Dept. of Math. University of Oslo
Statistical Research Report No 3
ISSN 0806–3842 June 2010

Bayesian Assessment of Availabilities and Unavailabilities of Multistate Monotone Systems

Bent Natvig · Jørund Gåsemyr · Trond Reitan

June 29, 2010

Abstract In the present paper we consider a multistate monotone system of multistate components. Following a Bayesian approach, the ambition is to arrive at the posterior distributions of the system availabilities and unavailabilities to the various levels in a fixed time interval based on both prior information and data on both the components and the system. We argue that a realistic approach is to start out by describing our uncertainty on the component availabilities and unavailabilities to the various levels in a fixed time interval, based on both prior information and data on the components, by the moments up till order m of their marginal distributions. From these moments analytic bounds on the corresponding moments of the system availabilities and unavailabilities to the various levels in a fixed time interval are arrived at. Applying these bounds and prior system information we may then fit prior distributions of the system availabilities and unavailabilities to the various levels in a fixed time interval. These can in turn be updated by relevant data on the system. This generalizes results given in (Natvig and Eide 1987) considering a binary monotone system of binary components at a fixed point of time. Furthermore, considering a simple network system, we show that the analytic bounds can be slightly improved by straightforward simulation techniques.

Keywords availabilities · Bayesian assessment · multistate monotone systems · unavailabilities

AMS 2000 Classification 62NO5, 90B25

B. Natvig · J. Gåsemyr
Department of Mathematics, University of Oslo,
P.O. Box 1053 Blindern, N–0316, Oslo, Norway
e-mail: bent@math.uio.no
e-mail: gaasemyr@math.uio.no

T. Reitan
Department of Biology, University of Oslo,
P.O. Box 1066 Blindern, N–0316, Oslo, Norway
e-mail: trond.reitan@bio.uio.no

1. Basic definitions and ideas

Let $S = \{0, 1, \ldots, M\}$ be the set of states of the system; the $M + 1$ states representing successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Furthermore, let $C = \{1, \ldots, n\}$ be the set of components and in general S_i, $i = 1, \ldots, n$ the set of states of the ith component. We require $\{0, M\} \subseteq S_i \subseteq S$. Hence, the states 0 and M are chosen to represent the endpoints of a performance scale that might be used for both the system and its components. Note that in most applications there is no need for the same detailed description of the components as for the system.

Let x_i, $i = 1, \ldots, n$ denote the state or performance level of the ith component at a fixed point of time and $\boldsymbol{x} = (x_1, \ldots, x_n)$. It is assumed that the state, ϕ, of the system at the fixed point of time is a deterministic function of $\boldsymbol{x}$; i.e. $\phi = \phi(\boldsymbol{x})$. Here $\boldsymbol{x}$ takes values in $S_1 \times S_2 \times \cdots \times S_n$ and ϕ takes values in S. The function ϕ is called the structure function of the system. We often denote a multistate system by (C, ϕ).

We start by giving a series of basic definitions.

Definition 1 *A system is a multistate monotone system (MMS) iff its structure function ϕ satisfies:*

(i) $\phi(\boldsymbol{x})$ *is non-decreasing in each argument*

(ii) $\phi(\mathbf{0}) = 0$ *and* $\phi(\boldsymbol{M}) = M \quad \mathbf{0} = (0, \ldots, 0)$, $\boldsymbol{M} = (M, \ldots, M)$.

Definition 2 *The monotone system (A, χ) is a module of the monotone system (C, ϕ) iff*

$$\phi(\boldsymbol{x}) = \psi[\chi(\boldsymbol{x}^A), \boldsymbol{x}^{A^c}],$$

where ψ is a monotone structure function and $A \subseteq C$.

Intuitively, a module is a monotone subsystem that acts as if it were just a supercomponent.

Definition 3 *A modular decomposition of a monotone system (C, ϕ) is a set of disjoint modules $\{(A_k, \chi_k)\}_{k=1}^r$ together with an organizing monotone structure function ψ, i.e.*

(i) $C = \cup_{i=1}^r A_i$ *where* $A_i \cap A_j = \emptyset \quad i \neq j$,

(ii) $\phi(\boldsymbol{x}) = \psi[\chi_1(\boldsymbol{x}^{A_1}), \ldots, \chi_r(\boldsymbol{x}^{A_r})] = \psi[\boldsymbol{\chi}(\boldsymbol{x})]$.

Making a modular decomposition of a system is a way of breaking it into a collection of subsystems which can be dealt with more easily.

In the following $\boldsymbol{y} < \boldsymbol{x}$ means that $y_i \leq x_i$ for $i = 1, \ldots, n$, and $y_i < x_i$ for some i.

Definition 4 *Let ϕ be the structure function of an MMS and let $j \in \{1, \ldots, M\}$. A vector $\boldsymbol{x}$ is said to be a path vector to level j iff $\phi(\boldsymbol{x}) \geq j$. The corresponding path set is given by $C_\phi^j(\boldsymbol{x}) = \{i | x_i \geq 1\}$. A minimal path vector to level j is a path vector $\boldsymbol{x}$ such that $\phi(\boldsymbol{y}) < j$ for all $\boldsymbol{y} < \boldsymbol{x}$. The corresponding path set is also said to be minimal.*

Definition 5 *Let ϕ be the structure function of an MMS and let $j \in \{1, \ldots, M\}$. A vector $\boldsymbol{x}$ is said to be a cut vector to level j iff $\phi(\boldsymbol{x}) < j$. The corresponding cut set is given by $D_\phi^j(\boldsymbol{x}) = \{i | x_i < M\}$. A minimal cut vector to level j is a cut vector $\boldsymbol{x}$ such that $\phi(\boldsymbol{y}) \geq j$ for all $\boldsymbol{y} > \boldsymbol{x}$. The corresponding cut set is also said to be minimal.*

We now consider the relation between the stochastic performance of the system (C, ϕ) and the stochastic performances of the components. Let τ be an index set contained in $[0, \infty)$.

Definition 6 *The performance process of the ith component, $i = 1, \ldots, n$ is a stochastic process $\{X_i(t), t \in \tau\}$, where for each fixed $t \in \tau$, $X_i(t)$ is a random variable which takes values in S_i. $X_i(t)$ denotes the state of the ith component at time t. The joint performance process for the components $\{\boldsymbol{X}(t), t \in \tau\} = \{(X_1(t), \ldots, X_n(t)), t \in \tau\}$ is the corresponding vector stochastic process. The performance process of an MMS with structure function ϕ is a stochastic process $\{\phi(\boldsymbol{X}(t)), t \in \tau\}$, where for each fixed $t \in \tau, \phi(\boldsymbol{X}(t))$ is a random variable which takes values in S. $\phi(\boldsymbol{X}(t))$ denotes the system state at time t.*

We assume that the sample functions of the performance process of a component are continuous from the right on τ. It then follows that the sample functions of $\{\phi(\boldsymbol{X}(t)), t \in \tau\}$ are also continuous from the right on τ. Now consider a time interval $I = [t_A, t_B] \subset [0, \infty)$ and let $\tau(I) = \tau \cap I$.

Definition 7 *The marginal performance processes $\{X_i(t), t \in \tau\}$, $i = 1, \ldots, n$ are independent in the time interval I iff, for any integer m and $\{t_1, \ldots, t_m\} \subset \tau(I)$ the random vectors $\{X_1(t_1), \ldots, X_1(t_m)\}, \ldots, \{X_n(t_1), \ldots, X_n(t_m)\}$ are independent.*

Definition 8 *The joint performance process for the components $\{\boldsymbol{X}(t), t \in \tau\}$ is associated in the time interval I iff, for any integer m and $\{t_1, \ldots, t_m\} \subset \tau(I)$ the random variables in the array*

$$\begin{array}{l} X_1(t_1) \ldots X_1(t_m) \\ \vdots \\ X_n(t_1) \ldots X_n(t_m) \end{array}$$

are associated.

For an introduction to the theory of associated random variables we refer to (Barlow and Proschan 1975).

Definition 9 *Let $i = 1, \ldots, n, j = 1, \ldots, M$. The availability, $p_i^{j(I)}$, and the unavailability, $q_i^{j(I)}$, to level j in the time interval I of the ith component are given by*

$$p_i^{j(I)} = P[X_i(s) \geq j \;\; \forall s \in \tau(I)] \qquad q_i^{j(I)} = P[X_i(s) < j \;\; \forall s \in \tau(I)].$$

The availability, $p_\phi^{j(I)}$, and the unavailability, $q_\phi^{j(I)}$, to level j in the time interval I for an MMS with structure function ϕ are given by

$$p_\phi^{j(I)} = P[\phi(\boldsymbol{X}(s)) \geq j \;\; \forall s \in \tau(I)] \qquad q_\phi^{j(I)} = P[\phi(\boldsymbol{X}(s)) < j \;\; \forall s \in \tau(I)].$$

Let for $i = 1, \ldots, n, j = 0, \ldots, M,$

$$r_i^{j(I)} = p_i^{j(I)} - p_i^{j+1(I)} = P[\min_{s \in \tau(I)} X_i(s) = j]$$
$$r_\phi^{j(I)} = p_\phi^{j(I)} - p_\phi^{j+1(I)} = P[\min_{s \in \tau(I)} \phi(\boldsymbol{X}(s)) = j].$$

Introduce for $i = 1, \ldots, n$ *the component availability and unavailability vectors*

$$\boldsymbol{p}_i^{(I)} = \left\{p_i^{j(I)}\right\}_{j=1,\ldots,M} \qquad \boldsymbol{q}_i^{(I)} = \left\{q_i^{j(I)}\right\}_{j=1,\ldots,M},$$

the $n \times M$ *component availability and unavailability matrices*

$$\boldsymbol{P}_\phi^{(I)} = \left\{p_i^{j(I)}\right\}_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}} \qquad \boldsymbol{Q}_\phi^{(I)} = \left\{q_i^{j(I)}\right\}_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}}$$

and the system availability and unavailability vectors

$$\boldsymbol{p}_\phi^{(I)} = \left\{p_\phi^{j(I)}\right\}_{j=1,\ldots,M} \qquad \boldsymbol{q}_\phi^{(I)} = \left\{q_\phi^{j(I)}\right\}_{j=1,\ldots,M}.$$

Finally, introduce for $i = 1, \ldots, n$ *the component parameter vectors*

$$\boldsymbol{r}_i^{(I)} = \left\{r_i^{j(I)}\right\}_{j=0,\ldots,M},$$

the $n \times (M+1)$ *parameter matrix*

$$\boldsymbol{R}_\phi^{(I)} = \left\{r_i^{j(I)}\right\}_{\substack{i=1,\ldots,n\\ j=0,\ldots,M}}$$

and the system parameter vector

$$\boldsymbol{r}_\phi^{(I)} = \left\{r_\phi^{j(I)}\right\}_{j=0,\ldots,M}.$$

When $I = [t, t]$*, we just drop* I *from the notation and use reliability and unreliability instead of respectively availability and unavailability.*

Note that for $i = 1, \ldots, n$

$$p_i^{j(I)} + q_i^{j(I)} \leq 1 \qquad p_\phi^{j(I)} + q_\phi^{j(I)} \leq 1. \tag{1}$$

Suppose now that we run K_i independent experiments for component i registering $x_i^{(k)}(s) \quad \forall s \in \tau(I)$ in the kth experiment, $k = 1, \ldots, K_i, i = 1, \ldots, n$. Let for $j = 1, \ldots, M, i = 1, \ldots, n$

$$D_i^{1j(I)} = \sum_{k=1}^{K_i} I[x_i^{(k)}(s) \geq j \ \ \forall s \in \tau(I)] \quad D_i^{2j(I)} = \sum_{k=1}^{K_i} I[x_i^{(k)}(s) < j \ \ \forall s \in \tau(I)],$$

and for $j = 0, \ldots, M, i = 1, \ldots, n$

$$D_i^{j(I)} = \sum_{k=1}^{K_i} I[\min_{s \in \tau(I)} x_i^{(k)}(s) = j].$$

Let for $r = 1, 2$ $\boldsymbol{D}_i^{r(I)} = (D_i^{r1(I)}, \ldots, D_i^{rM(I)})$, $\boldsymbol{D}^{r(I)} = (\boldsymbol{D}_1^{r(I)}, \ldots, \boldsymbol{D}_n^{r(I)})$. Furthermore, let $\boldsymbol{D}_i^{(I)} = (D_i^{0(I)}, \ldots, D_i^{M(I)})$ and $\boldsymbol{D}^{(I)} = (\boldsymbol{D}_1^{(I)}, \ldots, \boldsymbol{D}_n^{(I)})$.

Suppose also that we run K independent experiments on the system level registering $\phi(\boldsymbol{x}^{(k)}(s)) \ \ \forall s \in \tau(I)$ in the kth experiment, $k = 1, \ldots, K$. Let for $j = 1, \ldots, M$

$$D_\phi^{1j(I)} = \sum_{k=1}^{K} I[\phi(\boldsymbol{x}^{(k)}(s)) \geq j \ \ \forall s \in \tau(I)]$$
$$D_\phi^{2j(I)} = \sum_{k=1}^{K} I[\phi(\boldsymbol{x}^{(k)}(s)) < j \ \ \forall s \in \tau(I)],$$

and for $j = 0, \ldots, M$

$$D_\phi^{j(I)} = \sum_{k=1}^{K} I[\min_{s \in \tau(I)} \phi(\boldsymbol{x}^{(k)}(s)) = j].$$

Let for $r = 1, 2$ $\boldsymbol{D}_\phi^{r(I)} = (D_\phi^{r1(I)}, \ldots, D_\phi^{rM(I)})$. Furthermore, let $\boldsymbol{D}_\phi^{(I)} = (D_\phi^{0(I)}, \ldots, D_\phi^{M(I)})$. When $I = [t, t]$, we also drop I from the notation in all these data variables and data vectors.

Assume that the prior distribution of respectively the component availability and unavailability matrices, before running any experiment on the component level, $\pi(\boldsymbol{P}_\phi^{(I)})$ and $\pi(\boldsymbol{Q}_\phi^{(I)})$, can be written as

$$\pi(\boldsymbol{P}_\phi^{(I)}) = \prod_{i=1}^{n} \pi_i(\boldsymbol{p}_i^{(I)}) \qquad \pi(\boldsymbol{Q}_\phi^{(I)}) = \prod_{i=1}^{n} \pi_i(\boldsymbol{q}_i^{(I)}),$$

where for $i = 1, \ldots, n$ $\pi_i(\boldsymbol{p}_i^{(I)})$ is the prior marginal distribution of $\boldsymbol{p}_i^{(I)}$and $\pi_i(\boldsymbol{q}_i^{(I)})$ is the prior marginal distribution of $\boldsymbol{q}_i^{(I)}$. Hence, we assume that the components have independent prior component availability vectors and independent prior component unavailability vectors.

Note that before the experiments are carried through $D_i^{1j(I)}$ is binomially distributed with parameters K_i and $p_i^{j(I)}$, and $D_i^{2j(I)}$ binomially distributed with parameters K_i and $q_i^{j(I)}$. We assume that given $\boldsymbol{P}_\phi^{(I)}$, $\boldsymbol{D}_1^{1(I)}, \ldots, \boldsymbol{D}_n^{1(I)}$ are independent and that given $\boldsymbol{Q}_\phi^{(I)}, \boldsymbol{D}_1^{2(I)}, \ldots, \boldsymbol{D}_n^{2(I)}$ are independent. Hence, since we have assumed that the components have independent prior availability vectors, using Bayes' theorem the posterior distribution of the component availability matrix, $\pi(\boldsymbol{P}_\phi^{(I)} | \boldsymbol{D}^{1(I)})$, can be written as

$$\pi(\boldsymbol{P}_\phi^{(I)} | \boldsymbol{D}^{1(I)}) = \frac{\pi(\boldsymbol{D}^{1(I)} | \boldsymbol{P}_\phi^{(I)}) \pi(\boldsymbol{P}_\phi^{(I)})}{\int \pi(\boldsymbol{D}^{1(I)} | \boldsymbol{P}_\phi^{(I)}) \pi(\boldsymbol{P}_\phi^{(I)}) d\boldsymbol{P}_\phi^{(I)}}$$
$$= \frac{\prod_{i=1}^{n} \pi_i(\boldsymbol{D_i}^{1(I)} | \boldsymbol{p}_i^{(I)}) \pi_i(\boldsymbol{p}_i^{(I)})}{\prod_{i=1}^{n} \int \pi_i(\boldsymbol{D_i}^{1(I)} | \boldsymbol{p}_i^{(I)}) \pi_i(\boldsymbol{p}_i^{(I)}) d\boldsymbol{p}_i^{(I)}} = \prod_{i=1}^{n} \frac{\pi_i(\boldsymbol{D_i}^{1(I)} | \boldsymbol{p}_i^{(I)}) \pi_i(\boldsymbol{p}_i^{(I)})}{\pi_i(\boldsymbol{D}_i^{1(I)})}$$
$$= \prod_{i=1}^{n} \pi_i(\boldsymbol{p}_i^{(I)} | \boldsymbol{D}_i^{1(I)}),$$

where $\pi_i(\boldsymbol{p}_i^{(I)} | \boldsymbol{D}_i^{1(I)})$ is the posterior marginal distribution of $\boldsymbol{p}_i^{(I)}$. Similarly, the posterior distribution of the component unavailability matrix, $\pi(\boldsymbol{Q}_\phi^{(I)} | \boldsymbol{D}^{2(I)})$, can be written as

$$\pi(\boldsymbol{Q}_\phi^{(I)}|\boldsymbol{D}^{2(I)}) = \prod_{i=1}^{n} \pi_i(\boldsymbol{q}_i^{(I)}|\boldsymbol{D}_i^{2(I)}).$$

Hence, the posterior component availability vectors are independent given $\boldsymbol{D}^{1(I)}$ and the posterior component unavailability vectors are independent given $\boldsymbol{D}^{2(I)}$.

Now specialize $I = [t, t]$ and assume that the component states $X_1, \ldots, X_n$ are independent given $\boldsymbol{P}_\phi$. Since in this case $\boldsymbol{p}_\phi$ is a function of $\boldsymbol{P}_\phi$, the distribution, $\pi(\boldsymbol{p}_\phi(\boldsymbol{P}_\phi)|\boldsymbol{D}^1)$, can then be arrived at. Based on prior knowledge on the system level this may be adjusted to the prior distribution of the system reliability vector, $\pi_0(\boldsymbol{p}_\phi(\boldsymbol{P}_\phi)|\boldsymbol{D}^1)$. Note that before the experiments are carried through D_ϕ^{1j} is binomially distributed with parameters K and p_ϕ^j. Including the data $\boldsymbol{D}_\phi^1$, we end up with the posterior distribution of the system reliability vector, $\pi(\boldsymbol{p}_\phi(\boldsymbol{P}_\phi)|\boldsymbol{D}^1, \boldsymbol{D}_\phi^1)$, for $j = 1, \ldots, M$.

When considering the case $I = [t, t]$, we can instead of $\boldsymbol{P}_\phi$ as well consider the parameter matrix $\boldsymbol{R}_\phi$ and assume that the components have independent prior vectors $\boldsymbol{r}_i, i = 1, \ldots, n$, each having a Dirichlet distribution being the natural conjugate prior. Furthermore, we assume that given $\boldsymbol{R}_\phi$, $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ are independent. Note that before the experiments are carried through $\boldsymbol{D}_i$ is multinomially distributed with parameters K_i and $\boldsymbol{r}_i$. Hence, the posterior marginal distribution of $\boldsymbol{r}_i$ given the data $\boldsymbol{D}_i$, $\pi_i(\boldsymbol{r}_i|\boldsymbol{D}_i)$, is Dirichlet. Furthermore, we have

$$\pi(\boldsymbol{R}_\phi|\boldsymbol{D}) = \prod_{i=1}^{n} \pi_i(\boldsymbol{r}_i|\boldsymbol{D}_i).$$

Hence, life can be made easy at the component level. Assume that the component states $X_1, \ldots, X_n$ are independent given $\boldsymbol{R}_\phi$. The distribution, $\pi(\boldsymbol{r}_\phi(\boldsymbol{R}_\phi)|\boldsymbol{D})$, is tried to be arrived at. If this is successful, based on prior knowledge on the system level, it is adjusted to $\pi_0(\boldsymbol{r}_\phi(\boldsymbol{R}_\phi)|\boldsymbol{D})$. This may be possible for simple systems. Note that before the experiments are carried through, $\boldsymbol{D}_\phi$ is multinomially distributed with parameters K and $\boldsymbol{r}_\phi$. Hence, if $\pi_0(\boldsymbol{r}_\phi(\boldsymbol{R}_\phi)|\boldsymbol{D})$, as in a dream, ended up as a Dirichlet distribution, the posterior distribution, $\pi(\boldsymbol{r}_\phi(\boldsymbol{R}_\phi)|\boldsymbol{D}, \boldsymbol{D}_\phi)$, also would be a Dirichlet distribution. Do not forget this was a dream, also based on independent components given $\boldsymbol{R}_\phi$! So life will at least not be easy at the system level even when $I = [t, t]$.

For an arbitrary I $\boldsymbol{p}_\phi^{(I)}$ is not a function of just $\boldsymbol{P}_\phi^{(I)}$, and $\boldsymbol{q}_\phi^{(I)}$ not a function of just $\boldsymbol{Q}_\phi^{(I)}$. Hence, the approach above for the case $I = [t, t]$ can not be extended.

In Section 2 we discuss two different approaches to the computation of posterior moments for component availabilities and unavailabilities, the first one generalizing an approach given in (Mastran and Singpurwalla 1978). In Section 3 we start out by describing our uncertainty on the component availabilities and unavailabilities to the various levels in a fixed time interval, based on both prior information and data on the components, by the moments up till order m of their marginal distributions. From these moments analytic bounds on the corresponding moments of the system availabilities and unavailabilities to the various levels in a fixed time interval are arrived at. Applying these bounds and prior system information we may then fit prior distributions of the system availabilities and unavailabilities to the various levels in a fixed time interval. These can in turn be updated by relevant data on the system. This generalizes results given in (Natvig and Eide 1987) considering a binary monotone system of binary components at a fixed point of time. In Section 4 we present a straightforward simulation technique for obtaining bounds that improve the analytic bounds. Considering a simple network system, we show that the former bounds are slightly better than the latter.

2. Moments for posterior component availabilities and unavailabilities

Based on the experiences of the previous section we reduce our ambitions. We start by specifying marginal moments $E\{(p_i^{j(I)})^s\}$ and $E\{(q_i^{j(I)})^s\}$ for $s = 1,\ldots,m+K_i, j = 1,\ldots,M$ of $\pi_i(\boldsymbol{p}_i^{(I)})$ and $\pi_i(\boldsymbol{q}_i^{(I)}), i = 1,\ldots,n$. We will first illustrate how these can be updated to give posterior moments $E\{(p_i^{j(I)})^s|D_i^{1j(I)}\}$ and $E\{(q_i^{j(I)})^s|D_i^{2j(I)}\}$ for $s = 1,\ldots,m, j = 1,\ldots,M$ by using Lemma 1 in (Mastran and Singpurwalla 1978). Note that we loose information by conditioning on $D_i^{1j(I)}$ instead of $\boldsymbol{D}_i^{1(I)}$ and $D_i^{2j(I)}$ instead of $\boldsymbol{D}_i^{2(I)}$. However, such improved conditioning does not work with this approach. We have

$$E\{(p_i^{j(I)})^s|D_i^{1j(I)}\} \propto \int_0^1 (p_i^{j(I)})^s (p_i^{j(I)})^{D_i^{1j(I)}} (1-p_i^{j(I)})^{K_i - D_i^{1j(I)}} \pi_i(p_i^{j(I)}) dp_i^{j(I)}$$

$$= \int_0^1 (p_i^{j(I)})^{s+D_i^{1j(I)}} \sum_{r=0}^{K_i - D_i^{1j(I)}} \binom{K_i - D_i^{1j(I)}}{r} (-1)^r (p_i^{j(I)})^r \pi_i(p_i^{j(I)}) dp_i^{j(I)}.$$

Hence,

$$E\{(p_i^{j(I)})^s|D_i^{1j(I)}\} = \frac{\sum_{r=0}^{K_i - D_i^{1j(I)}} \binom{K_i - D_i^{1j(I)}}{r} (-1)^r E\{(p_i^{j(I)})^{s+D_i^{1j(I)}+r}\}}{\sum_{r=0}^{K_i - D_i^{1j(I)}} \binom{K_i - D_i^{1j(I)}}{r} (-1)^r E\{(p_i^{j(I)})^{D_i^{1j(I)}+r}\}}.$$

A similar expression is valid for $E\{(q_i^{j(I)})^s|D_i^{2j(I)}\}$. The advantage of using this lemma is that it is applicable for general prior distributions $\pi_i(p_i^{j(I)})$ and $\pi_i(q_i^{j(I)})$. A serious drawback is, however, that to arrive at $E\{(p_i^{j(I)})^s|D_i^{1j(I)}\}$ and $E\{(q_i^{j(I)})^s|D_i^{2j(I)}\}$ for $s = 1,\ldots,m, j = 1,\ldots,M$ one must specify marginal moments up till order $m + K_i$ of the corresponding prior distributions $\pi_i(p_i^{j(I)})$ and $\pi_i(q_i^{j(I)})$. This may be completely unrealistic unless K_i is small.

A more realistic alternative is given in the following. Assume that the components have independent prior parameter vectors $\boldsymbol{r}_i^{(I)}, i = 1,\ldots,n$, each having a Dirichlet distribution being the natural conjugate prior. Note that before the experiments are carried through $\boldsymbol{D}_i^{(I)}$ is multinomially distributed with parameters K_i and $\boldsymbol{r}_i^{(I)}$. Hence, the posterior marginal distribution of $\boldsymbol{r}_i^{(I)}$ given the data $\boldsymbol{D}_i^{(I)}$, $\pi_i(\boldsymbol{r}_i^{(I)}|\boldsymbol{D}_i^{(I)})$, is Dirichlet. We now have

$$p_i^{j(I)} = \sum_{\ell=j}^{M} r_i^{\ell(I)} \quad D_i^{1j(I)} = \sum_{\ell=j}^{M} D_i^{\ell(I)}. \tag{2}$$

Hence, the posterior marginal distribution of $p_i^{j(I)}$ given the data $D_i^{1j(I)}$ is beta. Accordingly, we loose no information by conditioning on $D_i^{1j(I)}$ instead of $\boldsymbol{D}_i^{1(I)}$.

We now assume that the prior distribution $\pi_i(p_i^{j(I)})$ is beta with parameters $a_i^{j(I)}$ and $b_i^{j(I)}$. It then follows that $\pi_i(p_i^{j(I)}|D_i^{1j(I)})$ is beta with parameters $a_i^{j(I)} + D_i^{1j(I)}$ and $b_i^{j(I)} + K_i - D_i^{1j(I)}$. We have

$$E\{(p_i^{j(I)})^s|D_i^{1j(I)}\} = \int_0^1 (p_i^{j(I)})^s \frac{\Gamma(a_i^{j(I)} + b_i^{j(I)} + K_i)}{\Gamma(a_i^{j(I)} + D_i^{1j(I)})\Gamma(b_i^{j(I)} + K_i - D_i^{1j(I)})}$$
$$(p_i^{j(I)})^{a_i^{j(I)}+D_i^{1j(I)}-1}(1-p_i^{j(I)})^{b_i^{j(I)}+K_i-D_i^{1j(I)}-1}dp_i^{j(I)}$$
$$= \frac{\Gamma(a_i^{j(I)} + b_i^{j(I)} + K_i)\Gamma(a_i^{j(I)} + D_i^{1j(I)} + s)}{\Gamma(a_i^{j(I)} + D_i^{1j(I)})\Gamma(a_i^{j(I)} + b_i^{j(I)} + K_i + s)}$$
$$\int_0^1 \frac{\Gamma(a_i^{j(I)} + b_i^{j(I)} + K_i + s)}{\Gamma(a_i^{j(I)} + D_i^{1j(I)} + s)\Gamma(b_i^{j(I)} + K_i - D_i^{1j(I)})}$$
$$(p_i^{j(I)})^{a_i^{j(I)}+D_i^{1j(I)}+s-1}(1-p_i^{j(I)})^{b_i^{j(I)}+K_i-D_i^{1j(I)}-1}dp_i^{j(I)}$$
$$= \frac{\Gamma(a_i^{j(I)} + b_i^{j(I)} + K_i)\Gamma(a_i^{j(I)} + D_i^{1j(I)} + s)}{\Gamma(a_i^{j(I)} + D_i^{1j(I)})\Gamma(a_i^{j(I)} + b_i^{j(I)} + K_i + s)},$$

the integral being equal to 1 since we are integrating up a beta density with parameters $a_i^{j(I)} + D_i^{1j(I)} + s$ and $b_i^{j(I)} + K_i - D_i^{1j(I)}$. A similar expression is valid for $E\{(q_i^{j(I)})^s|D_i^{2j(I)}\}$.

3. Bounds for moments for system availabilities and unavailabilities

From the marginal moments $E\{(p_i^{\ell(I)})^s|D_i^{1\ell(I)}\}$ and $E\{(q_i^{\ell(I)})^s|D_i^{2\ell(I)}\}$, we derive lower bounds on the marginal moments $E\{(p_\phi^{j(I)})^s|\boldsymbol{D}^{1(I)}\}$ and upper bounds on the marginal moments $E\{(p_\phi^{j(I)})^s|\boldsymbol{D}^{2(I)}\}$ for $s = 1, \ldots, m$, $\ell = 1, \ldots, M, j = 1, \ldots, M$. Similarly, we derive lower bounds on the marginal moments $E\{(q_\phi^{j(I)})^s|\boldsymbol{D}^{2(I)}\}$ and upper bounds on the marginal moments $E\{(q_\phi^{j(I)})^s|\boldsymbol{D}^{1(I)}\}$. Note that we now do not necessarily need the marginal performance processes of the components to be independent in I. From these bounds and prior knowledge on the system level we may fit $\pi_0(\boldsymbol{p}_\phi^{(I)})$ and $\pi_0(\boldsymbol{q}_\phi^{(I)})$. This may finally be updated to give $\pi(\boldsymbol{p}_\phi^{(I)}|\boldsymbol{D}_\phi^{1(I)})$ and $\pi(\boldsymbol{q}_\phi^{(I)}|\boldsymbol{D}_\phi^{2(I)})$.

What we will concentrate on is how to establish the bounds on the marginal moments of system availabilities and unavailabilities from the marginal moments of component availabilities and unavailabilities. To simplify notation we drop the reference to the data $(\boldsymbol{D}^{1(I)}, \boldsymbol{D}^{2(I)})$ from experiments on the component level.

Let us just for a while return to the case $I = [t, t]$ and assume that the component states $X_1, \ldots, X_n$ are independent given $\boldsymbol{R}_\phi$. Then we get

$$p_\phi^j(\boldsymbol{R}_\phi) = \sum_{\boldsymbol{x}} I[\phi(\boldsymbol{x}) \geq j] \prod_{i=1}^n r_i^{x_i}.$$

Hence, since we assume that the components have independent prior vectors $\boldsymbol{r}_i$ for $i = 1, \ldots, n$, generalizing a result in (Natvig and Eide 1987), we get

$$E\{p_\phi^j(\boldsymbol{R}_\phi)\} = \sum_{\boldsymbol{x}} I[\phi(\boldsymbol{x}) \geq j] \prod_{i=1}^n E\{r_i^{x_i}\} = p_\phi^j(E\{\boldsymbol{R}_\phi\}),$$

where

$$E\{\boldsymbol{R}_\phi\} = \left\{E\{r_i^j\}\right\}_{\substack{i=1,\ldots,n \\ j=0,\ldots,M}}.$$

Accordingly, one can arrive at an exact expression for $E\{p_\phi^j(\boldsymbol{R}_\phi)\}$ for not too large systems. The point is, however, that there seems to be no easy way to extend the approach above to give exact expressions for higher order moments of $p_\phi^j(\boldsymbol{R}_\phi)$. Hence, even when $I = [t, t]$ and component states are independent given $\boldsymbol{R}_\phi$, one needs bounds on higher order moments of $p_\phi^j(\boldsymbol{R}_\phi)$.

We need the following theorem proved in (Natvig and Eide 1987).

Theorem 1. *If $Y_1, \ldots, Y_n$ are associated random variables such that $0 \leq Y_i \leq 1, i = 1, \ldots, n$, then for $\alpha > 0$*

$$E\{(\prod_{i=1}^n Y_i)^\alpha\} \geq \prod_{i=1}^n E\{(Y_i)^\alpha\} \tag{3}$$

$$E\{\coprod_{i=1}^n Y_i\} = E\{1 - \prod_{i=1}^n (1 - Y_i)\} \leq \coprod_{i=1}^n E\{Y_i\}. \tag{4}$$

In the special case of independent random variables Y_1 and Y_2 with $0 \leq Y_i \leq 1, i = 1, 2$, we have

$$E\{(\prod_{i=1}^2 Y_i)^2\} \geq \prod_{i=1}^2 E\{(Y_i)^2\}. \tag{5}$$

Proof: For the case $Y_1, \ldots, Y_n$ binary and $\alpha = 1$, Equations (3) and (4) are proved in Theorem 3.1, page 32 of (Barlow and Proschan 1975). The proof, however, also works when $0 \leq Y_i \leq 1, i = 1, \ldots, n$. Using this fact and that non-decreasing functions of associated random variables are associated we get

$$E\{(\prod_{i=1}^n Y_i)^\alpha\} = E\{\prod_{i=1}^n (Y_i)^\alpha\} \geq \prod_{i=1}^n E\{(Y_i)^\alpha\},$$

and Equation (3) is proved. Equation (4) is proved in the same way. Finally, Equation (5) follows since

$$(\prod_{i=1}^2 Y_i)^2 = \prod_{i=1}^2 Y_i^2 + 2(Y_1^2 - Y_1)(Y_2^2 - Y_2) \geq \prod_{i=1}^2 Y_i^2,$$

and that Y_1 and Y_2 are assumed to be independent.

Equation (5) reveals the unpleasant fact that a symmetry in Equations (3) and (4) seems only possible for $\alpha = 1$, when $Y_1, \ldots, Y_n$ are not binary.

In the following, considering an MMS (C,ϕ), for $j \in \{1,\ldots,M\}$ let $\boldsymbol{y}_k^j = (y_{1k}^j,\ldots,y_{nk}^j)$, $k=1,\ldots,n_\phi^j$ be its minimal path vectors to level j and $\boldsymbol{z}_k^j = (z_{1k}^j,\ldots,z_{nk}^j)$, $k=1,\ldots,m_\phi^j$ be its minimal cut vectors to level j and

$$C_\phi^j(\boldsymbol{y}_k^j), k=1,\ldots,n_\phi^j \text{ and } D_\phi^j(\boldsymbol{z}_k^j), k=1,\ldots,m_\phi^j$$

the corresponding minimal path and cut sets to level j.

The three following theorems are taken from (Natvig 2011).

Theorem 2. *Let (C,ϕ) be an MMS and let for $j=1,\ldots,M$*

$$\ell_\phi^{`j(I)}(\boldsymbol{P}_\phi^{(I)}) = \max_{1\le k\le n_\phi^j} \prod_{i\in C_\phi^j(\boldsymbol{y}_k^j)} p_i^{y_{ik}^j(I)}$$

$$\bar{\ell}_\phi^{`j(I)}(\boldsymbol{Q}_\phi^{(I)}) = \max_{1\le k\le m_\phi^j} \prod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} q_i^{z_{ik}^j+1(I)}.$$

If the joint performance process of the system's components is associated in I, or the marginal performance processes of the components are independent in I, then

$$\ell_\phi^{`j(I)}(\boldsymbol{P}_\phi^{(I)}) \le p_\phi^{j(I)} \le \inf_{t\in\tau(I)} \left[1-\bar{\ell}_\phi^{`j([t,t])}(\boldsymbol{Q}_\phi^{([t,t])})\right] \le 1-\bar{\ell}_\phi^{`j(I)}(\boldsymbol{Q}_\phi^{(I)}) \qquad (6)$$

$$\bar{\ell}_\phi^{`j(I)}(\boldsymbol{Q}_\phi^{(I)}) \le q_\phi^{j(I)} \le \inf_{t\in\tau(I)} \left[1-\ell_\phi^{`j([t,t])}(\boldsymbol{P}_\phi^{j([t,t])})\right] \le 1-\ell_\phi^{`j(I)}(\boldsymbol{P}_\phi^{j(I)}). \qquad (7)$$

Theorem 3. *Let (C,ϕ) be an MMS and let for $j=1,\ldots,M$*

$$\ell_\phi^{**j(I)}(\boldsymbol{P}_\phi^{(I)}) = \prod_{k=1}^{m_\phi^j} \coprod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} p_i^{z_{ik}^j+1(I)}$$

$$\bar{\ell}_\phi^{**j(I)}(\boldsymbol{Q}_\phi^{(I)}) = \prod_{k=1}^{n_\phi^j} \coprod_{i\in C_\phi^j(\boldsymbol{y}_k^j)} q_i^{y_{ik}^j(I)}.$$

If the marginal performance processes of the components are independent in I, then

$$\ell_\phi^{**j(I)}(\boldsymbol{P}_\phi^{(I)}) \le p_\phi^{j(I)} \le \inf_{t\in\tau(I)} \left[1-\bar{\ell}_\phi^{**j([t,t])}(\boldsymbol{Q}_\phi^{([t,t])})\right] \le 1-\bar{\ell}_\phi^{**j(I)}(\boldsymbol{Q}_\phi^{(I)}) \qquad (8)$$

$$\bar{\ell}_\phi^{**j(I)}(\boldsymbol{Q}_\phi^{(I)}) \le q_\phi^{j(I)} \le \inf_{t\in\tau(I)} \left[1-\ell_\phi^{**j([t,t])}(\boldsymbol{P}_\phi^{([t,t])})\right] \le 1-\ell_\phi^{**j(I)}(\boldsymbol{P}_\phi^{(I)}). \qquad (9)$$

Theorem 4. *Let (C,ϕ) be an MMS with modular decomposition given by Definition 3 and let for $j=1,\ldots,M$*

$$B_\phi^{*j(I)}(\boldsymbol{P}_\phi^{(I)}) = \max_{j\le k\le M}[\max[\ell_\phi^{`k(I)}(\boldsymbol{P}_\phi^{(I)}), \ell_\phi^{**k(I)}(\boldsymbol{P}_\phi^{(I)})]]$$

$$\bar{B}_\phi^{*j(I)}(\boldsymbol{Q}_\phi^{(I)}) = \max_{1\le k\le j}[\max[\bar{\ell}_\phi^{`k(I)}(\boldsymbol{Q}_\phi^{(I)}), \bar{\ell}_\phi^{**k(I)}(\boldsymbol{Q}_\phi^{(I)})].$$

Introduce the following $r\times M$ module availability and unavailability matrices

$$\boldsymbol{P}_\psi^{(I)} = \{p_{\chi_k}^{j(I)}\}_{\substack{k=1,\ldots,r\\ j=1,\ldots,M}} \qquad \boldsymbol{Q}_\psi^{(I)} = \{q_{\chi_k}^{j(I)}\}_{\substack{k=1,\ldots,r\\ j=1,\ldots,M}}, \qquad (10)$$

and correspondingly define the following $r\times M$ matrices $\boldsymbol{B}_\psi^{*(I)}(\boldsymbol{P}_\phi^{(I)}), \bar{\boldsymbol{B}}_\psi^{*(I)}(\boldsymbol{Q}_\phi^{(I)})$.

Assume the marginal performance processes of the components to be independent in the time interval I. Then for $j = 1, \ldots, M$

$$\begin{aligned}
&B_{\psi}^{*j(I)}(\boldsymbol{B}_{\psi}^{*(I)}(\boldsymbol{P}_{\phi}^{(I)})) \leq B_{\psi}^{*j(I)}(\boldsymbol{P}_{\psi}^{(I)}) \leq p_{\phi}^{j(I)} \\
&\leq \inf_{t \in \tau(I)} \left[1 - \bar{B}_{\psi}^{*j([t,t])}(\boldsymbol{Q}_{\psi}^{([t,t])})\right] \leq 1 - \bar{B}_{\psi}^{*j(I)}(\boldsymbol{Q}_{\psi}^{(I)}) \\
&\leq 1 - \bar{B}_{\psi}^{*j(I)}(\bar{\boldsymbol{B}}_{\psi}^{*(I)}(\boldsymbol{Q}_{\phi}^{(I)}). \qquad (11)
\end{aligned}$$

$$\begin{aligned}
&\bar{B}_{\psi}^{*j(I)}(\bar{\boldsymbol{B}}_{\psi}^{*(I)}(\boldsymbol{Q}_{\phi}^{(I)})) \leq \bar{B}_{\psi}^{*j(I)}(\boldsymbol{Q}_{\psi}^{(I)}) \leq q_{\phi}^{j(I)} \\
&\leq \inf_{t \in \tau(I)} \left[1 - B_{\psi}^{*j([t,t])}(\boldsymbol{P}_{\psi}^{([t,t])})\right] \leq 1 - B_{\psi}^{*j(I)}(\boldsymbol{P}_{\psi}^{(I)}) \\
&\leq 1 - B_{\psi}^{*j(I)}(\boldsymbol{B}_{\psi}^{*(I)}(\boldsymbol{P}_{\phi}^{(I)})). \qquad (12)
\end{aligned}$$

We are now ready to establish the bounds for the moments of system availabilities and unavailabilities. Introduce the $m \times n \times M$ arrays of component availability and unavailability moments

$$E\{(\boldsymbol{P}_{\phi}^{(I)})^m\} = \left\{E\{(p_i^{j(I)})^s\}\right\}_{\substack{s=1,\ldots,m \\ i=1,\ldots,n \\ j=1,\ldots,M}} \qquad (13)$$

$$E\{(\boldsymbol{Q}_{\phi}^{(I)})^m\} = \left\{E\{(q_i^{j(I)})^s\}\right\}_{\substack{s=1,\ldots,m \\ i=1,\ldots,n \\ j=1,\ldots,M}} . \qquad (14)$$

Theorem 5. *Let (C, ϕ) be an MMS. Assume that respectively the component availability vectors $\boldsymbol{p}_i^{(I)} i = 1, \ldots, n$ and the component unavailability vectors $\boldsymbol{q}_i^{(I)} i = 1, \ldots, n$ are independent. Let*

$$\ell_{\phi}^{`j(I)m}(E\{(\boldsymbol{P}_{\phi}^{(I)})^m\}) = \max_{1 \leq k \leq n_{\phi}^j} \prod_{i \in C_{\phi}^j(\boldsymbol{y}_k^j)} E\{(p_i^{y_{ik}^j(I)})^m\}$$

$$u_{\phi}^{`j(I)m}(E\{(\boldsymbol{Q}_{\phi}^{(I)})^m\}) = \min_{1 \leq k \leq m_{\phi}^j} \sum_{r=o}^{m} \binom{m}{r} (-1)^r$$

$$\prod_{i \in D_{\phi}^j(\boldsymbol{z}_k^j)} E\{(q_i^{z_{ik}^j+1([I])})^r\}$$

$$\bar{\ell}_{\phi}^{`j(I)m}(E\{(\boldsymbol{Q}_{\phi}^{(I)})^m\}) = \max_{1 \leq k \leq m_{\phi}^j} \prod_{i \in D_{\phi}^j(\boldsymbol{z}_k^j)} E\{(q_i^{z_{ik}^j+1(I)})^m\}$$

$$\bar{u}_{\phi}^{`j(I)m}(E\{(\boldsymbol{P}_{\phi}^{(I)})^m\}) = \min_{1 \leq k \leq n_{\phi}^j} \sum_{r=o}^{m} \binom{m}{r} (-1)^r$$

$$\prod_{i \in C_{\phi}^j(\boldsymbol{y}_k^j)} E\{(p_i^{y_{ik}^j([I])})^r\}.$$

If the joint performance process of the system's components is associated in I, or the marginal performance processes of the components are independent in I, then for $m = 1, 2, \ldots$

$$\ell_{\phi}^{`j(I)m}(E\{(\boldsymbol{P}_{\phi}^{(I)})^m\}) \leq E\{(p_{\phi}^{j(I)})^m\}$$

$$E\{(p_\phi^{j(I)})^m\} \leq u_\phi^{`j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \tag{15}$$

$$\begin{aligned} &\bar{\ell}_\phi^{`j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \leq E\{(q_\phi^{j(I)})^m\} \\ &E\{(q_\phi^{j(I)})^m\} \leq \bar{u}_\phi^{`j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}). \end{aligned} \tag{16}$$

Proof: From Equation (6) we have

$$\begin{aligned} &E\{(p_\phi^{j(I)})^m\} \geq E\{\max_{1\leq k\leq n_\phi^j} \prod_{i\in C_\phi^j(\boldsymbol{y}_k^j)} (p_i^{y_{ik}^j(I)})^m\} \geq E\{\prod_{i\in C_\phi^j(\boldsymbol{y}_k^j)} (p_i^{y_{ik}^j(I)})^m\} \\ &= \prod_{i\in C_\phi^j(\boldsymbol{y}_k^j)} E\{(p_i^{y_{ik}^j(I)})^m\}, \quad 1\leq k\leq n_\phi^j, \end{aligned}$$

having used the independence of the component availability vectors. Since the inequality holds for all $1 \leq k \leq n_\phi^j$, the lower bound of Equation (15) follows. Similarly from Equation (6)

$$\begin{aligned} &E\{(p_\phi^{j(I)})^m\} \leq E\{(\min_{1\leq k\leq m_\phi^j}[1-\prod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} q_i^{z_{ik}^j+1([I])}])^m\} \\ &\leq \min_{1\leq k\leq m_\phi^j} E\{(1-\prod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} q_i^{z_{ik}^j+1([I])})^m\} \\ &= \min_{1\leq k\leq m_\phi^j} E\{\sum_{r=o}^{m}\binom{m}{r}(-1)^r \prod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} (q_i^{z_{ik}^j+1([I])})^r\} \\ &= \min_{1\leq k\leq m_\phi^j} \sum_{r=o}^{m}\binom{m}{r}(-1)^r \prod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} E\{(q_i^{z_{ik}^j+1([I])})^r\}, \end{aligned}$$

having used the independence of the component unavailability vectors. Hence, the upper bound of Equation (15) is proved. The bounds of Equation (16) follow completely similarly from Equation (7).

Note that respectively $p_i^{j([t_1,t_1])}$ and $p_i^{j([t_2,t_2])}$, and $q_i^{j([t_1,t_1])}$ and $q_i^{j([t_2,t_2])}$ are dependent for $t_1 \in \tau(I)$, $t_2 \in \tau(I)$, $t_1 \neq t_2$. Hence,

$$\begin{aligned} &E(p_i^j|\boldsymbol{D}^{1(I)}) \neq E(p_i^j|\boldsymbol{D}^1) \\ &E(q_i^j|\boldsymbol{D}^{2(I)}) \neq E(q_i^j|\boldsymbol{D}^2). \end{aligned}$$

This means that we cannot apply the best upper bounds in Equations (6) and (7).

Theorem 6. *Let (C,ϕ) be an MMS. Assume that respectively the component availability vectors $\boldsymbol{p}_i^{(I)} i = 1,\ldots,n$ and the component unavailability vectors $\boldsymbol{q}_i^{(I)} i = 1,\ldots,n$ are independent. Let*

$$\ell_\phi^{**j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) = \prod_{k=1}^{m_\phi^j}\sum_{r=o}^{m}\binom{m}{r}(-1)^r$$

$$\prod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} \sum_{s=o}^{r} \binom{r}{s} (-1)^s E\{(p_i^{z_{ik}^j+1(I)})^s\}$$

$$u_\phi^{**j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\}) = \coprod_{k=1}^{n_\phi^j} \prod_{i\in C_\phi^j(\boldsymbol{y}_k^j)} (1 - E\{q_i^{y_{ik}^j([I])}\})$$

$$\bar{\ell}_\phi^{**j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) = \prod_{k=1}^{n_\phi^j} \sum_{r=o}^{m} \binom{m}{r} (-1)^r$$

$$\prod_{i\in C_\phi^j(\boldsymbol{y}_k^j)} \sum_{s=o}^{r} \binom{r}{s} (-1)^s E\{(q_i^{y_{ik}^j(I)})^s\}$$

$$\bar{u}_\phi^{**j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\}) = \coprod_{k=1}^{m_\phi^j} \prod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} (1 - E\{p_i^{z_{ik}^j+1([I])}\}).$$

If the marginal performance processes of the components are independent in I, then for $m = 1, 2, \ldots$

$$\ell_\phi^{**j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) \leq E\{(p_\phi^{j(I)})^m\} \tag{17}$$

$$E\{p_\phi^{j(I)}\} \leq u_\phi^{**j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\}) \tag{18}$$

$$\bar{\ell}_\phi^{**j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \leq E\{(q_\phi^{j(I)})^m\} \tag{19}$$

$$E\{q_\phi^{j(I)}\} \leq \bar{u}_\phi^{**j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\}). \tag{20}$$

Proof: From Equation (8) we have

$$E\{(p_\phi^{j(I)})^m\} \geq E\{(\coprod_{k=1}^{m_\phi^j} \coprod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} p_i^{z_{ik}^j+1(I)})^m\}$$

$$\geq \coprod_{k=1}^{m_\phi^j} E\{(\coprod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} p_i^{z_{ik}^j+1(I)})^m\},$$

having applied Equation (3). The random variables

$$\coprod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} p_i^{z_{ik}^j+1(I)}, \quad k = 1, \ldots, m_\phi^j,$$

are associated since independent random variables and non-decreasing functions of associated random variables are asssociated, having used the independence of the component availability vectors. Continuing the derivation we get

$$= \coprod_{k=1}^{m_\phi^j} E\{\sum_{r=o}^{m} \binom{m}{r} (-1)^r \prod_{i\in D_\phi^j(\boldsymbol{z}_k^j)} \sum_{s=o}^{r} \binom{r}{s} (-1)^s (p_i^{z_{ik}^j+1(I)})^s\}$$

$$= \prod_{k=1}^{m_\phi^j} \sum_{r=o}^{m} \binom{m}{r} (-1)^r \prod_{i \in D_\phi^j(\boldsymbol{z}_k^j)} \sum_{s=o}^{r} \binom{r}{s} (-1)^s E\{(p_i^{z_{ik}^j+1(I)})^s\},$$

again having applied the independence of the component availability vectors. Hence, Equation (17) follows. Similarly from Equation (8)

$$E\{p_\phi^{j(I)}\} \leq E\{\coprod_{k=1}^{n_\phi^j} \prod_{i \in C_\phi^j(\boldsymbol{y}_k^j)} (1 - q_i^{y_{ik}^j([I])})\}$$

$$\leq \coprod_{k=1}^{n_\phi^j} \prod_{i \in C_\phi^j(\boldsymbol{y}_k^j)} (1 - E\{q_i^{y_{ik}^j([I])}\}),$$

having used Equation (4), noting that the random variables

$$\prod_{i \in C_\phi^j(\boldsymbol{y}_k^j)} (1 - q_i^{y_{ik}^j([I])}),\ k = 1, \ldots, n_\phi^j,$$

are associated by the same argument as above, and the independence of the component unavailability vectors. Hence, Equation (18) is proved. The bounds of Equations (19) and (20) follow completely similarly from Equation (9).

Due to the lack of symmetry in Theorem 1, we have not been able to obtain corresponding upper bounds for $E\{(p_\phi^{j(I)})^m\}$ and $E\{(q_\phi^{j(I)})^m\}, m = 2, 3, \ldots$ in this theorem. As for Theorem 5 we cannot apply the best upper bounds in Equations (8) and (9).

Corollary 7. *Make the same assumptions as in Theorem 6 and let*

$$\begin{aligned}
&L_\phi^{*j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) \\
&= \max[\ell_\phi^{`j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}), \ell_\phi^{**j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\})] \\
&U_\phi^{*j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\}) \\
&= \min[u_\phi^{`j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\}), u_\phi^{**j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\})] \\
&\bar{L}_\phi^{*j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \\
&= \max[\bar{\ell}_\phi^{`j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}), \bar{\ell}_\phi^{**j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\})] \\
&\bar{U}_\phi^{*j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\}) \\
&= \min[\bar{u}_\phi^{`j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\}), \bar{u}_\phi^{**j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\})].
\end{aligned}$$

Then for $m = 1, 2, \ldots$

$$L_\phi^{*j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) \leq E\{(p_\phi^{j(I)})^m\} \tag{21}$$

$$E\{p_\phi^{j(I)}\} \leq U_\phi^{*j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\}) \tag{22}$$

$$\bar{L}_\phi^{*j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \leq E\{(q_\phi^{j(I)})^m\} \tag{23}$$

$$E\{q_\phi^{j(I)}\} \leq \bar{U}_\phi^{*j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\}). \tag{24}$$

Corollary 8. *Make the same assumptions as in Theorem 6 and let*

$$\begin{aligned}
B_\phi^{*j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) &= \max_{j\le k\le M}[L_\phi^{*k(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\})]\\
C_\phi^{*j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\}) &= \min_{1\le k\le j}[U_\phi^{*k(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\})]\\
D_\phi^{*j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) &= \min_{1\le k\le j}[u_\phi^{‘k(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\})]\\
\bar{B}_\phi^{*j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) &= \max_{1\le k\le j}[\bar{L}_\phi^{*k(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\})]\\
\bar{C}_\phi^{*j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\}) &= \min_{j\le k\le M}[\bar{U}_\phi^{*k(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\})]\\
\bar{D}_\phi^{*j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) &= \min_{j\le k\le M}[\bar{u}_\phi^{‘k(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\})].
\end{aligned}$$

Then for $m = 1, 2, \ldots$

$$L_\phi^{*j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) \le B_\phi^{*j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) \le E\{(p_\phi^{j(I)})^m\} \tag{25}$$

$$E\{p_\phi^{j(I)}\} \le C_\phi^{*j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\}) \le U_\phi^{*j(I)1}(E\{(\boldsymbol{Q}_\phi^{(I)})^1\}) \tag{26}$$

$$E\{(p_\phi^{j(I)})^m\} \le D_\phi^{*j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \le u_\phi^{‘j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \tag{27}$$

$$\bar{L}_\phi^{*j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \le \bar{B}_\phi^{*j(I)m}(E\{(\boldsymbol{Q}_\phi^{(I)})^m\}) \le E\{(q_\phi^{j(I)})^m\} \tag{28}$$

$$E\{q_\phi^{j(I)}\} \le \bar{C}_\phi^{*j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\}) \le \bar{U}_\phi^{*j(I)1}(E\{(\boldsymbol{P}_\phi^{(I)})^1\}) \tag{29}$$

$$E\{(q_\phi^{j(I)})^m\} \le \bar{D}_\phi^{*j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}) \le \bar{u}_\phi^{‘j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\}). \tag{30}$$

It is important to note that the bounds for $E\{(p_\phi^{j(I)})\}$ and $E\{(q_\phi^{j(I)})\}$ given in this section equal the bounds in Section 3.2 of (Natvig 2011) by replacing $\boldsymbol{P}_\phi^{(I)}$ by $E\{(\boldsymbol{P}_\phi^{(I)})^1\}$ and $\boldsymbol{Q}_\phi^{(I)}$ by $E\{(\boldsymbol{Q}_\phi^{(I)})^1\}$. However, this is not true for higher order moments.

4. A simulation approach and a case study

An objection against the bounds in Theorems 5 and 6 and Corollaries 7 and 8 is that they are based on knowing all minimal path and cut vectors of the system. It is natural to try to both improve the bounds and reduce the computational complexity by introducing modular decompositions.

Looking at the bounds for $E\{(p_\phi^{j(I)})^m\}$ and $E\{(q_\phi^{j(I)})^m\}$ for $m = 2, 3, \ldots$, it seems that only the lower bounds of Theorem 5 are of the form that fits into the machinery of Section 3.3 of (Natvig 2011). We now get the following theorem

Theorem 9. *Let* (C, ϕ) *be an MMS with modular decomposition given by Definition 3. Make the same assumptions as in Theorem 6. Then for* $j = 1, \ldots, M$, $m = 1, 2, \ldots$

$$\ell_{\phi}^{`j(I)}(E\{(\boldsymbol{P}_{\phi}^{(I)})^m\}) = \ell_{\psi}^{`j(I)}(\boldsymbol{\ell}_{\psi}^{`(I)}(E\{(\boldsymbol{P}_{\phi}^{(I)})^m\}) \leq E\{(p_{\phi}^{j(I)})^m\} \tag{31}$$

$$\bar{\ell}_{\phi}^{`j(I)}(E\{(\boldsymbol{Q}_{\phi}^{(I)})^m\}) = \bar{\ell}_{\psi}^{`j(I)}(\bar{\boldsymbol{\ell}}_{\psi}^{`(I)}(E\{(\boldsymbol{Q}_{\phi}^{(I)})^m\}) \leq E\{(q_{\phi}^{j(I)})^m\}. \tag{32}$$

Proof: Equation (31) follows from the lower bound of Equation (15) and from Equation (3.45) of (Natvig 2011), by in the last expression replacing $\boldsymbol{P}_{\phi}^{(I)}$ by $E\{(\boldsymbol{P}_{\phi}^{(I)})^m\}$. Note that in this case the latter array given by Equation (13) can be replaced by an $n \times M$ matrix by fixing s at m. Hence, an $n \times M$ matrix is replaced by an $n \times M$ matrix. Equation (32) follows by a duality argument.

Hence, our analytical bounds are not improved by using a modular decomposition. On the other hand the computational complexity is reduced since we have to find minimal path and cut vectors only for each module and for the organizing structure.

All analytical bounds on the marginal moments $E\{(p_{\phi}^{j(I)})^m|\boldsymbol{D}^{1(I)}\}$ and $E\{(q_{\phi}^{j(I)})^m|\boldsymbol{D}^{2(I)}\}$ for $m = 1, \ldots, j = 1, \ldots, M$ given in Section 2 can be improved by straightforward simulation techniques. Let us illustrate this on the lower bounds in Equation (15). As in the proof of Theorem 5, with full notation, we have from Equation (6)

$$E\{(p_{\phi}^{j(I)})^m|\boldsymbol{D}^{1(I)}\} \geq E\{\max_{1\leq k\leq n_{\phi}^j} \prod_{i\in C_{\phi}^j(\boldsymbol{y}_k^j)} (p_i^{y_{ik}^j(I)})^m|\boldsymbol{D}^{1(I)}\}. \tag{33}$$

For $i = 1, \ldots, n$ we simulate from the posterior marginal distribution of $\boldsymbol{r}_i^{(I)}$ given the data $\boldsymbol{D}_i^{(I)}, \pi_i(\boldsymbol{r}_i^{(I)}|\boldsymbol{D}_i^{(I)})$, assumed being Dirichlet. We then calculate $p_i^{j(I)}$ from Equation (2) for $i = 1, \ldots, n, j = 1, \ldots, M$. For each round of n simulations the quantity $\max_{1\leq k\leq n_{\phi}^j} \prod_{i\in C_{\phi}^j(\boldsymbol{y}_k^j)}(p_i^{y_{ik}^j(I)})^m$ is calculated, and the right hand side of Equation (33) is estimated by the average of the simulated quantities. Theoretically, as seen from the proof, this improves the lower bound of Equation (15) of Theorem 5. Similarly, we obtain a simulated lower bound which improves the lower bound of Equation (17) of Theorem 6. In practice, the analytic bounds may be marginally better due to simulation uncertainty.

This simulation technique can also be applied to arrive at improved bounds using modular decompositions. From Theorem 4 we for instance get the following inequalities as starting points for the simulations.

Corollary 10. *Let (C, ϕ) be an MMS with modular decomposition given by Definition 3. Assume the marginal performance processes of the components to be independent in the time interval I. Then for $j = 1, \ldots, M$*

$$\begin{aligned}
&E\{(B_{\psi}^{*j(I)}(\boldsymbol{B}_{\psi}^{*(I)}(\boldsymbol{P}_{\phi}^{(I)})))^m|\boldsymbol{D}^{1(I)}\} \leq E\{(p_{\phi}^j)^m|\boldsymbol{D}^{1(I)}\} \\
&E\{(p_{\phi}^j)^m|\boldsymbol{D}^{2(I)}\} \leq E\{(1 - \bar{B}_{\psi}^{*j(I)}(\bar{\boldsymbol{B}}_{\psi}^{*(I)}(\boldsymbol{Q}_{\phi}^{(I)})))^m|\boldsymbol{D}^{2(I)}\} && (34)\\
&E\{(\bar{B}_{\psi}^{*j(I)}(\bar{\boldsymbol{B}}_{\psi}^{*(I)}(\boldsymbol{Q}_{\phi}^{(I)})))^m|\boldsymbol{D}^{2(I)}\} \leq E\{(q_{\phi}^j)^m|\boldsymbol{D}^{2(I)}\} \\
&E\{(q_{\phi}^j)^m|\boldsymbol{D}^{1(I)}\} \leq E\{(1 - B_{\psi}^{*j(I)}(\boldsymbol{B}_{\psi}^{*(I)}(\boldsymbol{P}_{\phi}^{(I)})))^m|\boldsymbol{D}^{1(I)}\}. && (35)
\end{aligned}$$

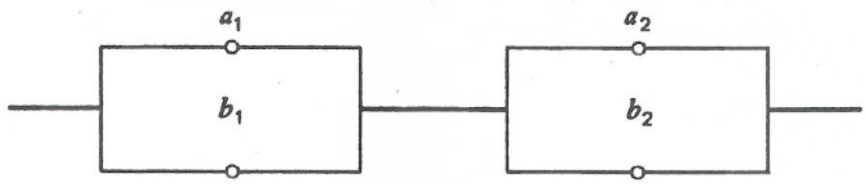

Figure 1 A simple network.

To illustrate the theory consider the simple network system depicted in Figure 1. Here module 1 is the parallel system of the components a_1 and b_1 and module 2 the parallel system of the components a_2 and b_2. We assume that the set of states of the ith component is given by $S_i = \{0, 3\}, i = 1, 2, 3, 4$, i.e. we have a binary description at the component level. Let for each module the state be 0 if neither of the components work, 1 if one component works and 3 if two components work. The states of the system are given in Table 1.

Table 1 States of the simple network system of Figure 1.

	3	0	2	3
Module 2	1	0	1	2
	0	0	0	0
		0	1	3
		Module 1		

Note for instance that the state 1 is critical both for each module and the system as a whole in the sense that the failing of a component leads to the 0 state.

The minimal path and cut vectors for the system of Figure 1 are given in respectively Tables 2 and 3.

Table 2 Minimal path vectors for the system of Figure 1.

Level	Component 1	Component 2	Component 3	Component 4
1	0	3	0	3
1	0	3	3	0
1	3	0	0	3
1	3	0	3	0
2	0	3	3	3
2	3	0	3	3
2	3	3	0	3
2	3	3	3	0
3	3	3	3	3

Table 3. Minimal cut vectors for the system of Figure 1.

Level	Component 1	Component 2	Component 3	Component 4
1, 2	0	0	3	3
1, 2	3	3	0	0
2	0	3	0	3
2	0	3	3	0
2	3	0	0	3
2	3	0	3	0
3	0	3	3	3
3	3	0	3	3
3	3	3	0	3
3	3	3	3	0

Following the argument leading to Equation (2) we assume that the posterior marginal distribution of $p_i^{j(I)}$ given the data $D_i^{1j(I)}$ is beta with parameters $\alpha E\{p_i^{j(I)}\}$ and $\alpha(1 - E\{p_i^{j(I)}\})$. Hence,

$$Var\{p_i^{j(I)}\} = E\{p_i^{j(I)}\}(1 - E\{p_i^{j(I)}\})/(\alpha + 1),$$

and the second order moment is given by

$$E\{(p_i^{j(I)})^2\} = E\{p_i^{j(I)}\}(1 + E\{p_i^{j(I)}\}\alpha)/(\alpha + 1).$$

$E\{p_i^{j(I)}\}$ is chosen to be equal to the value of $p_i^{j(I)}$ calculated by a standard deterministic analysis as given in (Natvig 2011). In these calculations the marginal performance processes of the two modules are assumed independent in the time interval I and also that the two components of each module fail and are repaired/replaced independently of each other. All components have the same instantaneous failure rate $\lambda = 0.001$ and repair/replacement rate $\mu = 0.01$.

In Table 4 the analytical lower bounds from Corollary 8, $B_\phi^{*j(I)m}(E\{(\boldsymbol{P}_\phi^{(I)})^m\})$, for $E\{(p_\phi^{j(I)})^m\}$ for $m = 1, 2$ and the corresponding simulated lower bounds, both not using and using modular decompositions, are calculated for the time interval I equal to $[100, 110]$, $[100, 200]$, $[1000, 1100]$, the parameter α equal to 1, 10, 1000 and for system level j equal to 1, 2, 3. Looking at the lower bounds for $E\{(p_\phi^{j(I)})\}$ there are just minor differences between the analytical and the simulated bounds that are not based on modular decompositions except for $\alpha = 1, j = 2$ and the two longest intervals $[100, 200]$ and $[1000, 1100]$, where the improvements are quite small. Correspondingly, there are just minor improvements of the simulated bounds that are based on modular decompositions compared to the ones that are not except for $\alpha = 10, 1000, j = 2$ and the two longest intervals $[100, 200]$ and $[1000, 1100]$, where the improvements are quite small. Furthermore, the analytical bounds do not not depend on the α parameter which is natural since $E\{p_i^{j(I)}\}$ is independent of this parameter. That these lower bounds are decreasing in the length of the interval I and the system state j is just a reflection of the fact that these properties hold for $E\{(p_\phi^{j(I)})\}$.

Turning to the lower bounds for $E\{(p_\phi^{j(I)})^2\}$ there are again just minor differences between the analytical and the simulated bounds that are not based on modular decompositions again except for $\alpha = 1, j = 2$ and the two longest intervals $[100, 200]$ and $[1000, 1100]$, where the improvements are quite small. Correspondingly, there are just minor improvements of the simulated bounds that are based on modular decompositions compared to the ones that are not except for $\alpha = 10, 1000, j = 2$ and the two longest intervals $[100, 200]$ and $[1000, 1100]$, where the improvements are quite small. Furthermore, these bounds are decreasing in the α parameter which is natural since $E\{(p_i^{j(I)})^2\}$ is decreasing in this parameter. That these lower bounds are decreasing in the length of the interval I and the system state j is just a reflection of the fact that these properties also hold for $E\{(p_\phi^{j(I)})^2\}$. It should also be noted that combining the lower bounds for $E\{(p_\phi^{j(I)})\}$ and $E\{(p_\phi^{j(I)})^2\}$ does not lead to a lower bound for $Var\{(p_\phi^{j(I)})\}$. However, for the analytical lower bounds it is revealing that this leads to positive variances. For the corresponding simulated lower bounds this is obviously the case. Finally, it should be remarked that in contrast to the analytical bounds, the simulated bounds are improved by using modular decompositions.

For computer code used in this section we refer to
http://folk.uio.no/trondr/system/.

References

Barlow RE, Proschan, F (1975) Statistical theory of reliability and life testing. Probability models. Holt, Rinehart and Winston, New York.

Mastran DV, Singpurwalla ND (1978) A Bayesian estimation of the reliability of coherent structures. Operat Res 26:663-672

Natvig B (2011) Multistate systems reliability theory with applications. Wiley, New York.

Natvig B, Eide H (1987) Bayesian estimation of system reliability. Scand J Statist 14:319-327

Table 4. Lower bounds of the simple network system of Figure 1.

			Analytical		Sim. minus m.d.		Sim. plus m.d.	
I	α	j	1. m.	2. m.	1. m.	2. m.	1. m.	2. m.
$[100, 110]$	1	1	0.9902	0.9833	0.9902	0.9833	0.9902	0.9833
$[100, 110]$	1	2	0.9710	0.9507	0.9726	0.9546	0.9730	0.9551
$[100, 110]$	1	3	0.7481	0.6487	0.7481	0.6487	0.7481	0.6487
$[100, 110]$	10	1	0.9902	0.9807	0.9902	0.9807	0.9902	0.9807
$[100, 110]$	10	2	0.9710	0.9433	0.9713	0.9446	0.9724	0.9465
$[100, 110]$	10	3	0.7481	0.5751	0.7481	0.5752	0.7481	0.5752
$[100, 110]$	1000	1	0.9902	0.9805	0.9902	0.9805	0.9902	0.9805
$[100, 110]$	1000	2	0.9710	0.9428	0.9710	0.9428	0.9722	0.9451
$[100, 110]$	1000	3	0.7481	0.5598	0.7481	0.5598	0.7481	05598
$[100, 200]$	1	1	0.9555	0.9263	0.9555	0.9262	0.9555	0.9262
$[100, 200]$	1	2	0.8723	0.7947	0.8857	0.8226	0.8884	0.8256
$[100, 200]$	1	3	0.5219	0.3821	0.5217	0.3819	0.5217	0.3819
$[100, 200]$	10	1	0.9555	0.9142	0.9555	0.9142	0.9555	0.9142
$[100, 200]$	10	2	0.8723	0.7641	0.8751	0.7731	0.8834	0.7865
$[100, 200]$	10	3	0.5219	0.2903	0.5219	0.2903	0.5219	0.2903
$[100, 200]$	1000	1	0.9555	0.9130	0.9555	0.9130	0.9555	0.9130
$[100, 200]$	1000	2	0.8723	0.7610	0.8723	0.7611	0.8819	0.7778
$[100, 200]$	1000	3	0.5219	0.2726	0.5219	0.2726	0.5219	0.2726
$[1000, 1100]$	1	1	0.9380	0.8986	0.9381	0.8987	0.9381	0.8987
$[1000, 1100]$	1	2	0.8254	0.7256	0.8460	0.7663	0.8500	0.7705
$[1000, 1100]$	1	3	0.4578	0.3157	0.4580	0.3159	0.4580	0.3159
$[1000, 1100]$	10	1	0.9380	0.8818	0.9380	0.8818	0.9380	0.8818
$[1000, 1100]$	10	2	0.8254	0.6857	0.8296	0.6987	0.8422	0.7178
$[1000, 1100]$	10	3	0.4578	0.2265	0.4578	0.2266	0.4578	0.2266
$[1000, 1100]$	1000	1	0.9380	0.8800	0.9380	0.8799	0.9380	0.8799
$[1000, 1100]$	1000	2	0.8254	0.6813	0.8254	0.6815	0.8400	0.7057
$[1000, 1100]$	1000	3	0.4578	0.2098	0.4578	0.2098	0.4578	0.2098