Using the Donsker Delta Function to Compute Hedging Strategies

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ABSTRACT. We use white noise calculus and the Donsker Delta Function to find explicit formulas for the replicating portfolios in a Black-Scholes market for a class of contingent T-claims.

1. Introduction

As a motivation for this paper we start by considering the following problem from mathematical economics:

Fix $T > 0$ and consider the following simple market model, with two securities:

1) A risk-free asset (e.g., a bank account), where the price A_t per unit at time t is given by the differential equation
$$dA_t = \rho(t)A_t dt \ , \ A_0 = 1 \tag{1.1}$$
2) A risky asset (e.g., a stock), where the price S_t per unit at time t is given by the stochastic differential equation
$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dB_t \ , \ S_0 > 0 \text{ constant} \tag{1.2}$$

Here $\rho(t), \mu(t)$ and $\sigma(t)$ are given deterministic functions with the property that

$$\int_0^T \left(|\rho(s)| + |\mu(s)| + \sigma^2(s)\right) ds < \infty \tag{1.3}$$

For simplicity we assume that σ is bounded away from zero. $B_t = B_t(\omega); t \geq 0, \omega \in \Omega$ denotes 1-dimensional Brownian motion starting at zero. The probability law of B_t is denoted by P and the σ-algebra generated by $\{B_s(\cdot)\}_{s\leq t}$ is denoted by $\mathcal{F}_t$. We refer to, e.g., [Ø] for more information about stochastic differential equations.

Now let $\phi : [0,T] \to \mathbf{R}$ be another deterministic function such that $\int_0^T \phi^2(t)dt < \infty$, and define

$$Z(t) = Z(t,\omega) = \int_0^t \phi(s)dB_s(\omega); \ 0 \leq t \leq T \tag{1.4}$$

Let $F(\omega)$ be a *contingent T-claim of Markovian type*, i.e., given by

$$F(\omega) = h(Z(T)) \tag{1.5}$$

where $h : \mathbf{R} \to \mathbf{R}$ is a bounded measurable function. Since this simple extension of the Black and Scholes market is complete, it is well known that the claim F can be hedged, i.e., there exist a *replicating (self-financing) portfolio* for F (see details below). The problem we study in this paper is:

PROBLEM 1.1

How do we find explicitly such a replicating portfolio for F?

We now want to describe this in more detail:

Let (ξ_t, η_t) be a *portfolio*, i.e., $\xi_t = \xi_t(\omega), \eta_t = \eta_t(\omega)$ are $\mathcal{F}_t$-adapted stochastic processes interpreted as the number of units held by a person at time t of assets #1 and #2, respectively. Then the *value* $V_t = V_t(\omega)$ of this portfolio at time t is defined by

$$V_t = \xi_t A_t + \eta_t S_t \tag{1.6}$$

The portfolio is called *self-financing* if

$$dV_t = \xi_t dA_t + \eta_t dS_t \tag{1.7}$$

This means that no external funds are added to the portfolio and that no funds are extracted from the portfolio as time evolves. From now on we consider only self-financing portfolios. From (1.6) we get

$$\xi_t = \frac{V_t - \eta_t S_t}{A_t} \tag{1.8}$$

Substituting this in (1.7) and using (1.1)–(1.2), we get

$$\begin{aligned} dV_t &= (V_t - \eta_t S_t)\frac{dA_t}{A_t} + \eta_t dS_t \\ &= \rho(t)V_t dt + \eta_t S_t\big((\mu(t) - \rho(t))dt + \sigma(t)dB_t\big) \end{aligned}$$

Since $\sigma(t) \neq 0$ for a.a. t, this can be written

$$dV_t = \rho(t)V_t dt + \sigma(t)\eta_t S_t\big(\alpha(t)dt + dB_t\big) \tag{1.9}$$

where

$$\alpha(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)} \tag{1.10}$$

Multiplying (1.9) by the integrating factor $e^{-\int_0^t \rho(s)ds}$, we get

$$d\left(e^{-\int_0^t \rho(s)ds} V_t\right) = e^{-\int_0^t \rho(s)ds}\sigma(t)\eta_t S_t\big(\alpha(t)dt + dB_t\big)$$

Hence

$$e^{-\int_0^T \rho(s)ds} V_T(\omega) = V_0 + \int_0^T e^{-\int_0^t \rho(s)ds}\sigma(t)\eta_t S_t\big(\alpha(t)dt + dB_t\big) \tag{1.11}$$

Now suppose that $F(\omega)$ is a given (European) contingent T-claim, i.e., $F(\omega)$ is a given $\mathcal{F}_T$-measurable, lower bounded random variable. To *hedge* such a claim means to find a constant V_0 and a self-financing portfolio (ξ_t, η_t) such that the corresponding value process V_t starts up with value V_0 for $t = 0$ and ends up with the value

$$V_T(\omega) = F(\omega) \ a.s. \tag{1.12}$$

at time T. V_0 is then the market value of F at time 0. We also require that the process $\{V_t\}_{t\in[0,T]}$ is (t,ω)-a.s. lower bounded. By (1.11) combined with (1.8) we see that it suffices to find V_0 and a process $u(t,\omega)$ such that

$$e^{-\int_0^T \rho(s)ds} F(\omega) = V_0 + \int_0^T u(t,\omega)\big(\alpha(t)dt + dB_t\big) \tag{1.13}$$

and

$$P\left[\int_0^T u^2(s,\omega)ds < \infty\right] = P\left[\int_0^T |u(s,\omega)\alpha(s)|ds < \infty\right] = 1 \tag{1.14}$$

and such that $\left\{\int_0^t u(t,\omega)\big(\alpha(t)dt + dB_t\big)\right\}_{t\in[0,T]}$ is lower bounded. If such a process $u(t,\omega)$ is found, we put

$$\eta_t = e^{\int_0^t \rho(s)ds}\sigma(t)^{-1}S_t^{-1}u(t,\omega) \tag{1.15}$$

and solve for ξ_t using (1.8). It is well known and easy to see by the Girsanov theorem that if

$$\int_0^T \alpha^2(s)ds < \infty \tag{1.16}$$

then V_0 is unique and given by

$$V_0 = E_Q\left[e^{-\int_0^T \rho(s)ds}F\right] \tag{1.17}$$

(provided this quantity is finite), where E_Q denotes expectation with respect to the measure Q defined on $\mathcal{F}_T$ by

$$dQ(\omega) = \exp\left[-\int_0^T \alpha(s)dB_s - \frac{1}{2}\int_0^T \alpha^2(s)ds\right]dP(\omega) \tag{1.18}$$

so that

$$\tilde{B}_t := \int_0^t \alpha(s)ds + B_t \tag{1.19}$$

is a Brownian motion with respect to Q. To find $u(t,\omega)$, there are several known methods:

a) If the claim $F(\omega)$ is of *Markovian type*, i.e.,

$$F(\omega) = h(S_T(\omega))$$

for some (deterministic) function $h : \mathbf{R} \to \mathbf{R}$, then $u(t,\omega)$ can (in principle) be found by solving a (deterministic) boundary value problem for a parabolic partial differential equation. See [BS], [M], and [D, Section 5D] for details.

b) For some not necessarily Markovian type claims $F(\omega)$ one can (in principle) apply the Clark-Ocone theorem (as extended by Karatzas and Ocone [KO]) to express $u(t,\omega)$ as follows:

$$u(t,\omega) = E_Q\left[D_tF|\mathcal{F}_t\right] \tag{1.20}$$

where D_tF is the Malliavin derivative of F at t. The problems with this formula are:

i) It is in general difficult to compute conditional expectations

and

ii) the Malliavin derivative D_tF only exists under additional restrictions on F. For example, it is *not* sufficient that $F \in L^2(\mathcal{F}_T, P)$ and it does not exist for the F given by (1.5) if h is not differentiable.

The purpose of this paper is to give an alternative approach based on white noise calculus and the Donsker delta function. We will show how this approach gives explicit formulas quickly and with easy, intuitive proofs, once the basic white noise calculus has been established. We illustrate this by using the method to solve Problem 1.1. See Theorem 3.9. and Corollary 3.10 together with the remarks following the corollary. Although Problem 1.1 could also be solved by Method a) and - with some additional work - by Method b), it is conceivable that the white noise approach can cover some cases which are not well adapted to Methods a) and b). Moreover, it may give new insights. See (3.32) and the corresponding remark.

In Section 2 we briefly recall some of the basic white noise theory. Then in Section 3 we give a special representation of the Donsker delta function and combine it with white noise calculus to compute explicitly the hedging strategies requested in Problem 1.1. In Section 4 we prove a similar formula for the n-dimensional case.

2. White noise, Hida distributions and the Wick product

Here we briefly recall some of the main concepts and results from white noise theory. For more information we refer the reader to [HKPS] and [HØUZ]. Our notation will follow that from [HØUZ].

From now on we will assume that our Brownian motion is constructed on a *white noise probability space* $(\Omega, \mathcal{F}, P)$ and we let $(\mathcal{S})$ and $(\mathcal{S})^*$ denote the space of *stochastic test functions* and the space of *stochastic distributions (Hida distributions)*, respectively.

Using the *Hermite functions* $e_1(x), e_2(x), \ldots$ (which form an orthonormal basis for $L^2(\mathbf{R})$) and the *Hermite polynomials* $h_n(x);\ n = 0, 1, 2, \ldots$, one constructs an orthogonal $L^2(P)$ basis

$$\{H_\alpha(\omega)\}_{\alpha \in \mathcal{I}}$$

where $\mathcal{I}$ denotes the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ of arbitrary but finite length, where $\alpha_1, \alpha_2, \ldots$ are non-negative integers. Thus every $X \in L^2(P)$ has a unique representation

$$X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega);\ c_\alpha \in \mathbf{R}$$

where

$$||X||^2_{L^2(P)} = E_P[X^2] = \sum_\alpha \alpha! c_\alpha^2 \tag{2.1}$$

and where $\alpha! = \alpha_1!\alpha_2! \cdots$ when $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}$.

The space $(\mathcal{S})$ of stochastic test functions can be described as the set of all $X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in L^2(P)$ such that

$$||X||^2_{0,k} := \sum_\alpha \alpha! c_\alpha^2 (2\mathbf{N})^{q\alpha} < \infty \quad \text{for all } q \in \mathbf{R} \tag{2.2}$$

where

$$(2\mathbf{N})^\beta = 2^{\beta_1}(2 \cdot 2)^{\beta_2} \cdots (2k)^{\beta_k} \cdots \quad \text{if } \beta = (\beta_1, \beta_2, \ldots) \in \mathcal{I}$$

Similarly, the space $(\mathcal{S})^*$ of Hida distributions can be described as the set of formal series $X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$ such that there exists $q \in \mathbf{R}$ such that

$$||X||^2_{0,-q} := \sum_\alpha \alpha! c_\alpha^2 (2\mathbf{N})^{-q\alpha} < \infty \tag{2.3}$$

Thus we have

$$(S) \subset L^2(P) \subset (S)^*$$

The family of seminorms $|| \cdot ||_{0,k}; k \in \mathbf{R}$ gives a natural projective topology on (S) and an inductive topology on $(S)^*$. With these topologies $(\mathcal{S})^*$ becomes the dual of $(\mathcal{S})$. The action of $F(\omega) = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})^*$ on $f(\omega) = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})$ is given by

$$< F, f >= \sum_\alpha \alpha! a_\alpha b_\alpha \tag{2.4}$$

One of the important features about the Hida space $(\mathcal{S})^*$ is that it contains the *singular white noise* $W_t(\omega)$ for all $t \in \mathbf{R}$. More precisely, if we define

$$W_t(\omega) = \sum_{i=1}^\infty e_i(t) H_{\epsilon_i}(\omega) \tag{2.5}$$

where $\epsilon_i = (0, 0, \ldots, 1, \ldots)$ with 1 on the ith place, then $W_t(\omega) \in (\mathcal{S})^*$ for each t and we have the crucial identities

$$\frac{d}{dt} B_t(\omega) = W_t \qquad (\text{in } (\mathcal{S})^*) \tag{2.6}$$

and

$$B_t(\omega) = \int_0^t W_s ds \qquad (\text{integration in } (\mathcal{S})^*) \tag{2.7}$$

The last identity can be generalized considerably by means of the *Wick product*:

DEFINITION 2.1
The Wick product $X \diamond Y$ *of* $X(\omega) = \sum_\alpha a_\alpha H_\alpha(\omega) \in (\mathcal{S})^*$ *and* $Y(\omega) = \sum_\alpha b_\beta H_\beta(\omega) \in (\mathcal{S})^*$ *is defined by*

$$\big(\mathcal{X} \diamond Y\big)(\omega) = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta}(\omega) = \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) H_\gamma(\omega)$$

For example we have

$$\left(B_t \diamond B_t\right)(\omega) = B_t^2(\omega) - t \tag{2.8}$$

and more generally

$$\left(\int_{\mathbf{R}} \phi(s) dB_s\right) \diamond \left(\int_{\mathbf{R}} \psi(s) dB_s\right) = \left(\int_{\mathbf{R}} \phi(s) dB_s\right) \cdot \left(\int_{\mathbf{R}} \psi(s) dB_s\right) - \int_{\mathbf{R}} \phi(s)\psi(s) ds \tag{2.9}$$

for all $\phi, \psi \in L^2(\mathbf{R})$. Some important properties of the Wick product are listed below:

$$X \in (\mathcal{S})^*, Y \in (\mathcal{S})^* \;\Rightarrow\; X \diamond Y \in (\mathcal{S})^* \tag{2.10}$$

$$X \in (\mathcal{S}), Y \in (\mathcal{S}) \;\Rightarrow\; X \diamond Y \in (\mathcal{S}) \tag{2.11}$$

$$X \diamond Y = Y \diamond X \qquad \text{(commutative law)} \tag{2.12}$$

$$X \diamond (Y \diamond Z) = (X \diamond Y) \diamond Z \qquad \text{(associative law)} \tag{2.13}$$

$$X \diamond (Y + Z) = X \diamond Y + X \diamond Z \qquad \text{(distributive law)} \tag{2.14}$$

$$X \diamond Y = X \cdot Y \qquad \text{if } X \text{ or } Y \text{ is } \textit{deterministic} \tag{2.15}$$

$$E[X \diamond Y] = E[X] \cdot E[Y] \qquad \text{(when defined)} \tag{2.16}$$

Using the associative law we can define Wick powers

$$X^{\diamond n} = X \diamond X \diamond \cdots \diamond X \qquad (n \text{ times})$$

More generally, if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is entire, i.e., an analytic function of the complex variable z in the complex plane $\mathbf{C}$, we can - for some $X \in (\mathcal{S})^*$ - define the *Wick version*

$$f^\diamond(X) = \sum_{k=0}^{\infty} a_k X^{\diamond k} \in (\mathcal{S})^* \tag{2.17}$$

For example, if $\phi \in L^2(\mathbf{R})$ is deterministic, then

$$\exp^\diamond \left[\int_{\mathbf{R}} \phi(s) dB_s\right] = \exp\left[\int_{\mathbf{R}} \phi(s) dB_s - \frac{1}{2}\int_{\mathbf{R}} \phi^2(s) ds\right] \tag{2.18}$$

We also mention the *chain rule* in $(\mathcal{S})^*$: Suppose $t \mapsto X_t : \mathbf{R} \to (\mathcal{S})^*$ is continuously differentiable and let $f : \mathbf{C} \to \mathbf{C}$ be entire such that $f(\mathbf{R}) \subset \mathbf{R}$ and $f^\diamond(X_t) \in (\mathcal{S})^*$ for all t, then

$$\frac{d}{dt}\left[f^\diamond(X_t)\right] = f'^\diamond(X_t) \diamond \frac{d}{dt} X_t \qquad \text{in } (\mathcal{S})^* \tag{2.19}$$

Finally we recall the following important connection between Ito integration and the Wick product:

Let $u(t,\omega)$ be an $\mathcal{F}_t$-adapted process such that $E[\int_a^b u^2(t,\omega) dt] < \infty$. Then $u(t,\omega) \diamond W_t$ is integrable in $(\mathcal{S})^*$ and

$$\int_a^b u(t,\omega) dB_t(\omega) = \int_a^b u(t,\omega) \diamond W_t(\omega) dt \tag{2.20}$$

(See [LØU], [B] and [HØUZ, Theorem 2.5.9] and the references therein). As a simple example to illustrate the above, first note that by the chain rule we have

$$\frac{d}{dt}\left[\exp^{\diamond}[B_t]\right] = \exp^{\diamond}[B_t] \diamond \frac{dB_t}{dt} = \exp^{\diamond}[B_t] \diamond W_t$$

and hence

$$\begin{aligned}\exp^{\diamond}[B_t] &= \exp^{\diamond}[B_0] + \int_0^t \exp^{\diamond}[B_s] \diamond W_s ds \\ &= 1 + \int_0^t \exp^{\diamond}[B_s] dB_s\end{aligned}$$

which is a direct proof (without using the Ito formula) that $\exp^{\diamond}[B_t]$ is a martingale.

The Hermite transform

In white noise analysis one makes use of several different transforms, the most popular being the $\mathcal{S}$-*transform* and the *Hermite transform*, $\mathcal{H}$. The construction of these transforms depends on the particular choice of Hermite functions as a basis for $L^2(\mathbf{R})$. When expanded along this basis, the $\mathcal{H}$-transform can be defined as follows

DEFINITION 2.2

Let $X(\omega) = \sum_\alpha a_\alpha H_\alpha(\omega) \in (\mathcal{S})^*$, then the Hermite transform of X (with respect to the basis $\{e_k\}_k$), denoted by $\mathcal{H}X$ or $\tilde{X}$, is defined by

$$\mathcal{H}X(z) = \tilde{X}(z) = \sum_\alpha a_\alpha z^\alpha \in \mathbf{C} \qquad \textit{(when convergent)} \tag{2.21}$$

where $z = (z_1, z_2, \ldots) \in \mathbf{C}^{\mathbf{N}}$ (the set of all sequences of complex numbers), and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \cdots \tag{2.22}$$

if $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}$, where $z_j^0 = 1$.

One can verify that the sum in (2.21) converges for all $z \in \mathbf{C}_c^{\mathbf{N}}$ (the set of all finite length sequences of complex numbers), and that any element in $(\mathcal{S})^*$ is uniquely characterized through its $\mathcal{H}$-transform. We recall the important relation

$$\mathcal{H}[X \diamond Y](z) = \mathcal{H}X(z) \cdot \mathcal{H}Y(z) \tag{2.23}$$

(2.23) can be extended to cover Wick-versions, so in general

$$\mathcal{H}[f^{\diamond}(X)](z) = f(\mathcal{H}[X](z)) \qquad \text{(when convergent)} \tag{2.24}$$

if $f : \mathbf{C} \to \mathbf{C}$ is entire, $f(\mathbf{R}) \subset \mathbf{R}$ and $f^{\diamond}(X) \in (\mathcal{S})^*$.

While the basis of Hermite functions is necessary for the definition of the topological structure in the Hida distribution space, it turns out that other bases may be convenient for computational purposes. If we remain within $L^2(P)$, the Wick product can be expanded along any orthonormal basis for $L^2(\mathbf{R})$, see [HØUZ, Appendix D: Base invariance of the Wick product]. In what follows we will sometimes specialize the theory to Wick powers of smoothed white noise. Within this context a different version of the $\mathcal{H}$-transform can be considered. By abuse of notation, we do not distinguish between the (strictly speaking, different) versions of the transform.

Given any $\phi \in L^2(\mathbf{R})$, we define the smoothed white noise, $w(\phi) = w(\phi, \omega)$, by

$$w(\phi, \omega) := \int_{\mathbf{R}} \phi(s) dB_s(\omega) \tag{2.25}$$

If we consider the context of random variables on the form $X(\omega) = \sum_{k=0}^{\infty} a_k w(\phi)^{\diamond k}$, then it is convenient to make use of the following formulation, see [GHLØUZ, §4.1]:

PROPOSITION 2.3

Let $\phi \in L^2(\mathbf{R})$ with $||\phi||_{L^2(\mathbf{R})} = 1$. Suppose $X(\omega) = \sum_{k=0}^{\infty} a_k w(\phi)^{\diamond k} \in (\mathcal{S})^*$, and define $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for $z \in \mathbf{C}$. Suppose $y \mapsto f(x + i\,y)$ is integrable with respect to the measure $e^{-y^2/2}dy$ on $\mathbf{R}$ for all $x \in \mathbf{R}$ and put

$$F(x) = \int_{-\infty}^{\infty} f(x + i\,y) e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$

Suppose $V(\omega) := F(w(\phi,\omega)) \in L^2(P)$. Then $X(\omega) = V(\omega)$ a.s., i.e.,

$$X(\omega) = \int_{-\infty}^{\infty} f(x + iy) e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} \Big|_{x = w(\phi,\omega)} \tag{2.26}$$

3. The Donsker delta function and the first main theorem

Donskers δ-function is a generalized white noise functional which have been treated in several papers within white noise analysis, see, e.g., [H], [K] and also [HKPS] and the references therein. For completeness we give an independent presentation here.

DEFINITION 3.1

Let $Y : \Omega \to \mathbf{R}$ be a random variable which also belongs to $(\mathcal{S})^*$. Then a continuous function

$$\delta_Y(\cdot) : \mathbf{R} \to (\mathcal{S})^*$$

is called a Donsker delta function of Y if it has the property that

$$\int_{\mathbf{R}} g(y)\delta_Y(y)dy = g(Y) \qquad a.s. \tag{3.1}$$

for all (measurable) $g : \mathbf{R} \to \mathbf{R}$ such that the integral converges.

PROPOSITION 3.2

Suppose Y is a normally distributed random variable with mean m and variance $v > 0$. Then δ_Y is unique and is given by the expression

$$\delta_Y(y) = \frac{1}{\sqrt{2\pi v}} \cdot \exp^{\diamond}\left[-\frac{(y-Y)^{\diamond 2}}{2v}\right] \in (\mathcal{S})^* \tag{3.2}$$

PROOF

Let $G_Y(y)$ denote the right hand side of (3.2). It follows from the characterization theorem for $(\mathcal{S})^*$ (see [PS]) that $G_Y(y) \in (\mathcal{S})^*$ for all y and that $y \mapsto G_Y(y)$ is continuous for $y \in \mathbf{R}$. We verify that G_Y satisfies (3.1), i.e., that

$$\int_{\mathbf{R}} g(y)G_Y(y)dy = g(Y) \qquad \text{a.s.} \tag{3.3}$$

First let us assume that g has the form

$$g(y) = e^{\lambda y} \qquad \text{for some } \lambda \in \mathbf{C} \tag{3.4}$$

Then by taking the Hermite transform of the left hand side of (3.3), we get

$$\begin{aligned} \mathcal{H}\left[\int_{\mathbf{R}} g(y)G_Y(y)dy\right] &= \int_{\mathbf{R}} e^{\lambda y}\mathcal{H}\left[G_Y(y)\right]dy \\ &= \int_{\mathbf{R}} e^{\lambda y}\frac{1}{\sqrt{2\pi v}}\exp\left[-\frac{(y-\tilde{Y})^2}{2v}\right]dy \end{aligned} \tag{3.5}$$

where $\tilde{Y} = \tilde{Y}(z)$ is the Hermite transform of Y at $z = (z_1, z_2, \ldots) \in \mathbf{C}^{\mathbf{N}}$. The expression (3.5) may be regarded as the expectation of $e^{\lambda Z}$ where Z is a normally distributed random variable with mean $\tilde{Y}$ and variance v. Now $Z := Y - m + \tilde{Y}$ is such a random variable. Hence (3.5) can be written as $E[e^{\lambda(Y-m+\tilde{Y})}]$, which by the well known formula for the characteristic function of a normal random variable is equal to $\exp[\lambda\tilde{Y} + \frac{1}{2}\lambda^2 v]$. We conclude that

$$\begin{aligned}\mathcal{H}\left[\int_{\mathbf{R}} g(y)G_Y(y)dy\right] &= \exp[\lambda\tilde{Y} + \frac{1}{2}\lambda^2 v] = \mathcal{H}\left[\exp^{\diamond}[\lambda Y + \frac{1}{2}\lambda^2 v]\right] \\ &= \mathcal{H}[\exp[\lambda Y]] = \mathcal{H}[g(Y)]\end{aligned}$$

This proves that (3.3) holds for functions g given by (3.4). Therefore (3.3) also holds for linear combinations of such functions. By a well known density argument, (3.3) holds for all g such that the integral in (3.1) converges.

It remains to prove uniqueness: If $H_1 : \mathbf{R} \to (\mathcal{S})^*$ and $H_2 : \mathbf{R} \to (\mathcal{S})^*$ are two continuous functions such that

$$\int_{\mathbf{R}} g(y)H_i(y)dy = g(Y);\ i = 1, 2 \tag{3.6}$$

for all g such that the integral converges, then in particular (3.6) must hold for all continuous functions with compact support. But then clearly we must have

$$H_1(y) = H_2(y) \qquad \text{for a.a. } y \in \mathbf{R}$$

and hence for all y by continuity.

⊓

LEMMA 3.3

Let $\psi : [0, T] \to \mathbf{R}, \phi : [0, T] \to \mathbf{R}$ *be deterministic functions and such that* $\int_0^T |\psi(s)|ds < \infty$ *and* $||\phi||^2_{[0,T]} := \int_0^T \phi^2(s)ds < \infty$. *Define*

$$Y(t) = \int_0^t \psi(s)ds + \int_0^t \phi(s)dB_s,\ 0 \le t \le T \tag{3.7}$$

Then

$$\begin{aligned}\exp^{\diamond}\left[-\frac{(y - Y(T))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] &= \exp^{\diamond}\left[-\frac{y^2}{2||\phi||^2_{[0,T]}}\right] \\ + \int_0^T \exp^{\diamond}\left[-\frac{(y - Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] &\diamond \frac{y - Y(t)}{||\phi||^2_{[0,T]}} \diamond (\psi(t) + \phi(t)W_t)dt\end{aligned} \tag{3.8}$$

PROOF

This is just an application of the fundamental theorem of calculus plus the chain rule in $(\mathcal{S})^*$: Define $H : [0, T] \to (\mathcal{S})^*$ by

$$H(t) = \exp^{\diamond}\left[-\frac{(y - Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right];\ 0 \le t \le T \tag{3.9}$$

Then

$$\begin{aligned} H(T) &= H(0) + \int_0^T \frac{dH}{dt} dt \\ &= \exp^{\diamond}\left[-\frac{y^2}{2||\phi||^2_{[0,T]}}\right] \\ &+ \int_0^T \exp^{\diamond}\left[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] \diamond \frac{d}{dt}\left[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] dt \\ &= \exp^{\diamond}\left[-\frac{y^2}{2||\phi||^2_{[0,T]}}\right] \\ &+ \int_0^T \exp^{\diamond}\left[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] \diamond \frac{y-Y(t)}{||\phi||^2_{[0,T]}} \diamond (\psi(t)+\phi(t)W_t)dt \end{aligned}$$

⊓

We are now ready for the first main result in this section:

THEOREM 3.4

Let $\phi : [0,T] \to \mathbf{R}, \alpha : [0,T] \to \mathbf{R}$ be deterministic functions such that

$$0 < ||\phi||^2_{[0,T]} := \int_0^T \phi^2(s)ds < \infty \quad \textit{and} \quad 0 \leq \int_0^T \alpha^2(s)ds < \infty \tag{3.10}$$

Define

$$Y(t) = Y(t,\omega) = \int_0^t \phi(s)dB_s + \int_0^t \phi(s)\alpha(s)ds;\ 0 \leq t \leq T \tag{3.11}$$

Let $f : \mathbf{R} \to \mathbf{R}$ be bounded. Then

$$f(Y(T)) = V_0 + \int_0^T u(t,\omega) \diamond (\alpha(t) + W_t)dt \tag{3.12}$$

where

$$V_0 = \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \exp\left[-\frac{y^2}{2||\phi||^2_{[0,T]}}\right] dy \tag{3.13}$$

and

$$u(t,\omega) = \phi(t) \cdot \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \exp^{\diamond}\left[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] \diamond \frac{y-Y(t)}{||\phi||^2_{[0,T]}} dy \tag{3.14}$$

PROOF

We now combine Proposition 3.2 and Lemma 3.3 to get, with $\psi(s) = \phi(s)\alpha(s)$

$$\begin{aligned} f(Y(T)) &= \int_{\mathbf{R}} f(y)\delta_{Y(T)}(y)dy = \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \exp^{\diamond}\left[-\frac{(y-Y(T))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] dy \\ &= \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \exp\left[-\frac{y^2}{2||\phi||^2_{[0,T]}}\right] dy + \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \cdot \\ &\left(\int_0^T \exp^{\diamond}\left[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] \diamond \frac{y-Y(t)}{||\phi||^2_{[0,T]}} \diamond (\phi(t)\alpha(t) + \phi(t)W_t)dt\right) dy \\ &= V_0 + \int_0^T \phi(t)\left(\int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \cdot \right. \\ &\left. \exp^{\diamond}\left[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] \diamond \frac{y-Y(t)}{||\phi||^2_{[0,T]}} dy\right) \diamond (\alpha(t) + W_t)dt \\ &= V_0 + \int_0^T u(t,\omega) \diamond (\alpha(t) + W_t)dt \end{aligned} \tag{3.15}$$

$\sqcap$

In the application of Theorem 3.4 the following result will be useful:

PROPOSITION 3.5

Let $p(x) = ax^2 + bx + c$, where a, b, c are real constants. Let ψ be as before and suppose that $2|a|||\psi||^2 < 1$, where $||\psi||^2 = \int_{\mathbf{R}} \psi^2(s)ds$. Define

$$Y(\omega) = \int_{\mathbf{R}} \psi(s)dB_s$$

Then

$$\exp^{\diamond}[aY^{\diamond 2} + bY + c] = K_{\psi}^{-1} \exp\left[K_{\psi}^{-2}\left(aY^2 + bY + c + \frac{(4ac - b^2)||\psi||^2}{2}\right)\right] \tag{3.16}$$

where the constant K_{ψ} is defined by

$$K_{\psi} := \sqrt{1 + 2a||\psi||^2} \tag{3.17}$$

PROOF

We expand the Wick product along a base with $\phi := \frac{\psi}{||\psi||}$ as its first base element, and note that $Y(\omega) = ||\psi||w(\phi)$. With reference to Proposition 2.3, consider

$$f(z) := e^{a||\psi||^2 z^2 + b||\psi||z + c} \tag{3.18}$$

Fix $x \in \mathbf{R}$. Then

$$\begin{aligned}
F(x) &:= \int_{-\infty}^{\infty} f(x + i\,y)e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\
&= \int_{-\infty}^{\infty} e^{a||\psi||^2(x^2 - y^2 + 2i\,xy) + b||\psi||(x + i\,y) + c}e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\
&= \int_{-\infty}^{\infty} e^{a||\psi||^2 x^2 + b||\psi||x + c}e^{i(2xa||\psi|| + b)||\psi||y - (\frac{1}{2} + a||\psi||^2)y^2}\frac{dy}{\sqrt{2\pi}} \\
&= e^{a||\psi||^2 x^2 + b||\psi||x + c} \cdot \frac{1}{\sqrt{1 + 2a||\psi||^2}}e^{-\frac{(2ax||\psi|| + b)^2||\psi||^2}{2 + 4a||\psi||^2}}
\end{aligned} \tag{3.19}$$

In this calculation we made use of the familiar formula

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{i\alpha t - \beta^2 t^2}dt = \frac{1}{\sqrt{2\beta}}e^{-\frac{\alpha^2}{4\beta}} \tag{3.20}$$

Hence

$$\begin{aligned}
&V(\omega) := F(w(\psi)) \\
&= e^{aw(\psi)^2 + bw(\psi) + c} \cdot \frac{1}{\sqrt{1 + 2a||\psi||^2}}e^{-\frac{(2aw(\psi) + b)^2||\psi||^2}{2 + 4a||\psi||^2}} \\
&= \frac{1}{\sqrt{1 + 2a||\psi||^2}}e^{aw(\psi)^2 + bw(\psi) + c}e^{-\frac{2a^2||\psi||^2 w(\psi)^2 + 2ab||\psi||^2 w(\psi) + \frac{1}{2}b^2||\psi||^2}{1 + 2a||\psi||^2}} \\
&= \frac{1}{\sqrt{1 + 2a||\psi||^2}}e^{\frac{aw(\psi)^2 + 2a^2||\psi||^2 w(\psi)^2 + bw(\psi) + 2ab||\psi||^2 w(\psi) + c + 2ac||\psi||^2 - 2a^2||\psi||^2 w(\psi)^2 - 2ab||\psi||^2 w(\psi) - \frac{1}{2}b^2||\psi||^2}{1 + 2a||\psi||^2}} \\
&= \frac{1}{\sqrt{1 + 2a||\psi||^2}}e^{\frac{1}{1 + 2a||\psi||^2}(aw(\psi)^2 + bw(\psi) + c + \frac{(4ac - b^2)||\psi||^2}{2})} \in L^2(P)
\end{aligned} \tag{3.21}$$

Therefore the result (3.16), (3.17) follows from Proposition 2.3.

⊓

COROLLARY 3.6
Under the same conditions as in the previous proposition, we have

$$\exp^{\diamond}[a(y-Y)^{\diamond 2}] = K_{\psi}^{-1}\exp[aK_{\psi}^{-2}(y-Y)^2] \tag{3.22}$$

PROOF
Just note that $b^2 - 4ac = 0$ in this case.

⊓

COROLLARY 3.7
Let $\phi(t), Y(t)$ be as in Theorem 3.4. Let $t < T$ and assume that

$$||\phi||^2_{[t,T]} := \int_t^T \phi^2(s)ds > 0$$

Then

$$\frac{1}{||\phi||_{[0,T]}}\exp^{\diamond}\left[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] = \frac{1}{||\phi||_{[t,T]}}\exp\left[-\frac{(y-Y(t))^2}{2||\phi||^2_{[t,T]}}\right] \tag{3.23}$$

PROOF
Put $\psi(s) = \phi(s)\mathcal{X}_{[0,t]}$ in Corollary 3.6. Then $a = -\frac{1}{2||\phi||^2_{[0,T]}}$, and we see that

$$2|a|||\psi||^2 = \frac{||\phi||^2_{[0,t]}}{||\phi||^2_{[0,T]}} = \frac{||\phi||^2_{[0,T]} - ||\phi||^2_{[t,T]}}{||\phi||^2_{[0,T]}} < 1 \tag{3.24}$$

by our assumptions. Moreover

$$K_{\psi} = \sqrt{1+2a||\psi||^2} = \sqrt{1 - \frac{||\phi||^2_{[0,t]}}{||\phi||_{[0,T]}{}^2}} = \frac{||\phi||_{[t,T]}}{||\phi||_{[0,T]}} \tag{3.25}$$

Hence

$$K_{\psi}^{-1} = \frac{||\phi||_{[0,T]}}{||\phi||_{[t,T]}} \quad \text{and} \quad aK_{\psi}^{-2} = -\frac{1}{2||\phi||^2_{[t,T]}} \tag{3.26}$$

Corollary 3.7 then follows directly from Corollary 3.6.

⊓

LEMMA 3.8
Let $\phi(t), Y(t)$ and $||\phi||_{[t,T]}$ be as in Corollary 3.7. Then

$$\begin{aligned}&\frac{1}{||\phi||_{[0,T]}}\exp^{\diamond}\left[-\frac{(y-Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right]\diamond\frac{y-Y(t)}{||\phi||^2_{[0,T]}}\\ =&\frac{1}{||\phi||_{[t,T]}}\exp\left[-\frac{(y-Y(t))^2}{2||\phi||^2_{[t,T]}}\right]\frac{y-Y(t)}{||\phi||^2_{[t,T]}}\end{aligned} \tag{3.27}$$

PROOF
If we differentiate both sides of (3.23) w.r.t. y, the result follows.

⊓

We can now give a more explicit (and familiar) representation than the one given in Theorem 3.4:

THEOREM 3.9

Let $\phi(t), Y(t)$ be as in Theorem 3.4 and assume that

$$||\phi||^2_{[t,T]} := \int_t^T \phi^2(s)ds > 0 \text{ for all } t < T \tag{3.28}$$

Let $f : \mathbf{R} \to \mathbf{R}$ be bounded. Then

$$f(Y(T)) = V_0 + \int_0^T g(t,\omega)\big(\alpha(t)dt + dB_t\big)$$

where

$$V_0 = \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \exp\left[-\frac{y^2}{||\phi||^2_{[0,T]}}\right] dy \tag{3.29}$$

and

$$g(t,\omega) = \phi(t) \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[t,T]}} \exp\left[-\frac{(y - Y(t))^2}{2||\phi||^2_{[t,T]}}\right] \frac{y - Y(t)}{||\phi||^2_{[t,T]}} dy \tag{3.30}$$

PROOF

We will apply Theorem 3.4, and therefore we consider

$$u(t,\omega) := \phi(t) \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \exp^{\diamond}\left[-\frac{(y - Y(t))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] \diamond \frac{y - Y(t)}{||\phi||^2_{[0,T]}} dy \tag{3.31}$$

By Lemma 3.8, $u(t,\omega) = g(t,\omega)$. Hence

$$E[\int_0^T u^2(t,\omega)dt] = E[\int_0^T g^2(t,\omega)dt] < \infty$$

and Theorem 3.4 gives with V_0 as in (3.29) (or (3.13)), that

$$\begin{aligned} f(Y(T)) &= V_0 + \int_0^T u(t,\omega) \diamond (\alpha(t) + W_t)dt \\ &= V_0 + \int_0^T g(t,\omega) \diamond (\alpha(t) + W_t)dt \\ &= V_0 + \int_0^T g(t,\omega)(\alpha(t)dt + dB_t) \end{aligned}$$

as claimed.

$\sqcap$

REMARK

The conclusion of Theorem 3.9 remains true without the assumption (3.28) if we interprete $g(t,\omega)$ as 0 when $||\phi||_{[t,T]} = 0$.

REMARK

Although the expression (3.30) clearly has a computational advantage to the Wick version (3.14), it should be noted that (3.14) may give some insight which is not evident from (3.30). For example, we might ask for the limiting behaviour as $t \to T$ of the replicating portfolio $g(t,\omega)$ in (3.30). If $\phi(t)$ is continuous at $t = T$, then by (3.14) we see that

$$\begin{aligned} &\lim_{t\to T} g(t,\omega) = \lim_{t\to T} u(t,\omega) \\ =&\phi(T) \int_{\mathbf{R}} \frac{f(y)}{\sqrt{2\pi}||\phi||_{[0,T]}} \exp^{\diamond}\left[-\frac{(y-Y(T))^{\diamond 2}}{2||\phi||^2_{[0,T]}}\right] \diamond \frac{y-Y(T)}{||\phi||^2_{[0,T]}} dy \end{aligned} \tag{3.32}$$

This limit clearly exists in $(\mathcal{S})^*$.

COROLLARY 3.10

For the digital payoff $F(\omega) = \mathcal{X}_{[K,\infty)}[Y(T)]$ *we have the representation*

$$\mathcal{X}_{[K,\infty)}(Y(T)) = V_0 + \int_0^T u(t,\omega)(\alpha(t)dt + dB_t)$$

where

$$V_0 = \int_K^{\infty} \frac{1}{\sqrt{2\pi}||\phi||_{[0,T]}} \exp\left[-\frac{y^2}{2||\phi||^2_{[0,T]}}\right] dy \tag{3.33}$$

and

$$u(t,\omega) = \frac{\phi(t)}{\sqrt{2\pi}||\phi||_{[t,T]}} \cdot \exp\left[-\frac{(K-Y(t))^2}{2||\phi||^2_{[t,T]}}\right] \tag{3.34}$$

PROOF

Here $f(y) = \mathcal{X}_{[K,\infty)}[y]$, so we see that (3.33) follows from (3.30) by performing the integration with respect to y.

$\sqcap$

REMARK

To be precise, the hedging procedure w.r.t. the contingent T-claim, $F(\omega) = h(Z(T))$, in (1.5), is carried out as follows: Put $Y(t) = Z(t) + \int_0^t \alpha(s)\phi(s)ds$ and let

$$f(x) := e^{-\int_0^T \rho(s)ds} h(x - \int_0^T \alpha(s)\phi(s)ds)$$

With these definitions $e^{-\int_0^T \rho(s)ds}F(\omega) = f(Y(T))$ and V_0 and $u(t,\omega)$ in (1.13) are then provided by the explicit expressions in Theorem 3.9.

4. The multi-dimensional case

In this section we generalize the results of the previous section to arbitrary dimension n. We let $B(t) = (B_1(t),\dots,B_n(t))^\top$ denote n-dimensional Brownian motion (where in general $M^\top$ denotes the transpose of the matrix M). Similarly $W(t) = (W_1(t),\dots,W_n(t))^\top$ is n-dimensional white noise.

DEFINITION 4.1
Let $Y = (Y_1, \ldots, Y_n) : \Omega \to \mathbf{R}^n$ be a random variable, each component of which belongs to $(\mathcal{S})^*$. Then a continuous function

$$\delta_Y(\cdot) : \mathbf{R}^n \to (\mathcal{S})^*$$

is called a *Donsker delta function of* Y if it has the property that

$$\int_{\mathbf{R}^n} g(y)\delta_Y(y)dy = g(Y) \text{ a.s.} \tag{4.1}$$

for all (measurable) $g : \mathbf{R}^n \to \mathbf{R}$ such that the integral converges. Here - and in the following - $dy = dy_1 \cdots dy_n$ denotes n-dimensional Lebesgue measure.

PROPOSITION 4.2
Suppose $Y : \Omega \to \mathbf{R}^n$ *is a normally distributed random variable with mean* $m = E[Y]$ *and covariance matrix* $C = [c_{ij}]_{1\le i,j\le n}$. *Suppose* C *is invertible with inverse* $A = [a_{ij}]_{1\le i,j\le n}$. *Then* $\delta_Y(y)$ *is unique and is given by the expression*

$$\delta_Y(y) = (2\pi)^{-n/2}\sqrt{|A|}\exp^{\diamond}\left[-\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y_i - Y_i) \diamond (y_j - Y_j)\right] \tag{4.2}$$

where $|A|$ *is the determinant of* A.

PROOF
Let $G_Y(y)$ denote the right hand side of (4.2). We verify that G_Y satisfies (4.1) , i.e., that

$$\int_{\mathbf{R}^n} g(y)\delta_Y(y)dy = g(Y) \text{ a.s.} \tag{4.3}$$

To this end let us first assume that g has the form

$$g(y) = e^{\lambda\cdot y} = e^{\lambda_1 y_1 + \cdots + \lambda_n y_n} \tag{4.4}$$

for some $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$. Then taking the $\mathcal{H}$-transform of the left hand side of (4.3), we get

$$\begin{aligned}\mathcal{H}\left[\int_{\mathbf{R}^n} g(y)G_Y(y)dy\right] &= \int_{\mathbf{R}^n} e^{\lambda\cdot y}\mathcal{H}[G_Y(y)]dy \\ &= \int_{\mathbf{R}^n} e^{\lambda\cdot y}(2\pi)^{-n/2}\sqrt{|A|}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y_i - \tilde{Y}_i) \diamond (y_j - \tilde{Y}_j)\right]\end{aligned} \tag{4.5}$$

where $\tilde{Y} = \tilde{Y}(z) = (\tilde{Y}_1(z), \ldots, \tilde{Y}_n(z))$ is the Hermite transform of $Y = (Y_1, \ldots, Y_n)$ at $z = (z_1, z_2, \ldots) \in \mathbf{C}^{\mathbf{N}}$. The expression (4.5) may be regarded as the expectation of $e^{\lambda\cdot Z}$ where Z is a normally distributed random variable with mean $\tilde{Y}$ and covariance matrix $C = A^{-1}$. Now $Z := Y - m + \tilde{Y}$ is such a random variable. Hence (4.5) can be written as $E[e^{\lambda\cdot(Y-m+\tilde{Y})}]$, which by the well known formula for the characteristic function of a normal random variable is equal to $\exp[\lambda\cdot\tilde{Y} + \frac{1}{2}\sum_{i,j=1}^{n} c_{ij}\lambda_i\lambda_j]$. We conclude that

$$\begin{aligned}\mathcal{H}\left[\int_{\mathbf{R}^n} g(y)G_Y(y)dy\right] &= \exp\left[\lambda\cdot\tilde{Y} + \frac{1}{2}\sum_{i,j=1}^{n} c_{ij}\lambda_i\lambda_j\right] \\ =&\mathcal{H}\left[\exp^{\diamond}\left[\lambda\cdot Y + \frac{1}{2}\sum_{i,j=1}^{n} c_{ij}\lambda_i\lambda_j\right]\right] = \mathcal{H}[\exp[\lambda\cdot Y]] = \mathcal{H}[g(Y)]\end{aligned}$$

This proves that (4.3) holds for all functions g given by (4.4). Hence using, e.g., the Fourier transform, we see that (4.3) holds in general. It remains to prove uniqueness: If $H_1 : \mathbf{R}^n \to (\mathcal{S})^*$ and $H_2 : \mathbf{R}^n \to (\mathcal{S})^*$ are two continuous functions such that

$$\int_{\mathbf{R}^n} g(y)H_i(y)dy = g(Y) \text{ for } i = 1, 2 \tag{4.6}$$

for all g such that the integral converges, then in particular (4.6) must hold for all continuous functions with compact support. But then we clearly must have

$$H_1(y) = H_2(y) \text{ for a.a. } y \in \mathbf{R}^n$$

and hence for all $y \in \mathbf{R}^n$ by continuity.

$\sqcap$

In the following we let $\psi : [0, T] \to \mathbf{R}^n, \phi : [0, T] \to \mathbf{R}^{n\times n}$ be deterministic functions such that

$$\int_0^T |\psi(s)|ds < \infty \quad \text{and} \quad ||\phi||^2 := \sum_{i,j=1}^{n} \int_0^T \phi_{ij}^2(s)ds < \infty \tag{4.7}$$

Define

$$\begin{aligned} Y(t) &= \int_0^t \phi(s)dB(s) + \int_0^t \psi(s)ds \\ &= \int_0^t (\phi(s)W(s) + \psi(s))ds \ ; \ 0 \le t \le T \end{aligned} \tag{4.8}$$

$$m = E[Y(T)] = \int_0^T \psi(s)ds \in \mathbf{R}^n \tag{4.9}$$

and, for $1 \le i, j \le n$

$$c_{ij} = E[(Y_i(T) - m_i)(Y_j(T) - m_j)] = \int_0^T (\phi\phi^\top)_{ij}(s)ds \tag{4.10}$$

Assume that the matrix $C = [c_{ij}]_{1\le i,j\le n}$ is invertible and put

$$A = [a_{ij}]_{1\le i,j\le n} = C^{-1} \tag{4.11}$$

Define

$$\begin{aligned} H(t) = H(t, y) &= \exp^\diamond \left[-\frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(y_i - Y_i(t)) \diamond (y_j - Y_j(t)) \right] \ ; \ 0 \le t \le T \\ &= \exp^\diamond \left[-\frac{1}{2}(y - Y(t))^\top \diamond (y - Y(t)) \right] \ ; \ 0 \le t \le T \end{aligned} \tag{4.12}$$

LEMMA 4.3

$$H(T) = H(0) + \int_0^T H(t) \diamond \Bigg(\frac{1}{2} \sum_{i,j=1}^n a_{ij} \Big((y_i - Y_i(t)) \diamond (\phi_j(t) W(t) + \psi_j(t)) + (y_j - Y_j(t)) \diamond (\phi_i(t) W(t) + \psi_i(t) \Big) \Bigg) dt \tag{4.13}$$

where ϕ_j is row number j of the matrix ϕ.

PROOF
By the chain rule

$$\begin{aligned} &H(T) \\ &= H(0) + \int_0^T \frac{dH}{dt} dt = H(0) + \int_0^T H(t) \diamond \frac{d}{dt} \left[-\frac{1}{2} \sum_{i,j=1}^n a_{ij} (y_i - Y_i(t)) \diamond (y_j - Y_j(t)) \right] dt \\ &= H(0) + \int_0^T H(t) \diamond \Bigg(\frac{1}{2} \sum_{i,j=1}^n a_{ij} \Big((y_i - Y_i(t)) \diamond \frac{d}{dt} Y_j(t) + (y_j - Y_j(t)) \diamond \frac{d}{dt} Y_i(t) \Big) \Bigg) dt \\ &= H(0) + \int_0^T H(t) \diamond \Bigg(\frac{1}{2} \sum_{i,j=1}^n a_{ij} \Big((y_i - Y_i(t)) \diamond (\phi_j(t) W(t) + \psi_j(t)) \\ &+ (y_j - Y_j(t)) \diamond (\phi_i(t) W(t) + \psi_i(t)) \Big) \Bigg) dt \end{aligned}$$

$\sqcap$

We can now prove the main result of this section:

THEOREM 4.4
Let $\alpha : [0, T] \to \mathbf{R}^n$ be a deterministic function such that

$$||\alpha||^2 = \int_0^T \alpha^2(s) ds < \infty \tag{4.14}$$

Let $\phi : [0, T] \to \mathbf{R}^{n \times n}$ be as in (4.7) and define

$$Y(t) = \int_0^t \phi(s) dB(s) + \int_0^t \phi(s) \alpha(s) ds \ ; \ 0 \le t \le T \tag{4.15}$$

Let $f : \mathbf{R}^n \to \mathbf{R}$ be bounded. Then

$$f(Y(T)) = V_0 + \int_0^T u(t, \omega) \diamond (\alpha(t) + W(t)) dt \tag{4.16}$$

where

$$V_0 = (2\pi)^{-n/2} \sqrt{|A|} \int_{\mathbf{R}^n} f(y) \exp[-\frac{1}{2} y^\top A y] dy \tag{4.17}$$

and

$$u(t, \omega) = (2\pi)^{-n/2} \sqrt{|A|} \int_{\mathbf{R}^n} f(y) \exp^\diamond \Big[-\frac{1}{2} (y - Y(t))^\top \diamond A (y - Y(t)) \Big] \diamond \Big((y - Y(t))^\top A \phi(t) \Big) dy \tag{4.18}$$

PROOF

We apply Proposition 4.2 and Lemma 4.3 with $\psi(t) = \phi(t)\alpha(t)$ and $a = \int_0^T \phi(t)\alpha(t)dt$ to get

$$\begin{aligned}
&f(Y(T)) = \int_{\mathbf{R}^n} f(y)\delta_{Y(T)}(y)dy \\
=&(2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y)\exp^{\diamond}\left[-\frac{1}{2}\sum_{i,j=1}^n a_{ij}(y_i - Y_i(T))\diamond(y_j - Y_j(T))\right]dy \\
=&(2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y)H(T,y)dy = (2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y)H(0,y)dy \\
+&(2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y)\int_0^T H(t,y)\diamond\left(\frac{1}{2}\sum_{i,j=1}^n a_{ij}\Big((y_i - Y_i(t))\diamond(\phi_j(t)W(t)+\psi_j(t))\right. \\
+&\left.(y_j - Y_j(t))\diamond(\phi_i(t)W(t)+\psi_i(t))\Big)\right)dt \\
=&(2\pi)^{-n/2}\sqrt{|A|}\int_{\mathbf{R}^n} f(y)H(0,y)dy \\
+&(2\pi)^{-n/2}\sqrt{|A|}\int_0^T\left(\int_{\mathbf{R}^n} f(y)H(t,y)\diamond(y-Y(t))^\top A\phi(t)dy\right)\diamond(\alpha(t)+W(t))dt
\end{aligned}$$

which by (4.12) is the same as (4.16)-(4.18).

$\sqcap$

LEMMA 4.5

Let $\phi(t), Y(t)$ be as in (4.7),(4.15). Let $0 \le t < T$ and define the $n \times n$ matrix

$$C_{[t,T]} = \int_t^T \phi(s)\phi^\top(s)ds \tag{4.19}$$

Assume that

$$|C_{[t,T]}| > 0 \tag{4.20}$$

and put

$$A_{[t,T]} = C_{[t,T]}^{-1} \tag{4.21}$$

Then

$$\begin{aligned}
&\sqrt{|A_{[0,T]}|}\exp^{\diamond}\left[-\frac{1}{2}(y-Y(t))^\top\diamond A_{[0,T]}(y-Y(t))\right] \\
=&\sqrt{|A_{[t,T]}|}\exp\left[-\frac{1}{2}(y-Y(t))^\top A_{[t,T]}(y-Y(t))\right]
\end{aligned}\tag{4.22}$$

PROOF

The proof uses the same method as the proof of Corollary 3.7. We omit the details.

$\sqcap$

COROLLARY 4.6

Let $\phi(t), Y(t)$ and $|A_{[t,T]}|$ be as in Lemma 4.5. Then

$$\begin{aligned} &\sqrt{|A_{[0,T]}|}\exp^{\diamond}\left[-\frac{1}{2}(y-Y(t))^{\top}\diamond A_{[0,T]}(y-Y(t))\right]\diamond(y-Y(t))^{\top}A_{[0,T]}\phi(t)\\ =&\sqrt{|A_{[t,T]}|}\exp\left[-\frac{1}{2}(y-Y(t))^{\top}A_{[t,T]}(y-Y(t))\right](y-Y(t))^{\top}A_{[t,T]}\phi(t)\end{aligned} \tag{4.23}$$

PROOF

Differentiate (4.22) with respect to $y_1, y_2, \dots, y_n$.

⊓

We can now give a more explicit formulation than the one given in Theorem 4.4:

THEOREM 4.7

Let $\phi(t), Y(t)$ and $|A_{[t,T]}|$ be as in Lemma 4.5. Assume that

$$|C_{[t,T]}| > 0 \qquad \text{for all } t \in [0,T] \tag{4.24}$$

Let $f : \mathbf{R}^n \to \mathbf{R}$ be bounded. Then

$$f(Y(T)) = V_0 + \int_0^T u(t,\omega)\,(\alpha(t)dt + dB(t)) \tag{4.25}$$

where V_0 is as in (4.17) and

$$\begin{aligned} u(t,\omega) = (2\pi)^{-n/2}\sqrt{|A_{[t,T]}|}\int_{\mathbf{R}} f(y)\exp\left[-\frac{1}{2}(y-Y(t))^{\top}A_{[t,T]}(y-Y(t))\right]\\ \cdot\,(y-Y(t))^{\top}A_{[t,T]}\phi(t)dy\end{aligned} \tag{4.26}$$

EXAMPLE 4.8

The general results in Theorem 4.4 and Theorem 4.7 can be used to study the replicating portfolios for more exotic options than those of the type (1.5). For example, one can study the portfolios of pathdependent options like a knock-out option of the form

$$F(\omega) = \mathcal{X}_{[K,\infty)}\left[\max_{0\le t\le T} Z(t,\omega)\right] \tag{4.27}$$

where $Z(t,\omega)$ is the (1-dimensional) process in (1.4). The idea is the following:

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be an equidistant partition of $[0,T]$, and define

$$\phi(t) = \begin{bmatrix} \mathcal{X}_{[0,t_1]}(t) & 0 & \cdots & 0\\ \mathcal{X}_{[0,t_2]}(t) & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{X}_{[0,t_n]}(t) & 0 & \cdots & 0 \end{bmatrix} \in \mathbf{R}^{n\times n} \tag{4.28}$$

Then

$$\phi\phi^{\top} = \begin{bmatrix} \mathcal{X}_{[0,t_1]} & \mathcal{X}_{[0,t_1]} & \cdots & \mathcal{X}_{[0,t_1]}\\ \mathcal{X}_{[0,t_1]} & \mathcal{X}_{[0,t_2]} & \cdots & \mathcal{X}_{[0,t_2]}\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{X}_{[0,t_1]} & \mathcal{X}_{[0,t_2]} & \cdots & \mathcal{X}_{[0,t_n]} \end{bmatrix} \tag{4.29}$$

and hence

$$C = \int_0^T \phi\phi^\top(t)dt = \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_n \end{bmatrix} \tag{4.30}$$

Since $|C| = t_1(t_2 - t_1)(t_3 - t_2)\cdots(t_n - t_{n-1}) = (\Delta t)^n$, where

$$\Delta t = t_i - t_{i-1} = \frac{T}{n} \neq 0;\ \ 0 \leq i \leq n-1$$

the matrix C is invertible. Hence Theorem 4.4 applies to

$$Y(t) := \int_0^t \phi(s)dB(s) + \int_0^t \phi(s)\alpha(s)ds = \begin{bmatrix} B_1(t\wedge t_1) + \int_0^{t\wedge t_1} \alpha_1(s)ds \\ \vdots \\ B_1(t\wedge t_n) + \int_0^{t\wedge t_n} \alpha_1(s)ds \end{bmatrix} \tag{4.31}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) : [0,T] \to \mathbf{R}^n$ is as in (4.14). In this case we see that

$$A = C^{-1} = \frac{n}{T}\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & \cdots & -1 & 1 \end{bmatrix} \tag{4.32}$$

Now let $f : \mathbf{R}^n \to \mathbf{R}$ be bounded. Then

$$f(Y(T,\omega)) = f(B_1(t_1) + \int_0^{t_1} \alpha_1(s)ds, \ldots, B_1(t_n) + \int_0^{t_n} \alpha_1(s)ds)$$

In particular, if $\alpha_1 = 0$, we get the following representation by Theorem 4.4:

$$f(B_1(t_1), \ldots, B_1(t_n)) = V_0 + \int_0^T u_1(t,\omega)dB_1(t)$$

where

$$\begin{aligned} &u_1(t,\omega) \\ &= \Big(\frac{n}{2\pi T}\Big)^{n/2} \int_{\mathbf{R}^n} f(y_1,\ldots,y_n) \exp^\diamond \Big[-\frac{n}{2T}\Big(2\sum_{i=1}^{n-1}(y_i - B_1(t\wedge t_i))^{\diamond 2} + (y_n - B_1(t))^{\diamond 2} \\ &-2\sum_{i=1}^{n-1}(y_i - B_1(t\wedge t_i)) \diamond (y_{i+1} - B_1(t\wedge t_{i+1}))\Big)\Big] \\ &\diamond \frac{n}{T}\Big(2\sum_{i=1}^{n-1}(y_i - B_1(t\wedge t_i))\mathcal{X}_{[0,t_i]}(t) + (y_n - B_1(t)) \\ &- \sum_{i=1}^{n-1}(y_i - B_1(t\wedge t_i))\mathcal{X}_{[0,t_{i+1}]}(t) - \sum_{i=1}^{n-1}(y_{i+1} - B_1(t\wedge t_{i+1}))\mathcal{X}_{[0,t_i]}(t)\Big)dy \end{aligned} \tag{4.33}$$

Thus we see that if F is the knock out option

$$F(\omega) = \mathcal{X}_{[K,\infty)}\Big[\max_{0\leq t\leq T} B_1(t,\omega)\Big]$$

then we obtain an approximation of the corresponding replicating portfolio $u(t,\omega)$ by choosing n large and $f(y_1,\ldots,y_n) = \max\{y_i | 1 \leq i \leq n\}$ in (4.33). With some extra work one could obtain a similar representation without Wick products using Theorem 4.7.

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