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Measures of component importance in repairable multistate systems - a numerical study

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Abstract

In [10] dynamic and stationary measures of importance of a component in a repairable multistate system were introduced. For multistate systems little has been published until now on such measures even in the nonrepairable case. According to the Barlow-Proschan type measures a component is important if there is a high probability that a change in the component state causes a change in whether or not the system state is above a given state. On the other hand, the Natvig type measures focus on how a change in the component state affects the expected system uptime and downtime relative to the given system state. In the present paper we first review these measures which can be estimated using the simulation methods suggested in [4]. Extending the work in [8] from the binary to the multistate case, a numerical study of these measures is then given for two three component systems, a bridge system and also applied to an offshore oil and gas production system. In the multistate case the importance of a component is calculated separately for each component state. Thus it may happen that a component is very important at one state, and less important, or even irrelevant at another. Unified measures combining the importances for all component states can be obtained by adding up the importance measures for each individual state. According to these unified measures a component can be important relative to a given system state but not to another. It can be seen that if the distributions of the total component times spent in the non complete failure states for the multistate system and the component lifetimes for the binary system are identical, the Barlow-Proschan measure to the lowest system state simply reduces to the binary version of the measure. The extended Natvig measure, however, does not have this property. This indicates that the latter measure captures more information about the system.

Keywords: Importance measures; Repairable systems; Discrete event simulation; Birnbaum measure; Barlow-Proschan measure; Natvig measure

1. Basic ideas, concepts and results

There seem to be two main reasons for coming up with a measure of importance of system components. Reason 1: it permits the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort. Reason 2: it may suggest the most efficient way to diagnose system failure by generating a repair checklist for an operator to follow. It should be noted that no measure of importance can be expected to be universally best irrespective of usage purpose. In this paper we will concentrate on what could be considered as allround measures of component importance. It extends the work in [8] from the binary to the multistate case.

In [10] dynamic and stationary measures of importance of a component in a repairable multistate system were introduced. For multistate systems little has been published until now on such measures

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even in the nonrepairable case. According to the Barlow-Proschan type measures a component is important if there is a high probability that a change in the component state causes a change in whether or not the system state is above a given state. On the other hand, the Natvig type measures focus on how a change in the component state affects the expected system uptime and downtime relative to the given system state. In the present paper we first review these measures which can be estimated using the simulation methods suggested in [4]. Extending the work in [8] from the binary to the multistate case, a numerical study of these measures is then given for two three component systems, a bridge system and also applied to an offshore oil and gas production system. In the multistate case the importance of a component is calculated separately for each component state. Thus it may happen that a component is very important at one state, and less important, or even irrelevant at another. Unified measures combining the importances for all component states can be obtained by adding up the importance measures for each individual state. According to these unified measures a component can be important relative to a given system state but not to another. It can be seen that if the distributions of the total component times spent in the non complete failure states for the multistate system and the component lifetimes for the binary system are identical, the Barlow-Proschan measure to the lowest system state simply reduces to the binary version of the measure. The extended Natvig measure, however, does not have this property. This indicates that the latter measure captures more information about the system.

Let $S = \{0, 1, \dots, M\}$ be the set of states of the system; the $M+1$ states representing successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Furthermore, let $C = \{1, \dots, n\}$ be the set of components and in general S_i, $i = 1, \dots, n$ the set of states of the ith component. We claim $\{0, M\} \subseteq S_i \subseteq S$. Hence, the states 0 and M are chosen to represent the endpoints of a performance scale that might be used for both the system and its components. Note that in most applications there is no need for the same detailed description of the components as for the system.

Let x_i, $i = 1, \dots, n$ denote the state or performance level of the ith component at a fixed point of time and $\boldsymbol{x} = (x_1, \dots, x_n)$. It is assumed that the state, ϕ, of the system at the fixed point of time is a deterministic function of $\boldsymbol{x}$; i.e. $\phi = \phi(\boldsymbol{x})$. Here $\boldsymbol{x}$ takes values in $S_1 \times S_2 \times \cdots \times S_n$ and ϕ takes values in S. The function ϕ is called the structure function of the system. We often denote a multistate system by (C, ϕ).

We now consider the relation between the stochastic performance of the system (C, ϕ) and the stochastic performances of the components. Introduce the random state $X_i(t)$ of the ith component at time t, $i = 1, \dots, n$ and the corresponding vector $\boldsymbol{X}(t) = (X_1(t), \dots, X_n(t))$. If ϕ is a multistate structure function, $\phi(\boldsymbol{X}(t))$ is the corresponding random system state at time t. Assume also that the stochastic processes $\{X_i(t), t \in [0, \infty)\}$, $i = 1, \dots, n$, are mutually independent. For the dynamic approach of the present paper this is a necessary assumption in order to arrive at explicit results.

In this and the subsequent section we consider the case where the components, and hence the system, can be repaired. In order to make things not too complex we assume that each component deteriorates by going through all of the states from the perfect functioning state until the complete failure state before being repaired to the perfect functioning state. Also at time $t = 0$ all components are in the perfect functioning state M. Let the ith component have an absolutely continuous distribution $F_i^k(t)$ of time spent in the state k, before jumping downwards to state $k-1$, with density $f_i^k(t)$, $\bar{F}_i^k(t) = 1 - F_i^k(t)$ and mean μ_i^k. Furthermore, let the ith component have an absolutely continuous repair time distribution $G_i(t)$ with density $g_i(t)$, $\bar{G}_i(t) = 1 - G_i(t)$ and mean μ_i^0. It is assumed that all these times spent in the various states are independent.

For $j = 0, \dots, M$ we introduce the notation:

$$P(X_i(t) = j) = a_i^j(t).$$

We also introduce the vector $\boldsymbol{a}(t)$ given by:

$$\boldsymbol{a}(t) = (a_1^1(t), \dots, a_1^M(t), a_2^1(t), \dots, a_n^M(t)).$$

At the system level we let:

$$P[\phi(\boldsymbol{X}(t)) \geq j] = p_\phi^j(\boldsymbol{a}(t)).$$

Note that:

$$p_\phi^j(\boldsymbol{a}(t)) = P[I(\phi(\boldsymbol{X}(t)) \geq j) = 1]$$

We denote $a_i^j(t)$ the availability of the ith component *at* level j at time t and $p_\phi^j(\boldsymbol{a}(t))$ the availability of the system at level j or above at time t. The corresponding stationary availabilities for $i = 1, \ldots, n$ and $j \in \{0, \ldots, M\}$ are given by:

$$a_i^j = \lim_{t\to\infty} a_i^j(t) = \frac{\mu_i^j}{\sum_{\ell=0}^M \mu_i^\ell}. \tag{1}$$

Introduce:

$$\boldsymbol{a} = (a_1^1, \ldots, a_1^M, a_2^1, \ldots, a_n^M),$$

and the M-dimensional row vectors:

$$\boldsymbol{e}^k = (1_k, \boldsymbol{0}),\ k = 1, \ldots, M \quad \boldsymbol{e}^0 = \boldsymbol{0}.$$

Now for $i = 1, \ldots, n$ and $k, j \in \{1, \ldots, M\}$ the generalized Birnbaum [2] measure in repairable systems is expressed as:

$$I_B^{(i,k,j)}(t) = p_\phi^j((\boldsymbol{e}^k)_i, \boldsymbol{a}(t)) - p_\phi^j((\boldsymbol{e}^{k-1})_i, \boldsymbol{a}(t)) \tag{2}$$

$$I_B^{(i,j)}(t) = \sum_{k=1}^M I_B^{(i,k,j)}(t) / \sum_{r=1}^n \sum_{k=1}^M I_B^{(r,k,j)}(t). \tag{3}$$

We obviously have:

$$\sum_{i=1}^n I_B^{(i,j)}(t) = 1, \qquad 0 \leq I_B^{(i,j)}(t) \leq 1. \tag{4}$$

For $i = 1, \ldots, n$ and $k, j \in \{1, \ldots, M\}$ the corresponding stationary measures are given by:

$$\begin{aligned} I_B^{(i,k,j)} &= \lim_{t\to\infty} I_B^{(i,k,j)}(t) \\ &= p_\phi^j((\boldsymbol{e}^k)_i, \boldsymbol{a}) - p_\phi^j((\boldsymbol{e}^{k-1})_i, \boldsymbol{a}). \end{aligned} \tag{5}$$

$$I_B^{(i,j)} = \sum_{k=1}^M I_B^{(i,k,j)} / \sum_{r=1}^n \sum_{k=1}^M I_B^{(r,k,j)}. \tag{6}$$

The generalized Barlow-Proschan [1] stationary measure is given by

$$\begin{aligned} I_{B-P}^{(i,j)} &= \lim_{t\to\infty} I_{B-P}^{(i,j)}(t) \\ &= \frac{\sum_{k=1}^M I_B^{(i,k,j)} / (\sum_{\ell=0}^M \mu_i^\ell)}{\sum_{r=1}^n \sum_{k=1}^M I_B^{(r,k,j)} / (\sum_{\ell=0}^M \mu_r^\ell)}. \end{aligned} \tag{7}$$

$I_{B-P}^{(i,j)}$ is the stationary probability that the jump downwards of the ith component is the cause of the system leaving the better states $\{j, \ldots, M\}$, given that the system has left these states, $j \in \{1, \ldots, M\}$.

We now turn to the Natvig [5], [6], [7], [9],[10] type measures and start by introducing some basic random variables for $i = 1, \ldots, n$, $k \in \{0, \ldots, M\}$, $m = 1, 2, \ldots$

$T_{i,k,m}$ = the time of the mth jump of the ith component into state k.

$D_{i,m}$ = the length of the mth repair time of the ith component.

We define $T_{i,M,0} = 0$ and have for $m = 1, 2, \ldots$:

$$T_{i,M,m} = T_{i,0,m} + D_{i,m}.$$

We argue that components that by deteriorating, strongly reduce the expected system time in the better states, are very important. In order to formalize this, we consider a time interval $[0, t]$ and introduce for $i = 1, \ldots, n$, $k \in \{0, \ldots, M-1\}$, $m = 1, 2, \ldots$:

$T'_{i,k,m}$ = the fictive time of the mth jump of the ith component into state k after a fictive minimal repair of the component at $T_{i,k,m}$; i.e., it is repaired to have the same distribution of remaining time in state $k+1$ as it had just before jumping downwards to state k,

as well as the intervals:

$$\begin{aligned}\tau_{i,k,m}(t) &= [T_{i,k,m},\ T'_{i,k,m}] \cap [0,t] \\ &= [\min(T_{i,k,m}, t),\ \min(T'_{i,k,m}, t)].\end{aligned}$$

Note that $\tau_{i,k,m}(t)$ represents the interval within $[0, t]$ where the mth fictive minimal repair of component i from level k to level $k+1$ is effective. We then define for $i = 1, \ldots, n$, $k, j \in \{1, \ldots, M\}$ and $m = 1, 2, \ldots$:

$Y^1_{i,k,j,m}$ = system time in state j or above in the interval $\tau_{i,k-1,m}(t)$ just *after* the jump downwards from state k to state $k-1$ of the ith component, which immediately undergoes a fictive minimal repair.

$Y^0_{i,k,j,m}$ = system time in state j or above in the interval $\tau_{i,k-1,m}(t)$ just *after* the jump downwards from state k to state $k-1$ of the ith component, assuming that the component stays in the latter state throughout this interval.

In order to arrive at a stochastic representation for the Natvig type measure we introduce the following random variables:

$$Z_{i,k,j,m} = Y^1_{i,k,j,m} - Y^0_{i,k,j,m}. \tag{8}$$

Thus, $Z_{i,k,j,m}$ can be interpreted as the fictive increase in system time in the states $\{j, \ldots, M\}$ in the interval $\tau_{i,k-1,m}(t)$ due to a fictive minimal repair of the ith component when jumping downwards from state k to state $k-1$.

Note that since the minimal repair is fictive, we have chosen to calculate the effect of this repair over the entire interval $\tau_{i,k-1,m}(t)$ even though this interval may extend beyond the time of the next jump of the ith component.

In order to summarize the effects of all the fictive minimal repairs, we have chosen to simply add up these contributions. Taking the expectation, we get for $i = 1, \ldots, n$, $j \in \{1, \ldots, M\}$ and $k \in \{1, \ldots, M-1\}$:

$$\begin{aligned}E\Big[\sum_{m=1}^{\infty} I(T_{i,k,m} \le t) Z_{i,k,j,m}\Big] &\stackrel{d}{=} EY_{i,k,j}(t), \\ E\Big[\sum_{m=1}^{\infty} I(T_{i,M,m-1} \le t) Z_{i,M,j,m}\Big] &\stackrel{d}{=} EY_{i,M,j}(t).\end{aligned} \tag{9}$$

We then suggest the following generalized Natvig measure, $I_N^{(i,j)}(t)$ of the importance of the ith component in the time interval $[0, t]$ in repairable systems:

$$I_N^{(i,j)}(t) = \sum_{k=1}^{M} EY_{i,k,j}(t) / \sum_{r=1}^{n} \sum_{k=1}^{M} EY_{r,k,j}(t), \tag{10}$$

tacitly assuming $EY_{i,k,j}(t) < \infty$, $i = 1, \ldots, n$, $k, j \in \{1, \ldots, M\}$. We obviously have:

$$\sum_{i=1}^{n} I_N^{(i,j)}(t) = 1, \qquad 0 \le I_N^{(i,j)}(t) \le 1. \tag{11}$$

From [7] it follows that for $k \in \{1, \ldots, M\}$:

$$\int_0^\infty \bar{F}_i^k(t)(-\ln \bar{F}_i^k(t))dt \tag{12}$$
$$= E(T'_{i,k-1,m} - T_{i,k-1,m}) \stackrel{d}{=} \mu_i^{k(p)}.$$

Accordingly, this integral equals the expected prolonged time in state k of the ith component due to a minimal repair. The following corresponding stationary measure is arrived at:

$$\begin{aligned} I_N^{(i,j)} &= \lim_{t\to\infty} I_N^{(i,j)}(t) \\ &= \frac{[\frac{\sum_{k=1}^M I_B^{(i,k,j)}}{(\sum_{\ell=0}^M \mu_i^\ell)}]\mu_i^{k(p)}}{\sum_{r=1}^n [\frac{\sum_{k=1}^M I_B^{(r,k,j)}}{(\sum_{\ell=0}^M \mu_r^\ell)}]\mu_r^{k(p)}}. \end{aligned} \tag{13}$$

We now also take a dual term into account so that components that by being repaired strongly reduce the expected system time in the worse states, are considered very important. In order to formalize this, we introduce for $i = 1, \ldots, n$, $m = 1, 2, \ldots$:

$T'_{i,M,m}$ = the fictive time of the mth jump of the ith component into state M after a fictive minimal complete failure of the component at $T_{i,M,m}$; i.e., it is put into state 0 with the same distribution of remaining time in this state as it had just before jumping upwards to state M,

as well as the intervals:

$$\begin{aligned} \tau_{i,M,m}(t) = &\; [T_{i,M,m},\ T'_{i,M,m}] \cap [0,t] \\ = &\; [\min(T_{i,M,m}, t),\ \min(T'_{i,M,m}, t)]. \end{aligned}$$

Note that $\tau_{i,M,m}(t)$ represents the interval within $[0,t]$ where the mth fictive minimal complete failure of component i from level M to level 0 is effective. We then define for $i = 1, \ldots, n, j \in \{1, \ldots, M\}, m = 1, 2, \ldots$:

$Y^1_{i,0,j,m}$ = system time in state $j-1$ or below in the interval $\tau_{i,M,m}(t)$ just *after* the jump upwards from state 0 to state M of the ith component, which immediately undergoes a fictive minimal complete failure.

$Y^0_{i,0,j,m}$ = system time in state $j-1$ or below in the interval $\tau_{i,M,m}(t)$ just *after* the jump upwards from state 0 to state M of the ith component, assuming that the component stays in the latter state throughout this interval.

Parallel to Eq.(8) we then introduce the following random variables

$$Z_{i,0,j,m} = Y^1_{i,0,j,m} - Y^0_{i,0,j,m}. \tag{14}$$

Thus, $Z_{i,0,j,m}$ can be interpreted as the fictive increase in system time in the states $\{0, \ldots, j-1\}$ in the interval $\tau_{i,M,m}(t)$ due to a fictive minimal complete failure of the ith component when jumping upwards from state 0 to state M.

Now adding up the contributions from the repairs at $T_{i,M,m}$, $m = 1, 2, \ldots$, and taking the expectation, we get for $i = 1, \ldots, n$ and $j \in \{1, \ldots, M\}$:

$$E\Big[\sum_{m=1}^\infty I(T_{i,0,m} \le t) Z_{i,0,j,m}\Big] \stackrel{d}{=} EY_{i,0,j}(t). \tag{15}$$

We then suggest the following dual generalized Natvig measure, $I_{N,D}^{(i,j)}(t)$, and extended generalized Natvig measure, $\bar{I}_N^{(i,j)}(t)$, of the importance of the ith component in the time interval $[0,t]$ in repairable systems

$$I_{N,D}^{(i,j)}(t) = EY_{i,0,j}(t)/\sum_{r=1}^{n} EY_{r,0,j}(t) \tag{16}$$

$$\bar{I}_N^{(i,j)}(t) = \sum_{k=0}^{M} EY_{i,k,j}(t)/\sum_{r=1}^{n}\sum_{k=0}^{M} EY_{r,k,j}(t), \tag{17}$$

tacitly assuming $EY_{i,k,j}(t) < \infty$, $i = 1, \dots, n, k \in \{0, \dots, M\}, j \in \{1, \dots, M\}$. Completely parallel to Eq.(12) we have:

$$\int_0^\infty \bar{G}_i(t)(-\ln \bar{G}_i(t))dt$$
$$= E(T'_{i,M,j} - T_{i,M,j}) \stackrel{d}{=} \mu_i^{0(p)}. \tag{18}$$

The corresponding stationary measures are given by:

$$I_{N,D}^{(i,j)} = \lim_{t\to\infty} I_{N,D}^{(i,j)}(t)$$
$$= \frac{[\frac{\sum_{k=1}^{M} I_B^{(i,k,j)}}{(\sum_{\ell=0}^{M}\mu_i^\ell)}]\mu_i^{0(p)}}{\sum_{r=1}^{n}[\frac{\sum_{k=1}^{M} I_B^{(r,k,j)}}{(\sum_{\ell=0}^{M}\mu_r^\ell)}]\mu_r^{0(p)}}. \tag{19}$$

$$\bar{I}_N^{(i,j)} = \lim_{t\to\infty} \bar{I}_N^{(i,j)}(t)$$
$$= \frac{[\frac{\sum_{k=1}^{M} I_B^{(i,k,j)}}{(\sum_{\ell=0}^{M}\mu_i^\ell)}](\mu_i^{k(p)} + \mu_i^{0(p)})}{\sum_{r=1}^{n}[\frac{\sum_{k=1}^{M} I_B^{(r,k,j)}}{(\sum_{\ell=0}^{M}\mu_r^\ell)}](\mu_r^{k(p)} + \mu_r^{0(p)})}. \tag{20}$$

Note that if $\mu_r^{k(p)}$, $r = 1, \dots, n, k = 1, \dots, M$ are all equal, which by Eq.(12) is the case when all components have the same distribution of the times spent in each of the non complete failure states, then Eq.(13) reduces to Eq.(7). Similarly, if $\mu_r^{0(p)}$, $r = 1, \dots, n$ are all equal, which by Eq.(18) is the case when all components have the same repair time distribution, then Eq.(19) reduces to Eq.(7). Similarly, if both $\mu_r^{k(p)}$, $r = 1, \dots, n, k = 1, \dots, M$ are all equal and $\mu_r^{0(p)}$, $r = 1, \dots, n$ are all equal, which by Eqs.(12) and (18) is the case when all components have the same distribution of the times spent in each of the non complete failure states and the same repair time distribution, then Eq.(20) reduces to Eqs.(7) and (6).

Now consider the special case where the component times to jumps downwards and repair times are Weibull distributed; i.e.

$$\bar{F}_i^k(t) = e^{-(\lambda_i^k t)^{\alpha_i^k}}, \qquad \lambda_i^k > 0,\ \ \alpha_i^k > 0$$
$$\bar{G}_i(t) = e^{-(\lambda_i^0 t)^{\alpha_i^0}}, \qquad \lambda_i^0 > 0,\ \ \alpha_i^0 > 0\,.$$

From Eqs.(12) and (18) we get as shown in [9] that $\mu_i^{k(p)} = \mu_i^k/\alpha_i^k$ for $k \in \{0, \dots, M\}$. Hence, Eq.(20) simplifies to

$$\bar{I}_N^{(i,j)} = \frac{[\frac{\sum_{k=1}^{M} I_B^{(i,k,j)}}{(\sum_{\ell=0}^{M}\mu_i^\ell)}](\mu_i^k/\alpha_i^k + \mu_i^0/\alpha_i^0)}{\sum_{r=1}^{n}[\frac{\sum_{k=1}^{M} I_B^{(r,k,j)}}{(\sum_{\ell=0}^{M}\mu_r^\ell)}](\mu_r^k/\alpha_r^k + \mu_r^0/\alpha_r^0)}. \tag{21}$$

Now for $k \in \{1, \dots, M\}$ assume that the shape parameter α_i^k is increasing and λ_i^k changing in such a way that μ_i^k is constant. Hence, according to (1) the availability a_i^k is unchanged. Then $\bar{I}_N^{(i,j)}$ is decreasing in α_i^k. This is natural since a large $\alpha_i^k > 1$ corresponds to a strongly increasing failure rate and the effect of a minimal repair is small. Hence, according to $\bar{I}_N^{(i,j)}$ the ith component is of less importance. If on the other hand $\alpha_i^k < 1$ is small, we have a strongly decreasing failure rate and the effect of a minimal repair is large. Hence, according to $\bar{I}_N^{(i,j)}$ the ith component is of higher importance. A completely parallel argument is valid for α_i^0.

In the examples given in the following three sections we will consider a multistate description of both the components and the system. For the components we let $S_i = \{0, 1, 2\}, i = 1, \ldots, n$. We then regard the system as a flow network and let the system state be the amount of flow that can be transported through the network. In order to determine this we start out by identifying the binary minimal cut sets of the network $K_\ell, \ell = 1, \ldots, m$; i.e. the minimal sets of components the removal of which will break the connection between the endpoints of the network. We then apply the well-known max-flow min-cut theorem, see [3], and get

$$\phi(\boldsymbol{X}(t)) = \min_{1 \leq \ell \leq m} \sum_{i \in K_\ell} X_i(t). \tag{22}$$

For system A treated in Section 2 and the offshore oil and gas production system treated in Section 4 there is at least one component in series with the rest of the system and we see that we also have $S = \{0, 1, 2\}$. This is in accordance with our general set up. However, for system B treated in Section 2 and the bridge structure treated in Section 3 there are no components in series with the rest of the system and we end up with $S = \{0, 1, 2, 3, 4\}$. Since our simulation program is written for handling systems that can be considered as flow networks, this does not cause any problems. Furthermore, in order to be able to compare with the binary description treated in [8] we let the distributions of the total component times spent in the non complete failure states for the multistate case be equal to the component lifetime distributions in the binary case except for the exponential distribution case treated in Section 4. For simplicity we also assume that the distributions of the times spent in each of the non complete failure states are the same for each component. Finally, we assume that the repair time distributions are the same in the binary and multistate cases.

2. Component importance in two three component systems

In this section we will simulate the component importance in two systems with three components. Figure 1 shows the systems we will be looking at.

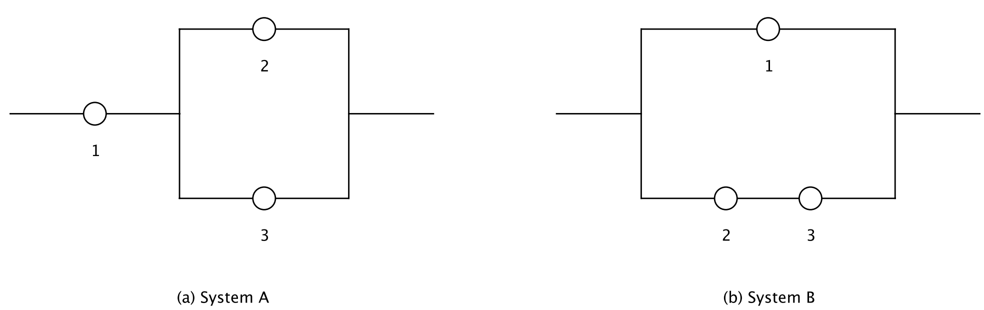

Figure 1: Systems of three components.

The times spent in each of the non complete failure states and the repair times are assumed to be gamma distributed. We will first see how an increasing variance in the distributions of the times spent in each of the non complete failure states of one of the components influences the component importances. The effect of a decreasing mean time to repair of one of the components will be investigated next.

Let first the components have the following distributions of times spent in state k before jumping to state $k-1$ for $k = 1, 2$ and of repair time:

Component 2: $\bar{F}_2^k(t) \sim \text{gamma}(2/c, 3c), k = 1, 2,\ \ \bar{G}_2(t) \sim \text{gamma}(4, 1/2)$,

Component i: $\bar{F}_i^k(t) \sim \text{gamma}(4, 1), k = 1, 2,\ \ \bar{G}_i(t) \sim \text{gamma}(4, 1/2),\ \ i = 1, 3$,

where c is a positive number. The mean times spent in a non complete failure state are $\mu_2^k = 6$ for component 2 and $\mu_i^k = 4$, $i = 1, 3$. All components have mean time to repair equal to $\mu_i^0 = 2$, $i = 1, 2, 3$. The variance associated with the times spent in a non complete failure state of component 2 is $18c$, while the variance associated with the corresponding times of components 1 and 3 is 4. The variances in the repair time distributions are 1 for all components. Tables 1 and 2 display the results from the simulations for all three versions of the Natvig measure and for $I_B^{(i,j)}(t)$ and $I_{B-P}^{(i,j)}(t)$ for system A for respectively system level $j = 1, 2$. Tables 3 - 6 do the same for system B for respectively system levels 1, 2, 3, 4.

c	i	$I_N^{(i,1)}(t)$	$I_{N,D}^{(i,1)}(t)$	$\tilde{I}_N^{(i,1)}(t)$	$I_B^{(i,1)}(t)$	$I_{B-P}^{(i,1)}(t)$
	1	0.773	0.810	0.785	0.780	0.810
1/2	2	0.136	0.095	0.123	0.128	0.095
	3	0.091	0.095	0.092	0.092	0.095
	1	0.730	0.809	0.754	0.780	0.810
1	2	0.185	0.095	0.157	0.128	0.095
	3	0.086	0.095	0.089	0.092	0.095
	1	0.676	0.810	0.716	0.780	0.810
2	2	0.244	0.095	0.200	0.128	0.095
	3	0.079	0.095	0.084	0.092	0.095
	1	0.617	0.809	0.670	0.780	0.810
4	2	0.311	0.095	0.251	0.128	0.095
	3	0.072	0.095	0.079	0.092	0.095

Table 1: *Simulations of System A with varying variance in the distributions of the times in a non complete failure state of component 2. Components 1 and 3 have identical distributions. System level is 1. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,2)}(t)$	$I_{N,D}^{(i,2)}(t)$	$\tilde{I}_N^{(i,2)}(t)$	$I_B^{(i,2)}(t)$	$I_{B-P}^{(i,2)}(t)$
	1	0.631	0.675	0.645	0.639	0.675
1/2	2	0.196	0.140	0.178	0.185	0.139
	3	0.174	0.186	0.178	0.176	0.186
	1	0.581	0.675	0.609	0.639	0.675
1	2	0.259	0.140	0.223	0.185	0.139
	3	0.160	0.186	0.168	0.176	0.186
	1	0.523	0.674	0.566	0.639	0.675
2	2	0.332	0.139	0.279	0.185	0.139
	3	0.144	0.186	0.156	0.176	0.186
	1	0.462	0.674	0.516	0.639	0.674
4	2	0.410	0.140	0.341	0.185	0.140
	3	0.128	0.186	0.143	0.176	0.186

Table 2: *Simulations of System A with varying variance in the distributions of the times in a non complete failure state of component 2. Components 1 and 3 have identical distributions. System level is 2. The time horizon is* $t = 20000$.

We first note that for both systems and all system levels $I_{N,D}^{(i,j)}(t)$ and $I_{B-P}^{(i,j)}(t)$ are practically equal. Since stationarity is reached and the repair time distributions are the same for all three components, this is in accordance with results given in Section 1. Furthermore, component 2's importance is increasing in c for both systems and all system levels both for the $I_N^{(i,j)}(t)$ and the extended measure. Hence, according to these measures the increased uncertainty associated with an increasing variance leads to increased importance of a component.

We will then look at how the extended measure ranks the components. For system A and system level 1 components 2 and 3 are equally important according to $I_{B-P}^{(i,1)}(t)$ irrespective of c. Since the variance of the distribution of the times in a non complete failure state of component 2 is greater than the corresponding one of component 3 for all c, the former component is more important than the latter according to the extended measure. Except for $c = 1/2$ this is also true for system level 2. However, for both system levels component 1 is not challenged as the most important for this

c	i	$I_N^{(i,1)}(t)$	$I_{N,D}^{(i,1)}(t)$	$\bar{I}_N^{(i,1)}(t)$	$I_B^{(i,1)}(t)$	$I_{B-P}^{(i,1)}(t)$
	1	0.479	0.524	0.493	0.487	0.524
1/2	2	0.261	0.191	0.239	0.248	0.190
	3	0.261	0.286	0.269	0.266	0.286
	1	0.429	0.523	0.457	0.486	0.524
1	2	0.337	0.191	0.294	0.248	0.191
	3	0.234	0.286	0.249	0.266	0.286
	1	0.376	0.524	0.415	0.487	0.524
2	2	0.419	0.190	0.359	0.248	0.190
	3	0.205	0.286	0.226	0.265	0.285
	1	0.322	0.524	0.369	0.486	0.523
4	2	0.502	0.190	0.429	0.248	0.191
	3	0.176	0.286	0.202	0.266	0.286

Table 3: *Simulations of System B with varying variance in the distributions of the times in a non complete failure state of component 2. Components 1 and 3 have identical distributions. System level is 1. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,2)}(t)$	$I_{N,D}^{(i,2)}(t)$	$\bar{I}_N^{(i,2)}(t)$	$I_B^{(i,2)}(t)$	$I_{B-P}^{(i,2)}(t)$
	1	0.491	0.536	0.505	0.500	0.537
1/2	2	0.254	0.186	0.233	0.242	0.185
	3	0.255	0.279	0.262	0.259	0.278
	1	0.442	0.537	0.470	0.500	0.537
1	2	0.329	0.186	0.287	0.241	0.185
	3	0.229	0.277	0.243	0.259	0.278
	1	0.388	0.537	0.428	0.500	0.537
2	2	0.411	0.185	0.351	0.242	0.185
	3	0.201	0.278	0.221	0.259	0.278
	1	0.334	0.537	0.382	0.500	0.537
4	2	0.494	0.185	0.420	0.241	0.185
	3	0.173	0.278	0.197	0.259	0.278

Table 4: *Simulations of System B with varying variance in the distributions of the times in a non complete failure state of component 2. Components 1 and 3 have identical distributions. System level is 2. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,3)}(t)$	$I_{N,D}^{(i,3)}(t)$	$\bar{I}_N^{(i,3)}(t)$	$I_B^{(i,3)}(t)$	$I_{B-P}^{(i,3)}(t)$
	1	0.400	0.445	0.414	0.408	0.444
1/2	2	0.300	0.222	0.276	0.286	0.222
	3	0.300	0.333	0.310	0.306	0.333
	1	0.354	0.444	0.379	0.408	0.445
1	2	0.381	0.222	0.336	0.286	0.222
	3	0.265	0.334	0.285	0.306	0.333
	1	0.305	0.444	0.340	0.408	0.444
2	2	0.467	0.222	0.405	0.286	0.222
	3	0.228	0.334	0.255	0.306	0.333
	1	0.257	0.445	0.299	0.408	0.444
4	2	0.551	0.222	0.477	0.286	0.222
	3	0.193	0.333	0.224	0.306	0.333

Table 5: *Simulations of System B with varying variance in the distributions of the times in a non complete failure state of component 2. Components 1 and 3 have identical distributions. System level is 3. The time horizon is* $t = 20000$.

measure being in series with the rest of the system and by far the most important according to $I_B^{(i,j)}(t)$.

For system B and all system levels component 3 is more important than component 2 according to $I_{B-P}^{(i,j)}(t)$ irrespective of c. This is also true for the extended measure for $c = 1/2$. However, as the variance of the distribution of the times in a non complete failure state of component 2 increases, this component gets increasingly more important according to the extended measure, and finally the most important one for all system levels.

We will now turn our attention to the case where one of the components experiences a decreasing mean time to repair (MTTR). First we will assume that this is the case for component 1, and that components 2 and 3 have identical distributions of times spent in each of the non complete failure states and of repair times. Then the roles of components 1 and 2 are interchanged. More specifically,

c	i	$I_N^{(i,4)}(t)$	$I_{N,D}^{(i,4)}(t)$	$\bar{I}_N^{(i,4)}(t)$	$I_B^{(i,4)}(t)$	$I_{B-P}^{(i,4)}(t)$
	1	0.334	0.375	0.347	0.341	0.375
1/2	2	0.333	0.250	0.307	0.318	0.250
	3	0.333	0.375	0.346	0.341	0.375
	1	0.291	0.375	0.315	0.341	0.375
1	2	0.418	0.251	0.371	0.318	0.250
	3	0.291	0.375	0.314	0.341	0.375
	1	0.247	0.375	0.279	0.341	0.375
2	2	0.506	0.250	0.442	0.318	0.250
	3	0.247	0.375	0.279	0.341	0.375
	1	0.205	0.375	0.242	0.341	0.375
4	2	0.589	0.250	0.516	0.318	0.250
	3	0.206	0.375	0.242	0.341	0.375

Table 6: *Simulations of System B with varying variance in the distributions of the times in a non complete failure state of component 2. Components 1 and 3 have identical distributions. System level is 4. The time horizon is* $t = 20000$.

let the components have the following distributions of times spent in state k before jumping to state $k-1$ for $k = 1, 2$ and of repair time:

Component 1: $\bar{F}_1^k(t) \sim \text{gamma}(12, 1), k = 1, 2,\ \ \bar{G}_1(t) \sim \text{gamma}(\frac{24}{c^2}, c)$,

Component i: $\bar{F}_i^k(t) \sim \text{gamma}(12, 1), k = 1, 2,\ \ \bar{G}_i(t) \sim \text{gamma}(4, \frac{1}{2}),\ \ i = 2, 3$,

where c is a positive number. Component 1's mean time to repair is $\mu_1^0 = 24/c$, while components 2 and 3 have constant mean times to repair equal to $\mu_i^0 = 2$, $i = 2, 3$. The mean times spent in the non complete failure states are equal for all components, thus $\mu_i^k = 12$, $i = 1, 2, 3$.

The variances are constant in all distributions. In the distributions of times spent in each of the non complete failure states the variances are 12 for all components. The variance in the repair time distribution of component 1 is 24, whereas the variance in the repair time distributions of components 2 and 3 equals 1.

Tables 7 and 8 display the results from the simulations for component 1 for all three versions of the Natvig measure and for $I_B^{(1,j)}(t)$ and $I_{B-P}^{(1,j)}(t)$ for system A for respectively system levels 1 and 2. Tables 9 - 12 do the same for system B for respectively system levels 1, 2, 3, 4. Since components 2 and 3 have interchangeable positions both in System A and B, and identical distributions of times spent in each of the non complete failure states and of repair times, they have the same importance for each of the five measures. These importances are easily found given the ones for component 1. Now we note for both systems and all system levels that $I_N^{(1,j)}(t)$ and $I_{B-P}^{(1,j)}(t)$ are practically equal. Since stationarity is reached and the distributions of times spent in each of the non complete failure states are the same for all three components, this is in accordance with results given in Section 1.

c	$I_N^{(1,1)}(t)$	$I_{N,D}^{(1,1)}(t)$	$\bar{I}_N^{(1,1)}(t)$	$I_B^{(1,1)}(t)$	$I_{B-P}^{(1,1)}(t)$
1	0.875	0.971	0.928	0.928	0.875
2	0.875	0.971	0.929	0.906	0.875
3	0.875	0.972	0.930	0.896	0.875
4	0.876	0.972	0.931	0.890	0.875
6	0.875	0.972	0.930	0.883	0.875

Table 7: *Simulations of System A with decreasing MTTR of component 1. Components 2 and 3 have identical distributions. System level is 1. The time horizon is* $t = 20000$.

c	$I_N^{(1,2)}(t)$	$I_{N,D}^{(1,2)}(t)$	$\bar{I}_N^{(1,2)}(t)$	$I_B^{(1,2)}(t)$	$I_{B-P}^{(1,2)}(t)$
1	0.650	0.898	0.775	0.774	0.650
2	0.651	0.900	0.778	0.720	0.651
3	0.650	0.902	0.779	0.696	0.650
4	0.651	0.903	0.781	0.682	0.651
6	0.650	0.903	0.780	0.667	0.650

Table 8: *Simulations of System A with decreasing MTTR of component 1. Components 2 and 3 have identical distributions. System level is 2. The time horizon is* $t = 20000$.

c	$I_N^{(1,1)}(t)$	$I_{N,D}^{(1,1)}(t)$	$\bar{I}_N^{(1,1)}(t)$	$I_B^{(1,1)}(t)$	$I_{B-P}^{(1,1)}(t)$
1	0.080	0.290	0.138	0.138	0.080
2	0.148	0.457	0.246	0.194	0.148
3	0.207	0.562	0.331	0.243	0.207
4	0.258	0.634	0.399	0.286	0.258
6	0.342	0.722	0.498	0.359	0.342

Table 9: *Simulations of System B with decreasing MTTR of component 1. Components 2 and 3 have identical distributions. System level is 1. The time horizon is* $t = 20000$.

c	$I_N^{(1,2)}(t)$	$I_{N,D}^{(1,2)}(t)$	$\bar{I}_N^{(1,2)}(t)$	$I_B^{(1,2)}(t)$	$I_{B-P}^{(1,2)}(t)$
1	0.315	0.685	0.460	0.460	0.316
2	0.380	0.749	0.536	0.460	0.381
3	0.409	0.774	0.568	0.460	0.409
4	0.425	0.787	0.586	0.460	0.425
6	0.442	0.798	0.602	0.460	0.442

Table 10: *Simulations of System B with decreasing MTTR of component 1. Components 2 and 3 have identical distributions. System level is 2. The time horizon is* $t = 20000$.

c	$I_N^{(1,3)}(t)$	$I_{N,D}^{(1,3)}(t)$	$\bar{I}_N^{(1,3)}(t)$	$I_B^{(1,3)}(t)$	$I_{B-P}^{(1,3)}(t)$
1	0.401	0.758	0.552	0.552	0.400
2	0.400	0.764	0.556	0.480	0.400
3	0.400	0.767	0.559	0.451	0.400
4	0.400	0.770	0.561	0.435	0.400
6	0.401	0.769	0.561	0.418	0.400

Table 11: *Simulations of System B with decreasing MTTR of component 1. Components 2 and 3 have identical distributions. System level is 3. The time horizon is* $t = 20000$.

c	$I_N^{(1,4)}(t)$	$I_{N,D}^{(1,4)}(t)$	$\bar{I}_N^{(1,4)}(t)$	$I_B^{(1,4)}(t)$	$I_{B-P}^{(1,4)}(t)$
1	0.334	0.702	0.480	0.480	0.334
2	0.333	0.709	0.485	0.409	0.333
3	0.333	0.713	0.487	0.381	0.334
4	0.334	0.715	0.489	0.366	0.333
6	0.332	0.713	0.488	0.349	0.333

Table 12: *Simulations of System B with decreasing MTTR of component 1. Components 2 and 3 have identical distributions. System level is 4. The time horizon is* $t = 20000$.

For system A for both system levels all measures are practically constant in c for component 1 except $I_B^{(1,j)}(t)$ which is decreasing in c. The latter fact follows from Eq.(5) and Eq.(6) since the component is in series with the rest of the system and its stationary availability, a_1^j, increases due to the decreasing mean time to repair, μ_1^0. For $I_{B-P}^{(1,j)}(t)$ it follows from Eq.(7) that the increase in the asymptotic failure rate $1/(\sum_{\ell=0}^{M}\mu_1^\ell)$ as μ_1^0 decreases, compensates for the decrease in $I_B^{(1,j)}(t)$.

For system B component 1 is in parallel with the rest of the system and the increase in a_1^1 as μ_1^0 decreases leads to an increasing $I_B^{(1,1)}(t)$. Since $1/(\sum_{\ell=0}^{M}\mu_1^\ell)$ increases as well, $I_{B-P}^{(1,j)}(t)$ also increases for $j = 1, 2$. As for system A $I_N^{(1,j)}(t)$ behaves much like $I_{B-P}^{(1,j)}(t)$.

If one is comparing the results in Tables 1, 3, 7 and 9 covering system level 1 by the corresponding results for the binary case given in [8], the Birnbaum measure, the Barlow-Proschan measure and the dual Natvig measure are mostly identical. The reason is that these measures are not affected by the fictive minimal repairs of the components when jumping downwards from one of the non complete failure states. Especially, the extended Natvig measure, however, is different. This indicates that the latter measure captures more information about the system.

Now, we consider the case where components 1 and 3 are assumed to have identical distributions of times spent in each of the non complete failure states and of repair times. More specifically, let the components have the following distributions of times spent in state k before jumping to state $k-1$ for $k = 1, 2$ and of repair time:

Component 2: $\bar{F}_2^k(t) \sim \text{gamma}(4, 1), k = 1, 2,\ \ \bar{G}_2(t) \sim \text{gamma}(\frac{6}{c^2}, \frac{c}{2})$,

Component i: $\bar{F}_i^k(t) \sim \text{gamma}(4, 1), k = 1, 2,\ \ \bar{G}_i(t) \sim \text{gamma}(6, \frac{1}{2}),\ \ i = 1, 3$,

where c is once again a positive number. In this example all components have identical distributions of times spent in each of the non complete failure states with mean times $\mu_i^k = 4$, $i = 1, 2, 3$. Moreover, the variances associated with these distributions are all equal to 4. The repair time distributions are identical for components 1 and 3 as well. The mean time to repair of these components are $\mu_i^0 = 3$, $i = 1, 3$, while the mean time to repair of component 2 is $\mu_2^0 = 3/c$. The variances in the repair time distributions are 3/2 for all components.

c	i	$I_N^{(i,1)}(t)$	$I_{N,D}^{(i,1)}(t)$	$\bar{I}_N^{(i,1)}(t)$	$I_B^{(i,1)}(t)$	$I_{B-P}^{(i,1)}(t)$
$\frac{1}{2}$	1	0.654	0.656	0.655	0.634	0.654
	2	0.115	0.113	0.114	0.142	0.115
	3	0.231	0.231	0.231	0.224	0.231
$\frac{3}{4}$	1	0.681	0.683	0.682	0.673	0.682
	2	0.137	0.135	0.136	0.147	0.136
	3	0.182	0.182	0.182	0.180	0.182
1	1	0.701	0.700	0.700	0.700	0.700
	2	0.150	0.150	0.150	0.150	0.150
	3	0.150	0.150	0.150	0.150	0.150
$\frac{3}{2}$	1	0.722	0.720	0.721	0.733	0.722
	2	0.167	0.170	0.168	0.154	0.167
	3	0.111	0.110	0.111	0.113	0.111
2	1	0.735	0.731	0.734	0.753	0.735
	2	0.176	0.181	0.178	0.156	0.176
	3	0.088	0.088	0.088	0.090	0.088
$\frac{5}{2}$	1	0.744	0.739	0.742	0.767	0.744
	2	0.183	0.188	0.185	0.158	0.183
	3	0.073	0.073	0.073	0.075	0.073

Table 13: *Simulations of System A with decreasing MTTR of component 2. Components 1 and 3 have identical distributions. System level is 1. The time horizon is* $t = 20000$.

Tables 13 and 14 display the results from the simulations for all three versions of the Natvig measure and for $I_B^{(i,j)}(t)$ and $I_{B-P}^{(i,j)}(t)$ for system A for respectively system levels 1 and 2. Tables 15 - 18 do the same for system B for respectively system levels 1, 2, 3, 4.

As for the previous case $I_N^{(i,j)}(t)$ and $I_{B-P}^{(i,j)}(t)$ are practically equal for both systems and all system levels since the distributions of times spent in each of the non complete failure states are the same for all three components. When $c = 1$, all components have identical distributions. Hence, in accordance with results given in Section 1 all measures give the same results in each system for each

c	i	$I_N^{(i,2)}(t)$	$I_{N,D}^{(i,2)}(t)$	$\bar{I}_N^{(i,2)}(t)$	$I_B^{(i,2)}(t)$	$I_{B-P}^{(i,2)}(t)$
	1	0.595	0.598	0.596	0.570	0.596
$\frac{1}{2}$	2	0.167	0.162	0.165	0.203	0.166
	3	0.238	0.239	0.239	0.228	0.238
	1	0.605	0.607	0.606	0.596	0.605
$\frac{3}{4}$	2	0.184	0.182	0.184	0.198	0.184
	3	0.210	0.211	0.211	0.207	0.210
	1	0.611	0.611	0.611	0.611	0.611
1	2	0.194	0.194	0.194	0.194	0.194
	3	0.195	0.195	0.195	0.195	0.195
	1	0.618	0.616	0.617	0.630	0.618
$\frac{3}{2}$	2	0.206	0.208	0.207	0.191	0.206
	3	0.176	0.175	0.176	0.180	0.176
	1	0.622	0.617	0.620	0.640	0.622
2	2	0.212	0.217	0.214	0.188	0.212
	3	0.166	0.166	0.166	0.171	0.166
	1	0.623	0.619	0.621	0.647	0.624
$\frac{5}{2}$	2	0.217	0.222	0.219	0.187	0.216
	3	0.160	0.159	0.160	0.166	0.160

Table 14: *Simulations of System A with decreasing MTTR of component 2. Components 1 and 3 have identical distributions. System level is 2. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,1)}(t)$	$I_{N,D}^{(i,1)}(t)$	$\bar{I}_N^{(i,1)}(t)$	$I_B^{(i,1)}(t)$	$I_{B-P}^{(i,1)}(t)$
	1	0.652	0.654	0.653	0.622	0.652
$\frac{1}{2}$	2	0.174	0.170	0.172	0.211	0.174
	3	0.174	0.175	0.175	0.166	0.174
	1	0.586	0.588	0.587	0.575	0.586
$\frac{3}{4}$	2	0.207	0.205	0.206	0.222	0.207
	3	0.207	0.207	0.207	0.203	0.207
	1	0.543	0.543	0.543	0.543	0.543
1	2	0.229	0.228	0.229	0.228	0.228
	3	0.228	0.228	0.228	0.228	0.228
	1	0.489	0.486	0.488	0.501	0.489
$\frac{3}{2}$	2	0.256	0.260	0.257	0.238	0.256
	3	0.255	0.254	0.254	0.261	0.255
	1	0.458	0.455	0.457	0.476	0.458
2	2	0.271	0.277	0.273	0.243	0.271
	3	0.271	0.268	0.270	0.281	0.271
	1	0.437	0.432	0.435	0.458	0.437
$\frac{5}{2}$	2	0.282	0.288	0.284	0.247	0.282
	3	0.282	0.279	0.281	0.295	0.282

Table 15: *Simulations of System B with decreasing MTTR of component 2. Components 1 and 3 have identical distributions. System level is 1. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,2)}(t)$	$I_{N,D}^{(i,2)}(t)$	$\bar{I}_N^{(i,2)}(t)$	$I_B^{(i,2)}(t)$	$I_{B-P}^{(i,2)}(t)$
	1	0.610	0.614	0.612	0.580	0.610
$\frac{1}{2}$	2	0.195	0.191	0.193	0.236	0.195
	3	0.195	0.196	0.195	0.185	0.195
	1	0.569	0.570	0.569	0.557	0.568
$\frac{3}{4}$	2	0.216	0.213	0.215	0.231	0.216
	3	0.216	0.216	0.216	0.212	0.216
	1	0.544	0.544	0.544	0.544	0.544
1	2	0.228	0.228	0.228	0.228	0.228
	3	0.228	0.228	0.228	0.228	0.228
	1	0.517	0.514	0.516	0.528	0.517
$\frac{3}{2}$	2	0.241	0.245	0.243	0.225	0.242
	3	0.242	0.241	0.241	0.247	0.242
	1	0.501	0.497	0.500	0.519	0.501
2	2	0.249	0.255	0.252	0.223	0.249
	3	0.249	0.248	0.249	0.258	0.249
	1	0.492	0.487	0.490	0.513	0.492
$\frac{5}{2}$	2	0.254	0.260	0.256	0.222	0.254
	3	0.254	0.253	0.254	0.265	0.254

Table 16: *Simulations of System B with decreasing MTTR of component 2. Components 1 and 3 have identical distributions. System level is 2. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,3)}(t)$	$I_{N,D}^{(i,3)}(t)$	$\bar{I}_N^{(i,3)}(t)$	$I_B^{(i,3)}(t)$	$I_{B-P}^{(i,3)}(t)$
$\frac{1}{2}$	1	0.400	0.404	0.402	0.370	0.400
	2	0.299	0.294	0.297	0.353	0.300
	3	0.300	0.303	0.301	0.277	0.300
$\frac{3}{4}$	1	0.400	0.402	0.401	0.389	0.400
	2	0.300	0.297	0.299	0.319	0.300
	3	0.300	0.301	0.300	0.292	0.300
1	1	0.400	0.400	0.400	0.400	0.400
	2	0.300	0.300	0.300	0.300	0.300
	3	0.300	0.300	0.300	0.300	0.300
$\frac{3}{2}$	1	0.400	0.397	0.399	0.411	0.400
	2	0.300	0.304	0.302	0.281	0.300
	3	0.300	0.298	0.299	0.308	0.300
2	1	0.400	0.397	0.399	0.417	0.400
	2	0.300	0.306	0.302	0.270	0.300
	3	0.300	0.298	0.299	0.313	0.300
$\frac{5}{2}$	1	0.400	0.396	0.399	0.421	0.400
	2	0.300	0.307	0.303	0.264	0.300
	3	0.300	0.297	0.299	0.315	0.300

Table 17: *Simulations of System B with decreasing MTTR of component 2. Components 1 and 3 have identical distributions. System level is 3. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,4)}(t)$	$I_{N,D}^{(i,4)}(t)$	$\bar{I}_N^{(i,4)}(t)$	$I_B^{(i,4)}(t)$	$I_{B-P}^{(i,4)}(t)$
$\frac{1}{2}$	1	0.334	0.336	0.334	0.306	0.333
	2	0.333	0.328	0.331	0.389	0.333
	3	0.333	0.336	0.334	0.305	0.333
$\frac{3}{4}$	1	0.333	0.334	0.333	0.323	0.333
	2	0.333	0.330	0.332	0.353	0.333
	3	0.334	0.336	0.335	0.324	0.334
1	1	0.333	0.333	0.333	0.333	0.333
	2	0.334	0.334	0.334	0.334	0.334
	3	0.333	0.333	0.333	0.333	0.333
$\frac{3}{2}$	1	0.333	0.331	0.333	0.344	0.333
	2	0.334	0.338	0.335	0.313	0.334
	3	0.333	0.331	0.332	0.343	0.333
2	1	0.333	0.330	0.332	0.349	0.334
	2	0.333	0.341	0.336	0.302	0.333
	3	0.333	0.330	0.332	0.349	0.333
$\frac{5}{2}$	1	0.333	0.329	0.331	0.352	0.333
	2	0.334	0.341	0.337	0.296	0.334
	3	0.333	0.329	0.332	0.352	0.333

Table 18: *Simulations of System B with decreasing MTTR of component 2. Components 1 and 3 have identical distributions. System level is 4. The time horizon is* $t = 20000$.

system level in this case. Irrespective of c the results are very similar for the three Natvig importance measures and $I_{B-P}^{(i,j)}(t)$ in each system for each system level. For System A and both system levels and all measures the importance of component 3 is decreasing in c, while the other components get increasingly more important except for $I_B^{(2,2)}(t)$ which is decreasing in c. For System B for all measures and system levels 1 and 2 the importance of component 1 is decreasing in c, while the other components get increasingly more important again except for $I_B^{(2,2)}(t)$ which is decreasing in c. For the three Natvig importance measures and $I_{B-P}^{(i,j)}(t)$ and system levels 3 and 4 the importance of all components are almost constant in c. Especially, for system level 4 according to these measures all components are almost equally important. On the other hand for $I_B^{(1,j)}(t)$ for system levels 3 and 4 the importance of component 1 is increasing in c.

We now turn to the ranking of the components according to the extended measure. We see that component 1 is ranked on top in both systems and for all system levels and almost all values of c, so we focus on the ranking of components 2 and 3. For both systems and all system levels component 3 is ranked before component 2 for $c = 1/2, 3/4$, whereas the ranking is opposite for $c = 3/2, 2, 5/2$. To explain this, however, we consider each system separately.

We first consider System A where the ranking is the same as for $I_B^{(i,j)}(t)$. When $c < 1$, component 2 has lower stationary availabilities than component 3. Since these components are connected in parallel, according to $I_B^{(i,j)}(t)$ component 3 is ranked before component 2. As soon as c gets larger than 1, the roles of component 2 and 3 change. Now component 2 has the higher stationary availabilities of the two, thus according to $I_B^{(i,j)}(t)$ this component is ranked before component 3. We then turn to System B. Here the ranking of the Natvig measures does not follow $I_B^{(i,j)}$ rather more $I_{B-P}^{(i,j)}(t)$ since the factor $1/\sum_{\ell=0}^{M}\mu_2^{\ell}$ is increasing in c due to μ_2^0 is decreasing in c.

3. Component importance in the bridge system

In this section we will investigate the bridge system depicted in Figure 2. As in the previous section the times spent in each of the non complete failure states and the repair times are assumed to be gamma distributed.

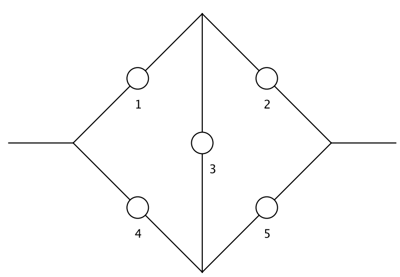

Figure 2: The bridge system.

We will again first see how an increasing variance in the distribution of the times spent in each of the non complete failure states of one of the components influences the component importances. More specifically, let the components have the following distributions of times spent in state k before jumping to state $k-1$ for $k = 1, 2$ and of repair time:

Component 1: $\bar{F}_1^k(t) \sim \text{gamma}(6/c, c), k = 1, 2,\ \ \bar{G}_1(t) \sim \text{gamma}(2, 1)$,

Component i: $\bar{F}_i^k(t) \sim \text{gamma}(3/2, 2), k = 1, 2,\ \ \bar{G}_i(t) \sim \text{gamma}(2, 1),\ \ i = 2, 3, 4$,

Component 5: $\bar{F}_5^k(t) \sim \text{gamma}(6, 1), k = 1, 2,\ \ \bar{G}_5(t) \sim \text{gamma}(2, 1)$,

where c is a positive number. The mean times spent in each of the non complete failure states of components 1 and 5 are $\mu_i^k = 6,\ i = 1, 5$ which is twice the size of the corresponding means of the other components. Thus, $\mu_i^k = 3,\ i = 2, 3, 4$. All the components have the same repair time distributions with expectation $\mu_i^0 = 2,\ i = 1, \dots, 5$. The variances in the distributions of times spent in each of the non complete failure states are 6 for all components except for the component 1 which has a variance of $6c$. In the repair time distributions the variances are 2.

The results of the simulations are shown in Tables 19 - 22 for all three versions of the Natvig measure and for $I_B^{(i,j)}(t)$ and $I_{B-P}^{(i,j)}(t)$ for respectively system levels 1, 2, 3, 4. As for the case presented in Tables 1 - 6 all components have the same repair time distribution. Hence, in accordance with results given in Section 1, $I_{N,D}^{(i,j)}(t)$ and $I_{B-P}^{(i,j)}(t)$ are practically equal. Parallel to the case mentioned above component 1's importance is increasing in c for all system levels both for the $I_N^{(1,j)}(t)$ and the extended measure. Hence, again according to these measures the increased uncertainty associated with an increasing variance leads to increased importance of a component.

The ranks of the extended measure shown in Table 23 change a lot as for System B in the case mentioned above. As c increases, and component 1 becomes more uncertain, it also becomes more important. The other components, however, are ranked in the order $2 \approx 4, 5, 3$. When $c = 3/2$ components 1 and 5 have swapped places in the ranking. Component 1 becomes the most important component when $c = 6$ for system levels 1 and 2, but not for system levels 3 and 4.

c	i	$I_N^{(i,1)}(t)$	$I_{N,D}^{(i,1)}(t)$	$\bar{I}_N^{(i,1)}(t)$	$I_B^{(i,1)}(t)$	$I_{B-P}^{(i,1)}(t)$
	1	0.147	0.202	0.168	0.271	0.202
	2	0.277	0.257	0.269	0.197	0.257
$\frac{1}{2}$	3	0.088	0.082	0.086	0.063	0.082
	4	0.277	0.257	0.269	0.197	0.257
	5	0.211	0.202	0.208	0.271	0.202
	1	0.199	0.202	0.200	0.271	0.202
	2	0.259	0.257	0.259	0.197	0.257
1	3	0.083	0.082	0.083	0.063	0.082
	4	0.260	0.257	0.259	0.197	0.257
	5	0.198	0.202	0.200	0.272	0.202
	1	0.235	0.202	0.223	0.272	0.202
	2	0.248	0.257	0.252	0.197	0.257
$\frac{3}{2}$	3	0.079	0.082	0.080	0.063	0.082
	4	0.248	0.256	0.251	0.197	0.257
	5	0.189	0.202	0.194	0.271	0.202
	1	0.262	0.202	0.241	0.271	0.202
	2	0.239	0.257	0.245	0.197	0.257
2	3	0.076	0.082	0.078	0.063	0.082
	4	0.239	0.257	0.245	0.197	0.257
	5	0.183	0.203	0.190	0.272	0.202
	1	0.385	0.202	0.329	0.271	0.202
	2	0.199	0.257	0.217	0.197	0.257
6	3	0.064	0.082	0.069	0.063	0.082
	4	0.199	0.257	0.217	0.197	0.257
	5	0.152	0.202	0.168	0.272	0.202

Table 19: *Simulations of the bridge system with increasing variance in the distributions of the times in a non complete failure state of component 1. System level is 1. The time horizon is* $t = 20000$.

We will now look at how a decreasing mean time to repair of one of the components influences the importance measures. More specifically, let the components have the following distributions of times spent in state k before jumping to state $k-1$ for $k = 1, 2$ and of repair time:

Component 1: $\bar{F}_1^k(t) \sim \text{gamma}(4, 1), k = 1, 2,\ \ \bar{G}_1(t) \sim \text{gamma}(6/c^2, c/2)$,

Component i: $\bar{F}_i^k(t) \sim \text{gamma}(4, 1), k = 1, 2,\ \ \bar{G}_i(t) \sim \text{gamma}(6, 1/2),\ \ i = 2, 3, 4, 5$,

where c as above is a positive number. In this example the distributions of the times spent in each of the non complete failure states are equal for all components with mean times $\mu_i^k = 4,\ i = 1, \dots, 5$. Moreover, the variances associated with these distributions are all equal to 4. Components $2, \dots, 5$ have mean time to repair equal to 3. The mean time to repair of component 1 is $3/c$, which is decreasing in c. The variance in the repair time distributions are 3/2 for all components.

The results of the simulations are shown in Tables 24 - 27 for all three versions of the Natvig measure and for $I_B^{(i,j)}(t)$ and $I_{B-P}^{(i,j)}(t)$ for respectively system levels 1, 2, 3, 4. As for the cases

c	i	$I_N^{(i,2)}(t)$	$I_{N,D}^{(i,2)}(t)$	$\bar{I}_N^{(i,2)}(t)$	$I_B^{(i,2)}(t)$	$I_{B-P}^{(i,2)}(t)$
	1	0.121	0.169	0.139	0.236	0.169
	2	0.315	0.296	0.308	0.237	0.297
$\frac{1}{2}$	3	0.074	0.070	0.072	0.056	0.070
	4	0.316	0.297	0.308	0.237	0.297
	5	0.174	0.169	0.172	0.236	0.169
	1	0.165	0.169	0.167	0.236	0.169
	2	0.299	0.296	0.298	0.237	0.296
1	3	0.070	0.070	0.070	0.056	0.070
	4	0.300	0.296	0.298	0.237	0.296
	5	0.165	0.169	0.167	0.236	0.169
	1	0.197	0.168	0.187	0.236	0.169
	2	0.288	0.297	0.291	0.237	0.296
$\frac{3}{2}$	3	0.068	0.070	0.069	0.056	0.070
	4	0.288	0.297	0.291	0.237	0.297
	5	0.159	0.168	0.162	0.236	0.169
	1	0.222	0.169	0.203	0.236	0.169
	2	0.279	0.296	0.285	0.237	0.296
2	3	0.065	0.069	0.067	0.056	0.070
	4	0.279	0.297	0.285	0.237	0.297
	5	0.154	0.169	0.159	0.236	0.169
	1	0.334	0.169	0.282	0.236	0.169
	2	0.239	0.296	0.257	0.237	0.296
6	3	0.056	0.070	0.060	0.056	0.070
	4	0.239	0.296	0.257	0.237	0.296
	5	0.132	0.169	0.144	0.236	0.169

Table 20: *Simulations of the bridge system with increasing variance in the distributions of the times in a non complete failure state of component 1. System level is 2. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,3)}(t)$	$I_{N,D}^{(i,3)}(t)$	$\bar{I}I_N^{(i,3)}(t)$	$I_B^{(i,3)}(t)$	$I_{B-P}^{(i,3)}(t)$
	1	0.114	0.160	0.132	0.226	0.160
	2	0.339	0.319	0.331	0.258	0.320
$\frac{1}{2}$	3	0.044	0.042	0.043	0.034	0.042
	4	0.339	0.320	0.331	0.258	0.320
	5	0.164	0.160	0.162	0.225	0.160
	1	0.157	0.160	0.158	0.226	0.160
	2	0.322	0.319	0.321	0.258	0.319
1	3	0.042	0.042	0.042	0.034	0.042
	4	0.323	0.320	0.321	0.258	0.320
	5	0.157	0.160	0.158	0.225	0.160
	1	0.187	0.160	0.177	0.226	0.160
	2	0.311	0.319	0.314	0.258	0.319
$\frac{3}{2}$	3	0.041	0.042	0.041	0.033	0.042
	4	0.311	0.320	0.314	0.258	0.319
	5	0.151	0.160	0.154	0.226	0.160
	1	0.211	0.160	0.193	0.225	0.160
	2	0.302	0.319	0.308	0.258	0.319
2	3	0.039	0.042	0.040	0.034	0.042
	4	0.302	0.320	0.308	0.258	0.320
	5	0.146	0.159	0.151	0.225	0.160
	1	0.320	0.160	0.269	0.226	0.160
	2	0.260	0.320	0.279	0.258	0.320
6	3	0.034	0.042	0.036	0.034	0.042
	4	0.260	0.319	0.279	0.258	0.319
	5	0.126	0.160	0.137	0.225	0.160

Table 21: *Simulations of the bridge system with increasing variance in the distributions of the times in a non complete failure state of component 1. System level is 3. The time horizon is* $t = 20000$.

presented in Tables 7 - 18 all components have the same distributions of the times spent in each of the non complete failure states. Hence, in accordance with results given in Section 1 $I_N^{(i,j)}(t)$ and $I_{B-P}^{(i,j)}(t)$ are practically equal. When $c = 1$, all components have identical distributions. Hence, in accordance with results given in Section 1 all measures give the same results for each system level in this case. As is seen from the tables, the results are very similar for all three Natvig importance measures and $I_{B-P}^{(i,j)}(t)$. The importance according to these measures of components 1 and 2 are increasing in c, while the ones of components 3 and 4 are decreasing in c for system levels 1 and 2, whereas these importances are approximately constant for system levels 3 and 4. The resulting ranks, which are identical for all the three Natvig measures and $I_{B-P}^{(i,j)}(t)$, are shown in Table 28. As

c	i	$I_N^{(i,4)}(t)$	$I_{N,D}^{(i,4)}(t)$	$\bar{I}_N^{(i,4)}(t)$	$I_B^{(i,4)}(t)$	$I_{B-P}^{(i,4)}(t)$
	1	0.120	0.167	0.138	0.234	0.167
	2	0.354	0.333	0.346	0.266	0.333
$\frac{1}{2}$	3	0.000	0.000	0.000	0.000	0.000
	4	0.354	0.333	0.346	0.266	0.333
	5	0.172	0.167	0.170	0.234	0.167
	1	0.163	0.166	0.164	0.233	0.167
	2	0.337	0.334	0.336	0.266	0.333
1	3	0.000	0.000	0.000	0.000	0.000
	4	0.337	0.333	0.335	0.267	0.333
	5	0.164	0.167	0.165	0.234	0.167
	1	0.194	0.167	0.184	0.233	0.167
	2	0.324	0.333	0.327	0.266	0.333
$\frac{3}{2}$	3	0.000	0.000	0.000	0.000	0.000
	4	0.324	0.333	0.328	0.267	0.334
	5	0.157	0.167	0.161	0.233	0.167
	1	0.220	0.167	0.201	0.234	0.167
	2	0.314	0.333	0.321	0.266	0.333
2	3	0.000	0.000	0.000	0.000	0.000
	4	0.314	0.333	0.321	0.267	0.333
	5	0.153	0.167	0.158	0.234	0.167
	1	0.331	0.167	0.279	0.234	0.167
	2	0.269	0.333	0.290	0.267	0.333
6	3	0.000	0.000	0.000	0.000	0.000
	4	0.269	0.334	0.290	0.267	0.333
	5	0.131	0.167	0.142	0.233	0.167

Table 22: *Simulations of the bridge system with increasing variance in the distributions of the times in a non complete failure state of component 1. System level is 4. The time horizon is $t = 20000$.*

c	Rank system levels 1,2	Rank system levels 3, 4
1/2	$2 \approx 4 > 5 > 1 > 3$	$2 \approx 4 > 5 > 1 > 3$
1	$2 \approx 4 > 1 \approx 5 > 3$	$2 \approx 4 > 1 \approx 5 > 3$
3/2	$2 \approx 4 > 1 > 5 > 3$	$2 \approx 4 > 1 > 5 > 3$
2	$2 \approx 4 > 1 > 5 > 3$	$2 \approx 4 > 1 > 5 > 3$
6	$1 > 2 \approx 4 > 5 > 3$	$2 \approx 4 > 1 > 5 > 3$

Table 23: *The ranks of the extended measure of component importance corresponding to the results in Tables 19 - 22*
.

in the previous case, component 3 is always the least important component. For all the importance measures there is a turning point at $c = 1$. For this c all components have identical distributions leaving components 1, 2, 4 and 5 equally important since they are similarly positioned in the system.

As a conclusion the results of the present section very much parallel the ones of the previous one.

c	i	$I_N^{(i,1)}(t)$	$I_{N,D}^{(i,1)}(t)$	$\bar{I}_N^{(i,1)}(t)$	$I_B^{(i,1)}(t)$	$I_{B-P}^{(i,1)}(t)$
	1	0.173	0.169	0.172	0.211	0.174
	2	0.198	0.199	0.199	0.189	0.198
$\frac{1}{2}$	3	0.074	0.074	0.074	0.071	0.074
	4	0.322	0.324	0.323	0.308	0.322
	5	0.232	0.233	0.233	0.222	0.232
	1	0.233	0.234	0.233	0.233	0.233
	2	0.234	0.234	0.234	0.234	0.234
1	3	0.066	0.066	0.066	0.066	0.066
	4	0.233	0.233	0.233	0.233	0.233
	5	0.234	0.233	0.234	0.233	0.233
	1	0.264	0.267	0.265	0.246	0.264
	2	0.251	0.250	0.251	0.258	0.252
$\frac{3}{2}$	3	0.062	0.062	0.062	0.064	0.062
	4	0.189	0.188	0.188	0.193	0.188
	5	0.234	0.233	0.234	0.240	0.234
	1	0.295	0.302	0.297	0.259	0.295
	2	0.269	0.267	0.268	0.283	0.269
$\frac{5}{2}$	3	0.059	0.058	0.058	0.062	0.059
	4	0.143	0.141	0.142	0.150	0.143
	5	0.235	0.232	0.234	0.246	0.235

Table 24: *Simulations of the bridge system with decreasing MTTR of component 1. System level is 1. The time horizon is $t = 20000$.*

c	i	$I_N^{(i,2)}(t)$	$I_{N,D}^{(i,2)}(t)$	$\bar{I}_N^{(i,2)}(t)$	$I_B^{(i,2)}(t)$	$I_{B-P}^{(i,2)}(t)$
	1	0.202	0.197	0.200	0.243	0.202
	2	0.217	0.219	0.218	0.206	0.217
$\frac{1}{2}$	3	0.070	0.070	0.070	0.066	0.070
	4	0.272	0.274	0.272	0.258	0.272
	5	0.239	0.240	0.239	0.227	0.239
	1	0.234	0.234	0.234	0.234	0.234
	2	0.235	0.234	0.235	0.234	0.234
1	3	0.063	0.063	0.063	0.063	0.063
	4	0.234	0.234	0.234	0.234	0.234
	5	0.234	0.234	0.234	0.234	0.234
	1	0.247	0.251	0.249	0.230	0.247
	2	0.241	0.240	0.241	0.247	0.241
$\frac{3}{2}$	3	0.060	0.060	0.060	0.061	0.060
	4	0.219	0.218	0.219	0.224	0.219
	5	0.233	0.232	0.232	0.238	0.232
	1	0.259	0.266	0.262	0.227	0.259
	2	0.247	0.245	0.247	0.258	0.247
$\frac{5}{2}$	3	0.058	0.057	0.057	0.060	0.057
	4	0.205	0.204	0.205	0.214	0.205
	5	0.230	0.228	0.230	0.241	0.230

Table 25: *Simulations of the bridge system with decreasing MTTR of component 1. System level 2. The time horizon is* $t = 20000$.

c	i	$I_N^{(i,3)}(t)$	$I_{N,D}^{(i,3)}(t)$	$\bar{I}_N^{(i,3)}(t)$	$I_B^{(i,3)}(t)$	$I_{B-P}^{(i,3)}(t)$
	1	0.242	0.237	0.240	0.289	0.242
	2	0.242	0.244	0.243	0.227	0.242
$\frac{1}{2}$	3	0.031	0.031	0.031	0.029	0.031
	4	0.242	0.244	0.243	0.227	0.242
	5	0.242	0.244	0.243	0.227	0.242
	1	0.242	0.243	0.242	0.242	0.242
	2	0.243	0.242	0.243	0.243	0.243
1	3	0.031	0.031	0.031	0.031	0.031
	4	0.242	0.242	0.242	0.242	0.242
	5	0.243	0.242	0.243	0.242	0.242
	1	0.242	0.246	0.244	0.225	0.243
	2	0.242	0.241	0.242	0.248	0.242
$\frac{3}{2}$	3	0.031	0.031	0.031	0.031	0.031
	4	0.243	0.241	0.242	0.248	0.242
	5	0.242	0.241	0.242	0.248	0.242
	1	0.242	0.248	0.245	0.211	0.242
	2	0.242	0.240	0.241	0.252	0.242
$\frac{5}{2}$	3	0.031	0.030	0.031	0.032	0.031
	4	0.242	0.240	0.241	0.252	0.242
	5	0.242	0.241	0.242	0.252	0.242

Table 26: *Simulations of the bridge system with decreasing MTTR of component 1. System level is 3. The time horizon is* $t = 20000$.

4. Application to an offshore oil and gas production system

We will now look at a West-African production site for oil and gas based on a memo [11]. Oil and gas are pumped up from one production well along with water. These substances are separated in a separation unit. We will assume this unit to function perfectly. After being separated the oil is run through an oil treatment unit, which is also assumed to function perfectly. Then the treated oil is exported through a pumping unit.

The gas is sent through two compressors which compress the gas. When both compressors are functioning, we get the maximum amount of gas. However, to obtain at least *some* gas production, it is sufficient that at least one of the compressors is functioning. If this is the case, the uncompressed gas is burned in a flare, which is assumed to function perfectly. The compressed gas is run through a unit where it is dehydrated. This is called a TEG (Tri-Ethylene Glycol) unit. After being dehydrated, the gas is ready to be exported. Some of the gas is used as fuel for the compressors.

The water is first run through a water treatment unit. This unit cleanses the water so that it legally can be pumped back into the wells to maintain the pressure, or back into the sea. If the water treatment unit fails, the whole production stops. The components in the system also need electricity which comes from two generators. At least one generator must function in order to produce some

c	i	$I_N^{(i,4)}(t)$	$I_{N,D}^{(i,4)}(t)$	$\bar{I}_N^{(i,4)}(t)$	$I_B^{(i,4)}(t)$	$I_{B-P}^{(i,4)}(t)$
	1	0.250	0.244	0.248	0.297	0.249
	2	0.251	0.252	0.251	0.234	0.250
$\frac{1}{2}$	3	0.000	0.000	0.000	0.000	0.000
	4	0.250	0.252	0.250	0.234	0.250
	5	0.250	0.252	0.251	0.235	0.250
	1	0.250	0.250	0.250	0.250	0.250
	2	0.250	0.250	0.250	0.250	0.250
1	3	0.000	0.000	0.000	0.000	0.000
	4	0.250	0.250	0.250	0.250	0.250
	5	0.250	0.250	0.250	0.250	0.250
	1	0.250	0.254	0.252	0.233	0.250
	2	0.249	0.248	0.249	0.255	0.249
$\frac{3}{2}$	3	0.000	0.000	0.000	0.000	0.000
	4	0.250	0.249	0.250	0.256	0.250
	5	0.251	0.249	0.250	0.256	0.251
	1	0.250	0.257	0.253	0.218	0.250
	2	0.250	0.248	0.249	0.260	0.250
$\frac{5}{2}$	3	0.000	0.000	0.000	0.000	0.000
	4	0.250	0.248	0.249	0.261	0.250
	5	0.250	0.248	0.249	0.261	0.250

Table 27: *Simulations of the bridge system with decreasing MTTR of component 1. System level is 4. The time horizon is* $t = 20000$.

c	Rank system levels 1, 2	Rank system levels 3, 4
1/2	$4 > 5 > 2 > 1 > 3$	$4 \approx 5 \approx 2 \approx 1 > 3$
1	$4 \approx 5 \approx 2 \approx 1 > 3$	$4 \approx 5 \approx 2 \approx 1 > 3$
3/2	$1 > 2 > 5 > 4 > 3$	$4 \approx 5 \approx 2 \approx 1 > 3$
5/2	$1 > 2 > 5 > 4 > 3$	$4 \approx 5 \approx 2 \approx 1 > 3$

Table 28: *The common ranks of the three Natvig measures of component importance corresponding to the results in Tables 24 - 27.*

oil and gas. If both generators are failed, the whole system is failed. The generators are powered by compressed and dehydrated gas. Thus, the simplified production site considered in the present

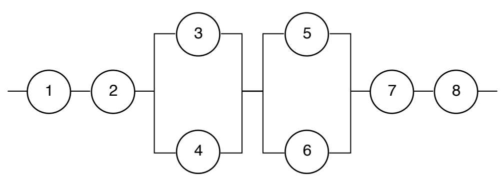

Figure 3: Model of oil and gas production site.

paper, consists of the following 8 relevant components, which are assumed to operate independently:

1. Well: A production well where the oil and gas come from.
2. Water cleanser: A component which cleanses the water which is pumped up from the production well along with the oil and gas.
3. Generator 1: Generator providing electricity to the system.
4. Generator 2: The same as Generator 1.
5. Compressor 1: A compressor which compresses the gas.
6. Compressor 2: The same as Compressor 1.
7. TEG: A component where the gas is dehydrated.
8. Oil export pump: An oil export pump.

The structure of the system is shown in Figure 3. The components 1, 2, 7 and 8 are all in series with the rest of the system, while the two generators, 3 and 4, operate in parallel with each other. Similarly, the two compressors, 5 and 6, operate in parallel with each other.

In [11] no explicit definition of the system is given. There are several different possible definitions, and in [8] we used the following: *The oil and gas production site is said to be functioning if it can produce some amount of both oil and gas. Otherwise the system is failed.*

Comp.	Bin. failure rate	μ_i^0	$\mu_i^1 + \mu_i^2$
1	$2.736 \cdot 10^{-4}$	7.000	3654.97
2	$8.208 \cdot 10^{-3}$	0.167	121.83
3 & 4	$1.776 \cdot 10^{-2}$	1.167	56.31
5 & 6	$1.882 \cdot 10^{-2}$	1.083	53.11
7	$1.368 \cdot 10^{-3}$	0.125	730.99
8	$5.496 \cdot 10^{-4}$	0.125	1819.51

Table 29: *Binary failure rates, mean repair times and mean total times spent in the non complete failure states of the components in the oil and gas production site.*

Table 29 shows the given binary failure rates, mean repair times and mean total times spent in the non complete failure states of the components in the system. The time unit is *days*. The mean total times spent in the non complete failure states are considerably larger than the mean repair times. For some components (the well, the TEG unit and the oil export pump) the former means are actually several years.

It should be stressed that the memo [11] does not discuss what might be reasonable life- and repair time distributions. Hence, the choice of the exponential and gamma distributions in the following may be difficult to justify empirically. Our aim is, however, to illustrate how different choices of such distributions influence the Natvig measures. In order to avoid making this sensitivity analysis too extensive, we have skipped the Weibull distribution.

4.1. Exponentially distributed repair times and times spent in each of the non complete failure states

In this subsection we assume that the components both have exponentially distributed times spent in each of the non complete failure states and exponentially distributed repair times. The failure rates in the former distributions are twice the inverses of the mean total times spent in each of the non complete failure states, while the repair rates are the inverses of the mean repair times. Thus, all the parameters needed in the simulations can be derived from Table 29. The time horizon t is set to 100000 days.

In Tables 30 and 31 we see that $I_N^{(i,j)}(t)$ is almost equal to its extended version $\bar{I}_N^{(i,j)}(t)$ for both system levels. This is because $E[Y_{i,k,j}(t)]$ for $k = 1, \ldots, M$ are very large compared to $E[Y_{i,0,j}(t)]$ for all components. Hence, the contributions of the latter terms in Eq.(17) are too small to make any difference.

The reason for this is that the repair times of the components are much shorter than the corresponding times spent in each of the non complete failure states. Hence, the fictive prolonged repair times of the components due to the fictive minimal complete failures are much shorter than the fictive prolonged times spent in each of the non complete failure states due the fictive minimal repairs. Especially, the fictive prolonged repair times will, due to the much longer times spent in each of the non complete failure states, mostly end long before the next real repair. Hence, it is very unlikely that the fictive minimal complete failure periods will overlap. As a conclusion it is very sensible for this case study that $I_N^{(i,j)}(t)$ is equal to $\bar{I}_N^{(i,j)}(t)$.

We also observe from Tables 30 and 31 that for the two almost equal measures the components 1, 2, 7 and 8 that are in series with the rest of the system have approximately the same importance for both system levels. This can be seen for system level 2 by the following argument. Since $t = 100000$ days we have reached stationarity. Furthermore, for the exponential distribution $\mu_i^{k(p)} = \mu_i^k$ for $k = 0, \ldots, M$. If components i and ℓ both are in series with the rest of the system by conditioning on the state of component ℓ and applying Eq.(1), the numerator of Eq.(13) equals $p_\phi^2((\boldsymbol{e}^2)_i, (\boldsymbol{e}^2)_\ell, \boldsymbol{a})a_i^2 a_\ell^2$. By a parallel argument this is also the numerator of $I_N^{(\ell,2)}$.

Note also that the remaining components that are parts of parallel modules have almost identical importances according to the two almost equal measures and are much less important than the ones in series with the rest of the system for both system levels. According to an argument in [8] the reason for this in the binary case is that all components 3, 4, 5 and 6 have almost identical availabilities as is seen from Table 29.

The ranks of the component importance for the three versions of the Natvig measure are given in Table 32. We suggest to apply the ranking based on the measure $\bar{I}_N^{(i,j)}(t)$ which is almost identical to the one of $I_N^{(i,j)}(t)$.

Comp.	$I_N^{(i,1)}(t)$	$I_{N,D}^{(i,1)}(t)$	$\bar{I}_N^{(i,1)}(t)$	$I_B^{(i,1)}(t)$	$I_{B-P}^{(i,1)}(t)$
1	0.244	0.438	0.244	0.247	0.024
2	0.250	0.318	0.250	0.247	0.731
3 & 4	0.007	0.002	0.007	0.007	0.005
5 & 6	0.005	0.092	0.005	0.005	0.033
7	0.249	0.039	0.248	0.247	0.122
8	0.246	0.016	0.246	0.247	0.049

Table 30: *Component importance using exponential distributions. System level is 1. The time horizon is* $t = 100000$.

Comp.	$I_N^{(i,2)}(t)$	$I_{N,D}^{(i,2)}(t)$	$\bar{I}_N^{(i,2)}(t)$	$I_B^{(i,2)}(t)$	$I_{B-P}^{(i,2)}(t)$
1	0.197	0.121	0.196	0.198	0.010
2	0.202	0.088	0.201	0.200	0.285
3 & 4	0.051	0.049	0.051	0.050	0.154
5 & 6	0.051	0.338	0.053	0.052	0.166
7	0.200	0.011	0.199	0.199	0.048
8	0.197	0.005	0.196	0.198	0.019

Table 31: *Component importance using exponential distributions. System level is 2. The time horizon is* $t = 100000$.

Measure	Rank system level 1
$I_N^{(i)}(t)$	$2 > 7 > 8 > 1 > 3 \approx 4 > 5 \approx 6$
$I_{N,D}^{(i)}(t)$	$1 > 2 > 5 \approx 6 > 7 > 8 > 3 \approx 4$
$\bar{I}_N^{(i)}(t)$	$2 > 7 > 8 > 1 > 3 \approx 4 > 5 \approx 6$
	Rank system level 2
$I_N^{(i)}(t)$	$2 > 7 > 8 \approx 1 > 3 \approx 4 \approx 5 \approx 6$
$I_{N,D}^{(i)}(t)$	$5 \approx 6 > 1 > 2 > 3 \approx 4 > 7 > 8$
$\bar{I}_N^{(i)}(t)$	$2 > 7 > 8 \approx 1 > 5 \approx 6 > 3 \approx 4$

Table 32: *The ranks of the component importance for system levels 1 and 2 for the three versions of the Natvig measure according to the results given in Tables 30 and 31 .*

4.2. Gamma distributed repair times and times spent in each of the non complete failure states

In this subsection we assume instead that the components have gamma distributed times spent in each of the non complete failure states and gamma distributed repair times. More specifically, we assume that for $i = 1, \ldots, 8$, the times spent in each of the non complete failure states of the ith component have the densities:

$$f_i(t) = \frac{1}{(\beta_i)^{\alpha_i/2}\Gamma(\alpha_i/2)} t^{\alpha_i/2-1} \exp(-t/\beta_i),$$

while the repair times of the ith component have the densities:

$$g_i(t) = \frac{1}{(\beta_i')^{\alpha_i'}\Gamma(\alpha_i')} t^{\alpha_i'-1} \exp(-t/\beta_i').$$

By choosing different values for the density parameters it is possible to alter the variances in the distributions of the times spent in each of the non complete failure states and still keep the expectations fixed. In order to see the effect of this on the importance measures, we focus on component 1 where we consider five different parameter combinations for the distribution of the times spent in each of the non complete failure states. For all these combinations the expected time spent in each of the non complete failure states is 1827.49 days, but the variance varies between $9.135 \cdot 10^2$ and $5.85 \cdot 10^5$. Table 33 lists these parameter combinations. For the remaining gamma densities we use the parameters listed in Tables 34 and 35. All parameters are chosen such that the expectations in the repair time distributions and in the total times spent in each of the non complete failure states match the corresponding values given in Table 29. We also use the same time horizon $t = 100000$ days as in the previous subsection.

Set	α_1	β_1	Variance
1	7309.940	0.500	$9.135 \cdot 10^2$
2	550.033	6.645	$1.215 \cdot 10^4$
3	101.493	36.012	$6.58 \cdot 10^4$
4	45.687	80.000	$1.462 \cdot 10^5$
5	11.422	319.994	$5.85 \cdot 10^5$

Table 33: *Parameter sets for the distribution of times spent in each of the non complete failure states for component 1.*

Comp.	α_i	β_i	Variance
2	30.000	4.062	$2.475 \cdot 10^2$
3 & 4	30.000	1.877	$5.285 \cdot 10^1$
5 & 6	10.000	5.311	$1.411 \cdot 10^2$
7	179.958	4.062	$1.485 \cdot 10^3$
8	218.219	8.338	$7.585 \cdot 10^3$

Table 34: *Parameters in the distribution of times spent in each of the non complete failure states for components* $2, \ldots, 8$.

Comp.	α_i'	β_i'	Variance
1	3.500	2.000	$1.400 \cdot 10^1$
2	0.668	0.250	$4.175 \cdot 10^{-2}$
3 & 4	3.000	0.389	$4.540 \cdot 10^{-1}$
5 & 6	1.500	0.722	$7.819 \cdot 10^{-1}$
7	1.000	0.125	$1.563 \cdot 10^{-2}$
8	1.000	0.125	$1.563 \cdot 10^{-2}$

Table 35: *Parameters in the repair time distributions of components* $1, \ldots, 8$.

Tables 36 - 40 display the results obtained from simulations using the parameters listed in Tables 33, 34 and 35. As for the case with exponentially distributed repair times and times spent in each of the non complete failure states $I_N^{(i,j)}(t)$ is almost equal to its extended version $\bar{I}_N^{(i,j)}(t)$ for both system levels.

We now observe that for these two almost equal measures the components 1, 2, 7 and 8 that are in series with the rest of the system have different importances for both system levels as opposed to the exponential case. However, the remaining components that are parts of parallel modules are still much less important for system level 1.

Furthermore, we see that for both system levels the extended component importance of component 1 is increasing with increasing variances, and decreasing shape parameters $\alpha_1/2$, all greater than 1, in its distribution of times spent in each of the non complete failure states. Since we have reached stationarity, this observation, based on simulations and assuming gamma distributed repair times and times spent in each of the non complete failure states, is in accordance with the theoretical result concerning the Weibull distribution described following Eq.(21).

Table 41 displays the ranks of the components according to the extended measure for both system levels. Along with the increased importance for both system levels, according to the extended measure, of component 1 as $\alpha_1/2$ decreases, we observe from this table a corresponding improvement

Comp.	$I_N^{(i,j)}(t)$	$I_{N,D}^{(i,j)}(t)$	$\bar{I}_N^{(i,j)}(t)$	$I_B^{(i,j)}(t)$	$I_{B-P}^{(i,j)}(t)$
1	0.032	0.249	0.035	0.245	0.023
2	0.519	0.414	0.518	0.245	0.694
3 & 4	0.010	0.058	0.011	0.005	0.030
5 & 6	0.018	0.080	0.019	0.005	0.031
7	0.206	0.042	0.204	0.245	0.115
8	0.186	0.016	0.183	0.245	0.046
1	0.020	0.062	0.021	0.196	0.010
2	0.309	0.095	0.300	0.201	0.284
3 & 4	0.078	0.071	0.082	0.052	0.155
5 & 6	0.139	0.244	0.144	0.052	0.165
7	0.122	0.010	0.117	0.198	0.047
8	0.113	0.004	0.109	0.199	0.019

Table 36: *Component importance using gamma distributions for system level 1 (upper part) and system level 2 (lower part). Variance of times spent in each of the non complete failure states of component:* $9.135 \cdot 10^2$.

Comp.	$I_N^{(i,j)}(t)$	$I_{N,D}^{(i,j)}(t)$	$\bar{I}_N^{(i,j)}(t)$	$I_B^{(i,j)}(t)$	$I_{B-P}^{(i,j)}(t)$
1	0.107	0.246	0.109	0.245	0.023
2	0.479	0.417	0.479	0.245	0.694
3 & 4	0.010	0.059	0.010	0.005	0.030
5 & 6	0.017	0.081	0.018	0.005	0.031
7	0.189	0.042	0.187	0.245	0.115
8	0.172	0.016	0.169	0.245	0.046
1	0.066	0.059	0.065	0.197	0.009
2	0.295	0.096	0.287	0.200	0.285
3 & 4	0.075	0.174	0.079	0.052	0.155
5 & 6	0.132	0.244	0.137	0.051	0.164
7	0.117	0.009	0.113	0.198	0.047
8	0.109	0.004	0.105	0.198	0.019

Table 37: *Component importance using gamma distributions for system level 1 (upper part) and system level 2 (lower part). Variance of times spent in each of the non complete failure states of component 1:* $1.215 \cdot 10^4$.

Comp.	$I_N^{(i,j)}(t)$	$I_{N,D}^{(i,j)}(t)$	$\bar{I}_N^{(i,j)}(t)$	$I_B^{(i,j)}(t)$	$I_{B-P}^{(i,j)}(t)$
1	0.219	0.241	0.220	0.245	0.023
2	0.419	0.418	0.419	0.245	0.694
3 & 4	0.008	0.058	0.009	0.005	0.030
5 & 6	0.015	0.081	0.016	0.005	0.031
7	0.166	0.044	0.164	0.245	0.115
8	0.150	0.018	0.148	0.245	0.046
1	0.141	0.056	0.138	0.197	0.009
2	0.272	0.095	0.265	0.201	0.285
3 & 4	0.068	0.172	0.072	0.051	0.155
5 & 6	0.122	0.245	0.126	0.052	0.165
7	0.107	0.010	0.104	0.199	0.047
8	0.100	0.004	0.097	0.198	0.019

Table 38: *Component importance using gamma distributions for system level 1 (upper part) and system level 2 (lower part). Variance of times spent in each of the non complete failure states of component 1:* $6.58 \cdot 10^4$.

Comp.	$I_N^{(i,j)}(t)$	$I_{N,D}^{(i,j)}(t)$	$\bar{I}_N^{(i,j)}(t)$	$I_B^{(i,j)}(t)$	$I_{B-P}^{(i,j)}(t)$
1	0.298	0.250	0.297	0.245	0.023
2	0.377	0.415	0.378	0.245	0.694
3 & 4	0.008	0.058	0.008	0.005	0.030
5 & 6	0.013	0.080	0.014	0.005	0.031
7	0.150	0.043	0.149	0.245	0.115
8	0.134	0.017	0.133	0.245	0.046
1	0.199	0.055	0.194	0.197	0.009
2	0.254	0.099	0.248	0.201	0.285
3 & 4	0.064	0.173	0.067	0.052	0.155
5 & 6	0.113	0.243	0.118	0.051	0.164
7	0.100	0.010	0.097	0.199	0.047
8	0.093	0.004	0.090	0.198	0.019

Table 39: *Component importance using gamma distributions for system level 1 (upper part) and system level 2 (lower part). Variance of times spent in each of the non complete failure states of component 1:* $1.462 \cdot 10^5$.

Comp.	$I_N^{(i,j)}(t)$	$I_{N,D}^{(i,j)}(t)$	$\bar{I}_N^{(i,j)}(t)$	$I_B^{(i,j)}(t)$	$I_{B-P}^{(i,j)}(t)$
1	0.466	0.249	0.464	0.245	0.023
2	0.286	0.416	0.287	0.245	0.694
3 & 4	0.006	0.058	0.006	0.005	0.030
5 & 6	0.010	0.081	0.011	0.005	0.031
7	0.114	0.041	0.113	0.245	0.115
8	0.103	0.016	0.102	0.245	0.046
1	0.337	0.056	0.329	0.197	0.009
2	0.210	0.097	0.207	0.200	0.285
3 & 4	0.053	0.173	0.056	0.051	0.155
5 & 6	0.094	0.244	0.098	0.051	0.164
7	0.083	0.010	0.081	0.199	0.047
8	0.077	0.004	0.075	0.198	0.019

Table 40: *Component importance using gamma distributions for system level 1 (upper part) and system level 2 (lower part). Variance of times spent in each of the non complete failure states of component 1:* $5.85 \cdot 10^5$.

in its rank. All the other components are ranked in the same order for every value of $\alpha_1/2$ both for system level 1 and 2. This is as expected from Eq.(20) since the ordering is determined by its numerator. For all components except component 1 the numerator depends on the distributions of repair time and the times spent in each of the non complete failure states of this component only through $a_1^j, j = 1, 2$, which is kept fixed when varying $\alpha_1/2$. We also see that the components that are in series with the rest of the system for both system levels are ranked according to the shape parameter $\alpha_i/2$, such that components with the smaller shape parameters are more important. Note also that according to the ranks components 1, 7 and 8 are more important on system level 1 than 2, while it is the other way round for components 5 and 6.

Comparing the ranks of the extended component importance in Table 41 for the gamma distributions to the ones in Table 32 for the exponential distributions, we see that in the former case for both system levels components 5 and 6 in one of the parallel modules are more important than components 3 and 4 in the other parallel module, whereas they are roughly equally important in the latter case. This is due to the fact that in the former case the repair times and the times spent in each of the non complete failure states of components 5 and 6 have a larger variance than the corresponding ones of components 3 and 4. In the case where exponential distributions were used, the corresponding variances were roughly similar. The ranking of the components 2, 7 and 8 that are in series with the rest of the system, are common for both system levels in these tables.

Table	Rank system level 1
36	$2 > 7 > 8 > 1 > 5 \approx 6 > 3 \approx 4$
37	$2 > 7 > 8 > 1 > 5 \approx 6 > 3 \approx 4$
38	$2 > 1 > 7 > 8 > 5 \approx 6 > 3 \approx 4$
39	$2 > 1 > 7 > 8 > 5 \approx 6 > 3 \approx 4$
40	$1 > 2 > 7 > 8 > 5 \approx 6 > 3 \approx 4$
	Rank system level 2
36	$2 > 5 \approx 6 > 7 > 8 > 3 \approx 4 > 1$
37	$2 > 5 \approx 6 > 7 > 8 > 3 \approx 4 > 1$
38	$2 > 1 > 5 \approx 6 > 7 > 8 > 3 \approx 4$
39	$2 > 1 > 5 \approx 6 > 7 > 8 > 3 \approx 4$
40	$1 > 2 > 5 \approx 6 > 7 > 8 > 3 \approx 4$

Table 41: *The ranks of the extended component importance according to the results given in Tables 36 - 40.*

5. Concluding remarks

In the present paper first a review of basic ideas, concepts and theoretical results, as treated in [10], for the Natvig measures of component importance for repairable multistate systems, and its extended version, has been given. Then two three component systems and the bridge system were analysed. We saw that an important feature of the Natvig measures is that they reflect the degree of uncertainty in the distributions of the repair time and of the times spent in each of the non complete failure states of the components. Finally, the theory was applied to an offshore oil and gas production system. First the times were assumed to be exponentially distributed and then gamma distributed both in accordance with the data given in the memo [11]. The time horizon was set at 100000 days so stationarity is reached.

A finding from the simulations of this case study is that the results for the original Natvig measure and its extended version, also taking a dual term into account, are almost identical. This is perfectly sensible since the dual term vanishes because the fictive prolonged repair times are much shorter than the fictive prolonged times spent in each of the non complete failure states. The weaknesses of this system are linked to the times spent in each of the non complete failure states and not the repair times.

Component 1 is the well being in series with the rest of the system. For this component we see that the extended component importance in the gamma case is increasing with increasing variances, and decreasing shape parameters, all greater than 1, in the distributions of times spent in each of the non complete failure states. This is in accordance with a theoretical result for the Weibull distribution. Along with this increased importance we also observed a corresponding improvement in its ranking.

Based on the presentation in [10] we feel that the Natvig measures of component importance for repairable multistate systems on the one hand represent a theoretical novelty. On the other hand the case study indicates a great potential for applications, especially due to the simulation methods applied, as presented in the companion paper [4].

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