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An anticipative stochastic calculus approach to pricing in markets driven by Lévy processes

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Abstract

We use the Itô-Ventzell formula for forward integrals and Malliavin calculus to study the stochastic control problem associated to utility indifference pricing in a market driven by Lévy processes. This approach allows us to consider general possibly non-Markovian systems, general utility functions and possibly partial information based portfolios. In the special case of the exponential utility function $U_\alpha = -\exp(-\alpha x)$; $\alpha > 0$, we obtain asymptotics properties for vanishing α. In the special case of full information based portfolios and no jumps, we obtain a recursive formula for the optimal portfolio in a non-Markovian setting.

1 Introduction

Consider a financial market with the following investment possibilities

(i) A *risk free asset*, where the unit price $S_0(t)$ at time t is:

$$S_0(t) = 1 \text{ for all } t \in [0, T], \tag{1.1}$$

where $T > 0$ is a fixed constant.

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(ii) A *risky asset*, where the unit price $S_1(t) = S(t)$ at time t is given by

$$dS(t) = S(t^-)\left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\right]. \tag{1.2}$$

Here $B(t)$ is a Brownian motion and $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is the compensated jump measure, $\tilde{N}(\cdot,\cdot)$, of an independent Lévy process $\eta(t) := \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds,dz)$, with jump measure $N(dt,dz)$ and Lévy measure $\nu(U) = E[N([0,1],U)]$ for $U \in B(\mathbb{R}_0)$ (i.e. U is a Borel set with closure $\bar{U} \subset \mathbb{R}_0 := \mathbb{R} - \{0\}$). The underlying probability space is denoted by $(\Omega, \mathcal{F}, P)$ and the σ-algebra generated by $\{B(s)\ ;\ s \le t, \eta(s)\ ;\ s \le t\}$ is denoted by $\mathcal{F}_t$.

The processes $\mu(t)$, $\sigma(t)$ and $\gamma(t,z)$ are assumed to be $\mathcal{F}_{t^-}$ predictable and satisfying

$$\int_0^T \left\{|\mu(t)| + \sigma^2(t) + \int_{\mathbb{R}} |\ln(1+\gamma(t,z)) - \gamma(t,z)|\nu(dz)\right\} dt < \infty \text{ a.s.} \tag{1.3}$$

and

$$\gamma(t,z) \ge -1 \text{ a.s. for all } z \in \mathbb{R}_0,\ t \in [0,T]. \tag{1.4}$$

Then, by the Itô formula for Itô-Lévy processes (see e.g. [12], Chapter 1) the solution of (1.2) is

$$S(t) = S(0)\exp\{\xi(t)\}\ ;\ t \in [0,T], \tag{1.5}$$

where

$$\xi(t) = \int_0^t \left\{\mu(s) - \frac{1}{2}\sigma^2(s) + \int_{\mathbb{R}_0} (\ln(1+\gamma(s,z)) - \gamma(s,z))\nu(dz)\right\} ds$$
$$\int_0^t \sigma(s)dB(s) + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\tilde{N}(dz,dz). \tag{1.6}$$

Let $\varphi(t) = (\varphi_0(t), \varphi_1(t))$ be an $\mathcal{F}_t$-predictable process representing a *portfolio* in this market, giving the number of units held in the risk free and the risky asset respectively, at time t. We will assume that φ is *self-financing*, in the sense that if

$$X(t) = X^\varphi(t) = \varphi_0(t)S_0(t) + \varphi_1(t)S_1(t) \tag{1.7}$$

is the *total value* of the investment at time t, then (since $dS_0(t) = 0$)

$$dX^\varphi(t) = \varphi_0(t)dS_0(t) + \varphi_1(t)dS_1(t) = \varphi_1(t)dS_1(t) \tag{1.8}$$

i.e.

$$X^\varphi(t) = x + \int_0^t u(s)dS(t), \qquad x = X^\varphi(0), \tag{1.9}$$

where $u(s) = \varphi_1(s)$.

In the following we let

$$\mathcal{E} \subseteq \mathcal{F}\ ;\ 0 \le t \le T$$

be a fixed subfiltration of $\{\mathcal{F}_t\}_{t\geq 0}$, representing the information available to the trader at time t. This means that we require that the portfolio $\varphi(t)$ must be $\mathcal{E}_t$-measurable for each $t \in [0,T]$.

For example, we could have

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+},$$

which models the situation when the trader has a delayed access to the information $\mathcal{F}_t$ from the market. This implies in particular that the control $\varphi(t)$ need not be Markovian.

If φ is self-financing and $\mathcal{E}$-adapted, and the value process $X^\varphi(t)$ is *lower bounded*, we say that φ is $\mathcal{E}$*-admissible*. The set of all $\mathcal{E}$-admissible controls is denoted by $\mathcal{A}_\mathcal{E}$.

If $\sigma \neq 0$ and $\gamma\nu \neq 0$ then it is well-known that the market is *incomplete*. This is already the case if $\mathcal{E}_t = \mathcal{F}_t$ for all $t \in [0,T]$, and even more so if $\mathcal{E}_t \subseteq \mathcal{F}_t$ for all $t \in [0,T]$.

Therefore the no-arbitrage principle is not sufficient to provide a unique price for a given European T-claim $G(\omega), \omega \in \Omega$. In this paper we will apply the *utility indifference principle* of Hodges and Neuberger [7] to find the price. In short , the principle is the following:

We fix a utility function $U : \mathbb{R} \to (-\infty, \infty)$. A trader with no final payment obligations faces the problem of maximizing the expected utility of the terminal wealth $X_x^{(\varphi)}(T)$ given that the initial wealth is $X_x^{(\varphi)}(0) = x \in \mathbb{R}$:

$$V_0(x) := \sup_{\varphi\in\mathcal{A}_\mathcal{E}} E\left[U\left(X_x^{(\varphi)}(T)\right)\right] = E\left[U\left(X_x^{(\hat\varphi)}(T)\right)\right], \tag{1.10}$$

where $\hat\varphi \in \mathcal{A}_\mathcal{E}$ is an optimal portfolio (if it exists).

If, on the other hand, the trader is also selling a guaranteed payoff $G(\omega)$ (a lower bounded $\mathcal{F}_T$-measurable random variable) and gets an initial payment $p > 0$ for this, the problem for the seller will be to find $V_G(x + p)$ and $u^* \in \mathcal{A}_\mathcal{E}$ (an optimal portfolio, if it exists), such that

$$\begin{aligned} V_G(x+p) &:= \sup_{u\in\mathcal{A}_\mathcal{E}} E\left[U\left(X_{x+p}^{(u)}(T) - G\right)\right] \\ &= E\left[U\left(X_{x+p}^{(u^*)}(T) - G\right)\right]. \end{aligned} \tag{1.11}$$

The *utility indifference pricing principles* states that the "right" price p of the European option with payoff G at time T is the solution p of the equation

$$V_G(x+p) = V_0(x). \tag{1.12}$$

This means that the seller is indifferent to the following two alternatives: Either

(i) receiving the payment p at time 0 and paying out $G(\omega)$ at time T, or

(ii) not selling the option at all, i.e. $p = G = 0$.

We see that in order to find the price p we need to solve the stochastic control problem (1.11) to find $V_G(x + p)$. Then we get $V_0(x)$ as a special case by putting $G = p = 0$.

In this paper we will use anticipative stochastic calculus (forward integrals) and Malliavin calculus to solve the problem (1.11). To the best of our knowledge this is the first time such an approach is used for this kind of problem. The motivations for our approach are the following:

(i) We want a method which applies to a wide class of utility functions, not just the exponential utility $U(x) = -\mathrm{e}^{-\alpha x}$; $\alpha > 0$, which seems to be almost the only one studied so far.

(ii) We are interested in the situation when the trader has only partial information $\mathcal{E}_t$ to her disposal. For example, if $\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}$, how does the information delay δ influence the price ?

(iii) Moreover, we want to allow more general payoffs $G(\omega)$ than the Markovian ones of the form $G = g(S(T))$. In particular, we want to allow path-dependent payoffs $G = g(\{S(t)\ ;\ t \leq T\})$.

In Section 4 we study the exponential utility case in more detail. Under some conditions we show that if $u_\alpha^{(G)}$ is an optimal portfolio corresponding to $U(x) = -\mathrm{e}^{-\alpha x}$ and terminal payoff G, then $\tilde{u}(t) := \lim_{\alpha \to 0} \alpha u_\alpha^{(G)}(t)$ is an optimal portfolio corresponding to $\alpha = 1$ and $G = 0$ (Theorems 4.4 and 4.5). In Theorem 4.6 we obtain a recursive formula for the optimal portfolio in a non-Markovian setting if $\mathcal{E}_t = \mathcal{F}_t$ and $\nu = 0$.

For more information and results about utility indifference pricing we refer to [2], [6], [7], [8], [10] and [17], and the references therein. For more information about stochastic calculus and financial markets with Lévy processes we refer to [1], [3] and [12].

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2 Some prerequisites on forward integrals and Malliavin calculus

In this section we give a brief summary of basic definitions and properties of forward integrals and Malliavin calculus for Lévy processes. General references to this section are [4], [5] and [14]. First we consider *forwards integrals*:

Definition 2.1 *[[14]] We say that a stochastic process* $\varphi(t)\ ;\ t \in [0,T]$*, is forward integrable over the interval* $[0,T]$ *with respect to Brownian motion* $B(\cdot)$ *if there exists a process* $I(t)\ ;\ t \in [0,T]$*, such that :*

$$\sup_{t \in [0,T]} \left(\int_0^t \varphi(s) \frac{B(s+\epsilon) - B(s)}{\epsilon} ds - I(t) \right) \to 0 \ \ as\ \varepsilon \to 0 \tag{2.1}$$

in probability. If this is the case we put

$$I(t) = \int_0^t \varphi(s) d^- B(s) \tag{2.2}$$

and call $I(t)$ *the forward integral of* φ *with respect to* $B(\cdot)$*.*

The forward integral is an extension of the Itô integral, in the sense that if φ is adapted and forward integrable, then the forward integral of φ coincides with the classical Itô integral.

Example 2.2 *[Simple integrands]*
If the process $\varphi(t)$ has the simple form

$$\varphi(t) = \sum_{j=1}^{m} a_j(\omega)\chi_{[t_j,t_{j+1})}(t) \ ; \ 0 \leq t_j, t \leq T \text{ for all } j$$

where $a_j(\omega)$ are arbitrary random variables, then φ is forward integrable and

$$\int_0^T \varphi(t)d^-B(t) = \sum_{j=1}^{m} a_j(\omega)(B(t_{j+1}) - B(t_j)).$$

Next we define the corresponding integral with respect to the compensated Poisson random measure $\tilde{N}(\cdot,\cdot)$:

Definition 2.3 *[[4] (Forward integrals with respect to $\tilde{N}(\cdot,\cdot)$)] We say that a stochastic process $\psi(t,z)$; $t \in [0,T]$, $z \in \mathbb{R}_0$ is forward integrable over $[0,T]$ with respect to $\tilde{N}(\cdot,\cdot)$ if there exists a process $J(t)$; $t \in [0,T]$, such that*

$$\sup_{t\in[0,T]} \left(\int_0^T \int_{\mathbb{R}_0} \psi(s,z) \mathbb{1}_{K_n}(z)\tilde{N}(ds,dz) - J(t) \right) \leftarrow 0 \text{ as } n \to \infty \tag{2.3}$$

in probability. Here $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of compact sets $K_n \subset \mathbb{R}_0$ with $\nu(K_n) < \infty$ such that $\bigcup_{n=1}^{\infty} = \mathbb{R}_0$ and we require that $J(t)$ does not depend on the sequence $\{K_n\}_{n=1}^{\infty}$ chosen.

If this is the case we put

$$J(t) = \int_0^t \int_{\mathbb{R}_0} \psi(s,z)\tilde{N}(d^-s,dz) \tag{2.4}$$

and we call $J(t)$ the forward integral of $\psi(\cdot,\cdot)$ with respect to $\tilde{N}(\cdot,\cdot)$.

Also in this case the forward integral coincides with the classical Itô integral if the integrand is $\mathcal{F}_t$-predictable.

We now combine the two concepts above and make the following definition:

Definition 2.4 *[Generalized forward processes]*
A (generalized) forward (Itô-Lévy) process is a stochastic process $Y(t)$; $t \in [0,T]$ of the form

$$Y(t) = Y(0) + \int_0^t \alpha(s)ds + \int_0^t \varphi(s)d^-B(s) + \int_0^t \int_{\mathbb{R}_0} \psi(s,z)\tilde{N}(d^-s,dz) \tag{2.5}$$

where $Y(0)$ is an $\mathcal{F}_T$-measurable random variable and $\varphi(s)$ and $\psi(s,z)$ are forward integrable processes. A shorthand notation for this is

$$d^-Y(t) = \alpha(t)dt + \varphi(t)d^-B(t) + \int_{\mathbb{R}_0} \psi(t,z)\tilde{N}(d^-t,dz)\ ;\ t \in (0,T) \tag{2.6}$$

$$Y(0) \text{ is } \mathcal{F}_T\text{-measurable} \tag{2.7}$$

Remark If $Y(0) = y \in \mathbb{R}$ is non-random, then the process $Y(t)$ is an Itô-Lévy process of the type discussed in [4]. The term "generalized" refers to the case when $Y(0)$ is random.

We will need an Itô formula for generalized forward processes. The following result is a slight extension of the Itô formula in [15], [16] (Brownian motion case) and [4] (Poisson random measure case). It may be regarded as a special case of the Itô-Ventzell formula given in [13]:

Theorem 2.5 *[[13] Special case of the Itô-Ventzell formula for forward processes]*
Let $Y(t)$ be a generalized forward process of the form (2.5) and assume that $\psi(t,z)$ is continuous in z near $z=0$ for a.a. t,ω and that

$$\int_0^T \int_{\mathbb{R}} \psi^2(t,z)\nu(dz)dt < \infty \text{ a.s.}$$

Let $f \in C^2(\mathbb{R})$ and define

$$Z(t) = f(Y(t)).$$

Then $Z(t)$ is a forward process given by

$$\begin{aligned} d^-Z(t) = \Big[& f'(Y(t))\alpha(t) + \frac{1}{2}f''(Y(t))\varphi^2(t) \\ & + \int_{\mathbb{R}_0} \{f(Y(t)+\psi(t,z)) - f(Y(t)) - f'(Y(t))\psi(t,z)\}\,\nu(dz)\Big]\,dt \\ & + f'(Y(t))d^-B(t) + \int_{\mathbb{R}} \big\{f(Y(t^-)+\psi(t,z)) - f(Y(t^-))\big\}\,\tilde{N}(d^-t,dz)\ ;\ t>0 \end{aligned} \tag{2.8}$$

$$Z(0) = f(Y(0)). \tag{2.9}$$

Next we give a short introduction to Malliavin calculus for Lévy processes.
Again it is natural to divide the presentation in two parts:
2.1 - Malliavin calculus for $B(\cdot)$.
2.2 - Malliavin calculus for $\tilde{N}(\cdot,\cdot)$.

For the case 2.1, we refer to [11] and [5] for proofs and more information. For the case 2.2, we refer to [4] and [5].

2.1 Malliavin calculus for $B(\cdot)$

A natural starting point is the *Wiener-Itô chaos expansion theorem*, which states that any $F \in L^2(\mathcal{F}_T^{(B)}, P)$ can be written

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{2.10}$$

for a unique sequence of symmetric, deterministic functions $f_n \in L^2(\lambda^n)$, where λ is Lebesgue measure on $[0, T]$ and

$$I_n(f_n) = n! \int_0^T \left(\int_0^{t_n} \cdots \left(\int_0^{t_2} f_n(t_1, \ldots, t_n) dB(t_1) \right) dB(t_2) \right) \cdots dB(t_n) \tag{2.11}$$

(the *n-times iterated integral* of f_n with respect to $B(\cdot)$) for $n = 1, 2, \ldots$ and $I_0(f_0) = f_0$ when f_0 is a constant. Here $\mathcal{F}_T^{(B)}$ is the σ-algebra generated by the random variables $\{B(s)\ ;\ 0 \leq s \leq T\}$.

Moreover, we have the isometry

$$E[F^2] = \|F\|_{L^2(P)}^2 = \sum_{n=0}^{\infty} n! \|F_n\|_{L^2(\lambda^n)}^2. \tag{2.12}$$

Definition 2.6 (Malliavin derivative D_t) *Let $\mathbb{D}_{1,2} = \mathbb{D}_{1,2}^{(B)}$ be the space of all $F \in L^2(\mathcal{F}_T^{(B)}, P)$ such that its chaos expansion* (2.10) *satisfies*

$$\|F\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=1}^{\infty} n n! \|f_n\|_{L^2(\lambda^n)}^2 < \infty. \tag{2.13}$$

For $F \in \mathbb{D}_{1,2}$ and $t \in [0, T]$, we define the Malliavin derivative of F at t (with respect to $B(\cdot)$), $D_t F$, by

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \tag{2.14}$$

where the notation $I_{n-1}(f_n(\cdot, t))$ means that we apply the $(n-1)$-times iterated integral to the first $n-1$ variables $t_1, \ldots, t_{n-1}$ of $f_n(t_1, t_2, \ldots, t_n)$ and keep the last variable $t_n = t$ as a parameter.

One can easily check that

$$E\left[\int_0^T (D_t F)^2 dt\right] = \sum_{n=1}^{\infty} n n! \|f_n\|_{L^2(\lambda^n)}^2 = \|F\|_{\mathbb{D}_{1,2}}^2. \tag{2.15}$$

Hence the map $(t, \omega) \to D_t F(\omega)$ belongs to $L^2(\lambda \times P)$.

Example 2.7 *If* $F = \int_0^T f(t)dB(t)$ *where* $f \in L^2(\lambda)$ *is deterministic, then*

$$D_t F = f(t) \text{ for a.a. } t \in [0,T].$$

More generally, if $u(s)$ *is Skorohod integrable,* $u(s) \in \mathbb{D}_{1,2}$ *for a.a.* s *and* $D_t u(s)$ *is Skorohod integrable for a.a.* t*, then*

$$D_t \left(\int_0^T u(s)\delta B(s) \right) = \int_0^T D_t u(s)\delta B(s) + u(t) \text{ for a.a. } (t,\omega), \tag{2.16}$$

where $\int_0^T \psi(s)\delta B(s)$ *denotes the Skorohod integral of* ψ *with respect to* $B(\cdot)$*. (See [5], Chapters 3 and 12 for a definition of Skorohod integrals and for more details).*

Some other basic properties of the Malliavin derivative D_t are the following:

Theorem 2.8 **(i) Chain rule** *([11], page 29)*

Suppose $F_1, \ldots, F_m \in \mathbb{D}_{1,2}$ *and that* $\varphi : \mathbb{R}^m \to \mathbb{R}$ *is* C^1 *with bounded partial derivatives. Then* $\varphi(F_1, \ldots, F_m) \in \mathbb{D}_{1,2}$ *and*

$$D_t \varphi(F_1, \ldots, F_m) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x_i}(F_1, \ldots, F_m) D_t F_i. \tag{2.17}$$

(ii) Integration by parts *([11], page 35)*

Suppose $u(t)$ *is* $\mathcal{F}_t$*-adapted with*

$$E\left[\int_0^T u^2(t)dt\right] < \infty$$

and let $F \in \mathbb{D}_{1,2}$*. Then*

$$E\left[F \int_0^T u(t)dB(t)\right] = E\left[\int_0^T u(t)D_t F dt\right]. \tag{2.18}$$

(iii) Duality formula for forward integrals *([15])*

Suppose $\beta(\cdot)$ *is forward integrable with respect to* $B(\cdot)$*,* $\beta(t) \in \mathbb{D}_{1,2}$ *and* $D_{t^+}\beta(t) := \lim_{s \to t^+} D_s \beta(t)$ *exists for a.a.* t *with*

$$E\left[\int_0^T |D_{t^+}\beta(t)|dt\right] < \infty.$$

Then

$$E\left[\int_0^T \beta(t)d^- B(t)\right] = E\left[\int_0^T D_{t^+}\beta(t)dt\right]. \tag{2.19}$$

2.2 Malliavin calculus for $\tilde{N}(\cdot)$

The construction of a stochastic derivative/Malliavin derivative in the pure jump martingale case follows the same lines as in the Brownian motion case. In this case the corresponding Wiener-Itô chaos expansion theorem states that any $F \in L^2(\mathcal{F}_T, P)$, (where in this case $\mathcal{F}_t = \mathcal{F}_t^{(\tilde{N})}$ is the the σ-algebra generated by $\eta(s) := \int_0^s \int_{\mathbb{R}_0} z\tilde{N}(dr, dz)$; $0 \leq s \leq t$), can be written

$$F = \sum_{n=0}^{\infty} I_n(f_n) \; ; \; f_n \in \hat{L}^2((\lambda \times \nu)^n). \tag{2.20}$$

Here $\hat{L}^2((\lambda \times \nu)^n)$ is the space of functions $f_n(t_1, z_1, \ldots, t_i, z_i)$; $t_1 \in [0, T]$, $z_i \in \mathbb{R}_0$ such that $f_n \in L^2((\lambda \times \nu)^n)$ and f_n is *symmetric* with respect to the pairs of variables $(t_1, z_1), \ldots, (t_n, z_n)$.

It is important to note that in this case the n-times iterated integral $I_n(f_n)$ is taken with respect to $\tilde{N}(dt, dz)$ and not with respect to $d\eta(t)$. Thus we define

$$I_n(f_n) = n! \int_0^T \int_{\mathbb{R}_0} \left(\int_0^{t_n} \int_{\mathbb{R}_0} \cdots \left(\int_0^{t_2} \int_{\mathbb{R}} f_n(t_1, z_1, \ldots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n) \right) \right) \tag{2.21}$$

for $f_n \in \hat{L}^2((\lambda \times \nu)^n)$.

The Itô isometry for stochastic integrals with respect to $\tilde{N}(dt, dz)$ then gives the following isometry for the chaos expansion:

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2((\lambda \times \nu)^n)}.$$

As in the Brownian motion case we use the chaos expansion to define the Malliavin derivative. Note that in this case there are two parameters t, z, where t represents time and $z \neq 0$ represents a generic jump size.

Definition 2.9 *[Malliavin derivative $D_{t,z}$] Let $\mathbb{D}_{1,2} = \mathbb{D}^{(\tilde{N})}_{1,2}$ be the space of all $F \in L^2(\mathcal{F}_T, P)$ such that its chaos expansion* (2.20) *satisfies*

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2((\lambda \times \nu)^n)} < \infty. \tag{2.22}$$

For $F \in \mathbb{D}_{1,2}$ we define the Malliavin derivative of F at t, z (with respect to $\tilde{N}(\cdot)$), $D_{t,z}F$, by

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t, z)), \tag{2.23}$$

where, similar to (2.14)*, $I_{n-1}(f_n(\cdot, t, z))$ means that we perform the $(n-1)$-times iterated integral with respect to $\tilde{N}$ of the first $n-1$ variable pairs $(t_1, z_1), \ldots, t_{n-1}, z_{n-1})$, keeping $(t_n, z_n) = (t, z)$ as a parameter.*

In this case we get the isometry

$$E\left[\int_0^T\int_{\mathbb{R}_0}(D_{t,z}F)^2\nu(dz)dt\right]=\sum_{n=0}^{\infty}nn!\|f_n\|^2_{L^2((\lambda\times\nu)^n)}=\|F\|^2_{\mathbb{D}^{(\tilde N)}_{1,2}} \tag{2.24}$$

(Compare with (2.15)*).*

Example 2.10 *If* $F=\int_0^T\int_{\mathbb{R}_0}f(t,z)\tilde N(dt,dz)$ *for some deterministic* $f(t,z)\in L^2(\lambda\times\nu)$*, then*

$$D_{t,z}F=f(t,z)\text{ for a.a. }t,z.$$

More generally, if $\psi(s,\zeta)$ *is Skorohod integrable with respect to* $\tilde N(\delta s,d\zeta)$*,* $\psi(s,\zeta)\in\mathbb{D}^{(\tilde N)}_{1,2}$ *for a.a.* s,ζ *and* $D_{t,z}\psi(s,\zeta)$ *is Skorohod integrable for a.a.* t,z *then*

$$D_{t,z}\left(\int_0^T\int_{\mathbb{R}_0}\psi(s,\zeta)\tilde N(\delta s,d\zeta)\right)=\int_0^T\int_{\mathbb{R}_0}D_{t,z}\psi(s,\zeta)\tilde N(\delta s,d\zeta)+\psi(t,z) \tag{2.25}$$

where $\int_0^T\int_{\mathbb{R}_0}\psi(s,z)\tilde N(\delta s,dz)$ *denotes the Skorohod integral of* ψ *with respect to* $\tilde N(\cdot,\cdot)$*. (See [4] for a definition of such Skorohod integrals and for more details).*

The properties of $D_{t,z}$ corresponding to the properties (2.17), (2.18) and (2.19) of D_t are the following :

Theorem 2.11 **(i) Chain rule** *([4]).*

Suppose $F_1,\dots,F_m\in\mathbb{D}^{(\tilde N)}_{1,2}$ *and that* $\varphi:\mathbb{R}^m\to\mathbb{R}$ *is continuous and bounded. Then* $\varphi(F_1,\dots,F_m)\in\mathbb{D}^{(\tilde N)}_{1,2}$ *and*

$$D_{t,z}\varphi(F_1,\dots,F_m)=\varphi(F_1+D_{t,z}F_1,\dots,F_m+D_{t,z}F_m)-\varphi(F_1,\dots,F_m). \tag{2.26}$$

(ii) Integration by parts*([4]).*

Suppose $\psi(t,z)$ *is* $\mathcal{F}_t$*-adapted and*

$$E\left[\int_0^T\int_{\mathbb{R}_0}\psi^2(t,z)\nu(dz)dt\right]<\infty$$

and let $F\in\mathbb{D}^{(\tilde N)}_{1,2}$*. Then*

$$E\left[\int_0^T\int_{\mathbb{R}_0}\psi(t,z)\tilde N(dt,dz)\right]=E\left[\int_0^T\int_{\mathbb{R}_0}\psi(t,z)D_{t,z}F\nu(dz)dt\right]. \tag{2.27}$$

(iii) Duality formula for forward integrals *([4]).*

Supposes $\theta(t,z)$ *is forward integrable with respect to* $\tilde{N}$, $\theta(t,z) \in \mathbb{D}_{1,2}^{(\tilde{N})}$ *and*

$$D_{t^+,z}\theta(t,z) := \lim_{s\to t^+} D_{s,z}\theta(t,z) \text{ exists for a.a. } t,z$$

with

$$E\left[\int_0^T \int_{\mathbb{R}_0} |D_{t^+,z}\theta(t,z)|\, \nu(dz)dt\right] < \infty.$$

Then

$$E\left[\int_0^T \int_{\mathbb{R}_0} \theta(t,z)\tilde{N}(d^-t,dz)\right] = E\left[\int_0^T \int_{\mathbb{R}_0} D_{t^+,z}\theta(t,z)\nu(dz)dt\right]. \tag{2.28}$$

Example 2.12 *[The European put]*

Let $S(t)$ *be the risky asset price given in* (1.2) *and* (1.5)-(1.6) *and let* $K > 0$ *be a constant. Define*

$$G = (K - S(T))^+ = \begin{cases} K - S(T) & \text{if } S(T) < K \\ 0 & \text{if } S(T) \geq K. \end{cases}$$

This is the payoff of a European put option with exercise price K *and exercise time* T.

For simplicity let us assume (in this example) that $\mu(s)$, $\sigma(s)$ and $\gamma(s,z)$ are *deterministic.* Then by a slight extension of the chain rule Theorem 2.8 we have

$$\begin{aligned} D_tG &= -\chi_{[0,K]}(S(T))D_tS(T) &\quad (2.29)\\ &= -\chi_{[0,K]}(S(T))S(T)\sigma(t). &\quad (2.30)\end{aligned}$$

And by the chain rule Theorem 2.11 we have

$$D_{t,z}G = (K - (S(T)) + D_{t,z}S(T))^+ - (K - S(T))^+,$$

where

$$\begin{aligned} D_{t,z}S(T) &= S(0)\exp(\xi(T) + D_{t,z}\xi(T)) - S(T) \\ &= S(T)(\exp(D_{t,z}\xi(T)) - 1) \\ &= S(T)(\exp(\ln\gamma(t,z)) - 1) = S(T)(\gamma(t,z) - 1). \end{aligned}$$

Hence

$$D_{t,z}G = (K - S(T)\gamma(t,z))^+ - (K - S(T))^+. \tag{2.31}$$

3 Solving the stochastic control problem

In this section we use forward integrals to solve the stochastic control problem (1.11). We will make the following assumptions:

$$U \in C^3(\mathbb{R}). \tag{3.1}$$

The payoff $G = G(\omega)$ is Malliavin differentiable both with respect to $B(\cdot)$ and with respect to $\tilde{N}(\cdot,\cdot)$. (3.2)

Choose $u \in \mathcal{A}_{\mathcal{E}}$, $x \in \mathbb{R}$ and consider

$$\begin{aligned} Y(t) &:= X(t) - G = X_x^{(u)}(t) - G = x - G + \int_0^t u(s)dS(s) \\ &= x - G + \int_0^t \mu(s)u(s)S(s)ds + \int_0^t \sigma(s)u(s)S(s)dB(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s)S(s^-)\gamma(s,z)\tilde{N}(ds,dz). \end{aligned} \tag{3.3}$$

By the Itô-Ventzell formula for forward integrals (Theorem 2.5) we have

$$\begin{aligned} d(U(Y(t))) &= U'(Y(t))[\mu(t)u(t)S(t)dt + \sigma(t)u(t)S(t)d^-B(t)] \\ &\quad + \frac{1}{2}U''(Y(t))\sigma^2(t)u^2(t)S^2(t)dt \\ &\quad + \int_{\mathbb{R}_0} \{U(Y(t) + u(t)S(t)\gamma(t,z)) - U(Y(t)) - u(t)S(t)\gamma(t,z)U'(Y(t))\}\nu(dz)dt \\ &\quad + \int_{\mathbb{R}_0} \{U(Y(t^-) + u(t)S(t^-)\gamma(t,z)) - U(Y(t^-))\}\tilde{N}(d^-t,dz). \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned} U(X(T) - G) = U(x - G) + \int_0^T \alpha(t)dt + \int_0^T \beta(t)d^-B(t) \\ + \int_0^T \int_{\mathbb{R}_0} \theta(t,z)\tilde{N}(d^-t,dz), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \alpha(t) &= U'(X(t) - G)u(t)S(t)\mu(t) + \frac{1}{2}U''(X(t) - G)u^2(t)S^2(t)\sigma^2(t) \\ &\quad + \int_{\mathbb{R}_0} \{U(X(t) + u(t)S(t)\gamma(t,z) - G) - U(X(t) - G) \\ &\quad - u(t)S(t)\gamma(t,z)U'(X(t) - G)\}\nu(dz) \end{aligned} \tag{3.6}$$

$$\beta(t) = U'(X(t) - G)u(t)S(t)\sigma(t) \tag{3.7}$$

and

$$\theta(t,z) = U(X(t^-) + u(t)S(t^-)\gamma(t,z) - G) - U(X(t^-) - G). \tag{3.8}$$

By the duality theorems for forward integrals (Theorem 2.8(iii) and Theorem 2.11(iii)) we have

$$E\left[\int_0^T \beta(t)d^-B(t)\right] = E\left[\int_0^T D_{t^+}\beta(t)dt\right] \tag{3.9}$$

and

$$E\left[\int_0^T \int_{\mathbb{R}_0} \theta(t,z)\tilde{N}(d^-t,dz)\right] = E\left[\int_0^T \int_{\mathbb{R}_0} D_{t^+,z}\theta(t,z)\nu(dz)dt\right]. \tag{3.10}$$

Since

$$D_{t^+}\beta(t) = u(t)S(t)\sigma(t)U''(X(t) - G)(-D_tG) \tag{3.11}$$

and

$$\begin{aligned} D_{t^+,z}\theta(t,z) &= U(X(t^-) + u(t)S(t)\gamma(t,z) - G - D_{t,z}G) \\ &\quad - U(X(t^-) + u(t)S(t)\gamma(t,z) - G) - U(X(t^-) - G - D_{t,z}G) + U(X(t^-) - G), \end{aligned} \tag{3.12}$$

we get by (3.5)-(3.10),

$$\begin{aligned} E[U(X(T) - G)] &= E[U(x - G)] + E\left[\int_0^T \left\{\alpha(t) + D_{t^+}\beta(t) + \int_{\mathbb{R}_0} D_{t^+,z}\theta(t,z)\nu(dz)\right\} dt\right] \\ &= E[U(x - G)] + E\left[\int_0^T \{u(t)S(t)[\mu(t)U'(X(t) - G) - \sigma(t)U''(X(t) - G)D_tG]\right. \\ &\quad + \frac{1}{2}u^2(t)S^2(t)\sigma^2(t)U''(X(t) - G) \\ &\quad + \int_{\mathbb{R}_0} [U(X(t) + u(t)S(t)\gamma(t,z) - G) - U(X(t) - G) \\ &\quad - u(t)S(t)\gamma(t,z)U'(X(t) - G) \\ &\quad + U(X(t) + u(t)S(t)\gamma(t,z) - G - D_{t,z}G) \\ &\quad - U(X(t) + u(t)S(t)\gamma(t,z) - G) \\ &\quad - U(X(t) - G - D_{t,z}G) + U(X(t) - G)]\nu(dz)\}dt\} \\ &= E[U(x - G)] + E\left[\int_0^T \{u(t)S(t)[\mu(t)U'(X(t) - G) - \sigma(t)U''(X(t) - G)D_tG\right] \\ &\quad + \frac{1}{2}u^2(t)S^2(t)\sigma^2(t)U''(X(t) - G) \\ &\quad + \int_{\mathbb{R}_0} [U(X(t) + u(t)S(t)\gamma(t,z) - G - D_{t,z}G) - U(X(t) - G - D_{t,z}G) \\ &\quad - u(t)S(t)\gamma(t,z)U'(X(t) - G)]\nu(dz)\}dt]. \end{aligned} \tag{3.13}$$

We may insert a conditional expectation with respect to $\mathcal{F}_t$ for each t in this integral and this gives:

$$\begin{aligned}
E[U(X(T)-G)] = E[U(x-G)] + E\bigg[\int_0^T \{&u(t)S(t)(\mu(t)E[U'(X(t)-G) \mid \mathcal{F}_t] \\
&- \sigma(t)E[U''(X(t)-G)D_tG \mid \mathcal{F}_t]) \\
&+ \frac{1}{2}u^2(t)S^2(t)\sigma^2(t)E[U''(X(t)-G) \mid \mathcal{F}_t] \\
&+ \int_{\mathbb{R}_0} E[(U(X(t)+u(t)S(t)\gamma(t,z)-G-D_{t,z}G)-U(X(t)-G-D_{t,z}G) \\
&- U(t)S(t)\gamma(t,z)U'(X(t)-G)) \mid \mathcal{F}_t]\nu(dz)dt. \qquad (3.14)
\end{aligned}$$

We conclude that our original stochastic control problem (1.11) is equivalent to a problem of the following type:

Problem 3.1 *Find Φ and $\hat{u} \in \mathcal{A}_\mathcal{E}$ such that*

$$\Phi := \sup_{u \in \mathcal{A}_\mathcal{E}} J(u) = J(\hat{u}) \qquad (3.15)$$

where

$$J(u) = E\left[\int_0^T f(t, X(t), u(t))dt + g(X(T))\right], \qquad (3.16)$$

with

$$\begin{aligned}
dX(t) &= b(t, X(t), u(t))dt + c(t, X(t), u(t))dB(t) \\
&\quad + \int_{\mathbb{R}_0} \theta(t, X(t), u(t), z)\tilde{N}(dt, dz)\ ;\ X(0) \in \mathbb{R}. \qquad (3.17)
\end{aligned}$$

In our case we have

$$b(t,x,u) = b(t,x,u,\omega) = uS(t)\mu(t), \qquad (3.18)$$

$$c(t,x,u) = c(t,x,u,\omega) = uS(t)\sigma(t), \qquad (3.19)$$

$$\theta(t,x,u,z) = \theta(t,x,u,z,\omega) = uS(t)\gamma(t,z), \qquad (3.20)$$

$$g = 0, \qquad (3.21)$$

and

$$\begin{aligned}
f(t,x,u) &= f(t,x,u,\omega) \\
&= uS(t)(\mu(t)E[U'(x-G) \mid \mathcal{F}_t] - \sigma(t)E[U''(x-G)D_tG \mid \mathcal{F}_t]) \\
&+ \frac{1}{2}u^2S^2(t)\sigma^2(t)E[U''(x-G) \mid \mathcal{F}_t] \\
&+ \int_{\mathbb{R}_0} E[(U(x+uS(t)\gamma(t,z)-G-D_{t,z}G)-U(x-G-D_{t,z}G) \\
&- uS(t)\gamma(t,z)U'(x-G)) \mid \mathcal{F}_t]\nu(dz). \qquad (3.22)
\end{aligned}$$

This is a partial information stochastic control problem of the type studied in [9]. We will use the stochastic maximum principle of that paper to study Problem 3.1.

To this end, we first briefly recall the general maximum principle of [9], using the notation of (3.15)-(3.17).

From now on, we make the following general assumptions:

- The functions $f(t,x,u), g(x), b(t,x,u), c(t,x,u)$ and $\theta(t,x,u,z)$ are C^1 with respect to x and u.
- For all $t, r \in (0,T)$, $t \leq r$, and all bounded $\mathcal{E}_t-$measurable random variables $\alpha = \alpha(\omega)$ the control
$$\beta_\alpha(s) = \alpha(\omega)\chi_{[t,r]}(s); \quad s \in [0,T] \tag{3.23}$$
belongs to $\mathcal{A}_\mathcal{E}$.
- For all $u, \beta \in \mathcal{A}_\mathcal{E}$ with β bounded, there exists $\delta > 0$ such that
$$u + y\beta \in \mathcal{A}_\mathcal{E} \quad \text{for all } y \in (-\delta, \delta)$$
and such that the family
$$\Big\{\frac{\partial f}{\partial x}(t, X^{u+y\beta}(t), u(t) + y\beta(t))\frac{d}{dy}X^{u+y\beta}(t) + \frac{\partial f}{\partial u}(t, X^{u+y\beta}(t), u(t) + y\beta(t))\beta(t)\Big\}_{y\in(-\delta,\delta)}$$
is $\lambda \times P$-uniformly integrable and the family
$$\Big\{g'(X^{u+y\beta}(T))\frac{d}{dy}X^{u+y\beta}(T)\Big\}_{y\in(-\delta,\delta)} \tag{3.24}$$
is P-uniformly integrable.
- For all $u, \beta \in \mathcal{A}_\mathcal{E}$ with β bounded the process $Y(t) = Y^{(\beta)}(t) = \frac{d}{dy}X^{(u+y\beta)}(t)|_{y=0}$ exists and satisfies the equation
$$\begin{aligned} dY(t) &= Y(t^-)\Big[\frac{\partial b}{\partial x}(t, X(t), u(t))dt + \frac{\partial \sigma}{\partial x}(t, X(t), u(t))dB(t) \\ &\quad + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(t, X(t^-), u(t^-), z)\tilde{N}(dt, dz)\Big] \\ &\quad + \beta(t^-)\Big[\frac{\partial b}{\partial u}(t, X(t), u(t))dt + \frac{\partial \sigma}{\partial u}(t, X(t), u(t))dB(t) \\ &\quad + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(t, X(t^-), u(t^-), z)\tilde{N}(dt, dz)\Big]; \\ Y(0) &= 0. \end{aligned} \tag{3.25}$$

- For all $u \in \mathcal{A}_{\mathcal{E}}$, the following processes

$$
\begin{aligned}
K(t) &:= g'(X(T)) + \int_t^T \frac{\partial f}{\partial x}(s, X(s), u(s))ds, \\
D_t K(t) &:= D_t g'(X(T)) + \int_t^T D_t \frac{\partial f}{\partial x}(s, X(s), u(s))ds, \\
D_{t,z} K(t) &:= D_{t,z} g'(X(T)) + \int_t^T D_{t,z} \frac{\partial f}{\partial x}(s, X(s), u(s))ds, \\
H_0(s,x,u) &:= K(s)b(s,x,u) + D_s K(s)\sigma(s,x,u) + \int_{\mathbb{R}_0} D_{s,z}K(s)\theta(s,x,u,z)\nu(dz), \\
G(t,s) &:= \exp\Big(\int_t^s \Big\{ \frac{\partial b}{\partial x}(r, X(r), u(r), \omega) - \tfrac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2 (r, X(r), u(r), \omega) \Big\} dr \\
&\quad + \int_t^s \frac{\partial \sigma}{\partial x}(r, X(r), u(r), \omega) dB(r) \\
&\quad + \int_t^s \int_{\mathbb{R}_0} \Big\{ \ln\Big(1 + \frac{\partial \theta}{\partial x}(r, X(r), u(r), z, \omega)\Big) - \frac{\partial \theta}{\partial x}(r, X(r), u(r), z, \omega) \Big\} \nu(dz)dr \\
&\quad + \int_t^s \int_{\mathbb{R}_0} \ln\Big(1 + \frac{\partial \theta}{\partial x}(r, X(r^-), u(r^-), z, \omega)\Big) \tilde{N}(dr, dz) \Big), \qquad (3.6) \\
p(t) &:= K(t) + \int_t^T \frac{\partial H_0}{\partial x}(s, X(s), u(s)) G(t,s) ds, \qquad (3.7) \\
q(t) &:= D_t p(t), \qquad (3.8) \\
r(t,z) &:= D_{t,z} p(t) \qquad (3.9)
\end{aligned}
$$

all exist for $0 \le t \le s \le T, \;\; z \in \mathbb{R}_0$.

Since $b(t,x,u) = b(t,u), \sigma(t,x,u) = \sigma(t,u)$ and $\theta(t,x,u,z) = \theta(t,u,z)$ do not depend on x this maximum principle gets a simpler form, which we now state, using the notation of (3.18)-(3.22):

Theorem 3.2 *[Stochastic maximum principle [9] (special case)] Suppose b, σ and θ do not depend on x. Put*

$$
K(t) = K^{(u)}(t) = \int_t^T \frac{\partial f}{\partial x}(s, X^{(u)}(s), u(s))ds + g'(X^{(u)}(T)) \tag{3.10}
$$

and define the Hamiltonian process $H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$
\begin{aligned}
H(t,x,u,\omega) &= f(t,x,u) + K(t)b(t,u) + D_t K(t) c(t,u) \\
&\quad + \int_{\mathbb{R}_0} D_{t,z} K(t) \theta(t,u,z) \nu(dz).
\end{aligned} \tag{3.11}
$$

Suppose $u = \hat{u} \in \mathcal{A}_{\mathcal{E}}$ is a critical point for

$$
J^{(G)}(u) := E[U(X^{(u)}(T) - G)], \tag{3.12}
$$

in the sense that

$$\frac{d}{dy}J(\hat{u}+y\beta)_{y=0}=0 \textit{ for all bounded } \beta\in\mathcal{A}_{\mathcal{E}}. \tag{3.13}$$

Then $\hat{u}$ is a conditional critical point for H, in the sense that

$$E\left[\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{u}(t)) \mid \mathcal{E}_t\right]=0 \textit{ for a.a.} t, \quad \omega \tag{3.14}$$

where $\hat{X}(t)=X^{(\hat{u})}(t)$, and H is evaluated at

$$K(t)=K^{(G)}(t)=\int_t^T \frac{\partial f}{\partial x}(s,\hat{X}(s),\hat{u}(s))ds+g'(\hat{X}(T)):=\hat{K}(t). \tag{3.15}$$

Conversely if (3.14) *holds then* (3.13) *holds.*

In our case we have, using (3.18)-(3.22),

$$\begin{aligned}K(t)=\int_t^T &\{u(s)S(s)(\mu(s)E[U''(X(s)-G)\mid\mathcal{F}_s]-\sigma(s)E[U'''(X(s)-G)D_sG\mid\mathcal{F}_s])\\ &+\frac{1}{2}u^2(s)S^2(s)\sigma^2(s)E[U'''(X(s)-G)\mid\mathcal{F}_s]\\ &+\int_{\mathbb{R}_0}E[(U'(X(s)+u(s)S(s)\gamma(s,z)-G-D_{s,z}G)-U'(X(s)-G-D_{s,z})\\ &-u(s)S(s)\gamma(s,z)U''(X(s)-G))\mid\mathcal{F}_s]\nu(dz)\}ds\end{aligned} \tag{3.16}$$

and, with $f(t,x,u)$ given by (3.22),

$$\begin{aligned}H(t,x,u)&=f(t,x,u)+K(t)uS(t)\mu(t)+D_tK(t)uS(t)\sigma(t)\\ &\quad+\int_{\mathbb{R}_0}D_{t,z}K(t)uS(t)\gamma(t,z)\nu(dz).\end{aligned} \tag{3.17}$$

Therefore, if $\hat{u}\in\mathcal{A}_{\mathcal{E}}$ is optimal then by Theorem 3.2:

$$\begin{aligned}0&=E\left[\frac{d}{du}H(t,\hat{X}(t),u)\mid\mathcal{E}_t\right]_{u=\hat{u}(t)}\\ &=\hat{u}(t)E[S^2(t)\sigma^2(t)U''(\hat{X}(t)-G)\mid\mathcal{E}_t]\\ &+E[\{S(t)\mu(t)(\hat{K}(t)+U'(\hat{X}(t)-G))+S(t)\sigma(t)(D_t\hat{K}-U''(\hat{X}(t)-G)D_tG)\\ &+S(t)\int_{\mathbb{R}_0}\gamma(t,z)[D_{t,z}\hat{K}(t)+U'(\hat{X}(t)+\hat{u}(t)S(t)\gamma(t,z)-G-D_{t,z}G)\\ &-U'(\hat{X}(t)-G)]\nu(dz)\}\mid\mathcal{E}_t]=0.\end{aligned} \tag{3.18}$$

We have proved

Theorem 3.3 *Suppose $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ is optimal for the stochastic control problem* (1.11)*. Then $\hat{u}(t)$ is a solution of equation* (3.18)*, with $\hat{K}(t) = K^{(\hat{u})}(t)$ given by* (3.16).

In particular, we get :

Theorem 3.4 *Suppose $\mathcal{E}_t = \mathcal{F}_t$ and $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is optimal for the problem* (1.11)*. Then $\hat{u}(t)$ is a solution of the equation*

$$\begin{aligned}
&\hat{u}(s)S(t)\sigma^2(t)E[U''(\hat{X}(t)-G) \mid \mathcal{F}_t] + \mu(t)E[\{\hat{K}(t) + U'(\hat{X}(t)-G)\} \mid \mathcal{F}_t] \\
&+ \sigma(t)E[\{D_t\hat{K}(t) - U''(\hat{X}(t)-G)D_tG\} \mid \mathcal{F}_t] \\
&+ \int_{\mathbb{R}_0} \gamma(t,z)E[\{D_{t,z}\hat{K}(t) + U'(\hat{X}(t) + \hat{u}(t)S(t)\gamma(t,z) - G - D_{t,z}G) \\
&\qquad - U'(\hat{X}(t)-G)\} \mid \mathcal{F}_t]\nu(dz) = 0,
\end{aligned} \tag{3.19}$$

with $\hat{K}(t) = K^{(\hat{u})}(t)$ given by (3.16).

To illustrate these results we look at some special cases :

Corollary 3.5 *Suppose $\nu = 0$ and $\mathcal{E}_t \subseteq \mathcal{F}_t$. If $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ is optimal, then*

$$\begin{aligned}
\hat{u}(t) = &\frac{E[S(t)\{\mu(t)U'(\hat{X}(t)-G) - \sigma(t)U''(\hat{X}(t)-G)D_tG\} \mid \mathcal{E}_t]}{-E[S^2(t)\sigma^2(t)U''(\hat{X}(t)-G) \mid \mathcal{E}_t]} \\
&+ \frac{E[S(t)\{\mu(t)\hat{K}(t) + \sigma(t)D_t\hat{K}(t)\} \mid \mathcal{E}_t]}{-E[S^2(t)\sigma^2(t)U''(\hat{X}(t)-G) \mid \mathcal{E}_t]}.
\end{aligned} \tag{3.20}$$

Corollary 3.6 *Suppose $\nu = 0$ and $\mathcal{E}_t = \mathcal{F}_t$. If $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is optimal, then*

$$\begin{aligned}
\hat{u}(t) = &\frac{\mu(t)E[U'(\hat{X}(t)-G) \mid \mathcal{F}_t] - \sigma(t)E[U''(\hat{X}(t)-G)D_tG \mid \mathcal{F}_t]}{-S(t)\sigma^2(t)E[U''(\hat{X}(t)-G) \mid \mathcal{F}_t]} \\
&+ \frac{\mu(t)E[\hat{K}(t) \mid \mathcal{F}_t] + \sigma(t)E[D_t\hat{K}(t) \mid \mathcal{F}_t]}{-S(t)\sigma^2(t)E[U''(\hat{X}(t)-G) \mid \mathcal{F}_t]}.
\end{aligned} \tag{3.21}$$

In both (3.20) and (3.21) we have

$$\begin{aligned}
\hat{K}(t) = &\int_t^T \{\hat{u}(s)S(s)(\mu(s)E[U''(\hat{X}(s)-G) \mid \mathcal{F}_s] - \sigma(s)E[U'''(\hat{X}(s)-G)D_sG \mid \mathcal{F}_s]) \\
&+ \frac{1}{2}\hat{u}^2(s)S^2(s)\sigma^2(s)E[U'''(X(s)-G) \mid \mathcal{F}_s]\}ds
\end{aligned} \tag{3.22}$$

(see (3.16)).

Corollary 3.7 *Suppose $\nu = G = 0$ and $\mathcal{E}_t = \mathcal{F}_t$. If $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is optimal and $\hat{X}(t) > 0$ for all $t \in [0,T]$, put*

$$\hat{\pi}(t) = \frac{\hat{u}(t)S(t)}{\hat{X}(t)} \; ; \; t \in [0,T]$$

i.e. $\hat{\pi}(t)$ *represents the fraction of the total wealth invested in the risky asset. Then* $\hat{\pi}(t)$ *solves the equation*

$$\hat{\pi}(t) = \frac{\mu(t)U'(\hat{X}(t))}{-\sigma^2(t)\hat{X}(t)U''(\hat{X}(t))} + \frac{\mu(t)E[\hat{K}(t) \mid \mathcal{F}_t] + \sigma(t)E[D_t\hat{K}(t) \mid \mathcal{F}_t]}{-\sigma^2(t)\hat{X}(t)U''(\hat{X}(t))} \tag{3.23}$$

where

$$\hat{K}(t) = \int_t^T \{\mu(s)\hat{\pi}(s)\hat{X}(s)U''(\hat{X}(s)) + \frac{1}{2}\sigma^2(s)\hat{\pi}^2(s)\hat{X}^2(s)U'''(\hat{X}(s))\}ds. \tag{3.24}$$

Corollary 3.8 *Suppose* $\nu = G = 0$ *and* $\mathcal{E}_t = \mathcal{F}_t$ *and that*

$$U(x) = \frac{1}{\lambda}x^\lambda \text{ for some } \lambda \in (-\infty, 1)\backslash\{0\}.$$

Then if $\hat{\pi} \in \mathcal{A}_{\mathcal{F}}$ *is optimal, we have*

$$\hat{\pi}(t) = \frac{\mu(t)}{(1-\lambda)\sigma^2(t)} + \frac{\mu(t)E[\hat{K}(t) \mid \mathcal{F}_t] + \sigma(t)E[D_t\hat{K}(t) \mid \mathcal{F}_t]}{(1-\lambda)\sigma^2(t)} \tag{3.25}$$

where

$$\hat{K}(t) = (\lambda - 1)\int_t^T \{\hat{\pi}(s)\hat{X}(s)^{\lambda-1}(\mu(s) + \frac{1}{2}(\lambda-2)\sigma^2(s)\hat{\pi}(s))\}ds. \tag{3.26}$$

In particular, if the coefficients $\mu(t)$ *and* $\sigma(t)$ *are deterministic, then the last term on the right hand side of* (3.25) *vanishes, and the formula for* $\hat{\pi}(t)$ *reduces to the classical Merton formula*

$$\hat{\pi}(t) = \frac{\mu(t)}{(1-\lambda)\sigma^2(t)}. \tag{3.27}$$

Thus (3.25) *gives a specification of the additional term needed in the case when the coefficients* $\mu(t)$ *and* $\sigma(t)$ *are random.*

4 The Exponential Utility Case

Although one of the motivations for this paper is to be able to handle a wide class of utility functions, it is nevertheless of interest to apply our general result to the widely studied exponential utility, i.e.

$$U(x) = -e^{-\alpha x} \ ; \ x \in \mathbb{R} \tag{4.1}$$

where $\alpha > 0$ is a constant.

4.1 The partial information case

We first consider the partial information case

$$\mathcal{E}_t \subseteq \mathcal{F}_t \text{ for all } t \in [0, T]. \tag{4.2}$$

For convenience we put

$$w(t) := u(t)S(t) \tag{4.3}$$

(the *amount* invested in the stock at time t). Then we get by (3.13)

$$\begin{aligned}
K(t) = \int_t^T & \{w(s)(\alpha^2\mu(s)E[U(X(s) - G) \mid \mathcal{F}_s] + \alpha^3\sigma(s)E[U(X(s) - G)D_sG \mid \mathcal{F}_s]) \\
& - \frac{1}{2}\alpha^3 w^2(s)\sigma^2(s)E[U(X(s) - G) \mid \mathcal{F}_s] \\
- \alpha \int_{\mathbb{R}_0} & E[(U(X(s) + w(s)\gamma(s,z) - G - D_{s,z}G) - U(X(s) - G - D_{s,z}G) \\
& + \alpha w(s)\gamma(s,z)U(X(s) - G)) \mid \mathcal{F}_s]\nu(dz)\}ds \\
= -\alpha \int_t^T & \exp(-\alpha X(s))\{\alpha\mu(s)w(s)E[\exp(\alpha G) \mid \mathcal{F}_s] \\
& + \alpha^2\sigma(s)w(s)E[\exp(\alpha G)D_sG \mid \mathcal{F}_s] \\
- \int_{\mathbb{R}_0} & (\exp(-\alpha w(s)\gamma(s,z))E[\exp(\alpha G + \alpha D_{s,z}G) \mid \mathcal{F}_s] \\
- E[\exp(&\alpha G + \alpha D_{s,z}G) \mid \mathcal{F}_s] + \alpha w(s)\gamma(s,z)E[\exp(\alpha G) \mid \mathcal{F}_s])\nu(dz)\}ds.
\end{aligned} \tag{4.4}$$

Equation (3.18) becomes:

$$\begin{aligned}
& -\alpha^2\hat{u}(t)E[S^2(t)\sigma^2(t)\exp(-\alpha\hat{X}(t) + \alpha G) \mid \mathcal{E}_t] \\
& + E[\{S(t)\mu(t)[\hat{K}(t) + \alpha\exp(-\alpha\hat{X}(t) + \alpha G)] \\
& + S(t)\sigma(t)[D_t\hat{K}(t) - \alpha^2\exp(-\alpha\hat{X}(t) + \alpha G)D_tG] \\
& + S(t)\int_{\mathbb{R}_0}\gamma(t,z)[D_{t,z}\hat{K}(t) - \alpha\exp(-\alpha\hat{X}(t) - \alpha\hat{u}(t)S(t)\gamma(t,z) + \alpha G + \alpha D_{t,z}G) \\
& + \alpha\exp(-\alpha\hat{X}(t) + \alpha G)]\nu(dz)\} \mid \mathcal{E}_t] = 0
\end{aligned} \tag{4.5}$$

If we write

$$X(t) = y + X_0(t), \tag{4.6}$$

where

$$X_0(t) = \int_0^t u(s)dS(s) = \int_0^t w(s)[\mu(s)ds + \sigma(s)dB(s) + \int_{\mathbb{R}_0}\gamma(s,z)\tilde{N}(ds,dz)] \tag{4.7}$$

we see from (4.4) that $K(t)$ has the form

$$K(t) = \exp(-\alpha y)K_0(t)$$

where $K_0(t)$ does not depend on y. Similarly we can factor out $\exp(-\alpha y)$ from the equation (4.5). This proves the following result:

Proposition 4.1 *Let $\mathcal{E}_t \subseteq \mathcal{F}_t$. Suppose there exists an optimal portfolio $\hat{u}(t)$ for Problem* (1.11)*, with $U(x) = -e^{-\alpha x}$. Then $\hat{u}(t)$ does not depend on the initial wealth $y = x + p$. Therefore*

$$V_G(x+p) = -e^{-\alpha(x+p)} V_G(0). \tag{4.8}$$

Similarly

$$V_0(x) = -e^{-\alpha x} V_0(0), \tag{4.9}$$

and hence the utility indifference price p is given by

$$p = \frac{1}{\alpha} \log \frac{V_0(0)}{V_G(0)}. \tag{4.10}$$

Remark 4.2 *This result was proved in [17] under more restrictive conditions: Markovian system, Markovian payoff G and conditions necessary for the application of a Girsanov transformation. Moreover, in [17] only the full information case is considered. Proposition 4.1 holds in the general partial information case $\mathcal{E}_t \subseteq \mathcal{F}_t$.*

4.2 Asymptotic behaviour of the optimal portfolio for vanishing α.

Suppose an optimal portfolio $u_\alpha(t) = u_\alpha^{(G)}(t)$ exists for the problem

$$\sup_{u \in \mathcal{A}_\mathcal{E}} E[-\exp(-\alpha(\int_0^T u(t)dS(t) - G))]$$

Let $u_\alpha^{(0)}(t)$ be the corresponding optimal portfolio when $G = 0$ and $\psi_\alpha := u_\alpha^{(G)}(t) - u_\alpha^{(0)}(t)$ the difference. In the full information case ($\mathcal{E}_t = \mathcal{F}_t$), it has been proved, see e.g. [8], [17] and the references therein, that $\psi_\alpha(t)$ is itself an optimal portfolio for the problem

$$\sup_{\psi} E^*[-\exp(-\alpha(\int_0^T \psi(t)dS(t) - G))]$$

where E^* denotes the expectation with respect to the *minimal entropy martingale measure.* Moreover $\lim_{\alpha\to 0} \psi_\alpha(t)$ exists in some sense. It is also of interest to study the limiting behaviour of $u_\alpha^{(G)}$. We show below that, under some conditions,

$$\lim_{\alpha \to 0} \alpha u_\alpha^{(G)}(t) = u_1^{(0)}(t) \text{ a.s. } t \in [0,T],$$

where $u_1^{(0)}$ is the optimal portfolio for $\alpha = 1$ and $G = 0$. It follows that

$$|u_\alpha^{(G)}(t)| \to \infty \text{ as } \alpha \to 0.$$

This shows that $u_\alpha^{(G)}(t)$ and $u_\alpha^{(0)}(t)$ have the same singularity at $\alpha = 0$, which is cancelled by substraction. This result holds in the general non-Markovian, partial information setting. We now explain this in more detail. We use our results from the previous section to study the behaviour of the optimal portfolio $u_\alpha(t)$ corresponding to $U(x) = -\mathrm{e}^{-\alpha x}$ when $\alpha \to 0$. If we divide (4.5) by α we get

$$\begin{aligned}
&- \alpha u_\alpha(t) E[S^2(t)\sigma^2(t)\exp(-\alpha X_\alpha(t) + \alpha G) \mid \mathcal{E}_t] \\
&+ E[\{S(t)\mu(t)[\frac{K_\alpha(t)}{\alpha} + \exp(-\alpha X_\alpha(t) + \alpha G)] \\
&+ S(t)\sigma(t)[\frac{D_t K_\alpha(t)}{\alpha} - \alpha \exp(-\alpha X_\alpha(t) + \alpha G) D_t G] \\
&+ S(t)\int_{\mathbb{R}_0} \gamma(t,z)[\frac{D_{t,z}K_\alpha(t)}{\alpha} - \exp(-\alpha X_\alpha(t) - \alpha u_\alpha(t) S(t)\gamma(t,z) + \alpha G + \alpha D_{t,z} G) \\
&+ \exp(-\alpha X_\alpha(t) + \alpha G)]\nu(dz)\} \mid \mathcal{E}_t] = 0,
\end{aligned} \tag{4.11}$$

where $K_\alpha(t), X_\alpha(t)$ are given by (4.4) and (4.6)-(4.7) with $u = u_\alpha$, i.e.

$$\begin{aligned}
\frac{K_\alpha(t)}{\alpha} = &\int_t^T \exp(-\alpha X_\alpha(s))\{\alpha u_\alpha(s)S(s)\mu(s)E[\mathrm{e}^{\alpha G} \mid \mathcal{F}_s] \\
&+ \alpha^2 \sigma(s) u_\alpha(s) S(s) E[\exp(\alpha G) D_s G \mid \mathcal{F}_s] \\
&- \int_{\mathbb{R}_0} (\exp(-\alpha u_\alpha(s) S(s)\gamma(s,z)) E[\exp(\alpha G + \alpha D_{s,z} G) \mid \mathcal{F}_s] \\
&- E[\exp(\alpha G + \alpha D_{s,z}G) \mid \mathcal{F}_s] + \alpha u_\alpha(s) S(s)\gamma(s,z) E[\exp(\alpha G) \mid \mathcal{F}_s])\nu(dz)\} ds
\end{aligned} \tag{4.12}$$

and

$$\alpha X_\alpha(t) = \alpha x + \int_0^t \alpha u_\alpha(s) S(s)[\mu(s)ds + \sigma(s)dB(s) + \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz)] \tag{4.13}$$

From this we deduce the following:

Lemma 4.3 *Suppose an optimal portfolio $u_\alpha(t) = u_\alpha^{(G)}(t)$ exists for all $\alpha > 0$, and that*

$$\tilde{u}(t) := \lim_{\alpha \to 0} \alpha\, u_\alpha(t) \tag{4.14}$$

exists in $L^2(d\lambda \times dP)$, where λ denotes the Lebesgue measure on $[0,T]$. Then $\tilde{u}(t)$ is a solution of the equation

$$\begin{aligned}
&- \tilde{u}(t) E[S^2(t)\sigma^2(t) e^{-\tilde{X}(t)} \mid \mathcal{E}_t] \\
&+ E[\{S(t)\mu(t)(\tilde{K}(t) + e^{-\tilde{X}(t)}) + S(t)\sigma(t) D_t \tilde{K}(t) \\
&+ S(t)\int_{\mathbb{R}_0} \gamma(t,z)[D_{t,z}\tilde{K}(t) + e^{-\tilde{X}(t)}(1 - e^{-\tilde{u}(t)S(t)})]\nu(dz)\} \mid \mathcal{E}_t] = 0,
\end{aligned} \tag{4.15}$$

where

$$\tilde{K}(t)=\int_t^T e^{-\tilde{X}(s)}\{\mu(s)\tilde{u}(s)S(s)-\int_{\mathbb{R}_0}(e^{-\tilde{u}(s)S(s)\gamma(s,z)}-1+\tilde{u}(s)S(s)\gamma(s,z))\nu(dz)\}ds \quad (4.16)$$

and

$$\tilde{X}(t)=\int_0^t \tilde{u}(s)S(s)[\mu(s)ds+\sigma(s)dB(s)+\int_{\mathbb{R}_0}\gamma(s,z))\tilde{N}(ds,dz)] \quad (4.17)$$

Let us now compare with the optimal portfolio $u_1^0(t)$ corresponding to $\alpha=1$ and $X(0)=G=0$. By (4.5) $u_1^0(t)$ is a solution of the equation

$$\begin{aligned}&-u_1^0(t)E[S^2(t)\sigma^2(t)\mathrm{e}^{-\hat{X}(t)}\mid\mathcal{E}_t]\\&+E[\{S(t)\mu(t)(\tilde{K}(t)+\mathrm{e}^{-\tilde{X}(t)})+S(t)\sigma(t)D_t\hat{K}(t)\\&\quad+S(t)\int_{\mathbb{R}_0}\gamma(t,z)[D_{t,z}\tilde{K}(t)+\mathrm{e}^{-\hat{X}(t)}(1-\mathrm{e}^{-u_1^{(0)}S(t)\gamma(t,z)})\nu(dz)\}\mid\mathcal{E}_t]=0,\end{aligned} \quad (4.18)$$

where

$$\hat{K}(t)=\int_t^T \mathrm{e}^{-\tilde{X}(s)}\{\mu(s)u_1^{(0)}(s)S(s)-\int_{\mathbb{R}_0}(\mathrm{e}^{-u_1^{(0)}(s)S(s)\gamma(s,z)}-1+u_1^{(0)}(s)S(s)\gamma(s,z))\nu(dz)\}ds \quad (4.19)$$

and

$$\hat{X}(t)=\int_0^t u_1^{(0)}(s)[\mu(s)ds+\sigma(s)dB(s)+\int_{\mathbb{R}_0}\gamma(s,z)\tilde{N}(ds,dz)]. \quad (4.20)$$

We see that the two systems of equations (4.15)-(4.17) in the unknown $\tilde{u}(t)$ and (4.18)-(4.20) in the unknown $u_1^{(0)}(t)$ are identical. Therefore we get

Theorem 4.4 *[***The limit of** $\alpha u_\alpha(t)$ **when** $\alpha\to 0$.*] Suppose an optimal portfolio $u_\alpha(t)=u_\alpha^{(G)}(t)$ exists for all $\alpha>0$ and that*

$$\tilde{u}(t)=\lim_{\alpha\to 0}\alpha u_\alpha(t) \quad (4.21)$$

exists in $L^2(d\lambda\times dP)$. Moreover, suppose that the system (4.15)-(4.17) has a unique solution $\tilde{u}(\cdot)$. Then $\tilde{u}(t)$ coincides with the optimal portfolio $u_1^{(0)}(t)$ corresponding to $\alpha=1$ and $G=0$.

Alternatively we get

Theorem 4.5 *Suppose (4.21) holds. Then $u=\tilde{u}(\cdot)$ is a critical point for the performance functional*

$$J^{(0)}(u):=E[-\exp(-X_0^{(u)}(T))];\ u\in\mathcal{A}_{\mathcal{E}},\ X_0^{(u)}(0)=0. \quad (4.22)$$

4.3 The complete information case ($\mathcal{E}_t = \mathcal{F}_t$)

Finally, let us look at the situation when we have complete information ($\mathcal{E}_t = \mathcal{F}_t$ for all t) and exponential utility: $U(x) = -e^{-\alpha x}$; $\alpha > 0$ constant. As before let us put

$$w(t) = u(t)S(t).$$

Define

$$L(t) = K(0) - K(t).$$

Then by (4.4)

$$\begin{aligned} L(t) = &\int_0^t e^{-\alpha X(s)}\{-\alpha\mu(s)w(s)E[e^{\alpha G} \mid \mathcal{F}_s] \\ &+ \alpha^2\sigma(s)w(s)E[e^{\alpha G}D_sG) \mid \mathcal{F}_s] \\ &- \int_{\mathbb{R}_0} ((\exp(-\alpha w(s)\gamma(s,z)) - 1)E[e^{\alpha(G+D_{s,z}G)} \mid \mathcal{F}_s] \\ &+ \alpha w(s)\gamma(s,z)E[e^{\alpha G} \mid \mathcal{F}_s])\nu(dz)\}ds. \end{aligned} \tag{4.23}$$

Since $\mathcal{E}_t = \mathcal{F}_t$ equation (4.5) simplifies to

$$\begin{aligned} &-\alpha^2 w(t)\sigma^2(t)e^{-\alpha X(t)}E[e^{\alpha G} \mid \mathcal{F}_t] + \mu(t)\{E[K(t) \mid \mathcal{F}_t] + \alpha e^{-\alpha X(t)}E[e^{\alpha G} \mid \mathcal{F}_t]\} \\ &+ \sigma(t)\{E[D_tK(t) \mid \mathcal{F}_t] - \alpha^2 e^{-\alpha X(t)}E[e^{\alpha G}D_tG \mid \mathcal{F}_t]\} \\ &+ \int_{\mathbb{R}_0} \gamma(t,z)\{E[D_{t,z}K(t) \mid \mathcal{F}_t] - \alpha e^{-\alpha X(t)}e^{-\alpha w(t)S(t)}E[e^{\alpha(G+D_{t,z}G)} \mid \mathcal{F}_t] + \alpha e^{-\alpha X(t)}E[e^{\alpha G} \mid \mathcal{F}_t]\}\nu(dz) \\ &= 0. \end{aligned} \tag{4.24}$$

Now assume that

$$\gamma(t,z) = 0 \text{ and } \sigma(t) \neq 0. \tag{4.25}$$

Then (4.24) can be written

$$E[D_tK(t) \mid \mathcal{F}_t] = -a(t)E[K(t) \mid \mathcal{F}_t] + b(t)w(t) + c(t), \tag{4.26}$$

where

$$a(t) = \frac{\mu(t)}{\sigma(t)} \tag{4.27}$$

$$b(t) = \alpha^2\sigma(t)e^{-\alpha X(t)}E[e^{\alpha G} \mid \mathcal{F}_t] \tag{4.28}$$

and

$$c(t) = e^{-\alpha X(t)}(\alpha^2 E[e^{\alpha G}D_tG \mid \mathcal{F}_t] - \alpha\frac{\mu(t)}{\sigma(t)}E[e^{\alpha G} \mid \mathcal{F}_t]). \tag{4.29}$$

Then by the Clark-Ocone theorem

$$
\begin{aligned}
L(T) &= E[L(T)] + \int_0^T E[D_s L(T) \mid \mathcal{F}_s] dB(s) \\
&= E[L(T)] + \int_0^T E[D_s K(0) \mid \mathcal{F}_s] dB(s) \\
&= E[L(T)] + \int_0^T E[D_s K(s) \mid \mathcal{F}_s] dB(s).
\end{aligned} \tag{4.30}
$$

It follows that if we define the martingale

$$
M(t) = E[L(T) \mid \mathcal{F}_t] = L(t) + E[K(t) \mid \mathcal{F}_t],
$$

then

$$
\begin{aligned}
M(t) &= E[L(T)] + \int_0^t E[D_s K(s) \mid \mathcal{F}_s] dB(s) \\
&= E[L(T)] + \int_0^t \{-a(s)E[K(s) \mid \mathcal{F}_s] + b(s)w(s) + c(s)\} dB(s) \\
&= E[L(T)] + \int_0^t \{-a(s)(E[L(T) \mid \mathcal{F}_s] - L(s)) + b(s)w(s) + c(s)\} dB(s).
\end{aligned}
$$

Hence $M(t)$ satisfies the equation

$$
dM(t) = -a(t)M(t)dB(t) + f_w(t)dB(t) \tag{4.31}
$$

where

$$
f_w(t) = a(t)L(t) + b(t)w(t) + c(t). \tag{4.32}
$$

Define

$$
J(t) = \exp(\int_0^t a(s)dB(s) + \frac{1}{2}\int_0^t a^2(s)ds) \ ; \ t \geq 0. \tag{4.33}
$$

Then

$$
dJ(t) = a(t)J(t)dB(t) + J(t)a^2(t)dt
$$

and hence, by (4.31)

$$
\begin{aligned}
d(J(t)M(t)) &= J(t)dM(t) + M(t)dJ(t) + dJ(t)dM(t) \\
&= J(t)dM(t) + M(t)J(t)[a(t)dB(t) + a^2(t)dt] \\
&\quad + J(t)[a(t)dB(t) + a^2(t)dt][-a(t)M(t)dB(t) + f_w(t)dB(t)] \\
&= J(t)dM(t) + J(t)a(t)M(t)dB(t) + J(t)a(t)f_w(t)dt
\end{aligned} \tag{4.34}
$$

Therefore, if we multiply (4.31) by $J(t)$ and use (4.34) we get

$$
d(J(t)M(t)) = J(t)f_w(t)\{dB(t) + a(t)dt\}.
$$

Integrating this we arrive at

$$M(t) = J^{-1}(t)[M(0) + \int_0^t J(s)f_w(s)\{dB(s) + a(s)ds\}] \tag{4.35}$$

where, by (4.26)

$$M(0) = E[L(T)] = E[K(0)] = E[K(0) \mid \mathcal{F}_0] = \frac{b(0)w(0) + c(0)}{a(0)} \tag{4.36}$$

Hence

$$\begin{aligned} E[K(t) \mid \mathcal{F}_t] &= M(t) - L(t) \\ &= J^{-1}(t)[E[L(T)] + \int_0^t J(s)f_w(s)\{dB(s) + a(s)ds\}] - L(t). \end{aligned} \tag{4.37}$$

This determines $E[K(t) \mid \mathcal{F}_t]$ as a function of the previous values

$$w(s)\ ;\ s \leq t$$

of our control process w.

Hence

$$D_t E[K(t) \mid \mathcal{F}_t] = E[D_t K(t) \mid \mathcal{F}_t]$$

is determined by $w(s)\ ;\ s \leq t$ also. Going back to equation (4.24), we see that we have now obtained a recursive equation for $w(t)$ in terms of previous values.Hence we have proved the following, which is one of the main results of this paper:

Theorem 4.6 *[***Optimal portfolio***]*

Suppose $\mathcal{E}_t = \mathcal{F}_t$, $\gamma(t,z) = 0$ and $\sigma(t) \neq 0$ for all $t \in [0,T]$. Suppose $\hat{u}(t) = \frac{\hat{w}(t)}{S(t)}$ is an optimal portfolio for the problem

$$\sup_{u \in \mathcal{A}} E[-\exp(-\alpha(X_u(T) - G))],$$

where

$$dX_u(t) = u(t)S(t)[\mu(t)dt + \sigma(t)dB(t)]\ ;\ X_u(0) = x.$$

Suppose $G \in \mathbb{D}_{1,2}^B$ is $\mathcal{F}_T$-measurable, $e^{\alpha G} \in L^2(P)$. Then $\hat{w}(t)$ is given recursively by

$$\begin{aligned} &\alpha^2 \hat{w}(t)\sigma^2(t)e^{-\alpha\hat{X}(t)}E[e^{\alpha G} \mid \mathcal{F}_t] \\ &= \mu(t)\{E[\hat{K}(t) \mid \mathcal{F}_t] + \alpha e^{-\alpha\hat{X}(t)}E[e^{\alpha G} \mid \mathcal{F}_t]\} \\ &+ \sigma(t)\{E[D_t\hat{K}(t) \mid \mathcal{F}_t] - \alpha^2 e^{-\alpha\hat{X}(t)}E[e^{\alpha G}D_tG \mid \mathcal{F}_t]\}, \end{aligned} \tag{4.38}$$

where $E[\hat{K}(t) \mid \mathcal{F}_t]$ is given by (4.36)-(4.37), together with (4.27)-(4.29) and (4.33), with $w = \hat{w}$, and

$$E[D_t\hat{K}(t) \mid \mathcal{F}_t] = D_t E[\hat{K}(t) \mid \mathcal{F}_t].$$

Remark 4.7 *Note that we do not require that the terminal payoff G or the market coefficients $\mu(t)$, $\sigma(t)$ are of Markovian type.*

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