Categories of domains with totality

Dag Normann

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Abstract

We investigate domains with totality where density in general does not hold.We define three categories of domains X with totality $\bar{X}$ satisfying certain structural properties. We then define the category of evaluation structures. These will induce domains with totality. We show that the category of evaluation structures is closed under dependent sums and products, under a universe constructor and under direct limits. This is applied to domains with totality defined by induction.

We investigate the topological properties of domains with totality induced from evaluation structures.

1 Introduction

1.1 Domain Theory

Domain theory is in essence the theory about the relation between finitary information bits and the completions. The finitary bits, which we will call *compact*, will be as concrete objects as they come in mathematics, hereditarily finite objects. The completions will consist of ideal objects; actually ideals in a suitably preordered set of compacts. In a way we can say that an ideal is the direct limit of a partially ordered set of finite information bits closed under the union of information.

It is the traditional domain theory originating from the work of Scott that has influenced our applications. Scott introduced a model for untyped

λ-calculus using a domain construction. This is impossile using true functions in the wellfounded set hierarchy. The key to Scott's success is that wellfoundedness is not preserved under direct limit constructions, while the algebraic properties of function application and abstraction are. Thus self-application is possible in the limit because one part of a function may operate on another part of the same function.

We will assume familiarity with domain theory as developed in e.g. Stoltenberg-Hansen & al. [24].

1.2 Totality

Since domains can be used to give a semantics for untyped λ-calculus it can in particular be used to give a semantics for typed λ-calculus, and it was reasonable to assume that domain theory can be used to give semantics for transfinite type theories known as *Intuitionistic Type Theory* or *Martin-Löf Type Theory*, see Martin-Löf [12]. The author first took an interest in such theories through his contact with Anne Salvesen [22, 23], at the time believing that the methodology of the countable functionals could be put to use. It was clear from the beginning that the objects of real interest are total in some sense.

An important constructor in type theory is the dependent sum $\Sigma(A, B)$ where the interpretation of B depends on the choice of an $a \in A$. The objects are pairs (a, b) with $a \in A$ and $b \in B(a)$.

It turned out to be a nontrivial task to describe the correct notion of 'dependent family' in order to prove desirable mathematical properties for dependent sums and products. It was clear that a conceptual analysis of 'totality' was needed in order to at least motivate the seemingly ad hoc properties one had to asssume to make the inductive proofs work.

Lill Kristiansen [7] considered an alternative situation. Girard [5] had used qualitative domains in order to give a semantics for system F introduced by himself [4] fifteen years earlier. In the Girard-implementation each closed second order type is interpreted as a qualitative domain X with a set $\bar{X}$ of total objects. Each term of the type will be interpreted as a total object. Kristiansen studied qualitative domains with totality from a conceptual point of view. Some of her results will be mentioned later in the text. Most of her results are published in one of the papers [8, 9, 10].

The author [13] gave the first attempt to analyse totality conceptually.

The essence in the analysis is that an object x is total if for some pregiven set Q of questions, $x(q)$ provides an atomic, finitary answer to each question in Q. This is then transfered to a technical definition of totality.

1.3   Domains with totality

Independently of this process. Palmgren and Stoltenberg-Hansen [21] constructed a model for partial type theory. They used the category of domains to give a sound definition of dependent families and they showed how each provable judgement of the theory can be interpreted as a correct statement about the implementation.

Berger [1, 2] suggested that domain theory could be used to give a semantics for total type theory. He (independently of the author of this paper) argued that in the interpretation we must distinguish between the first and second class citizens, the first class citizens are the total ones and are to be considered as the true objects. He also felt the need to give an abstract definition of totality. To Berger, a subset of a domain is a set of total objects if there is a set of Boolean valued tests that terminate for each object in the set and that can separate every finite separable set of compacts in the domain. Berger proves a general density theorem and a generalisation of the Kreisel-Lacombe-Shoenfield theorem in the setting of domains.

Inspired from the work of Palmgren, Stoltenberg-Hansen and Berger, the author realised that domains form a better tool than Kleene-associates in the attempts to construct transfinite systems of total objects that can be approximated by finite parts.

1.4   Density

If X is a domain and $\bar{X} \subseteq X$, then $\bar{X}$ is dense in X if each compact x_0 in X can be extended to an element in $\bar{X}$. Much of the work on domains with totality done so far is concerned with density or applications of density. Berger [3] is the most uppdated and general exposition of the theory of domains with dense totality. Various applications of density can be found in Normann [15, 17, 18, 19].
The general construction of universes developed in this paper is originally due to Berger [3] within the setting of domains with uniformly dense totality.

1.5 Non-density

Since there are vacant types (non-correct types) in type theory, a semantics for total type theory cannot satisfy density. Normann [16] gave the first ad hoc construction of a well behaved hierarchy of domains with totality where density does not hold in general. Waagbø [26] fulfilled one of the original aims of the investigation of domains with totality. He constructed an interpretation of one particular intuitionistic type theory, interpreting each type as an equivalence class of domains with totality and each object of that type as an equivalence class of total objects. He also gave a characterisation of his model as an inductively defined hierarchy of limit spaces.
Later Normann, Palmgren and Stoltenberg-Hansen [20] constructed an isomorphic model using Fréchet-products of finite type structures.

1.6 This paper

In this paper we will investigate domains with totality where density is not an essential property. In the first part we will isolate certain structural properties that the total elements may have and show that these properties are preserved through the constructions of dependent sums, dependent products and universes.

In the second part we take the view from Normann [13] that an object only can be called 'total' if there is some given set of calculations or evaluations on which the object terminates. Thus we restrict ourselves to objects in function spaces $E \to A_\perp$ where E is a domain and $A_\perp$ is a countable flat domain. At domain level the structures we will investigate will be retracts of such function spaces.

One of the main challenges has been to find out in what respect the elements of a dependent sum are total from a conceptual point of view. We show at domain level that the dependent sum is isomorphic to a retract of a function space.

Introducing totality it will again be the dependent sum that will cause the difficulties. It is not sufficient to isolate a set $\bar{E} \subseteq E$ of total elements. We will have to determine totality via a possibly transfinite deterministic process:

We first test partial totality for a part of x. Dependent on the outcome of this test we test partial totality for other parts

and so on. If x terminates on every test in this process,
then x will be total.

This is formalised in the notion of *evaluation structure*. We show that all the structural properties investigated in the first part are satisfied by domains with totality induced from evaluation structures. We also show that basic type constructors like dependent sums and products and the universe constructor can be seen as constructions of evaluation structures.

In section 5 we view type streams as certain well behaved direct limits of evaluation structures based on dependent sums and products. A consequence of our results is that domains with totality defined via a strictly positive inductive definition can be realised in a natural way as an evaluation structure. This covers case like

$$W(A, B) = \Sigma(A, (B \to W(A, B))),$$

the W-operator in intuitionistic type theory.

In section 6 we restrict ourselves to a subcategory and investigate general positive induction. Again the total elements of the limit can be seen as the total elements in some retract of a function space. We cannot combine dependent sums and general positive induction.

In the appendix we produce certain counterexamples showing that some of our results are the best possible, and we show that the restriction to retracts of function spaces is a real restriction.

1.7 Acknowledgements

2 Categories of domains with explicit totality

In this section we will consider some categories of domains with totality. We will introduce a base category K of domains with some sets of total objects. Over this category we will define notions like Σ-*constructions*, Π-*constructions*, *Parameterizations* and *Universes*. All these basic definitions are taken from Stoltenberg-Hansen and Palmgren [21] and from Berger [3], and we will use results from those papers.

We will then restrict ourselves to smaller categories where the total objects have some reasonable structural properties, and we will show that these categories are closed under the operators considered.

2.1 Basic Definitions

2.1.1 Prerequisites

We will let a *domain* be what is known as an algebraic domain, or a Scott-Ershov domain as defined in e.g. Stoltenberg-Hansen, Lindström and Griffor [24]. We assume that the reader is familiar with domain theory and will use standard concepts from that theory without further explanation.

If X is a domain, we will use $\sqsubseteq$ for the ordering, $\sqcup$ for the least upper bound and $\sqcap$ for the greatest lower bound, normally without an index X.

We let DOM be the category of domains, where the morphisms from X to Y are *projection pairs* $f = (f^+, f^-)$ such that $f^+ : X \to Y$, $f^- : Y \to X$ are continuous with

$$f^- \circ f^+ = id_X$$

and

$$f^+ \circ f^- \sqsubseteq id_Y$$

We assume familiarity with this category, see e.g. [24].

2.1.2 Parameterisations

Each domain can be considered to be a trivial category with one morphism $i_{x,y}$ from x to y exactly when $x \sqsubseteq y$.

Definition 2.1 A *parameterisation* in DOM will be a pair (X, F) where X is a domain and F is a continuous functor from X to DOM (F commutes with direct limits).

We will use the following conventions:
If (X, F) is a parameterisation and $x \sqsubseteq y$ are two elements in X, then $F(i_{x,y})$ is a projection pair $(f^+_{x,y}, f^-_{x,y})$. Following Palmgren and Stoltenberg-Hansen we will use the notation

$\quad z^y$ for $f^+_{x,y}(z)$ when $z \in F(x)$.

$\quad z_x$ for $f^-_{x,y}(z)$ when $z \in F(y)$.

Following Berger [3] we organize the class of parameterisations to a category PAR as follows:

Definition 2.2 Let (X, F) and (Y, G) be two parameterisations.
A *morphism* from (X, F) to (Y, G) is a pair (f, π) where f is a morphism from X to Y and π is a natural transformation from (X, F) to $(X, G \circ f^+)$

Definition 2.3 a) Let (X, F) be a parameterisation. We let $\Sigma(X, F)$ be the domain
$$\{(x, y) \mid x \in X \wedge y \in F(x)\}$$
with
$$(x, y) \sqsubseteq (x', y') \Leftrightarrow x \sqsubseteq x' \wedge y^{x'} \sqsubseteq y'$$

b) Let (X, F) be a parameterisation. We let $\Pi(X, F)$ be the domain of continuous functions f defined on X with $f(x) \in F(x)$ for all $x \in X$. f is *continuous* if f

is monotone: $(f(x))^y \sqsubseteq f(y)$ when $x \sqsubseteq y$

respects direct limits: If x is the least upper bound of a directed set $\{x_i\}_{i \in I}$, then $f(x)$ is the least upper bound of the directed set $\{(f(x_i))^x\}_{i \in I}$ in $F(x)$.

c) If (X, F) is a parameterisation, then *a parameterisation G over (X, F)* will be a parameterisation over $\Sigma(X, F)$.

d)   If G is a parameterisation over (X, F), we let $\Sigma(X, F, G)$ be the parameterisation

$$(X, \lambda x \in X.\Sigma(F(x), \lambda y \in F(x).G(x, y))).$$

e)   If G is a parameterisation over (X, F), we let $\Pi(X, F, G)$ be the parameterisation

$$(X, \lambda x \in X.\Pi(F(x), \lambda y \in F(x).G(x, y))).$$

Remark 2.1 These definitions are taken from [21] and from [3]. In [21] it is shown that the Σ and Π constructions lead to new domains. Berger [3] shows that the extensions of the Σ- and Π-operators to operators on parameterisations over parameterisations lead to parameterisations. We give a brief description here.

Definition 2.4 Let $(f, \pi) : (X, F) \to (X_1, F_1)$.

a)   We define $g = \Sigma(f, \pi)$ as follows:
$g^+(x, y) = (f^+(x), \pi(x)^+(y))$
$g^-(x_1, y_1) = (f^-(x_1), \pi(f^-(x_1))^-(F_1(i_{f^+(f^-(x_i)), x_1}))^-(y_1)).$

b)   We define $h = \Pi(f, \pi)$ as follows:
$h^+(z) = \lambda y \in X_1.\pi(f^-(y))^+(z(f^-(y)))$
$h^-(u) = \lambda x \in X.\pi(x)^-(u(f^+(x))).$

We may now extend the notion of morphisms from parameterisations to parameterisations over parameterisations.

Definition 2.5 a) If (X, F, G) and (X_1, F_1, G_1) are two parameterisations over parameterisations, we let a *morphism* from (X, F, G) to (X_1, F_1, G_1) be at tripple (f, π, ν) such that (f, π) is a morphism from (X, F) to (X_1, F_1) and $(\Sigma(f, \pi), \nu)$ is a morphism from $(\Sigma(X, F), G)$ to $(\Sigma(X_1, F_1), G_1)$.

b)   If (f, π, ν) is as in a), we let

$$\Sigma(f, \pi, \nu) = (f, \lambda x.\Sigma(\pi(x), \lambda y \in F(x).\nu(x, y)))$$

c)   If (f, π, ν) is as in a), we let

$$\Pi(f, \pi, \nu) = (f, \lambda x.\Pi(\pi(x), \lambda y \in F(x).\nu(x, y)))$$

Remark 2.2 Berger [3] shows that this defines continuous functors from the category of parameterisations over parameterisations to the category of parameterisations.

2.1.3   Universes

The Σ- and Π-constructions are examples of functorial operators on parameterisations. Now, if $\Phi : PAR \rightarrow DOM$ is any continuous functor, we can make the analogue extension of Φ to an operator

$$\Phi^* : PAR(PAR) \rightarrow PAR$$

by

$$\Phi^*(X, F, G) = \lambda x \in X.\Phi(F(x), \lambda y \in F(x).G(x, y))$$

Likewise, any functor $\Phi : PAR \rightarrow PAR$ can be extended to a functor over the category of parameterisations of parameterisations.
In Berger [3] general universes are considered based on operators $\Phi : PAR \rightarrow PAR$. In this paper we will restrict ourselves to the situation where we construct domains from parameterisations, and investigate the universes. We will be interested in the structural properties preserved under this universe operator. In case one is interested in iterations of these operators, like e.g. in [3], the proper extensions of these definitions have to be made.

Definition 2.6 Let $\Phi_1, \ldots, \Phi_n$ be continuous functors from PAR to DOM. Let (X, F) be a parameterisation. We define the *universe parameterisation*

$$(U, X, F, \Phi_1, \ldots, \Phi_n)$$

as the parameterisation

$$(S, I) = (S_{(X,F)}, I_{(X,F)})$$

which is the least solution to the following set of domain equations:

$*$ $B \in S$ with $I(B) = X$ (B is just a formal symbol for *Base*).

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∗ For $x \in X$, we let $(p, x) \in S$ with $I(p, x) = F(x)$.

∗ If $s \in S$ and $H : I(s) \to S$ is continuous, we let $(i, s, H) \in S$ with

$$I(i, s, H) = \Phi_i(I(s), \lambda x \in I(s).I(H(s)))$$

Remark 2.3 Berger [3] shows that this universe operator extends in a natural way to a continuous functor on the category PAR. We give the definition here but do not verify the properties.

Definition 2.7 Let $(f, \pi) : (X, F) \to (Y, G)$ be a morphism. We define a morphism

$$(h, \nu) : (S_{(X,F)}, I_{(X,F)}) \to (S_{(Y,G)}, I_{(Y,G)})$$

as the least solution to the following set of equations:

Base:
   $h^+(B) = B$
   $h^-(B) = B$
   $\nu(B) = f$

Base parameters:
   $h^+(p, x) = (p, f^+(x))$
   $h^-(p, y) = (p, f^-(y))$
   $\nu(p, x) = \pi(x)$

Operator Φ_i:
   $h^+(i, s, H) = (i, h^+(s), \lambda y \in I_{(Y,G)}(h^+(s)).h^+(H((\nu(s))^-(y))))$
   $h^-(i, t, H') = (i, h^-(t), \lambda x \in I_{(X,F)}(h^-(t)).h^-(H'((\nu(h^-(t)))^+(x))))$
   $\nu(i, s, H) = \Phi_i(\nu(s), \lambda x \in I_{(X,F)}(s).\nu(H(x)))$

2.2 The base category of domains with totality

2.2.1 Domains and parameterisations

In this section we will introduce the category K of domains with totality, and we will see how the operators of the previous sections can be extended to operators on this category. In the categories K_1, K_2 and K_3 to be defined

later, we will just restrict the objects considered, the morphisms will remain the same. We will show that these categories are closed under the Σ- and Π-operators. We will further discuss when the universe obtained by closing under some operators will be within one of these categories.

Definition 2.8 We define the category K as follows:

a)   The objects will be pairs $\mathbf{X} = (X, \bar{X})$ where X is a domain and $\bar{X} \subseteq X$ is any subset on X. $\bar{X}$ is called *a totality on X* or *the total elements in X*.

b)   If $\mathbf{X} = (X, \bar{X})$ and $\mathbf{Y} = (Y, \bar{Y})$ are two objects, a *morphism* will be a morphism f from X to Y such that $f^+(x) \in \bar{Y}$ whenever $x \in \bar{X}$.

c)   A morphism $f : \mathbf{X} \to \mathbf{Y}$ is *strong* if we in addition have that $f^-(y) \in \bar{X}$ whenever $y \in \bar{Y}$.

d)   We let K^s be the category of domains with totality and strong morphisms.

The notion of totality can be extended to parameterisations as follows:

Definition 2.9 a) Let $\mathbf{X} = (X, \bar{X})$ be an object in K.
A *total parameterisation* $\mathbf{F} = (F, \bar{F})$ over $\mathbf{X}$ will satisfy

    i) (X, F) is a parameterisation.

    ii) $\bar{F}(x)$ is defined for all $x \in \bar{X}$ and $\bar{F}(x)$ is then a totality on $F(x)$.

    iii)   If $x \in \bar{X}$, $y \in \bar{X}$ and $x \sqsubseteq y$, then $F(i_{x,y})$ is a strong morphism from $(F(x), \bar{F}(x))$ to $(F(y), \bar{F}(y))$.

b)   If $\mathbf{F}$ is a parameterisation over $\mathbf{X}$ and $\mathbf{G}$ is a parameterisation over $\mathbf{Y}$, then a morphism from $(\mathbf{X}, \mathbf{F})$ to $(\mathbf{Y}, \mathbf{G})$ is a morphism (f, π) from (X, F) to (Y, G) such that f is a morphism from $\mathbf{X}$ to $\mathbf{Y}$ and π is a natural transformation from $(\mathbf{X}, \mathbf{F})$ to $(\mathbf{X}, \mathbf{G} \circ f^+)$.
The morphism is *weakly strong* if $\pi(x)$ is a strong morphism for all $x \in \bar{X}$.
The morphism is *strong* if in addition f is strong.

Remark 2.4 This is the first time (a,iii) that we did not make the obvious choice in the definitions, we demanded that $F(i_{x,y})$ is a strong morphism and not just a morphism. This is because we want the two domains $F(x)$ and $F(y)$ to have essentially the same totality. If x is total, then it determines the totality completly, we permit no room for existing objects to become total via extending x.

2.2.2 Sums and products

Definition 2.10 a) Let $(\mathbf{X}, \mathbf{F})$ be a total parameterisation.
We define $\mathbf{Y} = \Sigma(\mathbf{X}, \mathbf{F})$ by

    i) $Y = \Sigma(X, F)$

    ii) $(x, y) \in \bar{Y}$ if $x \in \bar{X}$ and $y \in \bar{F}(x)$

b)   Let $(\mathbf{X}, \mathbf{F})$ be a total parameterisation.
We define $\mathbf{Y} = \Pi(\mathbf{X}, \mathbf{F})$ by

    i) $Y = \Pi(X, F)$

    ii) $y \in \bar{Y}$ if $y(x) \in \bar{F}(x)$ for all $x \in \bar{X}$

c)   A tripple $(\mathbf{X}, \mathbf{F}, \mathbf{G})$ is a *total parameterisation over a total parameterisation* if $(\mathbf{X}, \mathbf{F})$ and $(\Sigma(\mathbf{X}, \mathbf{F}), \mathbf{G})$ both are total parameterisations.

Remark 2.5 The extension of the Σ- and Π-operators to total parameterisations over total parameterisations is now trivial, and we do not write out the formal definition. The definition of morphisms between total parameterisations of total parameterisations is also canonical, and we leave out the details.

    The sum and product operators are still functorial:

Lemma 2.1 *Let $(f, \pi) : (\mathbf{X}, \mathbf{F}) \rightarrow (\mathbf{Y}, \mathbf{G})$ be a morphism.*

a)   *$\Sigma(f, \pi)$ is a morphism from $\Sigma(\mathbf{X}, \mathbf{F})$ to $\Sigma(\mathbf{Y}, \mathbf{G})$.*

b)   *If f in addition is a strong morphism from $\mathbf{X}$ to $\mathbf{Y}$ we have that $\Pi(f, \pi)$ is a morphism from $\Pi(\mathbf{X}, \mathbf{F})$ to $\Pi(\mathbf{Y}, \mathbf{G})$.*

c)   *If (f, π) is strong, then both $\Sigma(f, \pi)$ and $\Pi(f, \pi)$ are strong.*

Proof
The proof is by simple calculation and is left for the reader.

Lemma 2.2 *Let* $(\mathbf{X}_1, \mathbf{F}_1, \mathbf{G}_1)$ *and* $(\mathbf{X}_2, \mathbf{F}_2, \mathbf{G}_2)$ *be two total parameterisations over total parameterisations, and let* (f, π, ν) *be a morphism from* $(\mathbf{X}_1, \mathbf{F}_1, \mathbf{G}_1)$ *to* $(\mathbf{X}_2, \mathbf{F}_2, \mathbf{G}_2)$. *Then*

a) $\Sigma(f, \pi, \nu)$ *is a morphism from* $\Sigma(\mathbf{X}_1, \mathbf{F}_1, \mathbf{G}_1)$ *to* $\Sigma(\mathbf{X}_2, \mathbf{F}_2, \mathbf{G}_2)$

b) *If in addition* (f, π) *is weakly strong, then* $\Pi(f, \pi, \nu)$ *is a morphism from* $\Pi(\mathbf{X}_1, \mathbf{F}_1, \mathbf{G}_1)$ *to* $\Pi(\mathbf{X}_2, \mathbf{F}_2, \mathbf{G}_2)$.

2.2.3 Totality in Universes

We have now shown how the Σ- and Π-operators can be extended to operators on domains with totality. If an extension like this can be made, we can close a set of base types with totality under these operators and obtain a well founded hierarchy of domains with totality. This construction first appeared in Normann [15] using just the natural numbers as the base type, and the Π-constructor as the only constructor. At this level of generality, the construction is first described by Berger in e.g. [3].

Definition 2.11 Let Φ be a continuous functor from PAR to DOM.
We say that Φ has *a total extension* $\bar{\Phi}$ if

i) $\bar{\Phi}(\bar{X}, \bar{F})$ is a totality on $\Phi(X, F)$ whenever $(\bar{X}, \bar{F})$ is a totality on (X, F).

ii) If $(f, \pi) : (\mathbf{X}, \mathbf{F}) \to (\mathbf{Y}, \mathbf{G})$ is a strong morphism, then

$$\Phi(f, \pi) : (\Phi(X, F), \bar{\Phi}(\bar{X}, \bar{F})) \to (\Phi(Y, G), \bar{\Phi}(\bar{Y}, \bar{G}))$$

is a strong morphism.

Remark 2.6 Assuming that Φ maps strong morphisms to strong morphisms is the best we can do, covering the constructions we want to cover. Consider the example
$$\Phi(X, F) = \Sigma(x \in X)\Pi(y \in F(x))\mathbb{N}_\perp$$
Let (id, π) be a morphism from $(\mathbf{X}, \mathbf{F})$ to $\mathbf{X}, \mathbf{G}$). If $\pi(x)$ is not strong, then $\Phi(id, \pi)$ will not in general be a morphism.

Definition 2.12 Let $\Phi_1, \ldots, \Phi_n$ be operators with extensions $\bar{\Phi}_1, \ldots \bar{\Phi}_n$.
Let (X, F) be a parameterisation with totality $(\bar{X}, \bar{F})$.
We define

$$\bar{S} = \bar{S}(X, F, \Phi_1, \ldots, \Phi_n)$$

(omitting the mentioning of the total elements in the notation, implicitly assuming that it is always canonically given) and $\bar{I}(s)$ for $s \in \bar{S}$ by simultanous recursion as follows:

$B \in \bar{S}$ with $\bar{I}(B) = \bar{X}$.

If $x \in \bar{X}$, then $(p, x) \in \bar{S}$ with $\bar{I}(p, x) = \bar{F}(x)$.

If $s \in \bar{S}$ and $H : I(s) \to S$ is continuous such that $H : \bar{I}(s) \to \bar{S}$, then $(i, s, H) \in \bar{S}$ and

$$\bar{I}(i, s, H) = \bar{\Phi}_i(\bar{I}(s), \lambda x \in \bar{I}(s).\bar{I}(H(x))).$$

The extension of totality to universes is functorial. This was first proved by Berger [3]. We formulate the functoriality as follows:

Lemma 2.3 *Let $\Phi_1, \ldots, \Phi_n$ be continuous functors with extensions*
$\bar{\Phi}_1, \ldots, \bar{\Phi}_n$.
Let (X, F), (Y, G) , (f, π) and (h, ν) be as in Definition 2.7.
Let $\bar{X}$, $\bar{Y}$, $\bar{F}$ and $\bar{G}$ be totalities on the respective domains and parameterisations.
If (f, π) is a strong morphism between total parameterisations , then (h, ν) is a strong morphism between total parameterisations.

Proof
Let $\bar{S}_{(X,F)}$ be the total elements in the universe generated from $\bar{X}$, $\bar{F}$ and $\bar{\Phi}_1, \ldots, \bar{\Phi}_n$ etc.
We use induction on the recursive definitions of $\bar{S}_{(X,F)}$ and $\bar{S}_{(Y,G)}$ and prove that the functions

$$h^+ : \bar{S}_{(X,F)} \to \bar{S}_{(Y,G)}$$

and

$$h^- : \bar{S}_{(Y,G)} \to \bar{S}_{(X,F)}$$

both preserve totality, and that

$$\nu(s) : (I_{(X,F)}(s), \bar{I}_{(X,F)}(s)) \to (I_{(Y,G)}(s), \bar{I}_{(Y,G)}(s))$$

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is a strong morphism when $s \in \bar{S}_{(X,F)}$.

The details are tedious but simple and are left for the reader.

2.3 Domains with structured totality

2.3.1 The category K_1

One basic intuition about totality is that an object is total if it in some sense contains complete information. Thus, if we ad some information to a total object in a consistent way, the result should still be total.

Definition 2.13 a) Let $\mathbf{X} \in K$. We let $\mathbf{X} \in K_1$ if

$$\forall x, y \in X(x \in \bar{X} \wedge x \sqsubseteq y \to y \in \bar{X})$$

We organise K_1 to the category K_1 and the category K_1^s by using morphisms and strong morphisms from K.

b) A K_1-*parameterisation* will be a parameterisation $(\mathbf{X}, \mathbf{F})$ where $\mathbf{X} \in K_1$ and $(F(x), \bar{F}(x)) \in K_1$ for all $x \in \bar{X}$.

Lemma 2.4 *If* $(\mathbf{X}, \mathbf{Y})$ *is a* K_1 *parameterisation, then* $\Sigma(\mathbf{X}, \mathbf{F}) \in K_1$ *and* $\Pi(\mathbf{X}, \mathbf{F}) \in K_1$.

The proof is trivial.

By this lemma, the definition of parameterisation of parameterisations extends to K_1 and we get:

Lemma 2.5 *If* $(\mathbf{X}, \mathbf{F}, \mathbf{G})$ *is a* K_1-*parameterisation of* K_1-*parameterisations, then* $\Sigma(\mathbf{X}, \mathbf{F}, \mathbf{G})$ *and* $\Pi(\mathbf{X}, \mathbf{F}, \mathbf{G})$ *are* K_1-*parameterisations.*

These results also extends to universe operators:

Lemma 2.6 *Let* $\Phi_1, \ldots, \Phi_n$ *be continuous functors from* PAR *to* DOM *with extensions* $\bar{\Phi}_1, \ldots, \bar{\Phi}_n$ *such that when* $(\mathbf{X}, \mathbf{F})$ *is a* K_1-*parameterisation, then for all* $i = 1, \ldots, n$ *we have that*

$$(\Phi_i(X, F), \bar{\Phi}_i(\bar{X}, \bar{F})) \in K_1.$$

Then, if $(\mathbf{X}, \mathbf{F}) \in K_1$ *we have that*

$$((S_{(X,F)}, \bar{S}_{(X,F)}), (I_{(X,F)}, \bar{I}_{(X,F)}))$$

is a K_1 *parameterisation.*

Proof
We drop the index (X, F) in this proof.
Let $s \in \bar{S}$, $s \sqsubseteq t \in S$. By induction on the rank of s we show

i) $(I(s), \bar{I}(s)) \in K_1$

ii) $t \in \bar{S}$

iii) The morphism from $I(s)$ to $I(t)$ is strong when the totalities are added.

If $s = B$ then $t = B$ and all three statements are trivial.
If $s = (p, x)$, then $t = (p, y)$. i) holds because $F(x) \in K_1$ and ii) holds because $(X, \bar{X}) \in K_1$. iii) holds because F is a total parameterisation.
If $s = (i, s_1, H)$, then $t = (i, t_1, H')$. By the induction hypothesis, (s_1, H) induces a K_1-parameterisation which is mapped to a K_1-object by Φ_i. i) follows.
In order to prove ii) we must show that (t_1, H') induces a total parameterisation. This is an easy consequence of the induction hypothesis.
In order to prove iii) observe that we by the induction hypothesis have a morphism from the parameterisation induced by (s_1, H) to the one induced by (t_1, H'), where the $I(s_1) \to I(t_1)$-part is strong. iii) then follows from the definition of extension (Definition 2.11).

2.3.2 The category K_2

In order to justify our next structural property we return to the idea that an object $x \in X$ is total when it represents ways to process some atomic outputs in a continuous way from a given set of inputs. If i is some input material and x and y deal with i in the same way, $x \sqcap y$ will also deal with i in the same way. On the other hand, if $x \sqcap y$ deals with i at all, x and y will do it in the same way. Thus if two total objects handle all relevant input material in the same way, the meet must be total, and this must again represent that two total objects essentially are the same. We isolate this property in the category K_2:

Definition 2.14 a) Let $(X, \bar{X}) \in K_1$. Define $\approx_X$ on $\bar{X}$ by

$$x \approx_X y \Leftrightarrow x \sqcap y \in \bar{X}$$

We let $(X, \bar{X}) \in K_2$ if $\approx_X$ is an equivalence relation.

b)   Let $(\mathbf{X}, \mathbf{F})$ be a K_1-parameterisation.
$(\mathbf{X}, \mathbf{F})$ is a K_2-*parameterisation* if $\mathbf{X} \in K_2$ and $(F(x), \bar{F}(x)) \in K_2$ for all $x \in \bar{X}$.

Lemma 2.7 *Let* $\mathbf{X} \in K_1$. *The following are equivalent*

i) $\mathbf{X} \in K_2$

ii) *If* $x \sqcap y \in \bar{X}$ *and* $y \sqcap z \in \bar{X}$ *then* $x \sqcap y \sqcap z \in \bar{X}$.

The proof is trivial.

Theorem 2.1 *Let* $(\mathbf{X}, \mathbf{F})$ *be a* K_2-*parameterisation.*

a) $\mathbf{Z} = \Sigma(\mathbf{X}, \mathbf{F}) \in K_2$ *and for* $(x,y), (u,v) \in \bar{Z}$ *we have*

$$(x,y) \approx_Z (u,v) \Leftrightarrow (x \approx_X u) \wedge (y_{x \sqcap u} \approx_{F(x \sqcap u)} v_{x \sqcap u})$$

b) $\mathbf{U} = \Pi(\mathbf{X}, \mathbf{F}) \in K_2$ *and for* $f, g \in \bar{U}$ *we have*
$f \approx_U g \Leftrightarrow$
$\forall x \in \bar{X}(f(x) \approx_{F(x)} g(x)) \Leftrightarrow$
$\forall x \in \bar{X} \forall y \in \bar{X}(x \approx_X y \rightarrow (f(x))_{x \sqcap y} \approx_{F(x \sqcap y)} (g(y))_{x \sqcap y})$

Proof
It is sufficient to prove the first equivalences in a) and b). The second equivalence in b) is trivial, and the fact that $\approx_Z$ and $\approx_U$ are equivalence relations follows easily, use Lemma 2.7 in a).
The equivalence in a) follows from the general equation

$$(x,y) \sqcap (u,v) = (x \sqcap u, y_{x \sqcap u} \sqcap v_{x \sqcap u})$$

which is easy and left for the reader.
The equivalence in b) follows since $(f \sqcap g)(x) = f(x) \sqcap g(x)$ for any x, a general and easy fact of domain theory.

Remark 2.7 As indicated in motivating the definition of K_2, we consider $\approx_X$ to represent extentional equality. The characterisation in Theorem 2.1 shows that this is preserved under dependent sums and products.

We will now show that the universe operator is an operator on K_2-parameterisations, provided the $\Phi_1, \ldots, \Phi_n$ send K_2-parameterisations to K_2-objects. The following theorem was first proved in Normann [16] just for the Σ-and Π-operators with flat domains as base objects :

Theorem 2.2 *Let $\Phi_1, \ldots, \Phi_n$ be continuous functors from PAR to DOM with extensions $\bar{\Phi}_1, \ldots, \bar{\Phi}_n$ such that if $(\mathbf{X}, \mathbf{F})$ is a K_2-parameterisation, then $(\Phi_i(X, F), \bar{\Phi}_i(\bar{X}, \bar{F})) \in K_2$ for $i = 1, \ldots, n$.*
Then, if $(\mathbf{X}, \mathbf{F})$ is a K_2-parameterisation, the universe with $(\mathbf{X}, \mathbf{F})$ as a base parameterisation and closed under $\Phi_1, \ldots, \Phi_n$ will also be a K_2-parametersation.

Proof
Let $(\mathbf{X}, \mathbf{F})$ be given, and let (S, I) with totality $(\bar{S}, \bar{I})$ be the parameterised universe.
By induction on the formation of $\bar{S}$ we will prove

Claim

i)   If $s \in \bar{S}$ then $(I(s), \bar{I}(s)) \in K_2$.

ii)   If $s, t \in \bar{S}$ with $s \sqcap t \in \bar{S}$, then s and t have the same rank.

iii) $\approx_S$ is an equivalence relation on objects of the same rank (and thus at large).

Proof of claim
We divide the proof into three cases:

Case 1 $s = B$
    Then $s \approx_S t \Leftrightarrow t = B$, and the claim is trivial.

Case 2 $s = (p, x)$
    Then $s \approx_S t$ if and only if $t = (p, y)$ where $x \approx_X y$
    Again the claim is trivial.

Case 3 $s = (i, s_1, H)$
    By the induction hypothesis, $(I(s_1), \lambda x \in I(s_1).I(H(x)))$ is a K_2-parameterisation when totality is added. This is sufficient for i). ii) is an immediate consequence of the induction hypothesis, observing that the total meet will have the same rank as the two equivalent

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elements.

If $s \approx_S t$ we must have that t is of the form (i, t_1, H'), and

$$s \sqcap t = (i, s_1 \sqcap t_1, (H \sqcap H')\lceil I(s_1 \sqcap t_1))$$

(We use $\lceil$ for restriction). We now use Lemma 2.7.
Assume that $s = (i, s_1, H_s)$, $t = (i, t_1, H_t)$ and $r = (i, r_1, H_r)$ where $s \sqcap t \in \bar{S}$ and $t \sqcap r \in \bar{S}$.
Then

$$s \sqcap t \sqcap r = (i, s_1 \sqcap t_1 \sqcap r_1, (H_s \sqcap H_t \sqcap H_r)\lceil I(s_1 \sqcap t_1 \sqcap r_1))$$

We will prove that this object is total. By the induction hypothesis, $s_1 \sqcap t_1 \sqcap r_1$ is total. Let $x \in \bar{I}(s_1 \sqcap t_1 \sqcap r_1)$. Then $x^{s_1 \sqcap t_1} \in \bar{I}(s_1 \sqcap t_1)$ and

$$(H_s \sqcap H_t)(x^{s_1 \sqcap t_1}) = (H_s \sqcap H_t)(x) = H_s(x) \sqcap H_t(x)$$

Following this argument we see that

$$(H_s \sqcap H_t \sqcap H_r)(x) = H(x^{s_1}) \sqcap H(x^{t_1}) \sqcap H(x^{r_1})$$

and the latter is in $\bar{S}$ by the induction hypothesis. This proves iii).

The theorem is a direct consequence of the claim, so this ends the proof.

2.3.3   The category K_3

Given an element $(X, \bar{X})$ in K_2 we may form the quotient space

$$Tp(X) = \bar{X}/\approx_X .$$

$Tp(X)$ will have a natural topology inherited from the domain X, the quotient of the domain topology restricted to $\bar{X}$.
In the Appendix 2 we will show that the Σ-construction will not preserve that this topology is Haussdorff. If we accept a domain with no total elements as one of our base types, we will find a non-Haussdorff space in the universe generated by Σ- and Π-constructions. Here we will see that the operators we have considered this far will preserve membership in the class T_1 of topological spaces. In these topologies, singletons will be closed sets. This is useful for us.

Definition 2.15 a) Let $(X, \bar{X}) \in K_2$. Let $Tp(X)$ be the set of
$\approx_X$ - equivalence classes with the induced topology.
We let $(X, \bar{X}) \in K_3$ if $Tp(X)$ is a T_1-topology.

b) A K_3-papameterisation is a K_2-parameterisation where all domains
with totality involved are K_3-objects.

Theorem 2.3 *Let* $(\mathbf{X}, \mathbf{F})$ *be a* K_3*-parameterisation.*

a) $\Sigma(\mathbf{X}, \mathbf{F}) \in K_3$.

b) $\Pi(\mathbf{X}, \mathbf{F}) \in K_3$.

Proof
We have to prove that each equivalence class is closed in the topology restricted to the total objects.
a): Let $\mathbf{Y} = \Sigma(\mathbf{X}, \mathbf{F})$ and let $(x, y) \in \bar{Y}$ and $(x_1, y_1) \in \bar{Y}$ be non-equivalent
objects. There are two cases
In case $x_1 \not\approx_X x$ we may use that $\mathbf{X} \in K_3$.
In case $x_1 \approx_X x$, then

$$(y_1)_{F(x_1 \sqcap x)} \not\approx_{F(x_1 \sqcap x)} y_{F(x_1 \sqcap x)}$$

and by the induction hypothesis there is a compact $q \sqsubseteq (y_i)_{F(x_1 \sqcap x)}$ such that
q cannot be extended inside $F(x_1 \sqcap x)$ to any object equivalent to $y_{F(x_1 \sqcap x)}$.
Let $p \sqsubseteq x_1 \sqcap x$ be such that $q \in F(p)$ (More precisely $((q)_p)^{F(x_1 \sqcap x)} = q$)).
Then $(p, (q)_p)$ is a compact, it can be extended to (x_1, y_1) but not to any
total object equivalent to (x, y). We leave the further details for the reader.
b): Let $\mathbf{Z} = \Pi(\mathbf{X}, \mathbf{F})$. Let $f \in \bar{Z}$ and $f_1 \in \bar{Z}$ be non-equivalent total objects.
Then there is an $x \in \bar{X}$ such that $f(x)$ and $f_1(x)$ are not equivalent in $F(x)$.
For some compact $q \sqsubseteq f_1(x)$, no total extension of q in $F(x)$ is equivalent to
$f(x)$.
Let $p \sqsubseteq x$ be compact such that $q \in F(p)$. Let

$$f_0(y) = \perp \text{ if } p \not\sqsubseteq y$$

$$f_0(y) = q^y \text{ if } p \sqsubseteq y.$$

Then f_0 is compact, f_0 can be extended to f_1, but f_0 cannot be extended to
any total g equivalent to f.
This ends the proof of the theorem.

Theorem 2.4 *Let $\Phi_1, \ldots, \Phi_n$ be functors from PAR to DOM and let $\bar{\Phi}_1, \ldots, \bar{\Phi}_n$ be extensions to functors from K_3-parameterisations to K_3-objects.*
Let $(\mathbf{X}, \mathbf{F})$ be a K_3-parameterisation, and let $\mathbf{S}, \mathbf{I}$ be the induced universe parameterisation.
Then $(\mathbf{S}, \mathbf{I})$ is a K_3-parameterisation.

Proof
We may use a simple proof by induction to show that each $(I(s), \bar{I}(s)) \in K_3$ when $s \in \bar{S}$, where no surprises occur.
It remains to show that $(S, \bar{S}) \in K_3$. In order to do so we prove that the equivalence class of any $s \in \bar{S}$ is closed by induction on the rank of s. The induction start is trivial. If $s = (i, t, G)$ and $s_1 = (i, t_1, G_1)$ are non equivalent, then either t and t_1 are non-equivalent, or they are equivalent and there is some $x \in \bar{I}(t_1 \sqcap t)$ such that $G(x)$ and $G_1(x)$ are nonequivalent. In the first case we use the induction hypothesis directly, in the second case we combine the induction hypothesis with the Π-case of the proof of the previous theorem.
This ends our proof of this theorem.

Remark 2.8 Waagbø [25] proved that every topological space occuring in his semantics for intuitionistic type theory [26] is T_1. The arguments we have given here are basicly his arguments.

3 Evaluation structures

3.1 Discussion

In section 2 we considered four categories of domains with totality and we proved that all four categories are closed under the Σ- and Π-constructions, and under the universe operator. These results indicate that at least as long as the total objects are defined by recursion, they may support a rich structure even if properties like density and co-density (see Berger [3] for a discussion) are not present.

In this section we will define a category of domains with totality where the property of being total will be determined via a possibly transfinite but deterministic process. The intuition is that we may determine if an object

x is total by evaluating x along certain *evaluation paths*, and then, if all these evaluations terminate, x will be total. Before continuing the general discussion, let us see how this view fits in with the Π-construction and with the Σ-construction.

Let $Y = \Pi(X, F)$ and assume that we have certain evaluation paths for determining totality in X and in $F(x)$ for each total $x \in X$. Now, in order to determine totality of $y \in Y$, we just have to evaluate y along every path starting with some $x \in \bar{X}$ and continuing along a path for $F(x)$. We observe that we do not even need any structure on X in this case.

Let $Z = \Sigma(X, F)$ under the same informal assumptions as above. Now an evaluation path for $z = (x, y)$ will either start by selecting x or by selecting y.

In the first case we continue with an evaluation path for x and thus we can decide if x is total.

In the second case we will continue with an evaluation path for y. In order to know which set of evaluation paths to consider, we must know $F(x)$, so we must know x. The observation is that we must perform the first group of evaluations before knowing which evaluations belong to the second group. With an iteration of the Σ-construction we see that we may have evaluation paths of different degree of dependence of each other.

We will construct two categories. The objects of the first category will consist of subdomains of function spaces, where the values will be in some flat domain $A_\perp$. Thus every domain we consider will in a sense be a domain of functions, which functions will be determined by the choice of the subdomain. This category will be the category of 'Retracts of Functions-structures'

The objects of the second category will be objects of the first category equipped with a totality and a so called *relevance structure*. The total objects will correspond to possible evaluation paths, and the relevance structure will limit the set of evaluation paths along which we must evaluate a certain object in order to verify its totality. This category will be the *category of relevance structures*. We will finally consider a subcategory, the *category of evaluation structures*.

3.2 RoF-structures

3.2.1 The objects

Throughout this section we will let A be a set of 'possible atomic values' with the corresponding flat domain $A_\perp$. We will assume that A is enumerable, though this is of no importance before we start with lifting procedures, see section 4.3. We will further assume that A contains all the possible atomic entities that we will need in the constructions of this paper. For all $a \in A$, let c_a be the function in $E \to A_\perp$ with constant value a.

Definition 3.1 An RoF-structure $\tilde{E}$ over A will consist of

a) A domain E of partial evaluation paths.

b) A retraction ϕ_E of the domain $E \to A_\perp$ to a subdomain X_E such that $\phi(c_a)(\perp_E) = \perp_A$.

The last requirement of b) is *the non-triviality assumption*

We will use the nontriviality assumption to prove that the category of RoF-structures has a least element, see Lemma 3.13.

We will produce several non-trivial examples of RoF-structures. Let us first consider one simple, but important class of examples.

Lemma 3.1 *Let E be a domain, and let $\{A_e\}_{e \in E}$ be a parameterisation of subdomains of $A_\perp$ parameterised over E, with $A_{\perp_E} = \{\perp\}$.*
Then E can be organised to an RoF-structure $\tilde{E}$ such that E_X and $\Pi(e \in E)A_e$ are isomorphic.

Proof
Let $f \in E \to A_\perp$. Let $\phi(f)(e) = a$ if $a \in A_e$ and $f(e) = a$, $\phi(f)(e) = \perp$ otherwise. It is easy to see that ϕ is a retraction, and that $\phi(f) = f$ exactly when $f \in \Pi(e \in E)A_e$.

Our constructions of examples of RoF-structures will often be by recursion, where the base will consist of flat domains. The following lemma shows that flat domains can be represented as RoF-structures:

Lemma 3.2 *Let $B_\perp$ be a subdomain of $A_\perp$.*
Then there is an RoF-structure inducing a domain isomorphic to $B_\perp$.

Proof
We use Lemma 3.1.
Let $E = \{\perp, \star\}$. Let $C_\perp = \{\perp_A\}$ and let $C_\star = B_\perp$.
Then $\Pi(e \in E)C_e$ has a canonical subdomain isomorphic to $B_\perp$ such that the non-triviality assumption is satisfied.

Remark 3.1 In Appendix 4 we will show that there are separable domains that are not induced from RoF-structures.

3.2.2 Morphisms and parameterisations

We will now organise the RoF-structures to a category RoF:

Definition 3.2 Let $\tilde{E} = (E, \phi)$ and $\tilde{D} = (D, \psi)$ be two RoF-structures, $f = (f^+, f^-)$ a morphism from E to D.
Let $g = (g^+, g^-)$ be the induced morphism from $E \to A_\perp$ to $D \to A_\perp$.
We let f be a morphism from $\tilde{E}$ to $\tilde{D}$ in the category of RoF-structures if the retractions commute with g^+ and g^-.
The restriction of g to the fix-point sets of ϕ and ψ is called *the morphism induced by f*.

In an RoF-structure $\tilde{E}$, it is the domain X_E that represents the real objects. Thus it is natural that when we define the notion of a parameterisation within this category, it will be parameterisations over X_E, and not over E that we consider.

Definition 3.3 A *parameterisation of RoF-structures* will be a pair $(\tilde{E}, F)$ where $\tilde{E}$ is an RoF-structure and F is a continuous functor from X_E to the category of RoF-structures. In section 5.1 we will prove that direct limits will always exist in RoF.
F will induce a parameterisation (X_E, G) of domains by

$G(x) = X_{F(x)}$ for $x \in X_E$.

$G(i_{x,y})$ is the morphism induced by $F(i_{x,y})$.

We will organise the class of parameterisations of RoF-structures to a category $PAR(RoF)$:

Definition 3.4 Let $(\tilde{E}_1, \tilde{F}_1)$ and $(\tilde{E}_2, \tilde{F}_2)$ be two parameterisations of RoF-structures.

A *morphism* from $(\tilde{E}_1, \tilde{F}_1)$ to $(\tilde{E}_2, \tilde{F}_2)$ is a pair (f, π) where $f : \tilde{E}_1 \to \tilde{E}_2$ is a morphism in RoF and π is a natural transformation from $\tilde{F}_1$ to $\tilde{F}_2 \circ g^+$ where g is the morphism induced by f.

3.2.3 Products and sums

We showed that any dependent product of subdomains of $A_\perp$ can be viewed as an RoF-structure. We will now show that RoF in a natural way is closed under products. In order to obtain this, we could however restrict ourselves to dependent products of flat domains. Later we will also construct dependent sums. This construction cannot be carried out within the category of dependent products of flat domains, so we need the extension of this category to the RoF-category.

Definition 3.5 Let $(\tilde{E}, F)$ be a parameterisation of RoF-structures.
We define

$$\tilde{D} = \Pi(\tilde{E}, F) = (D, \psi)$$

as follows:

$$D = \Sigma(X_E, F)$$

$$\psi(f)(x, e) = \phi_{F(x)}(\lambda d.f(x, d))(e)$$

Lemma 3.3 *If $(\tilde{E}, F)$ is a parameterisation of RoF-structures, then $\tilde{D} = \Pi(\tilde{E}, F)$ is an RoF-structure, and X_D is isomorphic to $\Pi(X_E, G)$, where G is the induced parameterisation, see Definition 3.3.*

Proof
For $f \in D \to A_\perp$ and $x \in X$, we let $f_x(e) = f(x, e)$, and for $g \in \Pi(x \in X)F(x)$, we let $f_g(x, e) = g(x)(e)$.
We have for each $x \in X$ that $(\psi(f))_x = \phi_{F(x)}(f_x)$. It follows that ψ is a retraction, and that the maps $f \mapsto \lambda x.f_x$ and $g \mapsto f_g$ are inverses of each other on the domains in question.

We observe that the domain E does not play any part in this construction. Thus we may as well define $\Pi(X, F)$ when F is a parameterisation of RoF-structures over a domain X.

The construction of products is functorial:

Definition 3.6 Let (f, π) be a morphism from the parameterisation $(\tilde{E}_1, \tilde{F}_1)$ to $(\tilde{E}_2, \tilde{F}_2)$. We define

$$\Pi(f, \pi) = h = (h^+, h^-) : \Pi(\tilde{E}_1, \tilde{F}_1) \to \Pi(\tilde{E}_2, \tilde{F}_2)$$

by

$$h^+(x, e) = (g^+(x), (\pi(x))^+(e))$$
$$h^-(x, e) = (g^-(x), (F_2(i_{g^+(g^-(x)), x}) \circ \pi(g^-(x)))^-(e)).$$

The following is easy and the proof is left for the reader:

Lemma 3.4 *Definition 3.6 extends the Π-constructor to a functor from $PAR(RoF)$ to RoF.*

The construction in the Σ-case is not that simple:

Definition 3.7 Let $(\tilde{E}, F)$ be a parameterisation of RoF-structures. We write E_x for $E_{F(x)}$ and ϕ_x for $\phi_{F(x)}$, we write X for X_E and ϕ for ϕ_E. We define

$$\tilde{D} = \Sigma(\tilde{E}, F) = (D, \psi)$$

as follows:

$$D = E \oplus \Sigma(x \in X)E_x$$

For $u \in D \to A_\perp$ let

$$u_0 = \phi(\lambda e.u(l(e)))$$

and let

$$u_1 = \phi_{u_0}(\lambda e \in E_{u_0}.u(u_0, e)).$$

Here l and r are the 'inleft'- and 'inrightt'-operators connected with $\oplus$. If $\phi(x) = x$ and $\phi_x(y) = y$, let

$$p(x, y)(l(e)) = x(e) \text{ for } e \in E$$

$$p(x, y)(r(z, e)) = y(e_{z \sqcap x}) \text{ for } z \in X \text{ and } e \in E_z.$$

Let $\psi(u) = p(u_0, u_1)$

Lemma 3.5 *Let $(\tilde{E}, F)$ be a parameterisation of RoF-structures, and let $(D, \psi) = \Sigma(\tilde{E}, F)$.*
ψ is a retraction of $D \to A_\perp$ to a subdomain Z. Moreover
If $z \in Z$, then $z = p(z_0, z_1)$, $z_0 \in X$ and $z_1 \in X_{z_0}$.
If $x \in X$ and $y \in X_x$, then $p(x, y) \in Z$.

Proof

We will state two claims that can be verified by direct calculation, and the lemma will follow.

Claim 1

If $\phi(x) = x$ and $\phi_x(y) = y$ then $\psi(p(x,y)) = p(x,y)$.

Claim 2

If $u : D \to A_\perp$ and $\psi(u) = u$, then $u = p(u_0, u_1)$.

This lemma shows that our definition of dependent sum is a sensible one. Given this, we may define parameterisations over parameterisations. We have

Definition 3.8 A *parameterisation of parameterisations of RoF-structures* is a tripple $(\tilde{E}, F, G)$ where $(\tilde{E}, F)$ and $(\Sigma(\tilde{E}, f), G)$ both are parameterisations of RoF-structures

We state the following without proof:

Theorem 3.1 *Let $(\tilde{E}, F, G)$ be a parameterisation of parameterisations of RoF-structures. Then*

$$(\tilde{E}, \lambda x \in X_E.\Sigma(F(x), \lambda y \in X_{F(x)}.G(x,y)))$$

and

$$(\tilde{E}, \lambda x \in X_E.\Pi(F(x), \lambda y \in X_{F(x)}.G(x,y)))$$

are RoF-parameterisations.

We will also extend the Σ-constructor to a functor:

Definition 3.9 Let (f, π) be a morphism from $(\tilde{E}_1, \tilde{F}_1)$ to $(\tilde{E}_2, \tilde{F}_2)$. Let

$$\Sigma(f, \pi) = h = (h^+, h^-) : \Sigma(\tilde{E}_1, \tilde{F}_1) \to \Sigma(\tilde{E}_2, \tilde{F}_2)$$

be defined as follows:

$$h^+(l(e)) = l(f^+(e))$$
$$h^+(r(x,e)) = r(g^+(x), (\pi(x))^+(e))$$
$$h^-(l(e)) = l(f^-(e))$$
$$h^-(r(x,e)) = r(g^-(x), (F_2(i_{g^+(g^-(x)),x}) \circ \pi(g^-(x)))^-(e)).$$

The following lemma is easy and the proof is left for the reader:

Lemma 3.6 *Definition 3.9 extends the Σ-constructor to a functor from $PAR(RoF)$ to RoF.*

3.2.4 Universes

The construction of universes is mainly of interest when we consider totality. As a preparation for the work in Section 3.4.5 we will construct a universe operator in RoF. We will of course prove that this construction commutes with the construction in section 2.1.3.

Now, an operator within RoF does not induce an operator on the category of domains as a whole. We of course only need to consider the subcategory of domains induced from RoF-structures. Then the following is trivial:

Lemma 3.7 a) *Let (E, ϕ) and (D, ψ) be two RoF-structures. If $X_E = X_D$, then $E = D$ and $\phi = \psi$.*

b) *Let Φ be a continuous functor from the category $PAR(RoF)$ of parameterisations of RoF-structures to RoF. Then Φ induces a a continuous functor from the subcategory of PAR to the subcategory of DOM obtained by restricting to objects induced from RoF-structures.*

Proof
b) is a simple consequence of a). a) is proved by observing that if X_E and X_D are (set-theoretical) equal, then E and D are the domains of the same set of functions. Further, a retraction is determined by its set of fix-points.

We are now ready to define the universe operator in RoF.

Definition 3.10 Let $\Phi_1, \ldots, \Phi_n$ be continuous functors from the category $PAR(RoF)$ to RoF.
Let $(\tilde{E}, \tilde{F}) \in PAR(RoF)$.
We have induced a parameterisation of domains and operators on the set of domains induced from RoF-structures from this given parameterisation and those given operators.

The universe S and the interpretations $I(s)$ for $s \in S$ are given as the least fixpoints of some equations. The equation for S is

$$S = \{B\}_\perp \oplus X \oplus \Sigma(s \in S)(I(s) \to S) \oplus \cdots \oplus \Sigma(s \in S)(I(s) \to S)$$

where we interpret the list of $\oplus$'s as a sum over $\{0, \ldots, n+1\}$.
These equations can be, via the construction of sums and products of RoF-structures, transcribed to equations for a domain D, a retraction ψ on

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$D \to A_\perp$ and interpretations $I(z)$ for any z with $\psi(z) = z$. We give the equation for D:

$$D = \{0\}_\perp \oplus$$

$$\sum_{i=0}^{n+1} (\{0\}_\perp, E, (D \oplus \Sigma(z \in Z)(I(z) \times D)), \ldots, (D \oplus \Sigma(z \in Z)(I(z) \times D)))$$

The definition of ψ follows the definitions of the retractions in the constructions of products and sums, and the definition of I is as the corresponding definition in the construction of universes of domains.

Theorem 3.2 *Let (D, ψ) with I be constructed as in Definition 3.10.*

a) *(D, ψ) is an RoF-structure.*

b) *X_D is isomorphic to the index-domain S in the construction of a universe, based on the operators induced from $\Phi_1, \ldots, \Phi_n$, via a morphism π with an inverse ν.*

c) *If $x \in X_D$, then $I(x)$ is isomorphic to the domain indexed by $\pi(x)$ in the same universe.*

Proof

Both S and (D, ψ) with interpretations are defined as the limits of approximations S_n and (D_n, ψ_n) resp. From the corresponding results for sums and products, we see that the theorem holds for each n, and thus it holds in the limit.

For each set of operators $\Phi_1, \ldots, \Phi_n$ we have constructed a universe operator

$$UNIV((\tilde{E}, \tilde{F}); \Phi_1, \ldots, \Phi_n) = (\tilde{D}, \tilde{I}).$$

This operator is indeed functorial, i.e. if $(f, \pi) : (\tilde{E}, \tilde{F}) \to (\tilde{E}_1, \tilde{F}_1)$, then there is a canonocal morphism

$$(h, \nu) = UNIV((f, \pi); \Phi_1, \ldots, \Phi_n)$$

between the respective universe parameterisations. Thus we can view $UNIV(\cdot \; ; \Phi_1, \ldots, \Phi_n)$ as a functor in the category $PAR(FoR)$. We will not need the result here, and thus leave the details of the construction and verifications for the reader.

3.3 Totality and Evaluation Structures

An evaluation structure will be an RoF-structure where some of the evaluation paths are considered to be total. If e is a total evaluation path, we will also include a relevance requirement (E_e, R_e) for E. E_e will be a set of evaluation paths of lower rank, and R_e will map E_e into A. We will require that x will match R_e on E_e before we demand the evaluation of x along the path e to terminate. We first define the more general *relevance structures*.

Definition 3.11 A *relevance structure*

$$\tilde{E} = (E, \phi, \bar{E}, R, \{E_e\}_{e \in \bar{E}})$$

will consist of

i)   An RoF-structure (E, ϕ)

ii)   A totality $\bar{E}$ on E such that $(E, \bar{E}) \in K_1$

iii)   A family $\{E_e\}_{e \in \bar{E}}$ of totalities on E such that
$(E, E_e) \in K_1$ for all $e \in \bar{E}$, and
if $e \in \bar{E}$ and $e \sqsubseteq e_1$, then $E_e = E_{e_1}$.
We call E_e the *restriction of relevance* for e.

iv)   A continuous map $R : \Sigma(e \in \bar{E})E_e \to A$

We let $R_e(d) = R(e, d)$ for $e \in \bar{E}$ and $d \in E_e$

Definition 3.12 We use the notation from Definition 3.11.
Let $x \in X_E$, $e \in \bar{E}$.

a) We say that e is *relevant for x* if

> **i)** d is relevant for x for all $d \in E_e$.
> **ii)** $x(d) = R_e(d)$ for all $d \in E_d$.

b) x is *total at e* if e is relevant for x and $x(e) \in A$.

c) x is *total* if $x(e) \in A$ for all e relevant for x.

Remark 3.2 The definition of relevance is inductive. We will draw the neccessary conclusion of this in the next definition and lemma.

Definition 3.13 We use the notation from the definitions of this section. By induction we define the following subset $D \subset \bar{E}$:

$$e \in D \Leftrightarrow \bar{E}_e \subseteq D$$

Lemma 3.8 *Let E, D and the rest be as above.*
If e is relevant for x, then $e \in D$.

The proof is trivial. The insight to be obtained is that if e is relevant for x, then x is total at all predecessors of e.

Lemma 3.9 *Let $\tilde{E}$ be a relevance structure. Let $x \in X_E$, $e \in \bar{E}$ and $e \sqsubseteq e'$.*

a) *If e is relevant for x, then e' is relevant for x.*

b) *If x is total at e, then x is total at e'.*

Proof
b) is a trivial consequence of a).
We use induction on the rank of $e \in \bar{E}$. Let $d \in E_{e'}$. Then $d \in E_e$, and since e is relevant for x, d is relevant for x and $x(d) = R_e(d) = R_{e'}(d)$ by the continuity of R.

Lemma 3.10 *Let X be induced from a relevance structure with the notation as above. Let $\bar{X}$ be the total elements in X. Then $(X, \bar{X}) \in K_1$.*

Proof
Let $x_1 \sqsubseteq x_2$. By induction on the rank of e we prove that if $x_1 \in \bar{X}$ and $e \in \bar{E}$, then e is relevant for x_1 if and only if e is relevant for x_2. The lemma will then follow. The proof is easy and is left for the reader.

There is one technical obstacle in proving that $(X, \bar{X}) \in K_2$ when $(X, \bar{X})$ is induced from a relevance structure. So far, none of the definitions of totality and related concepts have been concerned with the retraction ϕ, and we may consider the same concepts in an absolute way on $E \to A_\perp$. In defining K_2 we will need to know when $x \sqcap y$ is total. This meet is not absolute for X and $E \to A_\perp$. The connection is

$$x \sqcap_X y = \phi(x \sqcap y).$$

Our intuition behind ϕ is that it somehow prunes away functions taking uninteresting values in A. We do not want ϕ to prune away values $x(e) \in A$ that are compatible with other total fix-points of ϕ. These informal considerations lead us to the following definition:

Definition 3.14 Let $\tilde{E}$ be a relevance structure. Then $\tilde{E}$ is an *evaluation structure* if for all $x \in E \to A_\perp$ and $e \in \bar{E}$, if

    i) x is total at e

    ii) For some y, $\phi(y)$ is total at e and $\phi(y)(e) = x(e)$

then $\phi(x)$ is total at e.

Definition 3.15 Let X be induced from an evaluation structure as above. Let x and y be total in X.
We let $x \approx y$ if $x \sqcap y$ is total in $E \to A_\perp$.

Lemma 3.11 *Let $\tilde{E}$ be an evaluation structure inducing $(X, \bar{X})$.*
The following are equivalent

i) $x \approx y$

ii) $\forall e \in \bar{E}(e$ *is relevant for* $x \Leftrightarrow e$ *is relevant for* $y) \wedge \forall e \in \bar{E}(e$ *is relevant for* $x \Rightarrow x(e) = y(e))$.

iii) $\phi(x \sqcap y)$ *is total in* X.

Proof
i) $\Rightarrow$ ii) follows from Lemma 3.10 using $x \sqcap y \sqsubseteq x$ and $x \sqcap y \sqsubseteq y$.
Now assume that ii) holds. By induction on e we prove that if e is relevant for x (and y) then $x \sqcap y$ is total at e. The proof is trivial using the general fact that $x(e) \sqcap y(e) = (x \sqcap y)(e)$. iii) then follows by the definition of evaluation structures, comparing $x \sqcap y$ with x. iii) $\Rightarrow$ i) is trivial by monotonisity of totality.

Theorem 3.3 *Let $(X, \bar{X})$ be induced from an evaluation structure. Then $(X, \bar{X}) \in K_2$.*

Proof
The characterisation of $\approx$ from Lemma 3.11 ii) clearly shows that $\approx$ is an equivalence relation, and from iii) we see that $\approx$ coincides with $\approx_{(X, \bar{X})}$.

Remark 3.3 From now on we will let $x \sqcap y$ mean the meet in the subdomain X.

3.4 The category of Evaluation Structures

3.4.1 Morphisms and strong morphisms

The set of evaluation structures is defined relative to a fixed set A of possible atomic values, and when we organise this set to a category, we will use the same A for all objects in the category.

The basic ingredients in an evaluation structure is

A domain E

A retraction ϕ

A totality $\bar{E}$ on E.

Restrictions of relevances (E_e, R_e) for each $e \in \bar{E}$

Some of this information will be positive, i.e. the set of total objects depends in a positive way on this information, while some information will be negative. A morphism in the category of evaluation structures will be a morphism (f^+, f^-) in the category of RoF-structures such that f^+ preserves positive information and f^- preserves negative information. A morphism will be strong if both f^+ and f^- preserve all information. This leads us to the following definition:

Definition 3.16 Let

$$\tilde{E} = (E, \phi, \bar{E}, \{(E_e, R_e)\}_{e \in \bar{E}})$$

and

$$\tilde{E}' = (E', \phi', \bar{E}', \{(E'_e, R'_e)\}_{e \in \bar{E}'})$$

be two evaluation structures, $f = (f^+, f^-)$ a morphism between the underlying RoF-structures (E, ϕ) and (E', ϕ').

a) f is a morphism in the *category of evaluation structures* if

i) $e \in \bar{E}' \Rightarrow f^-(e) \in \bar{E}$

ii) $e \in \bar{E}' \wedge d \in E_{f^-(e)} \Rightarrow f^+(d) \in E'_e \wedge R_{f^-(e)}(d) = R'_e(f^+(d))$.

b) A morphism as in a) is *strong* if in addition

$$e \in \bar{E} \Rightarrow f^+(e) \in \bar{E}' \wedge \forall d \in E'_{f^+(e)}(f^-(d) \in E_e).$$

Lemma 3.12 *Let $\tilde{E}$ and $\tilde{E}'$ be as in Definition 3.16. Let f be a morphism from $\tilde{E}$ to $\tilde{E}'$. Let $e \in \bar{E}'$, $x \in E' \to A_\perp$ and g the morphism induced from f.*
If e is relevant for x, then $f^-(e)$ is relevant for $g^-(x)$.

Proof
We use induction on the rank of e.
Let e be relevant for x and let $d \in E_{f^-(e)}$.
Then $f^+(d) \in E'_d$ and $R_{f^-(e)}(d) = R'_e(f^+(d))$.
Further then $f^+(d)$ is relevant for x and $x(f^+(d)) = R'_e(f^+(d)) \in A$.
By the induction hypothesis, d is relevant for $g^-(x)$. Moreover

$$g^-(x)(d) = x(f^+(d)) = R'_e(f^+(d)) = R_{f^-(e)}(d) \in A.$$

This shows that $f^-(e)$ is relevant for $g^-(x)$.

Theorem 3.4 *Let $\tilde{E}$ and $\tilde{E}'$ be as in Definition 3.16.*
Let f be a morphism from $\tilde{E}$ to $\tilde{E}'$.

a) *Let $g : X \to X'$ be the induced projection pair (see Definition 3.2).*
  Then $g^+ : \bar{X} \to \bar{X}'$.

b) *If in addition f is strong, then $g^- : \bar{X}' \to \bar{X}$, i.e. g is a strong morphism.*

Proof
a) is a direct consequence of Lemma 3.12
In order to prove b) assume in that f is strong, and let $y \in \bar{X}_{E'}$. We prove that if e is relevant for $g^-(y)$, then $f^+(e)$ is relevant for y:
Let $d \in E'_{f^+(e)}$. Then $f^-(d) \in E_e$, so $f^-(d)$ is relevant for $g^-(y)$ and $g^-(y)(f^-(d)) = R_{f^-(e)}(f^-(d))$. By the induction hypothesis, $f^+(f^-(d))$ is relevant for y, and by Lemma 3.9 d is relevant for y.
y is total, so $y(f^+(f^-(d))) \in A$. Moreover, we have

$$y(d) = y(f^+(f^-(d))) = g^-(y)(f^-(d)) = R_e(f^-(d)) = R'_{f^+(e)}(d).$$

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This shows the claim.

It follows that if e is relevant for $g^-(y)$, then $y(f^+(e)) \in A$.

But $g^-(y)(e) = y(f^+(e))$ by definition. This shows that $g^-(y)$ is total.

When we will investigate inductive definitions, the following will be important

Lemma 3.13 *The category of evaluation structures has an initial object $\tilde{M}$.*

Proof

Let M be the trivial domain with one element $\perp_M$.

Let ϕ_M be the function that sends all functions in $M \to A_\perp$ to the constant $\perp$-function. Then for any RoF-structure (E, ϕ) there is a unique morphism f from M to E, and by the nontriviality assumption on ϕ, see Definition 3.1, we see that f is an RoF-morphism.

Let $\perp_M \in \bar{M}$ with $M_{\perp_M} = \emptyset$. Then, if (E, ϕ) is extended to an evaluation structure, f will be a morphism of evaluation structures. This proves the lemma.

3.4.2 Parameterisations

Definition 3.17 A *parameterisation of evaluation structures* will consist of

i) A parameterisation $(\tilde{E}, F)$ of RoF-structures

ii) An extension of $\tilde{E}$ to an evaluation structure, adding $\bar{E}$ and $\{(E_e, R_e)\}_{e \in \bar{E}}$.

iii) An extension $\bar{F}(x)$ of $F(x)$ to an evaluation structure for each $x \in \bar{X}_E$, adding $\bar{E}^x$ and $\{(E_e^x, R_e^x)\}_{e \in \bar{E}^x}$.

such that the morphisms induced by $F(i_{x,y})$ are strong morphisms in the category of evaluation structures when $x \in \bar{X}_E$ and $x \sqsubseteq y \in \bar{X}_E$.

We then have

Theorem 3.5 *Let*
$$(E, \phi, F, \bar{E}, \{(E_e, R_e)\}_{e \in \bar{E}}, \bar{F})$$
be a parameterisation of evaluation structures.

Let (X_E, G) be the parameterisation of domains induced by (E, ϕ, F), and for $x \in \bar{X}_E$, let $\bar{G}(x)$ be the set of total elements in $G(x)$.

Then $(X, \bar{X}, G, \bar{G})$ is a parameterisation of domains with totality.

Proof
This is trivial from our previous results, in particular see Theorem 3.4.

In a parameterisation of evaluation structures, there is a tight connection between the relevance structures of comparable parts of the parameterisation.

Definition 3.18 Let (X, F) be a parameterisation of domains. Let $x, y \in X$ and let $e \in F(x)$.
We will use the following notation:

$$e[y] = (e_{x \sqcap y})^y$$

Remark 3.4 From now on we will use the notation $(\tilde{E}, \tilde{F})$ for parameterisations of evaluation structures: Without further spesifications we may use the notation introduced in the paper so far, like X_E , ϕ_E, ϕ^x , $X_{F(x)}$ etc., normally using an upper index ^{x} to denote an object in $F(x)$ and an upper bar $^-$ to denote totality.

Lemma 3.14 *Let $(\tilde{E}, \tilde{F})$ be a parameterisation of evaluation structures. Let $x, y \in \bar{X}_E$ with $x \approx y$, i.e. $x \sqcap y \in \bar{X}_E$.*

a) *If $e \in \bar{E}^x$ then $e[y] \in \bar{E}^y$.*

b) *If $e \in \bar{E}^x$ and $d \in \bar{E}^y$, then*

$$d \in E^y_{e[y]} \Leftrightarrow d[x] \in E^x_e$$

and in this case
$$R^y_{e[y]}(d) = R^x_e(d[x]).$$

Proof
By transitivity it is sufficient to prove this when $x \sqsubseteq y$ or $y \sqsubseteq x$.
In both cases a) and b) follows from the fact that the morphism from $\tilde{E}^x$ to $\tilde{E}^y$ or from $\tilde{E}^y$ to $\tilde{E}^x$ is a strong one.

We will now organise the parameterisations of evaluation structures to a category. We will use the morphisms between parameterisations of RoF-structures introduced in Definition 3.4.

Definition 3.19 Let $(\tilde{E}_1, \tilde{F}_1)$ and $(\tilde{E}_2, \tilde{F}_2)$ be two parameterisations of evaluation structures.

a) A *morphism* from $(\tilde{E}_1, \tilde{F}_1)$ to $(\tilde{E}_2, \tilde{F}_2)$ is a pair (f, π) such that

   i) (f, π) is a morphism between the underlying RoF-structures.

   ii) f is a strong morphism from $\tilde{E}_1$ to $\tilde{E}_2$.

   iii) If $x \in \bar{X}_{E_1}$, then $\pi(x)$ is a morphism from $\tilde{F}_1(x)$ to $\tilde{F}_2(g^+(x))$

b) The morphism (f, π) is *strong* if we demand that $\pi(x)$ is a strong morphism for all total x.

Remark 3.5 This definition clearly organises the class of parameterisations of evaluation structures to a category. We are a bit more restrictive than in the case of the category K, demanding f to be strong. In the case of evaluation structures, we need this in order to show that the Σ-construction is functorial.

3.4.3   Sums and products

One of the key motivations for developing the concepts and machinery of evaluation structures is to find an abstract notion of 'function space of total objects' closed under the Σ-constructions, and one of the key questions has been *"In what sense are the elements of a Σ-type total?"*

In this section we will see that the category of evaluation structures supports a natural construction of dependent sums and dependent products. We will first consider the products, where we will not need the elaborate structures on the set of parameters, it is sufficient that it is a K_2-domain with totality. Recall Definition 3.5 of products of RoF-structures.

Definition 3.20 Let $(\tilde{E}, \tilde{F})$ be a parameterisation of evaluation structures,

$$(D, \psi) = \Pi((E, \phi), F)$$

We ad relevance to (D, ψ) as follows:

$(x, e) \in \bar{D}$ if $x \in \bar{X}_E$ and $e \in \bar{E}^x$.

$D_{(x,e)} = \{(y, d) \mid y \approx x \wedge d \in E^y_{e[y]}\}$.

$Q_{(x,e)}(y, d) = R^x_e(d[x])$.

Remark 3.6 The new evaluation paths along which we evaluate an f will start with an evaluation of f_x, and then applying f_x on an evaluation path in $\tilde{E}^x$.

Theorem 3.6 *Definition 3.20 defines an evaluation structure.*
Let Y be the domain induced from (D, ψ) with totality $\bar{Y}$. Then the isomorphism between Y and $\Pi(x \in X)X_{F(x)}$ will preserve totality both ways.

Proof
The proof requires a lot of tedious, but simple verifications. The details are left for the reader.

Lemma 3.15 *Let*
$$(f, \pi) : (\tilde{E}_1, \tilde{F}_1) \to (\tilde{E}_2, F_2)$$

be a morphism between parameterisations of evaluation structures.
Then $\Pi(f, \pi)$ is a morphism from $\Pi(\tilde{E}_1, \tilde{F}_1)$ to $\Pi(\tilde{E}_2, \tilde{F}_2)$.
If (f, π) is strong, then $\Pi(f, \pi)$ is strong.

The verifications are easy and are left for the reader.

Definition 3.21 Let $(\tilde{E}, \tilde{F})$ be a parameterisation of evaluation structures, and let
$$(D, \psi) = \Sigma((E, \phi), F)$$

We ad relevance to (D, ψ) as follows:

$$\bar{D} = \{l(e) \mid e \in \bar{E}\} \cup \{r(x, e) \mid x \in \bar{X}_E \wedge e \in \bar{E}^x\}$$

We let

$$D_{l(e)} = \{l(d) \mid d \in E_e\}$$
$$Q_{l(e)}(l(d)) = R_e(d).$$

We let

$$D_{r(x,e)} = \{l(d) \mid d \text{ is relevant for } x\} \cup \{r(x', d) \mid x' \approx_{X_E} x \wedge d \in E^{x1}_{e[x']}\}$$

$$Q_{r(x,e)}(l(d)) = x(d)$$
$$Q_{r(x,e)}(r(x', d)) = R^{x'}_{e[x']}(d).$$

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Theorem 3.7 *Definition 3.21 defines an evaluation structure.*
Let Y be the domain induced from (D, ψ) with totality $\bar{Y}$. Then the isomorphism between Y and $\Sigma(x \in X)X_{F(x)}$ will preserve totalities both ways.

Proof
We see that we have defined a relevance structure by simple checking.
To prove the extra requirement for evaluation structures is trivial for evaluations for the first coordinate. For evaluations for the second coordinate, observe that the first coordinate of $\psi(x)$ will be total and equivalent to the first coordinate of y. We then argue within the evaluation structure indexed by the meet of these two, to verify the property.
Now, let $x \in \bar{X}$ and $y \in \bar{X}_x$. Let e be relevant for $p(x, y)$.
If $e = l(e')$ we see by a simple induction that e' is relevant for x, so $x(e') \in A$. Then
$$p(x, y)(e) = x(e') \in A$$
If $e = r(z, e')$, then we must have that $z(d) \in A$ and $z(d) = x(d)$ for all d relevant for z. It follows that all d relevant for z are also relevant for x, so $z \approx_{E_X} x$.
We see by induction on the rank of e' that $e'[x]$ is relevant for y. We then have
$$p(x, y)(r(z, e')) = y(e'[x \sqcap z]) = y(e'[x]) \in A.$$

This shows that $p(x, y)$ is total.

The converse follows by the same kind of calculations, and is left for the reader.

Lemma 3.16 *Let*
$$(f, \pi) : (\tilde{E}_1, \tilde{F}_1) \to (\tilde{E}_2, \tilde{F}_2)$$

be a morphism between parameterisations of evaluation structures.
Then $\Sigma(f, \pi)$ is a morphism from $\Sigma(\tilde{E}_1, \tilde{F}_1)$ to $\Sigma(\tilde{E}_2, \tilde{F}_2)$.
If (f, π) is strong, then $\Pi(f, \pi)$ is strong.

The verifications are easy and are mainly left for the reader.
Let us just observe why we need f to be strong also in this case. There are two reasons:

$$\Sigma(f, \pi)^-(l(x, e)) = l(g^-(x), (F_2(i_{g^+(g^-(x)), x} \circ \pi(g^-(x)))^-(e)))$$

where g is induced from f. We need $g^-(x)$ to be total when x is total in order to preserve the total evaluations downwards.

Further, we let $l(d) \in D_{l(x,e)}$ when d is relevant for x. Functoriality requires that $f^+(d) \in D_{l(x,e)}$ when $d \in D_{l(g^-(x),f^-(e))}$, but we must then in particular have that $f^+(d) \in \bar{E}_2$ when $d \in \bar{E}_1$.

3.4.4 Parameterisations over parameterisations

Having made the proper Σ-construction, we can always define parameterisations over parameterisations. The definition is now obvious:

Definition 3.22 A *parameterisation of evaluations structures over a parameterisation of evaluation structures* is a tripple $(\tilde{E}, \tilde{F}, \tilde{G})$ where $(\tilde{E}, \tilde{F})$ is a parameterisation of evaluation structures, and $(\Sigma(\tilde{E}, \tilde{F}), \tilde{G})$ is a parameterisation of evaluation structures.

We then have the following result which we state without a proof. The proof is embedded in what has been done so far:

Theorem 3.8 *Let $(\tilde{E}, \tilde{F}, \tilde{G})$ be a parameterisation over a parameterisation of evaluation structures. Then*

a) $(\tilde{E}, \lambda x \in E_X.\Pi(\tilde{F}(x), \lambda y \in X_{F(x)}.\tilde{G}(x,y)))$ *is a parameterisation of evaluation structures.*

b) $(\tilde{E}, \lambda x \in E_X.\Sigma(\tilde{F}(x), \lambda y \in X_{F(x)}.\tilde{G}(x,y)))$ *is a parameterisation of evaluation structures.*

Remark 3.7 We trust that the reader accept and understand the notation in Theorem 3.8. A notation that would take any distinctions in these structures (e.g. what is defined only for total input and what is not) into account would be quite unreadable, and so would then the formulation of the theorem.

3.4.5 Universes

In Definition 3.10 we defined the universe operator in the category of RoF-structures. We will now assume that $\Phi_1, \ldots, \Phi_n$ actually are operators on evaluation structures. Since an evaluation structure is not determined by its set of total elements, we cannot guarantee that $\Phi_1, \ldots, \Phi_n$ induce operators

on any reasonable subcategory of the category of domains with totality. Thus we will restrict ourselves to operators which can be viewed as extensions of operators on domains with totality.

Definition 3.23 a) Let Φ be a functor in the category of evaluation structures with strong morphisms.
We call Φ *separable* if for any evaluation structure $\tilde{E}$, the underlying RoF-structure of $\Phi(\tilde{E})$ only depends on the underlying RoF-structure of $\tilde{E}$ and $(X_{\Phi(\tilde{E})}, \bar{X}_{\Phi(\tilde{E})})$ only depends on $(X_E, \bar{X}_E)$.

b) A functor from the category of parameterisations of evaluation structures with strong morphisms to the category of evaluation structures with strong morphisms is *separable* if the underlying RoF-structure of $\Phi(\tilde{E}, \tilde{G})$ only depends on the underlying RoF-parameterisation of $(\tilde{E}, \tilde{F})$ and $(X_{\Phi(\tilde{E}, \tilde{F})}, \bar{X}_{\Phi(\tilde{E}, \tilde{F})})$ only depends on the parameterisation of domains with totality induced from $(\tilde{E}, \tilde{F})$.

We trivially have

Lemma 3.17 *The operators Π and Σ are separable.*

Remark 3.8 We do not really need separability in order to construct a universe, but we need it in order to connect our construction of the universe to the construction in Definition 2.12.

Definition 3.24 Let $\bar{\Phi}_1, \ldots, \bar{\Phi}_n$ be separable operators as above that are extensions of the RoF-operators $\Phi_1, \ldots, \Phi_n$.
Let $(\tilde{E}, \tilde{F})$ be a parameterisation of evaluation structures.
Let (D, ψ) with interpretation I be the RoF-universe-parameterisation as constructed in Definition 3.10.
We will define an evaluation structure on D inductively. Simultanously we have to define $\bar{Z}$ by induction. This induction will be equal to the definition of $\bar{S}$ following the isomorphism between Z and S. Following the definition of totality in evaluation structures for sums and products we get

$$\bar{D} = \{0\} \oplus \sum_{i=0}^{n+1} (\{0\}, \bar{E}, (\bar{D} \oplus \Sigma(z \in \bar{Z})(\bar{I}(z) \times \bar{D})), \ldots$$

$$\ldots, (\bar{D} \oplus \Sigma(z \in \bar{Z})(\bar{I}(z) \times \bar{D}))).$$

The definitions of D_d and $Q_d(d')$ for $d \in \bar{D}$ and $d' \in D_d$ will be as in the corresponding definitions for sums and products.

Theorem 3.9 *Let $\tilde{D}$ be as in Definition 3.24, and let $(S, \bar{S})$ be the parameter domain of the universe. Then the isomorphism between S and Z will preserve totality both ways. Moreover, the total elements of the interpretations will be preserved by the isomorphism.*

The proof is by a simple induction on the rank of the total elements. The rank of a total element in Z defined from the evaluation structure will be the rank of the set of evaluations relevant for z in the induced ordering of total evaluations. We omit all details.

The universe operator is functorial as an operator on $PAR(RoF)$. This remains true when we ad relevance and evaluation-structures to the picture. We state the following theorem without proof. The proof will be by a grand induction on all totalities involved in a universe parameterisation seen as a parameterisation of evaluation structures.

Theorem 3.10 *Let $(f, \nu) : (\tilde{E}_1, \tilde{F}_1) \to (\tilde{E}_2, \tilde{F}_2)$ be a morphism in the category of parameterisations of evaluation structures with strong morphisms, and let $\Phi_1, \ldots, \Phi_n$ be functors in that category as in the universe construction. Then*

$$UNIV((f, \nu); \Phi_1, \ldots, \Phi_n)$$

is a strong morphism.

4 Topology

4.1 Evaluation structures are in K_3

In this section we will show that if $(X, \bar{X})$ is a domain with totality induced from an evaluation structure, then $(X, \bar{X}) \in K_3$. By definition of K_3 this means to show that the quotient topology on $\bar{X}/\approx_X$ is a T_1-topology, which again is the same as showing that each equivalence class in $\bar{X}$ is closed in $\bar{X}$.

In this section we will use the notations from the definitions, i.e. E and $\{(E_e, R_e)\}_{e \in E}$ with totality $\bar{E}$ is inducing $(X, \bar{X})$.

Definition 4.1 Let $e \in \bar{E}$. We let

$$D_e = \bar{E}_e \cup \bigcup \{D_d \mid d \in \bar{E}_e\}.$$

We let $\bar{R}_e(d) = R_{e'}(d)$ for $d \in \bar{E}_{e'}$, $e' \in D_e \cup \{e\}$.

Lemma 4.1 *Let $x \in \bar{X}$, $e \in \bar{E}$.*
Then e is relevant for x if and only if x is total on D_e and $x(d) = \bar{R}_e(d)$ for all $d \in D$.

The proof is trivial.

Lemma 4.2 *Let $e \in \bar{E}$. Then $\{x \mid e$ is relevant for $x\}$ is open in $\bar{X}$.*

Proof
Let $e \in \bar{E}$, $x \in \bar{X}$ and assume that e is irelevant for x. Then there must be a $d \in D$ of minimal rank which is relevant for x but with $\bar{R}_e(d) \neq x(d)$.
Now, for each $d \in D_e$, let

$$O_d = \{x \in \bar{X} \mid x(d) \in A \wedge x(d) \neq \bar{R}_e(d)\}.$$

Each O_d is an open subset of $\bar{X}$, and the union will consist of exactly those x for which e is irelevant, a property respected by $\approx$.

As a consequence we get the main result of this subsection:

Theorem 4.1 *Let $(X, \bar{X})$ be induced from an evaluation structure. Then $(X, \bar{X}) \in K_3$.*

Proof
From Lemma 4.2 we see that the equivalence class of a total x is closed as follows:

$$y \approx x \Leftrightarrow \forall e (e \text{ relevant for } x \rightarrow e \text{ relevant for } y \wedge y(e) = x(e)).$$

4.2 Evaluation structures as limit spaces

Somehow, the quotient topology is not the adequate structure on $\bar{X}/\approx$. In Appendix 3 we will see via an example of finite rank that the quotient topology permits too many functions to be continuos. If we want a correspondance between $X/\approx \rightarrow Y/\approx$ and $(X \rightarrow Y)/\approx$ we will have to stick to limit structures on the quotient spaces that are not inherited from any topologies in general.

Definition 4.2 Let $(X, \bar{X}) \in K_2$. Let $\{u_i\}_{i \leq \omega}$ be a sequence from $X/\approx$. We say that $u_\omega = \lim_{i \to \infty} u_i$ if there are $x_i \in u_i$ for all $i \leq \omega$ such that $x_\omega = \lim_{i \to \infty} x_i$.

Remark 4.1 There is no reason to believe that this always will define a limit-structure, see Definition 4.3 below. We will prove that it will do so when $(X, \bar{X})$ is induced from an evaluation structure.

Definition 4.3 (Kuratowski [11], see also Hyland [6].)
Let X be a set. A *limit structure* on X is a relation $x_\omega = \lim_{i \to \infty} x_i$ ($\{x_i\}_{i < \omega}$ *converges to* x_ω) on sequences $\{x_i\}_{i \leq \omega}$ from X satisfying:

i)   If $x_i = x_\omega$ for all but finitely many i, then $x_\omega = \lim_{i \to \infty} x_i$.

ii)   Any subsequence of a convergent sequence is convergent with the same limits.

iii) If $\neg(x_\omega = \lim_{i \to \infty} x_i)$, then there is a subsequence of $\{x_i\}_{i < \omega}$ such that no further subsequence converges to x_ω.

Remark 4.2 Properties i) and ii) are trivially satisfied by the construction above, it is property iii) that is the problem.

Lemma 4.3 *Let $U = \bar{X}/\approx$, where $\bar{X}$ is induced from the evaluation structure $\tilde{E}$.*
Let $\{u_i\}_{i \leq \omega}$ be a sequence from U. Then the following are equivalent:

i) $u_\omega = \lim_{i \to \infty} u_i$

ii)   *For all increasing sequences $\{i_k\}_{k \leq \omega}$ with $i_\omega = \omega$, whenever e_k is relevant for u_{i_k} for all $k \leq \omega$ and $e_\omega = \lim_{k \to \infty} e_{i_k}$, then $u_\omega(e_\omega) = \lim_{i \to \infty} u_{i_k}(e_k)$.*

Proof
i) $\Rightarrow$ ii) is trivial.
Now assume ii). It is sufficient to find a sequence $\{x_i\}_{i \leq \omega}$ from $E \to A_\perp$ of total objects such that x_i is equivalent to all objects in u_i for all $i \leq \omega$. We then use the final property of evaluation structures and let

$$\{x_i'\}_{i \leq \omega} = \{\phi(x_i)\}_{i \leq \omega}$$

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which will be a convergent sequence from $\bar{X}$.

Claim 1
Let $e \in \bar{E}$ be relevant for the elements of u_ω. Then there is a compact approximation e_0 to e such that for all but finitly many i:
If d is relevant for u_i and d extends e_0, then $u_\omega(e) = u_i(d)$.

Proof
If not, we could find a counterexample to ii) quite easily.

Now let Γ be the set of pairs (p, a) such that

i) p is a compact in E with an extension to an element in $\bar{E}$ relevant for u_ω.

ii) There is a number k such that for all i with $k \leq i \leq \omega$ and all total extensions e of p, if e is relevant for u_i, then $u_i(e) = a$.

Let $\{(p_i, a_i)\}_{i \in \mathbb{N}}$ be an enumeration of Γ.
Now, let p be compact in E and $a \in A$. We define Δ by:
$(p, a) \in \Delta$ if there is some $i \in \mathbb{N}$ such that

$$p_i \sqsubseteq p \text{ and } a = a_i.$$

If $j < i$ and $a_j \neq a_i$ then p and p_j are inconsistent.

Claim 2
Let (p, a) and (q, b) be two elements of Δ such that p and q are consistent. Then $a = b$.

Proof
Choose i_p and i_q for (p, a) and (q, b) resp. Without loss of generality, assume that $i_p < i_q$. If $a \neq b$, then by construction of Δ we have that q is inconsistent with $p_{i_p} \sqsubseteq p$, contradicting the assumption.

Claim 3
If e is relevant for u_ω there is a compact $p \sqsubseteq e$ such that $(p, u_\omega(e)) \in \Delta$.

Proof
By Claim 1 and the definition of Γ there will be an $i \in \mathbb{N}$ such that $p_i \sqsubseteq e$ and $a_i = u_\omega(e)$. Let $j < i$ such that $a_j \neq a_i$. If p_j is consistent with e, then $e \sqcup p_j \in \bar{E}$ and $e \sqcup p_j$ is relevant for u_ω. This is impossible, and the claim follows.

Each $(p, a) \in \Delta$ determines an object in $E \to A_\perp$. We let x_ω be the least upper bound of these objects. Then for each e relevant for u we have that $x_\omega(e) = u_\omega(e)$. By induction on the rank of e it then follows that e is relevant for x_ω if and only if e is relevant for u_ω.

Claim 4
For each $i \in \mathbb{N}$ we can find x_i equivalent in $E \to A_\perp$ to u_i such that $x_\omega = \lim_{i \to \infty} x_i$.

Proof
Now, let $\{(q_i, b_i)\}_{i \in \mathbb{N}}$ be an enumeration of Δ.
For each i, let k_i be the largest number $\leq i$ such that for all $j \leq k_i$ we have that whenever e extends q_j and is relevant for $u_{i'}$ for $i' \geq i$, then $u_{i'}(e) = b_j$. By construction of Δ, the sequence $\{k_i\}_{i \in \mathbb{N}}$ is increasing and unbounded. Let $\Gamma_i = \{(p, a) \mid$ for all $e \in \bar{E}$, if $p \sqsubseteq e$ and e is relevant for u_i, then $u_i(e) = a\}$. Clearly $(p_j, a_j) \in \Gamma_i$ if $j \leq k_i$, and we may enumerate Γ_i starting with $(p_0, a_0), \ldots, (p_{k_i}, a_{k_i})$. If we construct Δ_i from Γ_i as we constructed Δ from Γ, based on this enumeration, and then construct x_i from Δ_i as we constructed x_ω from Δ, we see that for any compact $\tau \sqsubset x_\omega$ there will be an i such that for every $j \geq i$, $\tau \sqsubseteq x_j$.
By repeating the proofs of Claims 2 and 3 for Δ_i we see that x_i is total and equivalent to u_i. Alltogether, this is exactly what we want, and the lemma is proved.

Theorem 4.2 *Let $U = \bar{X}/\approx$ be induced from a separable evaluation structure with a limit-structure $\lim_{n \to \infty}$ as defined in Definition 4.2.*
Then $(U, \lim_{n \to \infty})$ is a limit space (see Definition 4.3).

Proof
Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence from U and let $u \in U$ be such that

$$\neg(u = \lim_{i \to \infty} u_i).$$

By Lemma 4.3 there is a subsequence $\{u_{i_j}\}_{j \in \mathbb{N}}$ and a sequence $\{e_j\}_{j \leq \omega}$ relevant for u_{i_j} (with $u_\omega = u$) such that

$$\neg(u_\omega(e_\omega) = \lim_{j \to \infty} u_{i_j}(e_j)).$$

By selecting a further subsequence if needed, we may assume that $u_\omega(e_\omega) \neq u_{i_j}(e_j)$ for all j. Then clearly no subsequence of $\{u_{i_j}\}_{j \in \mathbb{N}}$ will have u as a limit.

4.3 The Lifting Theorem

Let $\tilde{E}$ be an evaluation structure, $U = \bar{X}/\approx$. Let $(Z, \bar{Z})$ be an arbitrary domain with totality.

If $F : Z \to X$ is continuous and total, then F induce a function $F/\approx : \bar{Z} \to U$. The lifting problem will be to reverse this, if $F : \bar{Z} \to U$, when is there a continuous, total $\hat{F}$ with $F = \hat{F}/\approx$?

Clearly, a neccessary condition is that F maps convergent sequences in $\bar{Z}$ to convergent sequences in U. We will see that under mild assumptions, this is also a sufficient condition. The proof will use the methods used in the proof of Lemma 4.3.

Theorem 4.3 The Lifting Theorem

Let $(Z, \bar{Z}) \in K_1$ where Z is separable.

Let $\tilde{E}$ be a separable evaluation structure (i.e. E is separable), and let $U = \bar{X}/\approx$ be the quotient space with the limit structure $\lim_{n \to \infty}$ as defined Definition 4.2.

If $F : \bar{Z} \to U$ maps every convergent sequence in $\bar{Z}$ to a convergent sequence in U, then there is a continuous, total function $\hat{F} : Z \to X$ such that for all $z \in \bar{Z}$,

$$\hat{F}(z) \in F(z).$$

Proof

Claim 1

Let $z \in \bar{Z}$ and let e be relevant for $F(z)$.

Then there is a compact $\xi \sqsubseteq z$ and a compact $p \sqsubseteq e$ such that for all $z_1 \in \bar{Z}$ and $e_1 \in \bar{E}$, if $\xi \sqsubseteq z_1$, $p \sqsubseteq e_1$ and e_1 is relevant for $F(z_1)$, then $F(z)(e) = F(z_1)(e_1)$.

Proof

If not, we find a counterexample to the assumption that F preserve limits of convergent sequences.

Now, let $\Gamma = \{(\xi, p, a) \mid$ for all $z \in \bar{Z}$ and $e \in \bar{E}$, if $\xi \sqsubseteq z$ and $p \sqsubseteq e$ and e is relevant for $F(z)$, then $F(z)(e) = a\}$.

Γ is countable, and we may enumerate Γ by

$$\Gamma = \{(\xi_i, p_i, a_i)\}_{i \in \mathbb{N}}.$$

We now construct Δ as follows:

$(\xi, p, a) \in \Delta$ if for some i

$\xi_i \sqsubseteq \xi$, $p_i \sqsubseteq p$, $a_i = a$ and for all $j \leq i$, if $a_j \neq a$ then ξ and ξ_j or p and p_j are inconsistent.

As in the proof of Lemma 4.3 we get

Claim 2
If $(\xi, p, a) \in \Delta$ and $(\xi', p', a') \in \Delta$, ξ and ξ' are consistent and p and p' are consistent, then $a = a'$.

Claim 3
Let $z \in \bar{Z}$ and $z \sqsubseteq z'$. Then $F(z) = F(z')$.

Proof
$z = \lim_{n \to \infty} z'$ so $F(z) = \lim_{n \to \infty} F(z')$
Since U as a topological space is T_1, and our limit structure is contained in the limit structure inherited from the topology, we have $F(z) = F(z')$.

We are now ready to end the proof.
Let $G(z, e) = a$ if for some $\xi \sqsubseteq z$ and $p \sqsubseteq e$ we have that $(\xi, p, a) \in \Delta$.
Let $\hat{F}(z) = \phi(\lambda e \in E.G(z, e))$. Using Claim 3 and the argument of Claim 3 in the proof of Lemma 4.3 we see that when $z \in \bar{Z}$, then $\lambda e \in E.G(z, e)$ is equivalent to $F(z)$, so $\hat{F}$ is total and satisfies all other requirements of a lifting.

5   Type streams

In this section and in the next we will investigate inductive definitions of evaluation structures and compare them with corresponding inductive definitions in the category of domains with totality. We have shown in Lemma 3.13 that the category has an initial object $\tilde{M}$. Thus any functor Φ will define a sequence
$$\tilde{M}, \Phi(\tilde{M}), \Phi(\Phi(\tilde{M})), \ldots$$

In order to be able to continue this process we will first show that the category of evaluation structures contains direct limits of any directed system of objects.
We will then define and investigate the so called *type streams*. A type stream is a generalisation of objects constructed via a strictly positive induction. In the unpublished Normann [14], and later in Kristiansen and Normann [10]

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explicit versions of type streams were constructed. In this paper we will define type streams as limits of certain well behaved directed systems.

5.1 Direct limits

5.1.1 Direct limits of RoF-structures

Throughout this section we will let $(I, \leq)$ be a directed set.

Let $\{\tilde{E}_i\}_{i \in I}$, $\{f_{ij}\}_{i \leq j}$ be a directed system of RoF-structures where $\tilde{E}_i = (E_i, \phi_i)$.

We will always use the expression 'lim' in the sense of a direct limit in this section. We also assume that the reader is familiar with the construction of direct limits in the category of domains.

Let

$$(E, \{f_i\}_{i \in I}) = \lim(\{E_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$$

in the category of domains. By standard domain theory we get

Lemma 5.1 *Let g_{ij} and g_i be the induced morphisms at $E \to A_\perp$-level. Then*

$$(E \to A_\perp, \{g_i\}_{i \in I}) = \lim(\{E_i \to A_\perp\}_{i \in I}, \{g_{ij}\}_{i \leq j}).$$

Definition 5.1 Let

$$\phi : (E \to A_\perp) \to (E \to A_\perp)$$

be defined by

$$\phi(x) = \bigsqcup_{i \in I} g_i^+(\phi_i(g_i^-(x)))$$

The following facts are simple, though some of them requires tedious, but straightforward calculations:

Lemma 5.2 a) $\phi^2 = \phi$.

b) *If $i \in I$, then $g_i^+ \circ \phi_i = \phi \circ g_i^+$.*

c) *If $i \in I$ then $g_i^- \circ \phi = \phi_i \circ g_i^-$.*

d) *If ψ is a retraction on $E \to A_\perp$ commuting with ϕ_i, g_i^+ and g_i^- as above for all $i \in I$, then $\phi = \psi$.*

Alltogether, the construction and this lemma shows

Theorem 5.1 *The category FoR contains direct limits.*

We will show that Σ- and Π-constructions commute with direct limits. In order for this to make sense we need:

Theorem 5.2 *Let $\{(\tilde{E}_i, \tilde{F}_i)\}_{i \in I}$, $\{(f_{ij}, \pi_{ij})\}_{i \leq j}$ be a directed system of parameterisations of RoF-structures. There is a direct limit of the system in this category.*

Proof
The minimality of the direct limit force us to suggest the following:
Let
$$(\tilde{E}, \{f_i\}_{i \in I}) = \lim(\{\tilde{E}_i\}_{i \in I}, \{f_{ij}\}_{i \leq j}).$$
For each $x \in X_E$, let $\nu_{ij} : \tilde{F}_i(g_i^-(x)) \to \tilde{F}_j(g_j^-(x))$ be the composition of $\pi_{ij}(g_i^-(x))$ and the internal morphism from $\tilde{F}_j(g_{ij}^+(g_i^-(x)))$ to $\tilde{F}_j(g_j^-(x))$.
This defines a directed system. Let $\tilde{F}(x)$ be the limit RoF-structure.
Then in particular, for $x \in X_{E_i}$ we have that
$$\tilde{F}(g_i^+(x)) = \lim(\{\tilde{F}(g_{ij}^+(x))\}_{i \leq j}, \{\pi_{j,k}(g_{ij}^+(x))\}_{i \leq j \leq k}).$$

Let
$$\pi_i(x) : \tilde{F}_i(g_i^+(x)) \to \tilde{F}(g_i^+(x))$$

be the limit map. It is now routine to prove that π_i defined this way is a natural transformation and that $(\tilde{E}, \tilde{F})$ with limits (f_i, π_i) is a direct limit.

5.1.2 Direct limits of evaluation structures

Now, let $(\{\tilde{E}_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ be a directed system of evaluation structures. We will extend the construction from the previous section.

First we ad a totality on E. We simply let $\bar{E}$ be the largest set satisfying that all the f_i^-'s will preserve totality:
Let
$$e \in \bar{E} \Leftrightarrow \forall i \in I(f_i^-(e) \in \bar{E}_i)$$

In constructing E_e and R_e we choose the least possible set.
Let
$$d \in E_e \Leftrightarrow \exists i \in I \exists d_i \in (E_i)_{f_i^-(e)}(f_i^+(d_i) \sqsubseteq d).$$

In this case we let

$$R_e(d) = (R_i)_{f_i^-(e)}(d_i).$$

Lemma 5.3 $R_e(d)$ *is uniquely defined.*

The proof is trivial.

Lemma 5.4 *The construction above defines a relevance structure, which is the direct limit of the given directed system in the category of relevance structures.*

The proof is easy and is left for the reader.

Theorem 5.3 *The category of evaluation structures contains direct limits.*

Proof
By the above construction and lemmas it is sufficient to verify that the limit in the category of relevance structures indeed is an evaluation structure. Thus we verify the extra property.

Let $x \in E \to A_\perp$, $e \in \bar{E}$ and assume that x is total in e and that for some y total at e we have $\phi(y) = y$ and $y(e) = x(e)$. We will prove that $\phi(x)$ is total at e. We use induction on e.

Since $x(e) = y(e) \in A$, there is an $i \in I$ such that

$$g_i^-(x)(f_i^-(e)) = g_i^-(y)(f_i^-(e)) \in A.$$

Since e is relevant for x and y we use Lemma 3.12 and obtain that $f_i^-(e)$ is relevant for both $g_i^-(x)$ and $g_i^-(y)$.
Then both $g_i^-(x)$ and $g_i^-(y)$ are total at $f_i^-(e)$. Since $\phi_i(g_i^-(y)) = g_i^-(y)$ we get that $\phi_i(g_i^-(x))$ is total at $f_i^-(e)$. It follows that $\phi(x)(e) \in A$. Since totality is preserved uppwards we get that $\phi(x)$ is total at e.
This ends the proof.

Remark 5.1 Also the category K will contain direct limits. If $\{(X, \bar{X})\}_{i \in I}$ is a directed system of domains with totality with morphisms g_{ij} and with limit-domain X and limit morphisms g_i, we may let $x \in \bar{X}$ if for some $i \in I$, $g_i^-(x) \in \bar{X}_i$. The 'total objects'-functor from the category of evaluation structures to the category of domains with totality will not commute with direct limits. We do however have:

Lemma 5.5 *Let* $(\{\tilde{E}_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ *be a directed system of evaluation structures with limit* $\tilde{E}, \{f_i\}_{i \in I}$ *where all morphisms* f_{ij} *are strong. Then*

a) *Each* f_i *is strong*

b) x *is total in* X_E
    if and only if $g_i^-(x)$ *is total in* X_{E_i} *for some* $i \in I$
    if and only if $g_i^-(x)$ *is total in* X_{E_i} *for all* $i \in I$.

where as usual g *with some index is induced from* f *with the same index.*

Proof
b) is a trivial consequence of a). To prove a), let $e \in \bar{E}_i$. We will show that $g_i^+(e) \in \bar{E}$, i.e. prove that $g_j^-(g_i^+(e))$ is total for all j. This is trivial using that the system is directed and the morphisms are strong.

5.1.3 Limits of parameterisations

In a sense, the limit construction is itself functorial. We will demonstrate this via the proof of the following:

Theorem 5.4 *The category of parameterisations of evaluation structures contains direct limits.*

Proof
In the proof of Theorem 5.2 we have constructed the direct limit in the category of RoF-structures, and we will use the notation from that proof.
It remains to show that if $x \in \bar{X}_E$, then $\tilde{F}(x)$ can be seen as an evaluation structure.
Now, $g_i^-(x) \in \bar{X}_{E_i}$ since each f_{ij} is strong, and thus f_i (and g_i) will be strong. Thus every RoF-structure appearing in the direct limit defining $\tilde{F}(x)$ will be an evaluation structure, and we actually have a directed system of evaluation structures. We of course use the limit of that system in the category of evaluation structures as an evaluation structure on $\tilde{F}(x)$. It is then easy to see that we get a limit of the system of parameterisations.

Theorem 5.5 *Let* $(\{(\tilde{E}_i, \tilde{F}_i)\}_{i \in I}, \{(f_{ij}, \pi_{ij})\}_{i \leq j})$ *be a directed system of parameterisations of evaluation structures with limit* $(\tilde{E}, \tilde{F})$.
Then $\Pi(\tilde{E}, \tilde{F}) = lim_{i \in I} \Pi(\tilde{E}_i, \tilde{F}_i)$ *and* $\Sigma(\tilde{E}, \tilde{F}) = \lim_{i \in I} \Sigma(\tilde{E}_i, \tilde{F}_i)$.

Proof
Let $\tilde{D} = \Pi(\tilde{E}, \tilde{F})$, $\tilde{D}_I = \Pi(\tilde{E}_i, \tilde{F}_i)$.
We will show that $\tilde{D}$ and $\lim_{i \in I}(\{\tilde{D}_i\}_{i \in I}, \{\Pi(f_{ij}, \pi_{ij})\}_{i \in I})$ are isomorphic.

The isomorphism is first established at RoF-level:

$$D = \{(x, e) \mid x \in X_E \wedge e \in F(x)\}$$

where $X_E = \lim_{i \in I} X_{E_i}$ and $F(x) = \lim_{i \in I} F_i(g_i^-(x))$. This pair of limits can be replaced by one limit of pairs. Thus we may actually identify D and $\lim_{i \in I} D_i$.
With this identification we will prove that $\tilde{D} = \lim_{i \in I} \tilde{D}_i$.

First we look at $\bar{D}$ and the total evaluations $\bar{D}'$ of $\lim_{i \in I} \tilde{D}$.
$D \subseteq D'$ because the directed system of the $\tilde{D}_i$'s can be embedded int $\tilde{D}$.
So, let $(x, e) \in \bar{D}'$. By construction of $\bar{D}'$, we have that $\Pi(f_i, \pi_i)^-(x, e)$ is total for every $i \in I$. It follows that $g_i^-(x) \in \bar{X}_{E_i}$ and $\nu_i(x)^-(e)$ is total in $\tilde{F}_i(g_i^-(x))$ for each $i \in I$. This establishes that $(x, e) \in \bar{D}$.

We then will see that the relevances are the same. Again, one direction is trivial and we concentrate on the other direction.
Let $(x, e) \in \bar{D}$ and let $(y, d) \in D_{(x,e)}$. This means that $x \approx y$ and that $d[x]$ is in the restriction of relevance for e in $\tilde{F}(x)$. We now use that $\tilde{F}(x)$ is a limit, so there must be an $i \in I$ such that $\nu_i^-(d[x])$ is in the restriction of relevance for $\nu_i^-(e)$ in $F_i(g_i^-(x))$. Let ν_i' be defined from y like ν_i is defined from x. Then $(g_i^-(y), \nu_i'^-(d))$ will be in the restriction of relevance for $(g_i^-(x), \nu_i^-(e))$. This is sufficcient to ensure that (y, d) is in the restriction of relevance for (x, e) in the limit structure.
The proof in the Σ-case is rather like the proof in the Π-case and is left for the reader.

5.2   Type streams

Intuitivly a type stream is a not neccessarily wellfounded tree of Π- and Σ-expressions, with leaves from a given set of types. One important aspect is that the parameter-set at each node is fixed as one of the base types, and not itself a type stream.
One basic example of a type stream is the type of well founded trees over a fixed set. Consider the equation

$$X = A \oplus (B \to X)$$

In a sense, we can write this as

$$X = A \oplus (B \to (A \oplus (B \to (A \oplus (B \to \ldots))))).$$

Another class of examples are the W-types of intuitionistic, transfinite type theory. Given a parameterisation (X, F) of types we will have the following equation for $W = W(X, F)$:

$$W = \Sigma(x \in X)(F(x) \to W).$$

This equation may be evaluated top-down as a non-wellfounded tree.
Common for both these examples is that the proper elements, or as we will say, the total elements, are defined via a standard, positive induction. The expression itself will just force certain objects to be elements of the interpretation.

In Normann [14] we used functions on integers to code such trees of expressions, and in Kristiansen and Normann [10] we used nonwellfounded objects in a domain of syntactic forms to represent type streams. In this paper we will take the view that these trees of infinite depth in a sense are the limits of a sequence of trees of finite depth. We will use the category of evaluation structures and the direct limit construction there to make this representation precise. The main result will be that the totality defined from the evaluation structure and the totality defined via the natural inductive definition will coincide.

Let us look once more at our two examples. In the case $X = A \oplus (B \to X)$ we may consider the inductively defined sequence

$X_0 = \tilde{M}$

$X_{n+1} = A \oplus (B \to X_n).$

The minimality of $\tilde{M}$ induce unique morphisms $f_n : X_n \to X_{n+1}$ in the category of evaluation structures. Let X be the direct limit in this category. It turns out that X is a fix-point of the operator in this category, and that the total elements of X as an evaluation structure coincides with the totality in the least fix-point in the category K. Thus a transfinite induction is replaced by an ω-limit in the category of evaluation structures.

In the same way, W can be defined as the direct limit of the canonical ω-sequence.

5.2.1 The technical definition

Definition 5.2 Let $(S, \bar{S})$ be a domain with totality.
Let $F : S \to RoF$ be continuous with an extension $\bar{F}$ defined on $\bar{S}$ extending $F(s)$ to an evaluation structure.

a) Let the domain T_n of *n-trees* with totality $\bar{T}_n$ be defined by recursion as follows:

    i) $T_0 = \{*\}_\perp \oplus S$ with $\bar{T}_0 = \{*\} \oplus \bar{S}$.

    ii) $T_{n+1} = (\Sigma(s \in S)(X_{F(s)} \to T_n) \oplus \Sigma(s \in S)(X_{F(s)} \to T_n)) \oplus S$
       with the obvious definition of $\bar{T}_{n+1}$.
       (We write $\Pi(s, G)$ for $l(l(s, G))$ and $\Sigma(s, G)$ for $l(r(s, G))$.)

b) By recursion on n we define the projection $\rho_n : T_{n+1} \to T_n$ as follows:

    i) $\rho_1(\perp) = \perp$
       $\rho_1(r(s)) = r(s)$
       $\rho_1(\Sigma(s, G)) = \rho_1(\Pi(s, G)) = l(*)$.

    ii) $\rho_{n+1}(\perp) = \perp$
       $\rho_{n+1}(r(s)) = r(s)$
       $\rho_{n+1}(\Sigma(s, G)) = \Sigma(s, \lambda x.\rho_n(G(x)))$
       $\rho_{n+1}(\Pi(s, G)) = \Pi(s, \lambda x.\rho_n(G(x)))$

    We say that $s \in T_{n+1}$ is *an extension* of $t \in T_n$ if $\rho_{n+1}(s) = t$.

The intuition is that we have a tree-structure with branches of length at most $n + 1$. The $*$ represents open leaves, places where the tree can be extended, either by a dependent sum or a dependent product.

Definition 5.3 Each $t \in \bar{T}_n$ can be given an interpretation $I(t)$ as an evaluation structure as follows:

$$I(l(*)) = \tilde{M}$$

$$I(r(s)) = F(s)$$

$$I(\Pi(s, G)) = \Pi(F(s), \lambda x.I(G(x)))$$

$$I(\Sigma(s, G)) = \Sigma(F(s), \lambda x.I(G(x))).$$

Lemma 5.6 *Let $t \in T_{n+1}$ be an extension of $s \in T_n$.*

a) *If t is total then s is total.*

b) *There is a morphism $f_{s,t}$ from $I(s)$ to $I(t)$ that is an evaluation structure morphism.*

Proof
a) is trivial. In order to prove b) we construct the morphism by recursion:

i) Let $s \in T_0$, $t \in T_1$.
If $s = *$, let $f_{s,t}$ be the unique morphism from $\tilde{M}$ to $I(t)$.
If $s = r(s_1)$, then $s = t$ and $f_{s,t}$ is the indentity on $F(s_1)$.
If $s = \bot$, then $t = \bot$ and $f_{s,t}$ is the identity on $\{\bot\}$.

ii) Let $s \in T_n$, $t \in T_{n+1}$, $s \geq 0$.
If $s = \bot$ or $s = r(s_1)$ we act as in i).
If $s = \Sigma(s_1, G)$, then $t = \Sigma(s_1, H)$.
We let
$$f_{s,t} = \Sigma(id, \lambda x \in X_{I(s_1)}.f_{G(x),H(x)})$$
If $s = \Pi(s_1, G)$ we use the analogue definition.

Definition 5.4 a) A *type stream* is a sequence $\vec{t} = \{t_n\}_{n \in \mathbb{N}}$ where $t_n \in \bar{T}_n$ for each n and t_{n+1} is an extension of t_n for each n.

b) If $\vec{t}$ is a type stream we interpret $\vec{t}$ as the evaluation structure

$$I(\vec{t}) = \lim(\{I(t_n)\}_{n \in \mathbb{N}}, \{f_{t_n,t_{n+1}}\}_{n \in \mathbb{N}}).$$

For the sake of simplicity we write f_n for $f_{t_n,t_{n+1}}$.

5.2.2 Decomposition of type streams

Let $\vec{t} = \{t_n\}_{n \in \mathbb{N}}$ be a type stream. If $t_0 = r(s)$, then $t_n = r(s)$ for all n, and the limit will (up to isomorphism) be $I(s)$. We call this a *base type stream*. If $\vec{t}$ is not a base type stream, then $t_0 = l(*)$ and every t_{n+1} is either of the form $\Sigma(s, G_n)$ or of the form $\Pi(s, G_n)$. By the definition, the form will

be the same, and the s will be the same. For each $x \in X_{I(s)}$, the sequence $\{G_n(x)\}_{n \in \mathbb{N}}$ will be a sequence where each item but the first is an extension of the previous one. In fact it is easy to see that if x is total, then $\{G_n(x)\}_{n \in \mathbb{N}}$ will be a type stream.

Lemma 5.7 *If $\vec{t} = \{t_n\}_{n \in \mathbb{N}}$ is a type stream, where $t_{n+1} = \Sigma(s, G_n)$ for each n, then*

$$(I(s), \lambda x \in X_{I(s)}.I(\{G_n(x)\}_{n \in \mathbb{N}}))$$

is a parameterisation of evaluation structures. The same will hold if $t_{n+1} = \Pi(s, G_n)$ for each n.

Proof
In the proofs of Theorems 5.2 and 5.4 We constructed the direct limit of a system of parameterisations. For this special case, it was just the parameterisation suggested here. Thus we may rely on those two theorems.

Theorem 5.6 a) *If $\vec{t}$ is a type stream of Σ-type that decomposes to $(s, \{\vec{r}(x)\}_{x \in X_{I(s)}})$, then*

$$I(\vec{t}) = \Sigma(I(s), \lambda x. \in X_{I(s)} I(\vec{r}(x)).$$

b) *If $\vec{t}$ is a type stream of Π-type that decomposes to $(s, \{\vec{r}(x)\}_{x \in X_{I(s)}})$, then*

$$I(\vec{t}) = \Pi(I(s), \lambda x \in X_{I(s)} I(\vec{r}(x)).$$

Proof
Immediate from Theorem 5.5.

5.2.3 Well founded totality

The decomposition of type streams shows that we can view a type stream as a base type or as a dependent sum or dependent product of other type streams. Our results so far show that the interpretations of type streams as domains with totality satisfy the standard connection between totality in a dependent sum or product and the totalities in the parameterisation. There may however be many allocations of totalities to type streams satisfying these standard connections.

In this section we will show that the totality defined from the evaluation structure will be the least possible.

Definition 5.5 Let $\vec{t}$ be a type stream with domain interpretation $X(\vec{t})$. By a simultaneous induction we define the *w-totality* $X_w(\vec{t})$:

If $\vec{t}$ is a base type s, let $X_w(\vec{t}) = \bar{X}(s)$.

If $\vec{t} = \Sigma(s, \vec{r}(x))$, let $(x, y) \in X_w(\vec{t})$ if $x \in \bar{X}(s)$ and $y \in X_w(\vec{r}(x))$.

If $\vec{t} = \Pi(s, \vec{r}(x))$, let $\gamma \in X_w(\vec{t})$ if for all $x \in \bar{X}(s)$ we have that
$\gamma(x) \in X_w(\vec{r}(x))$.

Theorem 5.7 *For each type stream $\vec{t}$ we have that*

$$X_w(\vec{t}) = \bar{X}(\vec{t})$$

Proof
The inclusion

$$X_w(\vec{t}) \subseteq \bar{X}(\vec{t})$$

is trivial.

Now let $x \notin X_w(\vec{t})$. We will find a total evaluation e in $I(\vec{t})$ relevant for x such that $x(e)$ does not terminate.
By induction on m we will construct $\vec{t}^m$, x^m not in $X_w(\vec{t}^m)$ and for some m the object y_m as follows:
$\vec{t}^0 = \vec{t}$ and $x^0 = x$.
If $\vec{t}^m$ is a base type stream we stop the construction there.
If $\vec{t}^m = \Pi(s_m, F_m)$, we choose a total y_{m+1} in $X_{I(s_m)}$ such that $x^{m+1} = x^m(y_{m+1})$ is not w-total in $X_{I(F_m(y_{m+1}))}$. We let $\vec{t}^{m+1} = F_m(y_{m+1})$.
If $\vec{t}^m = \Sigma(s_m, F_m)$, either $\pi_0(x^m)$ is not total in $X_{I(s_m)}$, or it is, but $\pi_1(x^m)$ is not total in $F_m(\pi_0(x^m))$.
In the first case we let $\vec{t}^{m+1}$ be the base type stream of s_m and $x^{m+1} = \pi_0(x^m)$.
In the second case we let $x^{m+1} = \pi_1(x^m)$ and $\vec{t}^{m+1} = F_m(\pi_0(x^m))$.

Now we will construct the promised evaluation e. There are two cases.

Case 1
The construction of the sequence terminates.

We will construct e^m by reversed induction as follows, starting where the construction terminates.
So assume first that $\vec{t}^m$ is a base type stream s where x^m is not w-total. In this case, w-totality coincides with evaluation structure totality, so let e^m

be a total evaluation in $I(\vec{t}^m)$ relevant for x^m such that $x^m(e^m)$ does not terminate.

Now assume that e^{m+1} is constructed such that e^{m+1} is a total evaluation in $I(\vec{t}^m)$ relevant for x^{m+1}, but such that $x^{m+1}(e^{m+1})$ does not terminate.

If $\vec{t}^m = \Sigma(s_m, F_m)$, then either $x^{m+1} = \pi_0(x^m)$ or $x^{m+1} = \pi_1(x^m)$. In the first case, let $e^m = l(e^{m+1})$. In the second case, let $e^m = r(\pi_0(x_m), e^{m+1})$.

If $\vec{t}^m = \Pi(s_m, F_m)$, then $x^{m+1} = x^m(y_m)$. Let $e^m = (y_m, e^{m+1})$.

In this case, e^0 will be total in $I(\vec{t})$, relevant for x, but $x(e^0)$ will not terminate.

Case 2

The definition of the sequence does not terminate.

In this case we construct a double-sequence $\{e_n^m\}_{n,m\in\mathbb{N}}$ of evaluations. For each m and n, e_n^m is an evaluation in $I(t_n^m)$.

Since the construction of the sequence does not terminate, we will have that $I(t_0^m) = \tilde{M}$ for all $m \in \mathbb{N}$. We let $e_0^m = \bot_M$ for all $m \in \mathbb{N}$.

Assume that e_n^m is constructed for all $m \in \mathbb{N}$.

We construct e_{n+1}^m from e_n^{m+1} as we constructed e^m from e^{m+1} in the first case.

Claim 1

Each e_n^m is a total evaluation in $I(t_n^m)$.

The proof is trivial by induction on n.

Claim 2

If f_n^m is the morphism from $I(t_n^m)$ to $I(t_{n+1}^m)$ then $(f_n^m)^-(e_{n+1}^m) = e_n^m$.

Proof

Easy if we prove this simultaneously for all m by induction on n.

Now, let $e^m = \lim_{n\to\infty} e_n^m$.

Then e^m is a total evaluation in $I(\vec{t}^m)$. Moreover, e^m is constructed from e^{m+1} as in the first case.

Claim 3

For each m, e^m is relevant for x^m.

Proof

Let d be in the restriction of relevance for e^m. Then by construction of the direct limit there is a d' and an n such that d' is in the restriction of relevance

for e_n^m in $I(t_n^m)$ with d' mapped into d by the limit map. By induction on n we show that the image of d' under the limit map is relevant for x^m and that $x^m(d')$ terminates.

If $n = 0$, then the very existence of d' ensures that t_0^m is a base type. This would have left us with the case of the terminating sequence, so this is impossible.

Let $n = k + 1$. We consider two cases.

If $\vec{t}^m = \Pi(s_m, F_m)$, then $e_n^m = (y_n, e_k^{m+1})$ and $d' = (z, d'')$ with $z \approx y_n$. Without loss of generality we may assume that $z = y_n$ and that d'' is in the restriction of relevance for e_k^{m+1}. The claim then follows from the induction hypothesis.

The other case is $\vec{t}^m = \Sigma(s_m, F_m)$.

If $d' = r(z, d'')$ we may argue as above.

If $d' = l(d'')$ we have that d'' is relevant for $\pi_0(x^m)$ and $\pi_0(x^m)(d'')$ terminates. This will be preserved in the limit.

This ends the proof of Claim 3.

As a consequence e^0 is relevant for x, but $x(e^0)$ will not terminate. This ends the proof of the theorem.

5.2.4 Strictly positive induction

In any situation where we have constructions like function-space, cartesian products and disjoint sums, we may define strictly positive operators:

Definition 5.6 We define the *strictly positive expressions* inductively as follows:

A constant A (from a given set of objects) is strictly positive.

A variable X is strictly positive.

If Γ_1 and Γ_2 are strictly positive and A is a constant, then $\Gamma_1 \oplus \Gamma_2$, $\Gamma_1 \times \Gamma_2$ and $A \to \Gamma_1$ are strictly positive.

If the constants A are evaluation structures and the variable X range over evaluation structures, a strictly positive expression will define a functor Γ^e on the category of evaluation structures commuting with direct limits, see Theorem 5.2. Γ^e will have a fixpoint defined as $\tilde{E}_\omega = \lim_{n \to \infty} \tilde{E}_n$ with $\tilde{E}_0 = \tilde{M}$ and $\tilde{E}_{n+1} = \Gamma^e(\tilde{E}_n)$.

A strictly positive expression Γ can also be interpreted as a functor Γ^K in the category K of domains with totality. This functor will not commute in general with direct limits, but will have a least fixpoint (X, X_{wf}). As a special case of Theorem 5.7 we get

Corollary 5.1 *Let Γ be a strictly positive expression with evaluation structures as parameters.*
The total elements of the least fixpoint of Γ^e in the category of evaluation structures coincides with the total elements in the least fixpoint of Γ^K in the category K.

5.2.5 The set of type streams as an evaluation structure

Now, let $(\tilde{E}, \tilde{F})$ be a parameterisation of evaluation structures over one evaluation structure. We will see that the type streams obtained from this parameterisation can be seen as a parameterisation of evaluation structures over an evaluation structure.

So far totality has been connected with well foundedness in some sense. Type streams are not well founded so one important aspect of this problem is that we need a general method for modelling totality even when the total objects are not inductively defined.

We will not need this result later in the paper, and we will rather give the intuition behind the argument than the detailed proof.

Definition 5.7 Let $(\tilde{E}, \tilde{F})$ be as above. Let T be the least solution to the domain equation

$$T = S \oplus \Sigma(s \in S)(F(s) \to T) \oplus \Sigma(s \in S)(F(s) \to T)$$

where the objects represent base typestreams, Π-forms or Σ-forms following the ordering of the sum.
We can represent T as an RoF-structure (D, ψ).

There are two ways to define totality on T, we could consider the well founded types T_{wf} or those representing the type streams. In the first case the totality can be seen as the direct limit of types of finite rank in analogy with a type stream. In the proof of Theorem 5.7 we constructed evaluations E for every $x \notin T_{\mathrm{wf}}$ such that $x(e)$ does not terminate. Every total evaluation may actually appear as one constructed as a part of that proof.

There were two cases in that proof, one where we construct a terminating sequence and one where we construct a non-terminating sequence. In the first case the object is not total because it evaluates down to a non-total element of a base type. In the second case the object is not total because there is an infinite path in the evaluation tree. If we remove those evaluations from the construction, such objects would turn total. This is exactly what we do with type streams:

Let T be as above.

Let $T_0 = T$ Let T_{n+1} consist of the base types and of the Σ-types and Π-types over parameterisations (s, H) where $s \in \bar{S}$ and $H(x) \in T_n$ for all x total in the sense of $I(s)$.

Let $T_\omega = \bigcap_{n \in \mathbb{N}} T_n$.

From the discussion above we see that T_ω can be obtained as the total elements in an evaluation structure on (D, ψ).

It is easy to construct a type stream $\vec{t}$ from a $t \in T_\omega$ such that all type streams are obtained, and this will induce a parameterisation of the set of interpretations of type streams.

6 Positive inductive definitions

In the previous sections we considered strictly positive induction generalised to type streams. In this section we will consider general positive induction.

Now, any operator $\Gamma(X_1, \ldots, X_n)$ on domains constructed from $\rightarrow$, $\times$ and $\oplus$ will be functorial in the category of domains in the sense that for any $f_i : X_i \rightarrow Y_i$, $i = 1, \ldots, n$, there is a canonical morphism

$$\Gamma(f_1, \ldots, f_n) : \Gamma(X_1, \ldots, X_n) \rightarrow \Gamma(Y_1, \ldots, Y_n).$$

Details can be seen in e.g. Stoltenberg-Hansen & al. [24]. Considering various categories $K_1, \ldots, K_n$ and K_r of domains with totality we say that an operator is functorial from $K_1 \times \cdots K_n$ to K_r if Γ maps morphisms in $K_1 \times \cdots \times K_n$ to a morphism in K_r. If some K_i is the category of evaluation structures or some subcategory of it, we will require that Γ respects the induced morphisms on the induced domains.

We have one negative result that will force us to restrict ourselves to a subcategory of the category of evaluation structures:

Lemma 6.1 *There is a positive operator that is not functorial in the category of evaluation structures.*

Proof
Let
$$\Gamma(X) = (X \to \mathbb{N}_\perp) \to X.$$

Let

$\tilde{E} = \Sigma(x \in \mathbb{N}_\perp \to \mathbb{N}_\perp)\tilde{M}$

$\tilde{D} = \Sigma(x \in \mathbb{N}_\perp \to \mathbb{N}_\perp)\mathbb{N}_\perp.$

If $f : \tilde{E} \to \tilde{D}$ is the canonical morphism, $\Gamma(f)$ will not be a morphism from $\Gamma(\tilde{E})$ to $\Gamma(\tilde{F})$. The problem is that $h = \Gamma(f)$ will not satisfy that $h^+(d)$ is in the restriction of relevance for e whenever d is in the restriction of relevanse for $h^-(e)$.

6.1 Function space structures

Definition 6.1 Let $\tilde{E}$ be an evaluation structure. We call $\tilde{E}$ a *function space structure* if for all $e \in \bar{E}$, $E_e = \emptyset$, i.e. there are no restrictions of relevance.

We have the following trivial observation:

Lemma 6.2 a) $\tilde{M}$ *is a function space structure.*

b)   *If $f : \tilde{E} \to \tilde{D}$ is an evaluation structure morphism and $\tilde{D}$ is a function space structure, then $\tilde{E}$ is a function space structure.*

c)   *If $\tilde{E} = \lim(\{\tilde{E}_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ is a direct limit of function space structures in the category of evaluation structures, then $\tilde{E}$ is a functoon space structure.*

We also have the following results, which we state without proof. They can be proved by simple adjustments of arguments given in this paper.

Theorem 6.1 *Let $\tilde{E}$ be a function space structure with $(X, \bar{X})$ as the induced domain with totality.*

a) *The quotient topology $\bar{X}/\approx$ is Haussdorff.*

b) *If E is separable, $(Z, \bar{Z}) \in K_1$ is separable and*

$$F : \bar{Z} \to X/\approx$$

is continuous, then there is a continuous lifting

$$\hat{F} : Z \to X$$

such that $\hat{F}(z) \in F(z)$ for all $z \in \bar{Z}$.

The representation of a flat domain as an evaluation structure is clearly a function space structure. Moreover, the dependent product of a parameterisation of function space structures will be a function space structure.

We have represented disjoint unions of evaluation structures as a dependent sum. In this section we will use another representation:

Lemma 6.3 *Let $\{\tilde{E}_b\}_{b \in B_\perp}$ be a parameterisation of function space structures over the flat domain $B_\perp$.*
Then $\Sigma(b \in B_\perp)\tilde{E}_b$ can be realised as a function space structure.

Proof
Let $D = \{*, \perp\} \cup \{(b, e) \mid b \in B \wedge e \in E_b\}$
Let $x : D \to A_\perp$.
We let

$\phi(x)(*) = x(*)$

$\phi(x)(b, e) = \phi_b(e)$ if $x(*) = b$.

$\phi(x)(b, e) = 0$ if $x(b, e) = 0$, $x(*) \in B$ but $x(*) \neq b$.

$\phi(x)(b, e) = \perp$ otherwise.

6.2 A digression

Above we showed that the closure properties of the category of function space structures are quite strong. In this section we will show that there is a lot of domains with totality constructed by use of the Σ-constructor that will be

describable as function space structures. In this digression we will have to assume familiarity with hierarchies of domains with density and co-density, see e.g. Normann [15], Kristiansen and Normann [8] or Berger [3]. We will not need these results for the rest of the paper, except that we increase our pool of possible parameters in positive definitions.

We let (S, I) be the universe generated from $\mathbb{N}_\perp$ closing under dependent sums and products. Let S_{wf} be the total elements of S and $\bar{I}(s)$ be the total elements of $I(s)$. This parameterisation will satisfy uniform density, something called uniform co-density and the following important property:

Proposition   *Uniformly in s and t in S there is a function $Tr(s, t)(x)$ mapping $I(s)$ to $I(t)$ such that if s and t are total, then $Tr(s, t)$ is total.*

We then can prove the following

Theorem 6.2 *Uniformly in $s \in S_{\mathrm{wf}}$ there is a function space structure $\tilde{E}_s$ such that X_{E_s} is isomorphic to $I(s)$ with an isomorphism preserving totality both ways.*

Proof
We first construct E_s for $s \in S$:

  If $s = 0$, then E_s is atomic.

  If $s = \Pi(s_1, F)$, then $E_s = \Sigma(x \in I(s))E_{F(x)}$.

  If $s = \Sigma(s_1, F)$ then $E_s = E_{s_1} \oplus \Sigma(x \in I(s))E_{F(x)}$.

For each $s \in S_{\mathrm{wf}}$ we have that E_s is a cannonical domain with totality within the hierarchy S_{wf} itself. We will use the function Tr in order to construct the alternative ϕ_s on $E_s \to A_\perp$:
$\phi_0(x)(*) = n$ if $x(*) = n$.
$\phi_0(x)(*) = \perp$ if $x(*) \notin \mathbb{N}$.
Let $s = \Pi(s_i, F)$.
Let
$$\phi_s(\gamma)(x, e) = \phi_{F(x)}(\lambda d.\gamma(x, d))(e).$$
Let $s = \Sigma(s_1, F)$. Let
$$\phi_s(\gamma)(l(e)) = \phi_{s_1}(\lambda d.\gamma(l(d)))(e).$$

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Let $\gamma_0 = \phi_{s_1}(\lambda d.\gamma(l(d)))$.

$$\phi_s(\gamma)(r(x,e)) = \phi_{F(\gamma_0)}(\lambda d.\gamma(\gamma_0, d))(Tr(F(x), F(\gamma_0))(e)).$$

with $\gamma_1 = \phi_{F(\gamma_0)}(\lambda d.\gamma(\gamma_0, d))$.

It is only in the Σ-case that we have altered anything from the original definition. In this case we have used a different method to transport information from one part of the parameterisation to another. The proof of the fact that pairing and depairing are inverses of each other will work as in the proof of Lemma 3.5. The point with this construction is that if $s = \Sigma(s_1, F)$, $\phi_{s_1}(\delta) = \delta$, $\phi_{F(\delta)}(\pi) = \pi$ and both δ and π are total, then

$$p(\delta, \pi)(x, e) = \pi(Tr(F(x), F(\delta))(e))$$

is defined whenever x is total and e is a total evaluation in $E_{F(x)}$. Thus we need no restriction of the relevance to determine when an object represents a total ordered pair in a dependent sum.

6.3   Three categories

In the discussion of positive induction we will need three categories.

Definition 6.2 a) Let K^+ be the category K_1 of domains with totality. We call K^+ the *positive category*.

b)   Let K^- have the same objects as K^+. Let $f : X \to Y$ be a K^--morphism if $F^-(y) \in \bar{X}$ whenever $y \in \bar{Y}$.

c)   Let K^F be the category of function space structures.

Lemma 6.4 a) *Let* $\Gamma_\to(X, Y) = X \to Y$.
Then $\Gamma_\to$ *is functorial both as an operator from* $K^- \times K^+$ *to* K^+ *and as an operator from* $K^+ \times K^-$ *to* K^-.

b)   *Let* $\Gamma_\times(X, Y) = X \times Y$.
Then $\Gamma_\times$ *is functorial both in* K^+ *and in* K^-.

c)   *Let* $\Gamma_\oplus(X, Y) = X \oplus Y$.
Then $\Gamma_\oplus$ *is functorial both in* K^+ *and in* K^-.

The proofs are trivial and are left for the reader.

Now, let $\Gamma(X_1, \ldots, X_n)$ be defined from objects in K^F using $\rightarrow$, $\times$ and $\oplus$. Each occurence of X_i is either positive (+) or negative (-). This is the signature. If all occurences of X_i have the same signature, we denote this by $\sigma(i)$. We let $\hat{\sigma}(i)$ denote the other signature. We then get

Corollary 6.1 *If $\Gamma(X_1, \ldots, \Gamma_n)$ is as above, then Γ induce a functor*

$$\Gamma^+ : K^{\sigma(1)} \times \cdots \times K^{\sigma(n)} \rightarrow K^+$$

and a functor

$$\Gamma^- : K^{\hat{\sigma}(1)} \times \cdots \times K^{\hat{\sigma}(n)} \rightarrow K^-.$$

Lemma 6.5 $\Gamma_\rightarrow$ *induce a functor*

$$\Gamma_\rightarrow^F : K^- \times K^F \rightarrow K^F.$$

Proof
The function space structure for $X \rightarrow Y$ depends on the total elements in X and the function space structure for Y. We need that $(X, \bar{X}) \in K_1$ to ensure that the total evaluations form a K_1-object.
The functoriality is trivial.

Now, let *the strict signature* of an occurence of X_i be - (negative), + (positive but not strictly positive) or F (strictly positive)

Lemma 6.6 *Assume that all occurences of each X_i have the same strict signature $\delta(i)$.*
Then Γ induce a functor

$$\Gamma^F : K^{\delta(1)} \times \cdots \times K^{\delta(n)} \rightarrow K^F.$$

The proof is trivial.

Lemma 6.7 *Let $\Gamma(X)$ be an operator in one variable where all occurences are positive (some may be strictly positive, others not).*
Then

a) Γ^+ *has a least fixpoint in K^+.*

b) Γ *will be functorial in K^F with a least fixpoint in K^F.*

Proof
Since we may form direct limits in both categories, and since the process in the category of domains will halt after ω many steps, the existence of the two fixpoints are trivial. Functoriality in b) is easy and is left for the reader.

The direct limit will not produce the same domain with totality in the two categories, in general we will produce more total elements in a direct limit in K^F than in K^+. In the case of strictly positive induction, the total elements of the two fixpoints will be the same, Corollary 5.1. We will show that this also holds for positive induction in general. The rest of the argument is due to Kristiansen [7], see also [10], in the setting of qualitative domains with totality.

Theorem 6.3 *Let $\Gamma(X)$ be a positive operator using K^F-objects as parameters. Then the total objects in the least fixpoints of Γ in K^+ and K^F are the same.*

Proof
Let $\Delta(Y, Z)$ be such that

Z occurs only strictly positive.

Y occurs positively but nowhere strictly positively.

$\Gamma(X) = \Delta(X, X)$.

Let U be the least fix point of Γ in K^+.
Let V be the least fixpoint of $\lambda Z.\Delta(U, Z)$ in K^F.
For simplicity we write $\bar{V}$ for the total elements in the domain induced by V.
We will prove that $\bar{U} = \bar{V}$.
First, let $U = \lim_{\alpha < \gamma} U_\alpha$ where $U_{\alpha+1} = \Gamma^+(U_\alpha)$ By induction on α we see that U_α is a substructure of the domain with totality induced from V.
We then use Corollary 5.1 and see that $\bar{V}$ is the fixpoint of the operator $\lambda Z.\Delta(U, Z)$ within K^+. Inside K^+ every step will be bounded by $\Delta(U, U) = U$, so $\bar{V} \subseteq \bar{U}$.

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Now, let $(X, \bar{X})$ be the domain with totality induced from the fixpoint of Γ in K^F. Clearly $\bar{U} \subseteq \bar{X}$. By induction on the steps in K^F towards $(X, \bar{X})$ we see that $\bar{X} \subseteq \bar{V}$. Consequently $\bar{X} = \bar{U}$, which is what we wanted to prove. This ends the proof of the theorem.

7 Appendices

In this paper we have defined certain categories of domains with totality and proved certain results about them. Via some examples we will show that these categories are different, and that at least some of our results are the best obtainable. This will be the purpose of the first three appendices.

Appendix 1
$K \not\subseteq K_1 \not\subseteq K_1 \not\subseteq K_3$

Consider the categories defined in section 2. We clearly have

$$K_3 \subseteq K_2 \subseteq K_1 \subseteq K$$

In this appendix we will show that the inclusions are proper. It is trivial to see that $K \not\subseteq K_1$.

Lemma 7.1 $K_1 \not\subseteq K_2$

Proof
Let X be the domain of all subsets of $\{0, 1\}$ ordered by inclusion, and let $\bar{X}$ be the nonempty subsets.
Then $\{0\} \approx_X \{0, 1\} \approx_X \{1\}$ but $\{0\} \not\approx_X \{1\}$.

Lemma 7.2 $K_2 \not\subseteq K_3$

Proof
We let X consist of all subsets A of $\mathbb{N}$ where all even numbers in A are greater than all uneven numbers in A. Let E be the set of even numbers and U be the set of uneven numbers.
We let $A \in \bar{X}$ if $A = U$ or if A contains almost all even numbers.
Clearly $(X, \bar{X})$ is in K_1. Moreover $A \approx_X B$ if $A = B = U$ or if both A and B contain almost all even numbers. This shows that $(X, \bar{X}) \in K_2$.
The equivalence class of E will contain objects arbitrarily close to U, so this equivalence class is not closed in the set of total objects. Thus $(X, \bar{X}) \notin K_3$.

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Appendix 2
A counterexample to the Haussdorff Topology property

We have proved that the quotient topology on the total elements of any evaluation structure is T_1. In this appendix we will show that it will not in general be Haussdorff. One might suspect that this is so because the axioms are not strong enough. We will however produce the counterexample using dependent products and sums, so the counterexample will be a structure that we want to cover by the axiomatisation.

The idea is to construct a Σ-type where we at one parameter have two inequivalent total objects, while we at arbitrarily close parameters have that all total objects are equivalent, and all objects are total.

Let O denote the domain $\{\bot\}$ with no total objects.

If $f \in \mathbb{N}_\bot \to \mathbb{N}_\bot$ and $n \in \mathbb{N}$, we let $B(f, n)$ be $\mathbb{N}_\bot$ if $f(n) = 0$, $B(f, n) = O$ otherwise.

Let

$$X = \Sigma(f \in \mathbb{N}_\bot \to \mathbb{N}_\bot)(\Pi(n \in \mathbb{N}_\bot)B(f, n) \to \mathbb{N}_\bot).$$

Let c_0 be function sending even $\bot$ to 0, c_1 the likewise constant 1 function. Let $f(n) = 0$ for all n, but $f(\bot) = \bot$. We then have that (f, c_0) and (f, c_1) are two nonequivalent elements of X.

Let O_1 and O_2 be open neighbourhoods of (f, c_0) and (f, c_1) resp. respecting the equivalence relation. Then there will be a $g \neq f$ such that $(g, c_0) \in O_1$ and $(g, c_1) \in O_2$.

Since $\Pi(n \in \mathbb{N}_\bot)B(g, n)$ has no total elements, (g, c_0) and (g, c_1) are equivalent. This equivalence class will thus be contained in both neighbourhoods. This shows that the quotient space is not Haussdorff.

One consequence of the lack of Haussdorff-property is that a convergent sequence may not have a unique limit. In the example above, this will also be the case for the limit structure defined in 4.2. If f is as above and $f = \lim_{n \to \infty} f_i$ where each $f_i \neq f$, then for any c_i and any total c we have that $[(f, c)] = \lim_{i \in \mathbb{N}}[(f_i, c_i)]$ (where $[\cdot]$ denotes the equivalence class).

Appendix 3
A counterexample to a topological lifting theorem

In this appendix we will show that the lifting property does not hold with respect to the topology on quotient spaces of evaluation structures. We will prove this by constructing one domain with totality using dependent sums and products and standard base domains, and by constructing a sequence of equivalence classes of total objects that will converge in the sense of the quotient topology, but with no lifting to a convergent sequence of total domain elements. Thus our counterexample will be one of the almost simplest continuous functions that exist, a convergent sequence.

Let $Y_f = \Pi(n \in \mathbb{N}_\perp)A_{f(n)}$. Then Y_f can be identified with $\mathbb{N} \to \mathbb{N}$ if f is constant zero, Y_f has no total objects otherwise.

Let $n(f) \in Y_f$ be constant n at those i where $f(i) = 0$. If $g : Y_f \to \mathbb{N}_\perp$, let $h(f, g)(n) = g(n(f))$

We are now ready to define our domain with totality where the counterexample takes place:
Let

$$X = \Sigma(f \in \mathbb{N}_\perp \to \mathbb{N}_\perp)\Sigma(g \in Y_f \to \mathbb{N}_\perp)(\Pi(y \in Y_f)Y_{h(f,g)} \to \mathbb{N}_\perp)$$

Lemma 7.3 X is a domain with totality.

Proof
The critical part is the parameterisation $\{Y_{h(f,g)}\}_{y \in Y_f}$, but either the left hand side has no total elements so the parameterisation is o.k., or the righthandside is a domain with totality because $h(f, g)$ is total.

Let $f : \mathbb{N} \to \mathbb{N}$ be constant zero, f_i constant zero exept for $f_i(i) = 1$.
Let $g(y) = 0$ when $y(0)$ is defined, and let g_i be the empty function on Y_{f_i}, which is total.
$h(f, g)$ will be the constant zero function.
Let c be the constant zero and let c_i be the constant 1.

Lemma 7.4 (f, g, c) is total and each (f_i, g_i, c_i) is total.

The proof is trivial.

We use $[\cdot]$ to denote equivalence classes in X.

Lemma 7.5 *In the topology of the set of equivalence classes of total elements in X we have*

$$[(f, g, c)] = \lim_{i \to \infty}[(f_i, g_i, c_i)]$$

Proof

Let $O \subseteq X$ be open such that O is closed under equivalence of total elements. Assume that $(f, g, c) \in O$.

Then there is a compact approximation (p, q, τ) to (f, g, c) such that the corresponding neighbourhood $B_{(p,q,\tau)}$ is a subset of O.

Since q is only defined for y's with $y(0)$ defined, we may extend q to a total g' such that $h(f, g')$ is not constant zero.

Then $\Pi(y \in Y_f)Y_{h(f,g')}$ has no total elements, so τ is total and equivalent to $\perp$. It follows that

$$(f, g', \perp) \in O.$$

Now there is some n such that for $i \geq n$ we have that $(f_i, g', \perp) \in O$, and then by extension, $(f_i, g', c_i) \in O$.

But Y_{f_i} has no total elements, so (f_i, g', c_i) is total, and equivalent to (f_i, g_i, c_i).

Thus, for $i \geq n$ we have that $(f_i, g_i, c_i) \in O$, and this was exactly what we aimed to prove.

Lemma 7.6 *If $x \in [(f, g, c)]$ and $x_i \in [(f_i, g_i, c_i)]$ for each $i \in \mathbb{N}$, we cannot have that $x = \lim_{i \to \infty} x_i$.*

Proof

Assume that this is possible, and let $x, \{x_i\}_{i \in \mathbb{N}}$ be an example.

Then x is of the form (f, g, d) and each x_i is of the form (f_i, g'_i, d_i), where $g = \lim_{i \to \infty} g'_i$.

Then, in the sense of the description as an element in S_{wf} we have

$$\Pi(y \in Y_f)Y_{h(f,g)} = \lim_{i \to \infty} \Pi(y \in Y_{f_i})Y_{h(f_i,g'_i)}$$

and in the latter domains every object is total.

Let z be total in $\Pi(y \in Y_f)Y_{h(f,g)}$ and let $z_i \in \Pi(y \in Y_{f_i})H_{h(f_i,g'_i)}$ such that $z = \lim_{i \to \infty} z_i$.

Then $d(z) = 0$, and by continuity

$$\lim_{i \to \infty} d_i(z_i) = d(z) = 0,$$

contradicting the choice of d_i (as total and equivalent to c_i).
This ends the proof of the lemma.

A characterisation of topological lifting

It Theorem 4.3 we proved that if a function $F : \bar{Z} \to U$ preserve convergency with respect to the limit structure imposed on U, then F has a lifting. Now, if F is just continuous, it follows that F has a lifting provided all sequences converging in the sense of the topology also converges with the same limits in the sense of the limit structure.

A convergent sequence can be viewed as a continuous map from the ordinal $\omega + 1$ with the order topology. It is not hard to see that this topological space can be realised as the total objects in a K_1-object. We thus have obtained the following:

Theorem 7.1 *Let $\tilde{E}$ be an evaluation structure with the topological space U of equivalence classes of total objects in $(X, \bar{X})$.*
Then the following are equivalent:

i) *For every K_1-object $(Z, \bar{Z})$ and every continuous funnction $F : \bar{Z} \to U$ there is a lifting $\hat{F} : Z \to X$ of F.*

ii) *Whenever $u \in U$ is a limit, in the sense of the topology, of the sequence $\{u_i\}_{i \in \mathbb{N}}$, then there is an $x \in u$ and $x_i \in u_i$ such that $x = \lim_{i \to \infty} x_i$.*

Appendix 4
Not every domain is induced from an RoF-structure

In this appendix we will prove a finite partition property of function spaces of the form $E \to A_\perp$ and derive a kind of compactness property for partitions of compacts in subdomains of $E \to A_\perp$. This property will not be shared by all separable domains. Thus there are separable domains that cannot be induced from an RoF-structure.

In this appendix we let E be a domain, $A_\perp$ a flat domain, $Y = E \to A_\perp$ and finaly we let X be a subdomain of Y.

It is well known that the compacts in Y can be represented by finite sets

$$p = \{(p_1, a_1), \ldots, (p_n, a_n)\}$$

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where $p_1, \ldots, p_n$ are compacts in E and $a_1, \ldots, a_n$ are elements in A.

If $a_i = a_j$ and $p_i < p_j$, then (p_j, a_j) is a redundant element of p. We will assume that we only use representations of compacts as above without redundancies.

Lemma 7.7 *The representation of a compact in Y is unique.*

Proof
Let $p = \{(p_1, a_1), \ldots, (p_n, a_n)\}$ and $q = \{(q_1, b_1), \ldots, (q_m, b_m)\}$ represent the same compact. We prove that $p \subseteq q$ and the lemma follows by symmetry.

Let $(p_i, a_i) \in p$. Then

$$a_i \leq lub\{b_j \mid q_j \leq p_i\}.$$

In particular, for some j we have $a_i = b_j$ and $q_j \leq p_i$.

By the same argument there is a k such that $a_k = b_j$ and $p_k \leq q_j$. Since there are no redundancies, $p_k = p_i$ so $q_j = p_i$. Thus $(p_i, a_i) \in q$.

From now on we identify a compact with its non-redundant representation. We use $\sqcap$ and $\sqcup$ for the domain theoretical operations and $\cap$ and $\cup$ for the set theoretical operations.

Lemma 7.8 *Let p, q and r be compacts in Y with $p = q \sqcup r$.*
Let $q' = q \cap p$ and $r' = r \cap p$.
Then $p = q' \sqcup r'$.

Proof
By assumption, p is equivalent to the compact given by the (possibly overloaded with redundancies) representation $q \cup r$. By the argument of Lemma 7.7 we have $p \subseteq q \cup r$. But then

$$p = (q \cap p) \cup (r \cap p) = q' \cup r'.$$

Lemma 7.9 *Let p be a compact in X and let $\{(q_i, r_i)\}_{i \in \mathbb{N}}$ be an infinite sequence of pairs of compacts in X such that $p = q_i \sqcup r_i$ for all $i \in \mathbb{N}$.*
Then there exist n and m with $n < m$ such that $p = q_n \sqcup r_m$.

Proof
Working inside Y, let q_i' and r_i' be constructed as above. Choose $n < m$ such that $q_n' = q_m'$ and $r_n' = r_m'$. The property follows.

Corollary 7.1 *Not every separable domain is isomorphic to a domain induced from an RoF-structure.*

Proof
The domain generated from the set of finite and cofinite subsets of $\mathbb{N}$ ordered by inclusion will not satisfy the property of Lemma 7.9.

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