of $D(\text{QAut}(C, \omega))$ is trivial.

1.3.3 Free unitary quantum groups

Take a natural number $N \geq 2$ and a matrix $F \in \mathrm{GL}_N(\mathbb{C})$ such that $\mathrm{Tr}(F^*F) = \mathrm{Tr}((F^*F)^{-1})$. Recall that the compact free unitary quantum group U_F^+ is defined as the universal unital C^* -algebra with generators u_{ij} , $1 \leq i, j \leq N$, such that the matrices $U = (u_{ij})_{i,j}$ and FU^cF^{-1} are unitary, where $U^c = (u_{ij}^*)_{i,j}$, equipped with the comultiplication

$$\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}.$$

The dual discrete quantum group is denoted by $\mathbb{F}U_F$.

The set $I = \mathrm{Irr}(U_F^+)$ of isomorphism classes of irreducible representations of U_F^+ is the free monoid on letters α and β , with α corresponding to U , β to \bar{U} and the unit e to the trivial representation. The involution $s \mapsto \bar{s}$ is the anti-automorphism of the monoid defined by $\bar{\alpha} = \beta$ and $\bar{\beta} = \alpha$.

We will use the conventions of [VV10] in that we write x instead of U_x for $x \in I$ whenever convenient. The fusion rules for the representations of U_F^+ are given by

$$x \otimes y \cong \bigoplus_{z \in I: x_0 z, y = \bar{z} y_0} x_0 y_0.$$

Therefore if the last letter of x is the same as the first letter of y , then $U_x \otimes U_y$ is irreducible and isomorphic to U_{xy} .

The Woronowicz character ρ of U_F^+ is determined by the property

$$\rho_U = (F^*F)^t \quad (\text{the transpose of } F^*F).$$

Hence $\dim_q U = \mathrm{Tr}(F^*F) \geq N$, and the equality holds if and only if F is unitary. Let $q \in (0, 1]$ be such that

$$\mathrm{Tr}(F^*F) = q + q^{-1}.$$

Then $q = 1$ if and only if F is a unitary 2-by-2 matrix.

Assume from now on that F is not a unitary 2-by-2 matrix, so that $q < 1$. Consider the tree with vertex set I such that different elements x and y of I are connected by an edge if and only if one of them is obtained from the other by adding or removing one letter on the left. Denote by \bar{I} the end compactification of I . The elements of \bar{I} are words in α and β that are either finite or infinite on the left, and the boundary $\partial I = \bar{I} \setminus I$ is the set of infinite words. The algebra $C(\bar{I})$ of continuous functions on \bar{I} can be identified with the algebra of functions $f \in \ell^\infty(I)$ such that

$$|f(yx) - f(x)| \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ uniformly in } y \in I.$$

In [VV10], Vaes and Vander Venet extended this construction to $\ell^\infty(\mathbb{F}U_F)$ as follows. (To be more precise, they consider words infinite on the right, while in order to be consistent with our conventions, we consider words infinite on the left.)

For all $x, y \in I$, fix an isometry $V(yx, y \otimes x) \in \text{Mor}(yx, y \otimes x)$. Define ucp maps

$$\psi_{yx,x}: B(H_x) \rightarrow B(H_{yx}) \quad T \mapsto V(yx, y \otimes x)^*(1 \otimes T)V(yx, y \otimes x).$$

They do not depend on any choices. Define $C(\overline{\mathbb{F}U_F})$ as the set of $a \in \ell^\infty(\mathbb{F}U_F)$ such that

$$\|a_{yx} - \psi_{yx,x}(a_x)\| \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ uniformly in } y \in I.$$

By [VV10, Theorem 3.2], this is a unital C^* -subalgebra of $\ell^\infty(\mathbb{F}U_F)$ containing $c_0(\mathbb{F}U_F)$. It can therefore be considered as the algebra of continuous functions on a (noncommutative) compactification of $\mathbb{F}U_F$. The corresponding boundary is then defined by $C(\partial\mathbb{F}U_F) = C(\overline{\mathbb{F}U_F})/c_0(\mathbb{F}U_F)$. The Yetter–Drinfeld structure on $\mathcal{R}(\ell^\infty(\mathbb{F}U_F))$ defines such a structure on $C(\overline{\mathbb{F}U_F})$ and $C(\partial\mathbb{F}U_F)$.

Take any generating finitely supported probability measure μ on I . Then by [VV10] the Poisson integral $\mathcal{P}_\omega: C(\partial\mathbb{F}U_F) \rightarrow \ell^\infty(\mathbb{F}U_F)$ extends to an isomorphism of $\pi_\omega(C(\partial\mathbb{F}U_F))''$ onto $H^\infty(\mathbb{F}U_F, \mu)$, where ω is the state on $C(\overline{\mathbb{F}U_F})$ vanishing on $c_0(\mathbb{F}U_F)$ that is the weak* limit of the states $\phi_\mu^{*n}|_{C(\overline{\mathbb{F}U_F})}$. By [MN19] we can further identify $C(\partial\mathbb{F}U_F)$ with the Martin boundary of $\mathbb{F}U_F$, but we will need only the classical precursor of this result, that the Martin boundary of the random walk defined by $P_\mu|_{\ell^\infty(I)}$ is ∂I .

By [VV10, Proposition 4.1], the state ω is faithful on $C(\partial\mathbb{F}U_F)$, hence \mathcal{P}_ω defines an isomorphism of $C(\partial\mathbb{F}U_F)$ onto a dense (in the ultraweak operator topology) Yetter–Drinfeld U_F^\pm - C^* -subalgebra of $H^\infty(\mathbb{F}U_F, \mu)$. Note that this subalgebra is strictly smaller than $\mathcal{R}(H^\infty(\mathbb{F}U_F, \mu))$, since the latter algebra contains a copy of $L^\infty(\partial I, \nu)$, where ν is the measure on ∂I defined by $\omega|_{C(\partial I)}$. It is probably unrealistic to have an explicit description of the Furstenberg–Hamana boundary in this case, but we at least have the following result.

Theorem I.3.21. *For any free unitary quantum group U_F^\pm , with F not a unitary 2-by-2 matrix, $C(\partial\mathbb{F}U_F)$ is a $D(U_F^\pm)$ -boundary.*

By Remark I.3.4, in order to prove this theorem it suffices to show that ω is the unique U_F^\pm -invariant ϕ_μ -stationary state on $C(\partial\mathbb{F}U_F)$, cf. [Kal+22, Section 7.2].

As $C(\overline{\mathbb{F}U_F}) \cap \ell^\infty(I) = C(\bar{I})$ by construction, we have $C(\partial\mathbb{F}U_F)^{U_F^\pm} = C(\partial I)$. It follows that every U_F^\pm -invariant state on $C(\partial\mathbb{F}U_F)$ is determined by its restriction to $C(\partial I)$. This implies that ϕ_μ -stationarity should be possible to formulate entirely in terms of such restrictions. Such a reformulation is not completely straightforward though, since the operator $(\iota \otimes \phi_\mu)\hat{\Delta}$ does not leave $C(\bar{I})$ invariant unless U_F^\pm is of Kac type, that is, unless F is unitary. In order to deal with this, define the states

$$\psi_x = (\dim_q U_x)^{-1} \text{Tr}(\cdot \pi_x(\rho)) \quad \text{on } B(H_x),$$

and put $\psi_\mu = \sum_{x \in I} \mu(x)\psi_x$. The states ψ_x are invariant with respect to the right action $T \mapsto U_x(T \otimes 1)U_x^*$ of U_F^\pm on $B(H_x)$, and as a consequence the

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Markov operators $Q_x = (\iota \otimes \psi_x) \hat{\Delta}$ are right U_F^+ -equivariant. By restriction they define operators on $C(\bar{I})$.

Given a probability measure λ on \bar{I} , we denote by $\lambda * \delta_x$ the measure corresponding to the state $\int Q_x(\cdot) d\lambda$ on $C(\bar{I})$. We let $Q_\mu = \sum_x \mu(x) Q_x$ and $\lambda * \mu = \sum_x \mu(x) \lambda * \delta_x$.

Lemma I.3.22. *If ψ is a U_F^+ -invariant ϕ_μ -stationary state on $C(\overline{\mathbb{F}U_F})$ and λ is the measure on \bar{I} defined by $\psi|_{C(\bar{I})}$, then λ is μ -stationary, that is, $\lambda * \mu = \lambda$.*

Proof. Put $Q_{\phi_\mu} = (\iota \otimes \phi_\mu) \hat{\Delta}$. As $\psi = \psi Q_{\phi_\mu}$ by assumption, it suffices to show that $\psi Q_{\phi_\mu} = \psi Q_\mu$ on $C(\bar{I})$. We can approximate ψ in the weak* topology by the restrictions of normal left U_F^+ -invariant states on $\ell^\infty(\mathbb{F}U_F)$. It follows that it is enough to show that $\phi_x Q_{\phi_\mu} = \phi_x Q_\mu$ on $\ell^\infty(I)$ for all $x \in I$. We have

$$\phi_x Q_{\phi_\mu} = \phi_\mu P_x, \quad \phi_x Q_\mu = \psi_\mu P_x.$$

This gives us what we need, since P_x leaves $\ell^\infty(I)$ invariant and $\phi_\mu = \psi_\mu$ on $\ell^\infty(I)$, cf. [INT06, Proposition 3.1]. \blacksquare

As a consequence, in order to prove Theorem I.3.21 it suffices to establish the following.

Proposition I.3.23. *For any generating finitely supported probability measure μ on I , the measure $\nu = \lim_n \mu^{*n}$ is the unique μ -stationary probability measure on ∂I .*

The proof follows the familiar strategy for random walks on free and, more generally, hyperbolic groups, see [Kai00, Theorem 2.4]. The key point is the following result.

Lemma I.3.24. *Assume λ is a probability measure on \bar{I} and $\{x_n\}_n$ is a sequence in I converging to $x \in \partial I$. Then $\lambda * \delta_{x_n} \rightarrow \delta_x$ in the weak* topology on $C(\bar{I})^*$.*

Proof. For every $y \in I$, denote by Δ_y the set of all words in I of the form uy . The closure $\bar{\Delta}_y$ of Δ_y in \bar{I} consists of all words (finite and infinite) of the form uy . For $z \in \bar{I}$ of length $|z| \geq N$, denote by $[z]_N$ the word consisting of the last N letters of z .

The clopen sets $\bar{\Delta}_{[x]_N}$ form a neighbourhood base at x . Hence, in order to prove the lemma, it suffices to show that for every $N \geq 0$ we have $(\lambda * \delta_{x_n})(\bar{\Delta}_{[x]_N}) \rightarrow 1$ as $n \rightarrow \infty$. We claim that if $y \in I$ is such that $[y]_{N+k} = [x]_{N+k}$ for some $N, k \geq 0$, then

$$(\lambda * \delta_y)(\bar{\Delta}_{[x]_N}) \geq 1 - \frac{q^{2(k+1)}}{1 - q^2}, \quad (\text{I.13})$$

which obviously implies the required convergence.

Since the finitely supported probability measures on I are weakly* dense in the probability measures on \bar{I} , it suffices to show (I.13) for the measures $\lambda = \delta_z$,

$z \in I$. We have

$$\delta_z * \delta_y = \sum_{u \in I: z=z_0u, y=\bar{u}y_0} \frac{d(z_0y_0)}{d(z)d(y)} \delta_{z_0y_0},$$

where d denotes the quantum dimension. In the above sum, if $|y_0| \geq N$, then $z_0y_0 \in \Delta_{[x]_N}$. Therefore

$$(\delta_z * \delta_y)(\Delta_{[x]_N}) \geq 1 - \sum_{\substack{u \in I: z=z_0u, y=\bar{u}y_0 \\ |y_0| < N}} \frac{d(z_0y_0)}{d(z)d(y)}.$$

By [VV10, Eq. (5)], if $z = z_0u$ and $y = \bar{u}y_0$, we have

$$d(z) \geq q^{-|u|}d(z_0), \quad d(y) \geq q^{-|u|}d(y_0).$$

We also have $d(z_0y_0) \leq d(z_0)d(y_0)$. Hence, as $|y| \geq N + k$ by assumption,

$$(\delta_z * \delta_y)(\Delta_{[x]_N}) \geq 1 - \sum_{\substack{u \in I: z=z_0u, y=\bar{u}y_0 \\ |y_0| < N}} q^{2|u|} \geq 1 - \sum_{i=0}^{N-1} q^{2(|y|-i)} \geq 1 - \frac{q^{2(k+1)}}{1-q^2},$$

proving (I.13) for $\lambda = \delta_z$. ■

Proof of Proposition I.3.23. The remaining proof is essentially identical to that of [Kai00, Theorem 2.4]. Consider the random walk on I defined by $P_\mu|_{\ell^\infty(I)}$. Let Ω be the corresponding path space and \mathbb{P}_e the Markov measure on Ω defined by the initial distribution δ_e , so that the push-forward of \mathbb{P}_e under the n -th projection $\Omega \rightarrow I$, $\underline{x} \rightarrow x_n$, is μ^{*n} . The Martin boundary of the random walk is ∂I , implying that \mathbb{P}_e -a.e. path \underline{x} converges to a point $x_\infty \in \partial I$ and ν is the push-forward of \mathbb{P}_e under the map $\underline{x} \mapsto x_\infty$.

Assume that λ is a μ -stationary probability measure on \bar{I} . Then, for all $f \in C(\bar{I})$ and $n \geq 1$, we have

$$\int_{\bar{I}} f d\lambda = \int_{\bar{I}} f d(\lambda * \mu^{*n}) = \int_{\Omega} d\mathbb{P}_e(\underline{x}) \int_{\bar{I}} f d(\lambda * \delta_{x_n}).$$

By Lemma I.3.24 and the dominated convergence theorem, the last expression converges, as $n \rightarrow \infty$, to

$$\int_{\Omega} f(x_\infty) d\mathbb{P}_e(\underline{x}) = \int_{\partial I} f d\nu.$$

Hence $\lambda = \nu$. ■

I.4 Categorical perspective

For C^* -categories we follow the conventions of [NT13]. In particular, all such categories are assumed to be small and closed under subobjects and finite direct sums. The C^* -tensor categories are assumed to be strict.

I.4.1 Categorification of equivariant maps

The results of the previous sections can be formulated at the level of the representation categories $\text{Rep } G$ and extended to C^* -tensor categories. First we need to understand what the categorical analogues of ucp and completely isometric maps are.

Let \mathcal{C} be a rigid C^* -tensor category with simple unit. As in Section I.3.2, the morphism spaces in \mathcal{C} will be usually denoted by $\mathcal{C}(U, V)$, and we will write $\mathcal{C}(U)$ for $\mathcal{C}(U, U)$. Let $\text{Irr}(\mathcal{C})$ be the set of isomorphism classes of simple objects in \mathcal{C} , and for every $s \in \text{Irr}(\mathcal{C})$ fix a representative U_s . The class of the unit object $\mathbb{1}$ is denoted by $e \in \text{Irr}(\mathcal{C})$.

Recall that a right \mathcal{C} -module category is a C^* -category \mathcal{D} together with a unitary tensor functor from \mathcal{C}^{op} into the category of unitary endofunctors of \mathcal{D} . Equivalently, we have a unitary bifunctor $\otimes: \mathcal{D} \times \mathcal{C} \rightarrow \mathcal{D}$ and unitary isomorphisms $X \otimes \mathbb{1} \cong X$ and $(X \otimes U) \otimes V \cong X \otimes (U \otimes V)$ satisfying standard axioms, see, e.g., [DY13, Section 2.3]. Unless explicitly stated otherwise, we will consider strict module categories, that is, the last two isomorphisms are assumed to be the identity morphisms.

We will be interested in pairs (\mathcal{D}, X) consisting of a nonzero right \mathcal{C} -module category \mathcal{D} and an object $X \in \mathcal{D}$ that is generating in the sense that every object Y in \mathcal{D} is a subobject of $X \otimes U$ for some $U \in \mathcal{C}$. In this case we will say, by slightly abusing the terminology, that the pair (\mathcal{D}, X) is a *singly generated right \mathcal{C} -module category*.

For any compact quantum group G , there is a one-to-one correspondence between the isomorphism classes of unital G - C^* -algebras and the equivalence classes of singly generated $(\text{Rep } G)$ -module categories [DY13; Nes14]. Specifically, the category \mathcal{D}_A associated with a unital G - C^* -algebra A is the category of finitely generated G -equivariant C^* -Hilbert A -modules and the generating object is $X_A = A$. Equivalently, and this is the picture we are going to use, \mathcal{D}_A can be defined in the same way as the category \mathcal{C}_A we considered in Section I.3.2 for Yetter–Drinfeld algebras, but now the morphisms $\iota \otimes T$ are defined only for the morphisms T in $\text{Rep } G$, so \mathcal{D}_A is only a right $(\text{Rep } G)$ -module category rather than a C^* -tensor category. In this picture the generating object is the unit $\mathbb{1} \in \text{Rep } G \subset \mathcal{D}_A$.

Assume A_1 and A_2 are two unital G - C^* -algebras. Consider the corresponding subalgebras $\mathcal{A}_i \subset A_i$ of regular elements and the $(\text{Rep } G)$ -module categories $\mathcal{D}_i = \mathcal{D}_{A_i}$. Then every G -equivariant linear map $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ defines linear maps

$$\Theta_\phi: \mathcal{D}_1(U, V) \rightarrow \mathcal{D}_2(U, V) \quad \text{by} \quad \Theta_\phi(T) = (\phi \otimes \iota)(T) \quad (\text{I.14})$$

for $T \in \mathcal{D}_1(U, V) \subset \mathcal{A}_1 \otimes B(H_U, H_V)$. As we will see shortly, the following gives an axiomatization of such maps.

Definition I.4.1. Given a rigid C^* -tensor category \mathcal{C} with simple unit, a \mathcal{C} -linear transformation $\Theta: (\mathcal{D}_1, X_1) \rightarrow (\mathcal{D}_2, X_2)$ between two singly generated right \mathcal{C} -module categories is a collection of linear maps

$$\Theta_{U,V}: \mathcal{D}_1(X_1 \otimes U, X_1 \otimes V) \rightarrow \mathcal{D}_2(X_2 \otimes U, X_2 \otimes V) \quad (U, V \in \mathcal{C})$$

such that

- (i) $\Theta_{U,V}((\iota \otimes S_1)T(\iota \otimes S_2)) = (\iota \otimes S_1)\Theta_{Y,Z}(T)(\iota \otimes S_2)$ for all $U, V, Y, Z \in \mathcal{C}$, $S_1 \in \mathcal{C}(Z, V)$, $S_2 \in \mathcal{C}(U, Y)$ and $T \in \mathcal{D}_1(X_1 \otimes Y, X_1 \otimes Z)$;
- (ii) $\Theta_{U \otimes Y, V \otimes Y}(T \otimes \iota_Y) = \Theta_{U,V}(T) \otimes \iota_Y$ for all $U, V, Y \in \mathcal{C}$ and $T \in \mathcal{D}_1(X_1 \otimes U, X_1 \otimes V)$.

Such a transformation is called a *cb transformation*, if the maps $\Theta_{U,V}$ are bounded and

$$\|\Theta\|_{\text{cb}} := \sup_{U, V \in \mathcal{C}} \|\Theta_{U,V}\| < \infty.$$

It is called a *cp* (resp., *ucp*, *completely isometric*) *transformation*, if the maps $\Theta_U := \Theta_{U,U}$ are positive (resp., unital positive, isometric).

We will usually write $\Theta(T)$ instead of $\Theta_{U,V}(T)$.

Particular cases of \mathcal{C} -linear transformations have appeared, under different names, in [NY17] and [PV15], see Examples I.4.6 and I.4.7 below. In full generality the \mathcal{C} -linear transformations have been introduced in [JP17, Definition 34] under the name of *multipliers*, which in our opinion should rather be reserved to some special cases.

Remark I.4.2.

(1) For nonstrict module categories the definition is basically the same, but we have to use associativity morphisms in \mathcal{D}_1 and \mathcal{D}_2 to make sense of condition (ii).

(2) Condition (i) means that the maps $\Theta_{U,V}$ are natural in U and V in the sense that they define a natural transformation between the bifunctors $\mathcal{C} \times \mathcal{C} \rightarrow \text{Set}$, $(U, V) \mapsto \mathcal{D}_i(X_i \otimes U, X_i \otimes V)$ ($i = 1, 2$).

(3) Since $\mathcal{D}_i(X_i \otimes U, X_i \otimes V) \otimes \text{Mat}_n(\mathbb{C}) \cong \mathcal{D}_i(X_i \otimes U^n, X_i \otimes V^n)$, each of the spaces $\mathcal{D}_i(X_i \otimes U, X_i \otimes V)$ has an operator space structure. It is then clear that if Θ is a cb transformation, then the maps $\Theta_{U,V}$ are completely bounded and $\|\Theta\|_{\text{cb}} = \sup_{U, V \in \mathcal{C}} \|\Theta_{U,V}\|_{\text{cb}}$. Similarly, if Θ is a completely isometric transformation, then the maps $\Theta_{U,V}$ are completely isometric.

Note that the same operator space structure on $\mathcal{D}_i(X_i \otimes U, X_i \otimes V)$ is defined by identifying $\mathcal{D}_i(X_i \otimes U, X_i \otimes V)$ with a corner of the C^* -algebra $\mathcal{D}_i(X_i \otimes (U \oplus V))$. If $U = V$, then this operator space structure also coincides with the one defined by the C^* -algebra structure on $\mathcal{D}_i(X_i \otimes U)$. If Θ is a cp transformation, then the maps Θ_U are cp.

(4) Any \mathcal{C} -linear transformation Θ is determined by the maps $\Theta_U: \mathcal{D}_1(X_1 \otimes U) \rightarrow \mathcal{D}_2(X_2 \otimes U)$, $U \in \mathcal{C}$. If we use Frobenius reciprocity and decompose into simple objects in \mathcal{C} , then we see that Θ is also determined by the maps $\Theta_{1, U_s}: \mathcal{D}_1(X_1, X_1 \otimes U_s) \rightarrow \mathcal{D}_2(X_2, X_2 \otimes U_s)$, $s \in \text{Irr}(\mathcal{C})$. Furthermore, it is not difficult to show that every collection of linear maps $\mathcal{D}_1(X_1, X_1 \otimes U_s) \rightarrow \mathcal{D}_2(X_2, X_2 \otimes U_s)$, $s \in \text{Irr}(\mathcal{C})$, arises this way, cf. [PV15, Proposition 3.6] and [JP17, Proposition 7]; for the representation categories this will also follow from Proposition I.4.3.

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(5) If a unital \mathcal{C} -linear transformation Θ respects composition of morphisms and involution, then it gives rise to a unitary \mathcal{C} -module functor $\mathcal{D}_1 \rightarrow \mathcal{D}_2$. Namely, by replacing \mathcal{D}_1 by an equivalent category we may assume that the objects $X_1 \otimes U$ ($U \in \mathcal{C}$) are all different. If we denote by $\tilde{\mathcal{D}}_1$ the full subcategory of \mathcal{D}_1 consisting of such objects, then Θ defines a strict unitary \mathcal{C} -module functor $\tilde{\mathcal{D}}_1 \rightarrow \mathcal{D}_2$ such that $X_1 \otimes U \mapsto X_2 \otimes U$. This functor can then be extended to a unitary \mathcal{C} -module functor $\mathcal{D}_1 \rightarrow \mathcal{D}_2$, since by assumption every object of \mathcal{D}_1 is a subobject of $\tilde{\mathcal{D}}_1$. Conversely, any unitary \mathcal{C} -module functor $\mathcal{D}_1 \rightarrow \mathcal{D}_2$ mapping X_1 into X_2 defines a unital \mathcal{C} -linear transformation that respects composition of morphisms and involution.

Proposition I.4.3. *Assume G is a compact quantum group, A_1 and A_2 are unital G - C^* -algebras. Consider the corresponding subalgebras $\mathcal{A}_i \subset A_i$ of regular elements and the right $(\text{Rep } G)$ -module categories $\mathcal{D}_i = \mathcal{D}_{A_i}$. Then the map $\phi \mapsto \Theta_\phi$ defined by (I.14) gives a one-to-one correspondence between the G -equivariant linear maps $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ and the $(\text{Rep } G)$ -linear transformations $(\mathcal{D}_1, \mathbb{1}) \rightarrow (\mathcal{D}_2, \mathbb{1})$. Furthermore, ϕ is cb (resp., cp, ucp, completely isometric) if and only if Θ_ϕ is, and we have*

$$\|\phi\|_{\text{cb}} = \|\Theta_\phi\|_{\text{cb}}. \quad (\text{I.15})$$

To be precise, we say that a G -equivariant linear map $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is cp, if it has a (necessarily unique) extension to a cp map $A_1 \rightarrow A_2$. It can be shown that this is equivalent to requiring $(\phi \otimes \iota)(T^*T)$ to be positive in $A_2 \otimes \text{Mat}_n(\mathbb{C})$ for all $T \in A_1 \otimes \text{Mat}_n(\mathbb{C})$, but we won't use this.

Proof of Proposition I.4.3. We have linear isomorphisms

$$\pi_i: \bigoplus_{s \in \text{Irr}(G)} \bar{H}_s \otimes \mathcal{D}_i(\mathbb{1}, U_s) \rightarrow \mathcal{A}_i, \quad \bar{\xi} \otimes T \mapsto (\iota \otimes \bar{\xi})(T),$$

for $\xi \in H_s$ and $T \in \mathcal{D}_i(\mathbb{1}, U_s) \subset \mathcal{A}_i \otimes B(\mathbb{C}, H_s) = \mathcal{A}_i \otimes H_s$, see [Nes14, Section 2]. Then

$$\phi \pi_1(\bar{\xi} \otimes T) = \pi_2(\bar{\xi} \otimes \Theta_\phi(T)).$$

This shows that the map $\phi \mapsto \Theta_\phi$ is injective. This also implies that given any $(\text{Rep } G)$ -linear transformation $\Theta: (\mathcal{D}_1, \mathbb{1}) \rightarrow (\mathcal{D}_2, \mathbb{1})$, we can define a G -equivariant linear map $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ such that $\Theta_\phi = \Theta$ on $\mathcal{D}_1(\mathbb{1}, U_s)$ for all $s \in \text{Irr}(G)$. But then $\Theta_\phi = \Theta$ by Remark I.4.2(4). Therefore the map $\phi \mapsto \Theta_\phi$ is indeed a bijection.

Next, it is clear by definition that if ϕ is cb, then Θ_ϕ is also cb and $\|\Theta_\phi\|_{\text{cb}} \leq \|\phi\|_{\text{cb}}$. Assume now that Θ_ϕ is cb. Let us show first that ϕ is bounded and $\|\phi\| \leq \|\Theta_\phi\|_{\text{cb}}$. Assume the G -action on A_i is given by $\alpha_i: A_i \rightarrow C(G) \otimes A_i$. Consider the right regular representation $V \in M(K(L^2(G)) \otimes C(G))$ of G . It has the property that

$$V_{31}^*(\alpha_i \otimes \iota)(T)V_{31} = T \quad \text{for all } T \in \alpha_i(A_i)_{21} \in A_i \otimes C(G),$$

where we view the last factor $C(G)$ as a subalgebra of $B(L^2(G))$. We can find a net of finite rank G -invariant projections $p_j \in B(L^2(G))$ converging strongly to 1. Let $V_j \in \text{Rep } G$ be the restriction of V to $p_j L^2(G)$. Then the above identity implies that

$$T_{ij}(a) := (1 \otimes p_j)\alpha_i(a)_{21}(1 \otimes p_j) \in \mathcal{D}_i(V_j) \quad \text{for all } a \in A_i.$$

By the G -equivariance of ϕ we also have

$$\Theta_\phi(T_{1j}(a)) = T_{2j}(\phi(a)) \quad \text{for all } a \in \mathcal{A}_1, \quad (\text{I.16})$$

and therefore $\|T_{2j}(\phi(a))\| \leq \|\Theta_\phi\|_{\text{cb}} \|T_{1j}(a)\|$. Passing to the limit we get $\|\phi(a)\| \leq \|\Theta_\phi\|_{\text{cb}} \|a\|$ for all $a \in \mathcal{A}_1$, as required.

Repeating the above argument with \mathcal{A}_i replaced by $\mathcal{A}_i \otimes \text{Mat}_n(\mathbb{C})$ and V by $V \oplus \cdots \oplus V$, we get that $\|\phi\|_{\text{cb}} \leq \|\Theta_\phi\|_{\text{cb}}$, finishing the proof of (I.15).

The same arguments as above show that ϕ is completely isometric if and only if Θ_ϕ is. It is also obvious that ϕ is unital if and only if Θ_ϕ is.

It remains to deal with complete positivity. It is again clear by definition that if ϕ is cp, then Θ_ϕ is also cp. Assume now that Θ_ϕ is cp. Then from the cb case we can already conclude that ϕ is cb and $\|\phi\|_{\text{cb}} = \|\Theta_\phi\|_{\text{cb}} = \|\phi(1)\|$. Hence ϕ extends to a cb map $A_1 \rightarrow A_2$, which we continue to denote by ϕ . Let us show that ϕ is positive. This can again be deduced from (I.16), but it is also possible to give a more direct algebraic proof as follows.

Take $a \in \mathcal{A}_1$. Then a is the sum of finitely many elements $(\iota \otimes \bar{\xi}_k)(T_k)$, with $\xi_k \in H_{s_k}$, $T_k \in \mathcal{D}_1(\mathbb{1}, U_{s_k})$. Put $U = \bigoplus_k U_{s_k}$. Then we can define $\xi \in H_U$ and $T \in \mathcal{D}_1(\mathbb{1}, U)$ such that $a = (\iota \otimes \bar{\xi})(T)$. It follows that $aa^* = (\iota \otimes \omega_{\xi, \xi})(TT^*)$, where $TT^* \in \mathcal{D}_1(U)$ and $\omega_{\xi, \xi} = (\cdot \xi, \xi)$ is a positive linear functional on $B(H_U)$. Hence

$$\phi(aa^*) = (\iota \otimes \omega_{\xi, \xi})\Theta_\phi(TT^*) \geq 0,$$

so that ϕ is positive. Passing to the matrix algebras over \mathcal{A}_i , a similar argument shows that ϕ is cp. \blacksquare

We are, however, more interested in braided-commutative Yetter–Drinfeld G - C^* -algebras A . In this case, as we discussed in Section I.3.2, the \hat{G} -action defines a tensor structure on \mathcal{D}_A , so that we get a C^* -tensor category \mathcal{C}_A containing $\text{Rep } G$.

Assume again that \mathcal{C} is a rigid C^* -tensor category with simple unit. If we are given a C^* -tensor category \mathcal{B} and a dominant unitary tensor functor $\mathcal{C} \rightarrow \mathcal{B}$, then we say that \mathcal{B} is a \mathcal{C} -tensor category. By replacing \mathcal{B} by an equivalent category we can and will tacitly assume that \mathcal{B} is strict and \mathcal{C} is simply a C^* -tensor subcategory of \mathcal{B} .

Definition I.4.4. A \mathcal{C} -linear transformation $\Theta: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between two \mathcal{C} -tensor categories is a collection of linear maps

$$\Theta_{U,V}: \mathcal{C}_1(U, V) \rightarrow \mathcal{C}_2(U, V) \quad (U, V \in \mathcal{C})$$

such that

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- (i) $\Theta(S_1 T S_2) = S_1 \Theta(T) S_2$ for all $U, V, Y, Z \in \mathcal{C}$, $S_1 \in \mathcal{C}(Z, V)$, $S_2 \in \mathcal{C}(U, Y)$ and $T \in \mathcal{C}_1(Y, Z)$;
- (ii) $\Theta(\iota_Z \otimes T \otimes \iota_Y) = \iota_Z \otimes \Theta(T) \otimes \iota_Y$ for all $U, V, Y, Z \in \mathcal{C}$ and $T \in \mathcal{C}_1(U, V)$.

In other words, a \mathcal{C} -linear transformation $\Theta: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a collection of maps $\Theta_{U,V}: \mathcal{C}_1(U, V) \rightarrow \mathcal{C}_2(U, V)$ that defines \mathcal{C} -linear transformations $(\mathcal{C}_1, \mathbb{1}) \rightarrow (\mathcal{C}_2, \mathbb{1})$ if we consider \mathcal{C}_i separately as left and right \mathcal{C} -module categories. Equivalently, it is a \mathcal{C} -linear transformation $(\mathcal{C}_1, \mathbb{1}) \rightarrow (\mathcal{C}_2, \mathbb{1})$ of right \mathcal{C} -module categories satisfying the extra condition

$$\Theta(\iota_Y \otimes T) = \iota_Y \otimes \Theta(T)$$

for all $U, V, Y \in \mathcal{C}$ and $T \in \mathcal{C}_1(U, V)$.

Proposition I.4.5. *Assume G is a compact quantum group, A_1 and A_2 are unital braided-commutative Yetter–Drinfeld G - C^* -algebras. Consider the corresponding subalgebras $\mathcal{A}_i \subset A_i$ of regular elements and the $(\text{Rep } G)$ -tensor categories $\mathcal{C}_i = \mathcal{C}_{\mathcal{A}_i}$. Then a G -equivariant linear map $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is \hat{G} -equivariant if and only if the corresponding $(\text{Rep } G)$ -linear transformation $\Theta_\phi: (\mathcal{C}_1, \mathbb{1}) \rightarrow (\mathcal{C}_2, \mathbb{1})$ of singly generated right $(\text{Rep } G)$ -module categories is a $(\text{Rep } G)$ -linear transformation $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ of $(\text{Rep } G)$ -tensor categories.*

Proof. This follows immediately from the definition of the tensor structure on \mathcal{C}_i in terms of the \hat{G} -action, see (I.10). ■

Example I.4.6. Given a \mathcal{C} -tensor category \mathcal{B} , a ucp transformation $\hat{\mathcal{C}} \rightarrow \mathcal{B}$ (see the next subsection for the definition of $\hat{\mathcal{C}}$) is the same thing as a right invariant mean on the functor $\mathcal{C} \rightarrow \mathcal{B}$ in the sense of [NY17, Definition 6.1].

Example I.4.7 (cf. [JP17, Example 15]). We can view \mathcal{C} as a \mathcal{C} -bimodule category, or equivalently, as a right $(\mathcal{C}^{\text{op}} \boxtimes \mathcal{C})$ -module category. Then a $(\mathcal{C}^{\text{op}} \boxtimes \mathcal{C})$ -linear transformation $(\mathcal{C}, \mathbb{1}) \rightarrow (\mathcal{C}, \mathbb{1})$ is the same thing as a multiplier on \mathcal{C} in the sense of [PV15, Definition 3.4] (see also [PV15, Proposition 3.6]). As an example consider the representation category $\mathcal{C} = \text{Rep } G$ of a compact quantum group G . Then the C^* -algebra corresponding to the bimodule category $\text{Rep } G$ and the generating object $\mathbb{1}$ is $C(G)$ equipped with its usual left and right actions of G by translations. Therefore by Proposition I.4.3 we get a one-to-one correspondence between the cb-multipliers (resp., cp-multipliers) on $\text{Rep } G$ and the G - G -equivariant cb (resp., cp) maps $C(G) \rightarrow C(G)$, thus recovering [PV15, Proposition 6.1], but also proving equality (I.15).

On the other hand, if we view \mathcal{C} either as a right or left \mathcal{C} -module category with generating object $\mathbb{1}$ or as a \mathcal{C} -tensor category, then $\mathcal{C}(\mathbb{1}, U_s) = 0$ for all $s \neq e$ and the only \mathcal{C} -linear transformations Θ on \mathcal{C} are the scalars, that is, $\Theta_{U,V}: \mathcal{C}(U, V) \rightarrow \mathcal{C}(U, V)$ is the multiplication by a scalar, the same one for all U and V .

Example I.4.8. Consider a compact quantum group G . The forgetful functor $\text{Rep } G \rightarrow \text{Hilb}_f$ allows us to view the tensor category Hilb_f of finite dimensional

Hilbert spaces as a right $(\text{Rep } G)$ -module category. The G - C^* -algebra corresponding to $(\text{Hilb}_f, \mathbb{C})$ is $C(G)$ equipped with the action of G by left translations. The G -equivariant linear maps $\mathbb{C}[G] \rightarrow \mathbb{C}[G]$ have the form $m_\phi = (\iota \otimes \phi)\Delta$ for linear functionals ϕ on $\mathbb{C}[G]$. If m_ϕ is cb (resp., cp), then ϕ is called a cb (resp., cp) multiplier, or a Herz–Schur multiplier, on \hat{G} . By Proposition I.4.3 we therefore get a one-to-one correspondence between the cb (resp., cp) multipliers on \hat{G} and the cb (resp., cp) $(\text{Rep } G)$ -linear transformations $(\text{Hilb}_f, \mathbb{C}) \rightarrow (\text{Hilb}_f, \mathbb{C})$.

It is known that $\phi \in \mathbb{C}[G]^*$ is a cp multiplier if and only if the linear functional ϕ is positive, see [DH85, Proposition 4.2], [Daw12, Theorem 5.2]. Let us give a quick proof of this fact. If ϕ is positive, then it extends to a positive linear functional on the universal completion $C_u(G)$ of $\mathbb{C}[G]$. As the comultiplication Δ on $\mathbb{C}[G]$ extends to a $*$ -homomorphism $C(G) \rightarrow C(G) \otimes C_u(G)$, it follows that $m_\phi = (\iota \otimes \phi)\Delta$ extends to a cp map on $C(G)$. Conversely, assume ϕ is a cp multiplier. Take $a \in \mathbb{C}[G]$. As in the last part of the proof of Proposition I.4.3, we can find an element $T \in \mathcal{D}_{C(G)}(\mathbb{1}, U)$ and a state ω on $B(H_U)$ such that $aa^* = (\iota \otimes \omega)(TT^*)$. Then $(m_\phi \otimes \iota)(TT^*)$ is a positive element of $\mathcal{D}_{C(G)}(U)$, so it has the form SS^* for some $S \in \mathcal{D}_{C(G)}(U) \subset \mathbb{C}[G] \otimes B(H_U)$. But then

$$\phi(aa^*) = \varepsilon(m_\phi(aa^*)) = \omega((\varepsilon \otimes \iota)(S)(\varepsilon \otimes \iota)(S)^*) \geq 0,$$

so that ϕ is positive.

We thus get a one-to-one correspondence between the positive linear functionals on $\mathbb{C}[G]$ and the cp $(\text{Rep } G)$ -linear transformations $(\text{Hilb}_f, \mathbb{C}) \rightarrow (\text{Hilb}_f, \mathbb{C})$. This recovers [JP17, Proposition 15].

I.4.2 Furstenberg–Hamana boundaries of monoidal categories

We continue to assume that \mathcal{C} is a rigid C^* -tensor category with simple unit. We say that a \mathcal{C} -tensor category \mathcal{A} is \mathcal{C} -*injective* if, given \mathcal{C} -tensor categories \mathcal{B} and \mathcal{D} , a completely isometric ucp transformation $\Phi: \mathcal{B} \rightarrow \mathcal{D}$ and a ucp transformation $\Psi: \mathcal{B} \rightarrow \mathcal{A}$, there is a ucp transformation $\tilde{\Psi}: \mathcal{D} \rightarrow \mathcal{A}$ making the diagram

$$\begin{array}{ccc} \mathcal{D} & & \\ \uparrow \Phi & \searrow \tilde{\Psi} & \\ \mathcal{B} & \xrightarrow{\Psi} & \mathcal{A} \end{array}$$

commutative, that is, $\tilde{\Psi}_{U,V} = \tilde{\Psi}_{U,V} \Phi_{U,V}$ for all $U, V \in \mathcal{C}$.

Definition I.4.9. A \mathcal{C} -tensor category \mathcal{A} is called a *Furstenberg–Hamana boundary* of \mathcal{C} , if \mathcal{A} is \mathcal{C} -injective and, for all \mathcal{C} -tensor categories \mathcal{B} , every ucp transformation $\mathcal{A} \rightarrow \mathcal{B}$ is completely isometric.

Let us introduce the following terminology. Given two \mathcal{C} -tensor categories \mathcal{A} and \mathcal{B} , by a \mathcal{C} -*tensor functor* $\mathcal{A} \rightarrow \mathcal{B}$ we mean a unitary tensor functor $\mathcal{A} \rightarrow \mathcal{B}$ such that its composition with the functor $\mathcal{C} \rightarrow \mathcal{A}$ is naturally unitarily

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monoidally isomorphic to the functor $\mathcal{C} \rightarrow \mathcal{B}$. If \mathcal{C} is a subcategory of \mathcal{A} and \mathcal{B} , as we usually assume, then a \mathcal{C} -tensor functor can be assumed to be identical on \mathcal{C} , see the discussion in [NY14, Section 2.1]. We say that \mathcal{A} and \mathcal{B} are equivalent as \mathcal{C} -tensor categories, if there is a \mathcal{C} -tensor functor $\mathcal{A} \rightarrow \mathcal{B}$ that is an equivalence of categories.

Theorem I.4.10. *For any rigid C^* -tensor category \mathcal{C} with simple unit, there is a unique up to equivalence Furstenberg–Hamana boundary of \mathcal{C} .*

We denote the Furstenberg–Hamana boundary of \mathcal{C} by $\partial_{\text{FH}}\mathcal{C}$.

In order to prove the theorem we need a categorical analogue of $\mathcal{R}(\ell^\infty(\hat{G}))$. This is the \mathcal{C} -tensor category $\hat{\mathcal{C}}$ introduced in [NY17]. For $U, V \in \mathcal{C}$, the morphism space $\hat{\mathcal{C}}(U, V)$ is defined as the space $\text{Nat}_b(\iota \otimes U, \iota \otimes V)$ of bounded natural transformations between the functors $\iota \otimes U$ and $\iota \otimes V$ of tensoring on the right on \mathcal{C} . The tensor product of natural transformations is defined by the following rules. Given $\nu = (\nu_X: X \otimes U \rightarrow X \otimes V)_{X \in \mathcal{C}} \in \hat{\mathcal{C}}(U, V)$, we have

$$(\nu \otimes \iota_Y)_X = \nu_X \otimes \iota_Y, \quad (\iota_Y \otimes \nu)_X = \nu_{X \otimes Y}.$$

The following is a (partial) categorical analogue of Proposition I.2.5.

Proposition I.4.11. *For any \mathcal{C} -tensor category \mathcal{A} , there is a one-to-one correspondence between the ucp transformations $\mathcal{A} \rightarrow \hat{\mathcal{C}}$ and the states on the C^* -algebra $\mathcal{A}(\mathbb{1}) = \text{End}_{\mathcal{A}}(\mathbb{1})$.*

We denote by \mathcal{P}_ψ the ucp transformation $\mathcal{A} \rightarrow \hat{\mathcal{C}}$ defined by a state ψ on $\mathcal{A}(\mathbb{1})$, and call it a *Poisson transformation*.

Proof. Assume $\mathcal{P}: \mathcal{A} \rightarrow \hat{\mathcal{C}}$ is a ucp transformation. Composing $\mathcal{P}: \mathcal{A}(\mathbb{1}) \rightarrow \hat{\mathcal{C}}(\mathbb{1}) \cong \ell^\infty(\text{Irr}(\mathcal{C}))$ with the evaluation at $e \in \text{Irr}(\mathcal{C})$, we get a state ψ on $\mathcal{A}(\mathbb{1})$. Take an object U in \mathcal{C} and choose a standard solution (R_U, \bar{R}_U) of the conjugate equations for U in \mathcal{C} . Then we can define a state ψ_U on $\mathcal{A}(U)$ by

$$\psi_U(T) = d^{\mathcal{C}}(U)^{-1} \psi(\bar{R}_U(T \otimes \iota) \bar{R}_U^*),$$

where $d^{\mathcal{C}}(U) = \|R_U\|^2 = \|\bar{R}_U\|^2$ is the intrinsic dimension of U in \mathcal{C} . The same argument as in the proof of [NY17, Lemma 4.1] shows that, as ψ_U is independent of the choice of standard solutions, $\mathcal{C}(U)$ is contained in the centralizer of ψ_U . Since $\mathcal{C}(U)$ is finite dimensional and ψ_U is faithful on it, it follows that there is a unique ψ_U -preserving conditional expectation $E_U: \mathcal{A}(U) \rightarrow \mathcal{C}(U)$.

We claim that \mathcal{P} is given by

$$\mathcal{P}(T)_X = E_{X \otimes U, X \otimes U}(\iota_X \otimes T): X \otimes U \rightarrow X \otimes U \quad (\text{I.17})$$

for all $U, X \in \mathcal{C}$, $T \in \mathcal{A}(U)$. In order to prove this, let us first of all observe that we have

$$\psi_U(T) = \text{tr}_U^{\mathcal{C}}(\mathcal{P}(T)_{\mathbb{1}}) \quad \text{for all } T \in \mathcal{A}(U),$$

since $\mathcal{P}(\bar{R}_U^*(T \otimes \iota_{\bar{U}})R_U)_{\mathbb{1}} = \bar{R}_U^*(\mathcal{P}(T)_{\mathbb{1}} \otimes \iota_{\bar{U}})\bar{R}_U$, where $\text{tr}^{\mathcal{C}}$ is the normalized categorical trace. Since we also have

$$\mathcal{P}(T)_X = \mathcal{P}(T)_{X \otimes \mathbb{1}} = (\iota_X \otimes \mathcal{P}(T))_{\mathbb{1}} = \mathcal{P}(\iota_X \otimes T)_{\mathbb{1}},$$

we get, for all $U, X \in \mathcal{C}$, $T \in \mathcal{A}(U)$ and $S \in \mathcal{C}(X \otimes U)$, that

$$\begin{aligned} \mathrm{tr}_{X \otimes U}^{\mathcal{C}}(S\mathcal{P}(T)_X) &= \mathrm{tr}_{X \otimes U}^{\mathcal{C}}(S\mathcal{P}(\iota_X \otimes T)_{\mathbf{1}}) \\ &= \mathrm{tr}_{X \otimes U}^{\mathcal{C}}(\mathcal{P}(S(\iota_X \otimes T))_{\mathbf{1}}) = \psi_{X \otimes U}(S(\iota_X \otimes T)), \end{aligned}$$

which implies (I.17). Therefore \mathcal{P} is completely determined by ψ .

Conversely, starting from an arbitrary state ψ on $\mathcal{A}(\mathbf{1})$ we define conditional expectations E_U as above and then define $\mathcal{P}: \mathcal{A}(U) \rightarrow \hat{\mathcal{C}}(U)$ by (I.17). By identifying $\mathcal{A}(U, V)$ with a corner of $\mathcal{A}(U \oplus V)$, we also get maps $\mathcal{A}(U, V) \rightarrow \hat{\mathcal{C}}(U, V)$. It is straightforward to check that \mathcal{P} is a ucp transformation, cf. [NY17, Lemma 4.5]. \blacksquare

Proof of Theorem I.4.10. Consider the convex semigroup S of ucp \mathcal{C} -linear transformations $\hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$. We can think of the elements of S as maps on the Banach space

$$\ell^\infty\text{-}\bigoplus_{U, V \in \mathcal{C}} \hat{\mathcal{C}}(U, V) \cong \ell^\infty\text{-}\bigoplus_{s \in \mathrm{Irr}(\mathcal{C}), U, V \in \mathcal{C}} \mathcal{C}(U_s \otimes U, U_s \otimes V),$$

which is dual to $\ell^1\text{-}\bigoplus_{s \in \mathrm{Irr}(\mathcal{C}), U, V \in \mathcal{C}} \mathcal{C}(U_s \otimes U, U_s \otimes V)^*$. It is not difficult to see that S is closed in the topology of pointwise weak* convergence. Hence, by Proposition I.2.1, there is a minimal idempotent $\Theta \in S$. We define a new \mathcal{C} -tensor category \mathcal{A} , with

$$\mathcal{A}(U, V) = \Theta(\hat{\mathcal{C}}(U, V)) \quad (U, V \in \mathcal{C})$$

and the composition of morphisms given by the Choi–Effros product $T_1 \cdot T_2 = \Theta(T_1 T_2)$, cf. [NY17, Section 2]. The rest of the argument, showing that \mathcal{A} is a unique up to equivalence Furstenberg–Hamana boundary of \mathcal{C} , is similar to the proof of Theorem I.2.4 (starting from Step 4), but now using Proposition I.4.11 instead of Proposition I.2.5. We skip the details. \blacksquare

Recall that by Corollary I.2.8 the Yetter–Drinfeld C^* -algebras $C(\partial_{\mathrm{FH}} D(G))$ are braided-commutative. Hence we have $(\mathrm{Rep} G)$ -tensor categories associated with them.

Proposition I.4.12. *For any compact quantum group G , the $(\mathrm{Rep} G)$ -tensor category associated with $C(\partial_{\mathrm{FH}} D(G))$ is equivalent to $\partial_{\mathrm{FH}}(\mathrm{Rep} G)$.*

Proof. In view of Propositions I.4.3 and I.4.5, we have a one-to-one correspondence between the $D(G)$ -equivariant ucp maps from one braided-commutative Yetter–Drinfeld G - C^* -algebra into another, and the ucp transformations between the associated $(\mathrm{Rep} G)$ -tensor categories. This implies that the $(\mathrm{Rep} G)$ -tensor category associated with $C(\partial_{\mathrm{FH}} D(G))$ has the defining properties of $\partial_{\mathrm{FH}}(\mathrm{Rep} G)$, hence it is equivalent to $\partial_{\mathrm{FH}}(\mathrm{Rep} G)$. \blacksquare

This result follows also from the construction of $C(\partial_{\mathrm{FH}} D(G))$ and $\partial_{\mathrm{FH}}(\mathrm{Rep} G)$, since the semigroups S used in both cases become literally the same, hence they

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have the same minimal idempotents, under the correspondence between the $D(G)$ -equivariant ucp maps on $\mathcal{R}(\ell^\infty(\hat{G}))$ and the ucp $(\text{Rep } G)$ -linear transformations on $\widehat{\text{Rep } G}$.

I.4.3 Monoidal categories with weakly amenable dimension functions

The Markov operators P_s introduced in Section I.3.1 have a categorical analogue [NY17]. Namely, using the notation of the previous subsection, for $\nu = (\nu_X)_X \in \hat{\mathcal{C}}(U, V)$, we define

$$P_s(\nu)_X = (\text{tr}_{U_s}^{\mathcal{C}} \otimes \iota)(\nu_{U_s \otimes X}) \in \mathcal{C}(X \otimes U, X \otimes V).$$

Therefore $P_s: \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$ is a ucp \mathcal{C} -linear transformation. Recall then that a probability measure μ on $\text{Irr}(\mathcal{C})$ is called *ergodic*, if the constants are the only P_μ -harmonic elements in $\hat{\mathcal{C}}(\mathbb{1}) \cong \ell^\infty(\text{Irr}(\mathcal{C}))$, where $P_\mu = \sum_{s \in \text{Irr}(\mathcal{C})} \mu(s)P_s$.

Theorem I.4.13. *Assume \mathcal{C} is a rigid C^* -tensor category with simple unit and weakly amenable intrinsic dimension function. Then there is a unique up to equivalence \mathcal{C} -tensor category $\partial_\Pi \mathcal{C}$ with simple unit such that the following conditions are satisfied:*

- (i) *the dimension function on \mathcal{C} defined by the intrinsic dimension function on $\partial_\Pi \mathcal{C}$ is amenable;*
- (ii) *if \mathcal{B} is another \mathcal{C} -tensor category with simple unit that defines the amenable dimension function on \mathcal{C} , then there is a \mathcal{C} -tensor functor $\partial_\Pi \mathcal{C} \rightarrow \mathcal{B}$, unique up to natural unitary monoidal isomorphism.*

Furthermore, if ϕ is the unique state on $\text{End}_{\partial_\Pi \mathcal{C}}(\mathbb{1}) \cong \mathbb{C}$, then the Poisson transformation $\mathcal{P}_\phi: \partial_\Pi \mathcal{C} \rightarrow \hat{\mathcal{C}}$ is completely isometric and, for all $U, V \in \mathcal{C}$, the image of $\text{Mor}_{\partial_\Pi \mathcal{C}}(U, V)$ under \mathcal{P}_ϕ is

$$\{\nu \in \hat{\mathcal{C}}(U, V) \mid P_s(\nu) = \nu \text{ for all } s \in \text{Irr}(\mathcal{C})\}.$$

If $\text{Irr}(\mathcal{C})$ is countable, then the last space coincides with the space of P_μ -harmonic bounded natural transformations $\iota \otimes U \rightarrow \iota \otimes V$ for any ergodic probability measure μ on $\text{Irr}(\mathcal{C})$.

Proof. When $\text{Irr}(\mathcal{C})$ is countable, this is a reformulation of Theorems 4.1 and 5.1 in [NY17]. The result is then extended to arbitrary categories similarly to the proof of Theorem I.3.7, by considering a net of full subcategories $(\mathcal{C}_i)_i$ with countable $\text{Irr}(\mathcal{C}_i)$ and weakly amenable intrinsic dimension functions. We omit the details. ■

Theorem I.4.14. *For any rigid C^* -tensor category \mathcal{C} with simple unit and weakly amenable intrinsic dimension function, the \mathcal{C} -tensor categories $\partial_{\text{FH}} \mathcal{C}$ and $\partial_\Pi \mathcal{C}$ are equivalent.*

Proof. The proof is similar to that of Theorem I.3.10. Let us show that $\partial_{\Pi}\mathcal{C}$ is \mathcal{C} -injective. Assume first that $\text{Irr}(\mathcal{C})$ countable. Choose an ergodic probability measure μ on $\text{Irr}(\mathcal{C})$, a free ultrafilter ω on \mathbb{N} and define an idempotent ucp transformation $E: \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$ by

$$E(\nu)_X = \lim_{n \rightarrow \omega} \frac{1}{n} \sum_{k=1}^n P_{\mu}^k(\nu)_X$$

for all $\nu \in \hat{\mathcal{C}}(U, V)$ and $X \in \mathcal{C}$. Then the image of E consists exactly of the P_{μ} -harmonic bounded natural transformations, hence, by Theorem I.4.13, it coincides with the image of \mathcal{P}_{ϕ} . As \mathcal{P}_{ϕ} is completely isometric and $\hat{\mathcal{C}}$ is \mathcal{C} -injective by Proposition I.4.11, it follows that $\partial_{\Pi}\mathcal{C}$ is \mathcal{C} -injective as well. The case when $\text{Irr}(\mathcal{C})$ is uncountable is dealt with similarly to the proof of Corollary I.3.8, we omit the details.

Next, assume $\Theta: \partial_{\Pi}\mathcal{C} \rightarrow \mathcal{B}$ is a ucp transformation for some \mathcal{C} -tensor category \mathcal{B} . Consider any ucp transformation $\Psi: \mathcal{B} \rightarrow \hat{\mathcal{C}}$. Then $\Psi\Theta = \mathcal{P}_{\phi}$ by Proposition I.4.11, as ϕ is the only state on $\text{End}_{\partial_{\Pi}\mathcal{C}}(\mathbb{1})$. Thus, $\Psi\Theta$ is completely isometric, so Θ must be completely isometric as well. Therefore $\partial_{\Pi}\mathcal{C}$ is a Furstenberg–Hamana boundary of \mathcal{C} . ■

The following result is a generalization of Corollary I.3.11.

Corollary I.4.15. *For any rigid C^* -tensor category \mathcal{C} with simple unit, the category $\partial_{\text{FH}}\mathcal{C}$ has simple unit if and only if \mathcal{C} has weakly amenable intrinsic dimension function.*

Proof. The “if” direction is immediate, as $\partial_{\Pi}\mathcal{C}$ has simple unit by definition. Conversely, assume $\partial_{\text{FH}}\mathcal{C}$ has simple unit. By the \mathcal{C} -injectivity there is a ucp transformation $\Theta: \hat{\mathcal{C}} \rightarrow \partial_{\text{FH}}\mathcal{C}$. In the terminology of [NY17] this means that the functor $\mathcal{C} \rightarrow \partial_{\text{FH}}\mathcal{C}$ is amenable. Then by [NY17, Lemma 6.4] the map $\Theta_{\mathbb{1}}$ is an invariant mean on $\ell^{\infty}(\text{Irr}(\mathcal{C}))$, so that the intrinsic dimension function on \mathcal{C} is weakly amenable. ■

The following is a generalization of Corollary I.3.12.

Corollary I.4.16. *For any rigid C^* -tensor category \mathcal{C} with simple unit, the Furstenberg–Hamana boundary of \mathcal{C} is trivial if and only if \mathcal{C} is amenable.*

The triviality here means that the \mathcal{C} -tensor category $\partial_{\text{FH}}\mathcal{C}$ is equivalent to \mathcal{C} .

Proof. Again, the “if” direction is immediate, as if \mathcal{C} is amenable, then \mathcal{C} has the defining properties of $\partial_{\Pi}\mathcal{C}$, so that we can take $\partial_{\text{FH}}\mathcal{C} = \partial_{\Pi}\mathcal{C} = \mathcal{C}$. Conversely, assume $\partial_{\text{FH}}\mathcal{C}$ is trivial. By the previous corollary, the intrinsic dimension function on \mathcal{C} is weakly amenable. But then $\partial_{\Pi}\mathcal{C}$ is equivalent to $\partial_{\text{FH}}\mathcal{C} \sim \mathcal{C}$, which means that the intrinsic dimension function on \mathcal{C} is amenable. ■

The q -deformations provide arguably the most important examples of nonamenable C^* -tensor categories with weakly amenable intrinsic dimension

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functions. For these categories we have, thanks to Proposition I.4.12, the following reformulation of Examples I.3.16, I.3.19 and I.3.20.

Example I.4.17. Assume G is a compact connected semisimple Lie group and consider its q -deformation G_q , $q > 0$. Let $T \subset G_q$ be the nondeformed maximal torus. Then, for $q \neq 1$, we have $\partial_{\text{FH}}(\text{Rep } G_q) = \text{Rep } T$, with the functor $\text{Rep } G_q \rightarrow \text{Rep } T$ being simply the forgetful functor. For $q = 1$, the Furstenberg–Hamana boundary of $\text{Rep } G$ is trivial.

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