

Optimal consumption and portfolio with both fixed and proportional transaction costs

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May 2, 2001

Abstract

We consider a market model with one riskfree and one risky asset, in which the dynamics of the risky asset is governed by a geometric Brownian motion. In this market we consider an investor who consumes from the bank account and who has the opportunity at any time to transfer funds between the two assets. We suppose that these transfers involve a fixed transaction cost $k > 0$, independent of the size of the transaction, plus a cost proportional to the size of the transaction.

The objective is to maximize the cumulative expected utility of consumption over a planning horizon. We formulate this problem as a combined stochastic control/impulse control problem, which in turn leads to a (nonlinear) quasivariational Hamilton-Jacobi-Bellman inequality (QVHJBI). We prove that the value function is the unique viscosity solution of this QVHJBI. Finally numerical results are presented.

1 Introduction

Let (Ω, \mathcal{F}, P) be a probability space with a given filtration $\{\mathcal{F}_t\}_{t \geq 0}$. We denote by $X(t)$ the amount of money the investor has in the bank at time t and by $Y(t)$ the amount of money invested in the risky asset at time t . We assume that in the absence of consumption and transactions the process $X(t)$ grows deterministically at exponential rate r , while $Y(t)$ is a geometric Brownian motion, i.e.

$$(1.1) \quad dX(t) = rX(t)dt; \quad X(0) = x$$

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$$(1.2) \quad dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t); \quad Y(0) = y$$

where $W(t)$ is a 1-dimensional \mathcal{F}_t -Brownian motion and $\alpha > r > 0$ and $\sigma \neq 0$ are constants.

Suppose that at any time t the investor is free to choose a *consumption rate* $c(t) \geq 0$. This consumption is automatically drawn from the bank account holding with no extra costs. Moreover, at any time the investor can decide to transfer money from the bank account to the stock and conversely. Assume that a purchase of size ℓ of stocks incurs a transaction cost consisting of a sum of a fixed cost $k > 0$ (independent of the size of the transaction) plus a cost $\lambda \ell$ proportional to the transaction ($\lambda \geq 0$). These costs are drawn from the bank account. Similarly a sale of size m of stocks incurs the fixed cost $K > 0$ plus the proportional cost μm ($\mu \geq 0$). For simplicity we will assume that $K = k$ and $\mu = \lambda$. In this context the investor will only change his portfolio finitely many times in any finite time interval. The control of the investor will consist of a combination of a regular *stochastic control* $c(t)$ and an *impulse control* $v = (\tau_1, \tau_2, \dots; \xi_1, \xi_2, \dots)$. Here $0 \leq \tau_1 < \tau_2 < \dots$ are \mathcal{F}_t -stopping times giving the times when the investor decides to change his portfolio and $\{\xi_j \in \mathbf{R}; j = 1, 2, \dots\}$ are \mathcal{F}_{τ_j} -measurable random variables giving the sizes of the transactions at these times. We assume that

$$(1.3) \quad c(t) \text{ is } \mathcal{F}_t\text{-adapted, } c(t, \omega) \geq 0 \text{ and } \lim_{j \rightarrow \infty} \tau_j = \infty \text{ a.s.}$$

(possibly $\tau_n = \infty$ a.s. for some $n < \infty$).

If such a control $w = (c, v)$ is applied to the system $(X(t), Y(t))$ it gets the form

$$(1.4) \quad dX(t) = (rX(t) - c(t))dt; \quad \tau_i \leq t < \tau_{i+1}$$

$$(1.5) \quad dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t); \quad \tau_i \leq t < \tau_{i+1}$$

$$(1.6) \quad X(\tau_{i+1}) = X(\tau_{i+1}^-) - k - \xi_{i+1} - \lambda|\xi_{i+1}|,$$

$$(1.7) \quad Y(\tau_{i+1}) = Y(\tau_{i+1}^-) + \xi_{i+1}.$$

Thus a positive value of ξ_{i+1} indicates that money is being taken from the bank account at time τ_{i+1} to buy stocks, and conversely if ξ_{i+1} is negative.

If our agent has the amount x on the bank account and y in stocks, his *net wealth* is given by

$$(1.8) \quad H(x, y) = \max\{x + y - \lambda|y| - k, \min\{x, y\}\}.$$

Therefore it is natural to define the *solvency region* \mathcal{S} by

$$(1.9) \quad \mathcal{S} = \{(x, y) \in \mathbf{R}^2; H(x, y) \geq 0\}$$

and we put

$$(1.10) \quad \tilde{\mathcal{S}} = \mathbf{R}^+ \times \mathcal{S}.$$

Define the line segments ℓ_1, ℓ_2 by

$$(1.11) \quad \ell_1 = \{(x, y); x + (1 - \lambda)y = k, x < 0\}$$

$$(1.12) \quad \ell_2 = \{(x, y); x + (1 + \lambda)y = k, y < 0\}$$

and let the points P, Q be the end points of these segments, i.e.

$$(1.13) \quad P = \left(0, \frac{k}{1 - \lambda}\right), \quad Q = (k, 0).$$

(See Figure 1 and also Remark 2.4.)

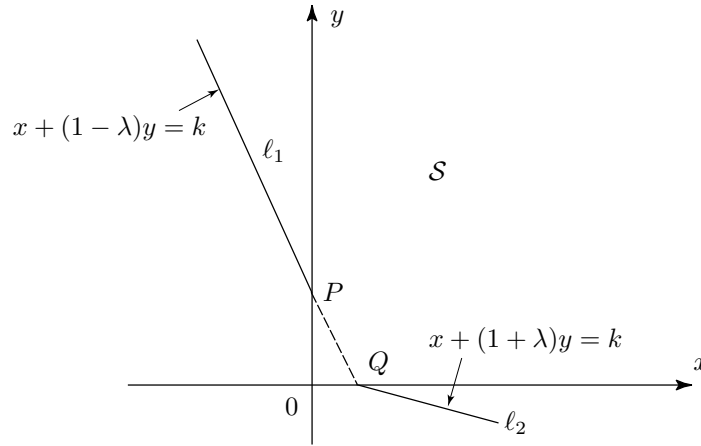


Figure 1

The investor's objective is to maximize over all combined controls $w = (c, v)$ the expression

$$(1.14) \quad J^w(s, x, y) = E^{s, x, y} \left[\int_0^\infty e^{-\delta(s+t)} \frac{c^\gamma(t)}{\gamma} dt \right] = e^{-\delta s} E^{x, y} \left[\int_0^\infty e^{-\delta t} \frac{c^\gamma(t)}{\gamma} dt \right]$$

where $\delta > 0$, $0 < \gamma < 1$ are constants ($1 - \gamma$ is the relative risk aversion coefficient), $E^{s, x, y}$ denotes the expectation with respect to the probability law $P^{s, x, y}$ of

$$(1.15) \quad Z(t) = Z^w(t) := (s + t, X(t), Y(t)); \quad t \geq 0$$

starting at $z = (s, x, y)$.

We seek the value function(s)

$$(1.16) \quad \Phi(s, x, y) = \sup_{w \in \mathcal{W}} J^w(s, x, y), \quad \Psi(x, y) = \Phi(0, x, y)$$

where $\mathcal{W} = \mathcal{W}(x, y)$ is the set of all *admissible* controls, i.e. all combined controls which satisfy (1.3) and which do not cause the process $Z(t)$ to exit from \mathcal{S} . Note that

$$(1.17) \quad J^w(s, x, y) = e^{-\delta s} J^w(0, x, y) \quad \text{and} \quad \Phi(s, x, y) = e^{-\delta s} \Phi(0, x, y) = e^{-\delta s} \Psi(x, y),$$

so the introduction of the s -variable is not really necessary. However, it turns out to be convenient in order to simplify the notation and the arguments in some of the proofs later.

We also seek a corresponding *optimal* control, i.e. a combined control w^* such that

$$(1.18) \quad \Phi(s, x, y) = J^{w^*}(s, x, y) = e^{-\delta s} \Psi(x, y).$$

This problem may be regarded as a generalization of optimal consumption and portfolio problems studied by Merton [M] and Davis & Norman [DN]. See also Shreve and Soner [SS]. [M] considers the case with no transaction costs ($\lambda = k = 0$), in which case the problem is no longer a combined control problem but a pure stochastic control problem. In this case it is proved in [M] that it is optimal to choose the portfolio such that

$$(1.19) \quad \frac{Y(t)}{X(t)} = \frac{\pi^*}{1 - \pi^*} \quad \text{for all } t,$$

(the Merton line) where

$$(1.20) \quad \pi^* = \frac{\alpha - r}{(1 - \gamma)\sigma^2}.$$

Moreover, the corresponding value function in the Merton case $\lambda = k = 0$ is given by

$$(1.21) \quad \Psi_0(x, y) = C_1(x + y)^\gamma$$

where

$$(1.22) \quad C_1 = \frac{1}{\gamma} C_0^{\gamma-1}, \quad \text{with} \quad C_0 = \frac{1}{1 - \gamma} \left[\delta - \gamma r - \frac{\gamma(\alpha - r)^2}{2\sigma^2(1 - \gamma)} \right]$$

provided that

$$(1.23) \quad \delta > \gamma \left[r + \frac{(\alpha - r)^2}{2\sigma^2(1 - \gamma)} \right].$$

See e.g. [DN, Section 2].

From now on we assume that (1.23) holds.

It is easy to see that

$$(1.24) \quad \Psi(x, y) \leq \Psi_0(x, y).$$

This is also pointed out in Corollary 2.2, to be proved later.

[DN] and [SS] consider the case with proportional transaction costs only ($k = 0$), in which case the problem can be formulated as a singular stochastic control problem. It is proved in [DN] and [SS] that under some conditions there exist two straight lines Γ_1, Γ_2

through the origin, bounding a cone NT , such that it is optimal to make no transactions if $(X(t), Y(t)) \in NT$ and make transactions corresponding to local time at $\partial(NT)$, resulting in reflections back to NT every time $(X(t), Y(t)) \in \partial(NT)$. Depending on the parameters the Merton line may or may not go between the lines Γ_1, Γ_2 (see the discussion in [AMS, Section 7.2]). For an extension of these results to markets with jumps see [FØS1] and [FØS2].

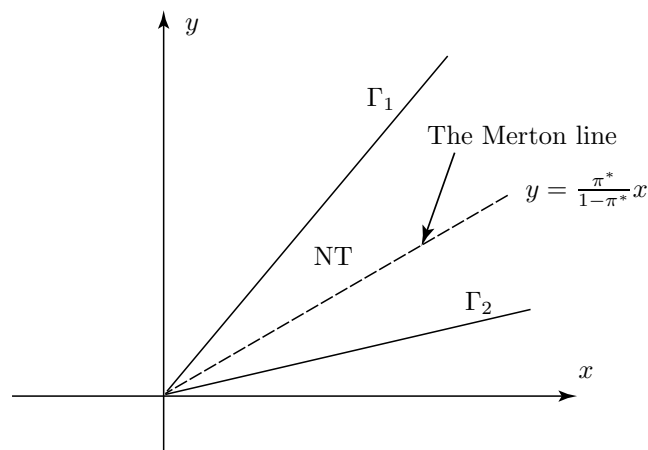


Figure 2

The first paper to model markets with fixed transaction costs $k > 0$ by impulse control theory seems to be [EH], but they do not consider optimal consumption.

Perhaps the paper which is closest to ours is [K]. Here optimal consumption in markets with fixed transaction costs are considered, but consumption is only allowed at the discrete times of the transactions. This makes it possible to put the problem within the framework of impulse control and quasi-variational inequalities.

In our paper we allow consumption to take place at any time, independent of the (discrete) times chosen for the transactions. As explained above we model this as a combined stochastic control and impulse control problem, or a *combined control* problem, for short.

In Section 2 we introduce quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI) associated to this combined control problem. We point out that if a function $\psi(x, y)$ satisfies these QVHJBI (and some additional smoothness conditions), then ψ coincides with the value function Ψ , defined by (1.16). (See Theorem 2.1).

In Section 3 we prove that the value function Ψ is the unique viscosity solution of the QVHJBI formulated in Section 2.

Finally in Section 4 we present some numerical estimates for Ψ and the optimal consumption-investment policy $w^* = (c^*, v^*)$.

For other recent papers on impulse control and combined control see e.g. [BØ], [MØ], [CZ1], [CZ2] and [BP] and the references therein. We refer to [BL] for a comprehensive treatment of the general theory of impulse control and their quasi-variational inequalities.

Remark 1.1 Another natural choice of solvency region would be the set

$$(1.25) \quad \mathcal{S}_+ := [0, \infty) \times [0, \infty) .$$

This choice models a situation where no borrowing or shortselling is allowed. We will mostly use the choice \mathcal{S} given by (1.9) in this paper, but we point out that the proofs carry over to the \mathcal{S}_+ case with only minor modifications. (Usually the \mathcal{S}_+ case is simpler.)

2 Quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI)

Let A^c be the generator of the process $Z^c(t) = (s + t, X^c(t), Y^c(t))$ when there are no transactions, i.e. A^c is the partial differential operator given by

$$(2.1) \quad (A^c f)(s, x, y) = \frac{\partial f}{\partial s} + (rx - c) \frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 f}{\partial y^2}$$

for any $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ and (s, x, y) such that the derivatives exist. In particular, if $f(s, x, y) = e^{-\delta s} g(x, y)$ then

$$(A^c f)(s, x, y) = e^{-\delta s} L^c g(x, y) ,$$

where

$$(2.2) \quad L^c g(x, y) = -\delta g + (rx - c) \frac{\partial g}{\partial x} + \alpha y \frac{\partial g}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 g}{\partial y^2} .$$

For $(x, y) \in \mathcal{S}$ and $\xi \neq 0$ put

$$(2.3) \quad x' = x'(\xi) = x - k - \xi - \lambda|\xi| , \quad y' = y'(\xi) = y + \xi .$$

We define the *intervention operator* \mathcal{M} by

$$(2.4) \quad \mathcal{M}h(x, y) = \sup\{h(x', y'); \xi \in \mathbf{R} \setminus \{0\}, (x', y') \in \mathcal{S}\}$$

for all locally bounded $h : \mathcal{S} \rightarrow \mathbf{R}^+$, $(x, y) \in \mathcal{S}$.

If $(x', y') \notin \mathcal{S}$ for all $\xi \in \mathbf{R} \setminus \{0\}$ we put $\mathcal{M}h(x, y) = 0$. If for all $(x, y) \in \mathcal{S}$ there exists $(x', y') = (x'(\xi), y'(\xi)) \in \mathcal{S}$ such that

$$\mathcal{M}h(x, y) = h(x', y')$$

then we put

$$(2.5) \quad \widehat{\xi}(x, y) = \widehat{\xi}_h(x, y) = (x', y')$$

(More precisely, we let $\widehat{\xi}(x, y)$ denote a measurable selection of the map $(x, y) \rightarrow (x', y')$.)

If Φ is the value function for our problem, then for each s we can interpret $\mathcal{M}\Phi(s, x, y)$ as the maximal value we can obtain by making an admissible transaction at (s, x, y) .

Following [BØ] we call a locally bounded function $h : \tilde{\mathcal{S}} \rightarrow \mathbf{R}^+$ *stochastically C^2* with respect to Z^c if $(A^c h)(z)$ exists for a.a. $z = (s, x, y)$ with respect to *the Green measure* (expected occupation time measure) $G(z_0, \cdot)$, and the generalized Dynkin formula holds for h , i.e.

$$E^{z_0} [h(Z^c(\tau'))] = E^{z_0} [h(Z^c(\tau))] + E^{z_0} \left[\int_{\tau}^{\tau'} (A^c h)(Z^c(t)) dt \right]$$

for all stopping times τ, τ' such that

$$(2.6) \quad \tau \leq \tau' \leq T_R := \inf\{t > 0, |Z^c(t)| \geq R\} \wedge R \quad \text{for some } R < \infty.$$

Recall that for each $z_0 \in \tilde{\mathcal{S}}$ the *Green measure* $G(z_0, \cdot)$ of the process Z^c in $\tilde{\mathcal{S}}$ is defined by

$$G(z_0, H) = E^{z_0} \left[\int_0^{\tau} \mathcal{X}_H(Z^c(t)) dt \right] \quad \text{for all Borel sets } H \subset \tilde{\mathcal{S}}$$

where $\tau = \inf\{t > 0; Z^c(t) \notin \tilde{\mathcal{S}}\}$ and $\mathcal{X}_H(z) = 1$ if $z \in H$, $\mathcal{X}_H(z) = 0$ if $z \notin H$.

If h is a function on \mathcal{S} we define

$$(2.7) \quad \mathcal{L}h(x, y) = \sup_{c \geq 0} \left\{ L^c h(x, y) + \frac{c^\gamma}{\gamma} \right\}; \quad (x, y) \in \mathcal{S}$$

and

$$(2.8) \quad \mathcal{L}_0 h(x, y) = L^0 h(0, y) = -\delta h + \alpha y \frac{\partial h}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 h}{\partial y^2}$$

for all points (x, y) where the partial derivatives of h involved in $L^c h$ exist.

We then put (see (1.11)–(1.13) for definitions of ℓ_1, ℓ_2 and P)

$$(2.9) \quad \mathcal{L}_1 h(x, y) = \begin{cases} \mathcal{L}h(x, y) & \text{for } (x, y) \in \mathcal{S} \setminus (\ell_1 \cup \ell_2) \setminus [0, P] \\ \mathcal{L}_0 h(x, y) & \text{for } (x, y) \in [0, P] \end{cases}$$

Note that at $[0, P]$ the only admissible consumption is $c = 0$.

By adapting Theorem 3.1 in [BØ] to our situation we get the following sufficient QVHJBI:

Theorem 2.1 *Let \mathcal{S} and $\tilde{\mathcal{S}}$ be as defined in (1.9) and put $U = \mathcal{S} \setminus (\ell_1 \cup \ell_2)$, $\tilde{U} = [0, \infty) \times U$.*

a) *Suppose we can find a locally bounded function $\psi : \mathcal{S} \rightarrow \mathbf{R}^+$ such that $\psi \in C^1(U)$ and*

$$(2.10) \quad \phi(s, x, y) := e^{-\delta s} \psi(x, y) \quad \text{is stochastically } C^2 \text{ with respect to } Z^c(t)$$

for all Markov controls $c = c(x, y)$

$$(2.11) \quad \mathcal{L}_1 \psi \leq 0 \quad \text{a.e. with respect to } G(z_0, \cdot) \text{ on } \tilde{U} \text{ for all } z_0 \in \tilde{U}$$

$$(2.12) \quad \psi(x, y) \geq \mathcal{M}\psi(x, y) \quad \text{for all } (x, y) \in U.$$

Then

$$\psi(x, y) \geq \Psi(x, y) \quad \text{for all } (s, x, y) \in \tilde{U}.$$

b) Define the continuation region

$$D = \{(x, y) \in U; \psi(x, y) > \mathcal{M}\psi(x, y)\}.$$

Suppose

$$(2.13) \quad \mathcal{L}_1\psi(x, y) = 0 \quad \text{on } D.$$

and that $\hat{\xi}(x, y) = \hat{\xi}_\psi(x, y)$ (defined in (2.5)) exists for all $(x, y) \in \mathcal{S}$. Define

$$c^*(x, y) = \begin{cases} \left(\frac{\partial\psi}{\partial x}\right)^{\frac{1}{\gamma-1}} & \text{for } (x, y) \in U \setminus [0, P] \\ 0 & \text{for } (x, y) \in [0, P] \end{cases}$$

and define the impulse control

$$v^* := (\tau_1^*, \tau_2^*, \dots; \xi_1^*, \xi_2^*, \dots)$$

as follows:

Put $\tau_0^* = 0$ and inductively

$$(2.14) \quad \tau_{k+1}^* = \inf\{t > \tau_k^*; (X^{(k)}(t), Y^{(k)}(t)) \notin D\}$$

$$(2.15) \quad \xi_{k+1}^* = \hat{\xi}(X^{(k)}(\tau_{k+1}^{*-}), Y^{(k)}(\tau_{k+1}^{*-}))$$

where $\hat{\xi}$ is as defined in (2.5) and $(X^{(k)}, Y^{(k)})$ is the process obtained by applying the combined control

$$w^{(k)} := (c^*, (\tau_1^*, \dots, \tau_k^*; \xi_1^*, \dots, \xi_k^*)); \quad k = 1, 2, \dots$$

Suppose $w^* := (c^*, v^*) \in \mathcal{W}$ and that

$$(2.16) \quad e^{-\delta t}\psi(X^{(w^*)}(t), Y^{(w^*)}(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ a.s.}$$

and that the family

$$(2.17) \quad \{e^{-\delta\tau}\psi(X^{(w^*)}(\tau), Y^{(w^*)}(\tau)); \tau \text{ stopping time}\}$$

is uniformly integrable. Then

$$(2.18) \quad \psi(x, y) = \Psi(x, y)$$

and w^* is optimal.

Proof. This follows by the proof of Theorem 3.1 in [BØ] with only minor modifications.

Note that the Hamilton-Jacobi-Bellman inequality (HJBI) (3.7) in [BØ] has the following form in our case, if $(x, y) \in U \setminus [0, P]$,

$$\mathcal{L}\psi(x, y) = \sup_{c \geq 0} \left\{ -\delta\psi + (rx - c)\frac{\partial\psi}{\partial x} + \alpha y\frac{\partial\psi}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2\psi}{\partial y^2} + \frac{c^\gamma}{\gamma} \right\} \leq 0 .$$

This can only hold if $\frac{\partial\psi}{\partial x} > 0$ and then the supremum of this expression is obtained when

$$(2.19) \quad c = c^* = \left(\frac{\partial\psi}{\partial x} \right)^{\frac{1}{\gamma-1}} .$$

If $(x, y) \in [0, P]$ then only the zero consumption $c = c^* = 0$ is admissible so again by the HJBI we get $L^0\psi(0, y) = 0$. \square

We can use this to prove the claim (1.24):

Corollary 2.2

a) As in (1.21)–(1.22) let

$$(2.20) \quad \Psi_0(x, y) = C_1(x + y)^\gamma$$

be the value function for the Merton problem ($k = \lambda = 0$). Then

$$(2.21) \quad \Psi(x, y) \leq \Psi_0(x, y) \quad \text{for all } (x, y) \in \mathcal{S} .$$

b) Let b be a constant such that

$$(2.22) \quad 1 - \lambda \leq b \leq 1 + \lambda$$

Suppose

$$(2.23) \quad \delta > \gamma\alpha .$$

Then there exists $K < \infty$ such that

$$(2.24) \quad \Psi(x, y) \leq K(x + by)^\gamma \quad \text{for all } (x, y) \in \mathcal{S} .$$

Proof. a) We verify that $\psi := \Psi_0$ satisfies the conditions of Theorem 2.1 a): First, $\phi(s, x, y) = e^{-\delta s}\psi(x, y)$ is C^2 and therefore trivially stochastically C^2 . Hence (2.10) holds. Second, ψ satisfies (2.11), because Ψ_0 satisfies the Hamilton-Jacobi-Bellman equation. Third, if we put, as in (2.3)

$$x' = x'(\xi) = x - \xi - k - \lambda|\xi| \quad \text{and} \quad y' = y'(\xi) = y + \xi ,$$

then $x' + y' \leq x + y$ for all x, y, ξ and therefore

$$(2.25) \quad \begin{aligned} \mathcal{M}\Psi_0(x, y) &= \sup_{\xi \neq 0} \Psi_0(x', y') = \sup_{\xi \neq 0} \{C_1(x' + y')^\gamma\} \\ &\leq C_1(x + y)^\gamma = \Psi_0(x, y) , \end{aligned}$$

where C_1 is defined in (1.22). Therefore (2.12) also holds. Hence (2.21) follows.

b) We proceed as in a), except now we choose $K < \infty$ and define

$$(2.26) \quad u(x, y) = K(x + by)^\gamma .$$

Now we get

$$x' + by' = \begin{cases} x + by - k - \xi(1 + \lambda - b) & \text{for } \xi > 0 \\ x + by - k - \xi(1 - \lambda - b) & \text{for } \xi < 0 \end{cases}$$

Thus in any case we have, by (2.22),

$$x' + by' \leq x + by$$

and this proves that

$$u(x, y) \geq \mathcal{M}u(x, y) .$$

It remains to verify that $\psi := u$ satisfies (2.11): By (2.26) we get

$$\begin{aligned} \mathcal{L}u(x, y) &= (x + by)^{\gamma-2} \left[\left(\frac{1-\gamma}{\gamma} (K\gamma)^{\frac{\gamma}{\gamma-1}} - \delta K \right) (x + by)^2 \right. \\ &\quad \left. + K\gamma(rx + \alpha by)(x + by) - \frac{1}{2}\sigma^2 K\gamma(1-\gamma)b^2 y^2 \right]. \end{aligned}$$

Hence $\mathcal{L}u(x, y) \leq 0$ for all $(x, y) \in \mathcal{S}$ if and only if

$$\left[\frac{1-\gamma}{\gamma} (K\gamma)^{\frac{\gamma}{\gamma-1}} - \delta K + K\gamma\alpha \right] (x + by)^2 \leq \frac{1}{2}\sigma^2 K\gamma(1-\gamma)b^2 y^2$$

for all $(x, y) \in \mathcal{S}$. This holds if and only if

$$(2.27) \quad \delta > \gamma\alpha + (1-\gamma)(K\gamma)^{\frac{1}{\gamma-1}} .$$

If (2.23) holds, then (2.27) holds for K large enough. Thus (2.24) follows from Theorem 2.1a). \square

Remark 2.3 Corollary 2.2 proves in particular that the value function Ψ is *finite*. Moreover, $\Psi(x, y)$ is *bounded* on every straight line in \mathcal{S} of the form

$$x + by = \text{constant},$$

for every constant $b \in [1 - \lambda, 1 + \lambda]$.

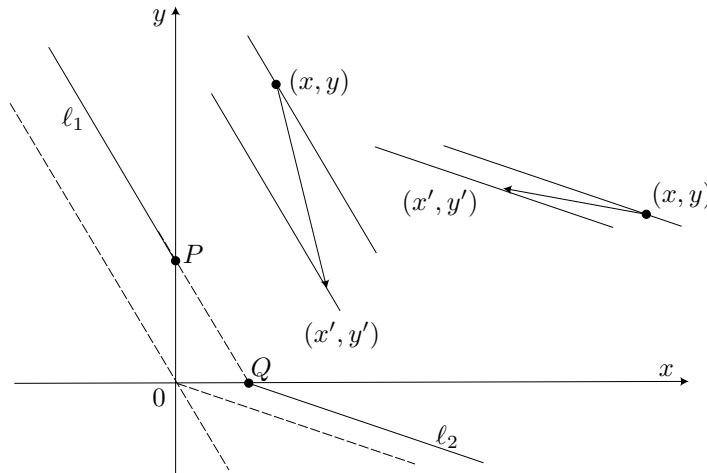


Figure 3

Remark 2.4 (Some comments on the boundary behaviour) Suppose the current position of the investor is a point $(x, y) \in \mathcal{S}$. If we make a transaction of size ξ at that instant, then after the transaction the new position is given by

$$(2.28) \quad \begin{cases} x' = x - \xi - \lambda|\xi| - k \\ y' = y + \xi \end{cases}$$

Hence

$$(2.29) \quad x' + (1 - \lambda)y' = x + (1 - \lambda)y - k - \lambda(|\xi| + \xi)$$

and

$$(2.30) \quad x' + (1 + \lambda)y' = x + (1 + \lambda)y - k - \lambda(|\xi| - \xi).$$

In particular, if we *sell* stocks ($\xi < 0$) then $x' + (1 - \lambda)y' = x + (1 - \lambda)y - k$, so (x, y) will move to a point (x', y') on the line parallel to ℓ_1 lying $\frac{k}{1-\lambda}$ units below the parallel of ℓ_1 through (x, y) . See Figure 3.

Similarly, if we *buy* stocks ($\xi > 0$) then $x' + (1 + \lambda)y' = x + (1 + \lambda)y - k$, so (x, y) will move to a point (x', y') on the line parallel to ℓ_2 lying $\frac{k}{1+\lambda}$ units below the parallel of ℓ_2 through (x, y) .

We now use this to deduce the boundary behaviour of the value function Ψ on $\partial\mathcal{S}$.

- (i) If $(x, y) \in \ell_1$ then we have to make an immediate transaction to avoid that the diffusion $Y(t)$ will take us out of \mathcal{S} . By the above we see that the only possibility is to *sell* stocks of such an amount that $(x', y') = (0, 0)$. We conclude that

$$(2.31) \quad \Psi(x, y) = \mathcal{M}\Psi(x, y) = 0 \quad \text{for } (x, y) \in \ell_1.$$

- (ii) If $(x, y) \in \ell_2$ we argue similarly: The only admissible action is to *buy* stocks immediately of such an amount that $(x', y') = (0, 0)$. Hence

$$(2.32) \quad \Psi(x, y) = \mathcal{M}\Psi(x, y) = 0 \quad \text{for } (x, y) \in \ell_2 .$$

- (iii) On the segment $0 < x < k$, $y = 0$ we are not allowed to make any transaction. There is no diffusion and all we can do is to consume optimally. Hence the HJB equation indicates that, with \mathcal{L} as in (2.7) we should have

$$(2.33) \quad \mathcal{L}\Psi = -\delta\Psi + rx\frac{\partial\Psi}{\partial x} + \frac{1-\gamma}{\gamma}\left(\frac{\partial\Psi}{\partial x}\right)^{\frac{\gamma}{\gamma-1}} = 0 \quad \text{for } x \in (0, k) ,$$

provided that Ψ is smooth enough (see Section 4).

- (iv) On the segment $x = 0$, $0 < y < \frac{k}{1-\lambda}$ we cannot consume because this would bring us outside \mathcal{S} . Hence the HJB equation indicates that

$$(2.34) \quad \mathcal{L}\Psi(0, y) = c^* = 0 \quad \text{for } 0 < y < \frac{k}{1-\lambda}$$

and hence that

$$(2.35) \quad \mathcal{L}_0\Psi = -\delta\Psi + \alpha y\frac{\partial\Psi}{\partial y} + \frac{1}{2}\sigma^2 y^2\frac{\partial^2\Psi}{\partial y^2} = 0 \quad \text{for } 0 < y < \frac{k}{1-\lambda} ,$$

provided that Ψ is smooth enough (see Section 4).

Summarizing we see that the boundary behaviour of Ψ on $\partial\mathcal{S}$ can be described by

$$(2.36) \quad \begin{cases} \Psi(x, y) = \mathcal{M}\Psi(x, y) = 0 & \text{for } (x, y) \in \ell_1 \cup \ell_2 \\ \mathcal{L}\Psi(x, y) = 0 & \text{for } 0 \leq x \leq k, y = 0 \text{ i.e., } (x, y) \in [0, Q] \\ \mathcal{L}_0\Psi(x, y) = 0 & \text{for } x = 0, 0 \leq y \leq y_1 = \frac{k}{1-\lambda} \text{ i.e., } (x, y) \in [0, P]. \end{cases}$$

Note that Ψ is not continuous on $\partial\mathcal{S}$: The points $(0, \frac{k}{1-\lambda})$ and $(k, 0)$ are points of discontinuity. However, Ψ is upper semicontinuous.

3 Viscosity solutions

Theorem 2.1 is a *verification theorem*, stating that if we can find a smooth enough function satisfying the required (quasi-) variational inequalities, then we have also found the value function of the problem. It is natural to ask if the converse is also true: Is the value function always a solution of the corresponding (quasi-) variational inequalities? The problem is that the value function need not be smooth enough for these inequalities to be welldefined in the strong sense. In fact, the value function is not even continuous at the points P and Q (see (2.36) and below). However, we shall see that the inequalities are satisfied in a weak sense: The value function is a *viscosity solution* of the (quasi-) variational inequalities.

We first recall the following concepts, which will be useful for us:

Definition 3.1 If C is a topological space and $u: C \rightarrow \mathbf{R}$ is a function, then the upper semi-continuous (*usc*) envelope $\bar{u}: C \rightarrow \mathbf{R}$ and the lower semi-continuous (*lsc*) envelope $\underline{u}: C \rightarrow \mathbf{R}$ of u are defined by

$$\bar{u}(x) = \limsup_{\substack{y \rightarrow x \\ y \in C}} u(y), \quad \underline{u}(x) = \liminf_{\substack{y \rightarrow x \\ y \in C}} u(y), \quad \text{respectively.}$$

We let $\text{USC}(C)$ and $\text{LSC}(C)$ denote the set of usc functions and lsc functions on C , respectively.

Note that in general we have

$$\underline{u} \leq u \leq \bar{u}$$

and that u is usc if and only if $u = \bar{u}$, u is lsc if and only if $u = \underline{u}$. In particular, u is continuous if and only if

$$\underline{u} = u = \bar{u}.$$

We establish some auxiliary results about the operator \mathcal{M} :

Lemma 3.2

- (i) If $u: \mathcal{S} \rightarrow \mathbf{R}$ is usc, then $\mathcal{M}u$ is usc.
- (ii) If $u: \mathcal{S} \rightarrow \mathbf{R}$ is continuous, then $\mathcal{M}u$ is continuous.

Proof. (i) Suppose $u: \mathcal{S} \rightarrow \mathbf{R}$ is usc. For $\zeta = (x, y) \in \mathcal{S}$ define

$$\ell(\zeta) = \ell(x, y) = \{(x'(\xi), y'(\xi)) \in \mathcal{S}; \quad \xi \in \mathbf{R} \setminus \{0\}\}$$

where x', y' are as in (2.3). Then $\ell(\zeta)$ is a union of two *closed finite* line segments and since u is usc there exists $\zeta^* \in \ell(\zeta)$ such that

$$\mathcal{M}u(\zeta) = \sup\{u(\zeta'); \zeta' \in \ell(\zeta)\} = u(\zeta^*).$$

Fix $\zeta_0 \in \mathcal{S}$ and let $\{\zeta_n\}_{n=1}^\infty$ be a sequence in \mathcal{S} such that $\zeta_n \rightarrow \zeta_0$ as $n \rightarrow \infty$. We must show that

$$\mathcal{M}u(\zeta_0) \geq \limsup_{n \rightarrow \infty} \mathcal{M}u(\zeta_n).$$

Let $\hat{\zeta}$ be a cluster point of $\{\zeta_n^*\}_{n=1}^\infty$, i.e. $\hat{\zeta}$ is the limit of some convergent subsequence $\{\zeta_{n_k}^*\}_{k=1}^\infty$ of $\{\zeta_n^*\}_{n=1}^\infty$. Since $\zeta_n \rightarrow \zeta_0$ we see that $\ell(\zeta_n) \rightarrow \ell(\zeta_0)$, in the natural sense. Hence, since $\zeta_{n_k}^* \in \ell(\zeta_{n_k})$ for all k , we conclude that $\hat{\zeta} = \lim_{k \rightarrow \infty} \zeta_{n_k}^* \in \ell(\zeta_0)$. Therefore

$$\mathcal{M}u(\zeta_0) \geq u(\hat{\zeta}) \geq \limsup_{n \rightarrow \infty} u(\zeta_n^*) = \limsup_{n \rightarrow \infty} \mathcal{M}u(\zeta_n).$$

(ii) Suppose $u: \mathcal{S} \rightarrow \mathbf{R}$ is continuous. Fix $\zeta_0 \in \mathcal{S}$ and let $\zeta_n \rightarrow \zeta_0$ as in (i). By (i) it suffices to show that

$$(*) \quad \mathcal{M}u(\zeta_0) \leq \liminf_{n \rightarrow \infty} \mathcal{M}u(\zeta_n).$$

Suppose not. Then $u(\zeta_0^*) = \mathcal{M}u(\zeta_0) > \liminf_{n \rightarrow \infty} \mathcal{M}u(\zeta_n) + \varepsilon = \liminf_{n \rightarrow \infty} u(\zeta_n^*) + \varepsilon$ for some $\varepsilon > 0$.

Since u is continuous there is a neighbourhood G of ζ_0^* such that

$$u(\zeta') \geq \liminf u(\zeta_n^*) + \varepsilon \quad \text{for all } \zeta' \in G .$$

But if n is big enough we have $\ell(\zeta_n) \cap G \neq \emptyset$ so since ζ_n^* is a maximum point of u on $\ell(\zeta_n)$ we have

$$u(\zeta_n^*) \geq u(\zeta') \quad \text{for } n \text{ big enough .}$$

This contradiction shows that (*) holds, and the proof is complete. \square

Lemma 3.3 (i) *Let $u : \mathcal{S} \rightarrow \mathbf{R}$. Then $\overline{\mathcal{M}u} \leq \mathcal{M}\bar{u}$.*

(ii) *Let $\psi : \mathcal{S} \rightarrow \mathbf{R}$ be such that $\psi \geq \mathcal{M}\psi$. Then $\underline{\psi} \geq \mathcal{M}\underline{\psi}$.*

Proof. (i): Choose $\zeta_0, \zeta_n \in \mathcal{S}$, $n = 1, 2, \dots$, such that $\zeta_n \rightarrow \zeta_0$ and $\mathcal{M}u(\zeta_n) \rightarrow \overline{\mathcal{M}u}(\zeta_0)$ as $n \rightarrow \infty$. Then by Lemma 3.2 (i) applied to the usc function \bar{u}

$$\overline{\mathcal{M}u}(\zeta_0) = \lim_{n \rightarrow \infty} \mathcal{M}u(\zeta_n) \leq \limsup_{n \rightarrow \infty} \mathcal{M}\bar{u}(\zeta_n) \leq \mathcal{M}\bar{u}(\zeta_0)$$

(ii): Choose $\zeta_0, \zeta_n \in \mathcal{S}$, $n = 1, 2, \dots$, such that $\zeta_n \rightarrow \zeta_0$ and $\mathcal{M}\psi(\zeta_n) \rightarrow \underline{\mathcal{M}\psi}(\zeta_0)$ as $n \rightarrow \infty$. Then

$$\underline{\psi}(\zeta_0) \geq \underline{\mathcal{M}\psi}(\zeta_0) = \lim_{n \rightarrow \infty} \mathcal{M}\psi(\zeta_n) \geq \liminf_{n \rightarrow \infty} \mathcal{M}\underline{\psi}(\zeta_n) \geq \mathcal{M}\underline{\psi}(\zeta_0) .$$

\square

Corollary 3.4 *Suppose $u : \mathcal{S} \rightarrow \mathbf{R}$ is usc and $u(\zeta_0) > \mathcal{M}u(\zeta_0) + \eta$ for some $\zeta_0 \in \mathcal{S}$, $\eta > 0$. Then $u(\zeta_0) > \overline{\mathcal{M}u}(\zeta_0) + \eta$.*

Proof. $u(\zeta_0) > \mathcal{M}u(\zeta_0) + \eta = \mathcal{M}\bar{u}(\zeta_0) \geq \overline{\mathcal{M}u}(\zeta_0) + \eta$ by Lemma 3.3 (i). \square

As in (2.7) we let \mathcal{L} be the differential operator

$$(3.1) \quad \mathcal{L}h(x, y) = \sup_{c \geq 0} \left\{ -\delta h + (rx - c) \frac{\partial h}{\partial x} + \alpha y \frac{\partial h}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 h}{\partial y^2} + \frac{c^\gamma}{\gamma} \right\}$$

and as in (2.2) we put

$$(3.2) \quad \mathcal{M}h(x, y) = \sup_{\xi \neq 0} \{h(x', y'); (x', y') \in \mathcal{S}\} \quad \text{for } (x, y) \in \mathcal{S} ,$$

where

$$(3.3) \quad x' = x - k - \xi - \lambda|\xi| , \quad y' = y + \xi .$$

The inequalities (2.11), (2.12) and (2.13) of Theorem 2.1 together with the boundary behaviour (2.36) can be combined into one equation as follows:

$$(3.4) \quad F(D^2\Psi(\zeta), D\Psi(\zeta), \Psi, \zeta) = 0 \quad \text{for all } \zeta = (x, y) \in \mathcal{S},$$

where

$$F: \mathbf{R}^{2 \times 2} \times \mathbf{R}^2 \times \mathbf{R}^{\mathcal{S}} \times \mathbf{R}^2 \rightarrow \mathbf{R}$$

is defined by

$$(3.5) \quad F(A, p, g, \zeta) = \begin{cases} \max\{\Lambda(A, p, g, \zeta), (\mathcal{M}g - g)(\zeta)\}; & \zeta \in \mathcal{S}^0 \\ \Lambda(A, p, g, \zeta); & \zeta \in [0, Q] \\ \Lambda_0(A, p, g, \zeta); & \zeta \in [0, P] \\ (\mathcal{M}g - g)(\zeta); & \zeta \in \ell_1 \cup \ell_2 \end{cases}$$

where

$$(3.6) \quad \Lambda(A, p, g, \zeta) = -\delta g + r\zeta_1 p_1 + \alpha\zeta_2 p_2 + \frac{1}{2}\sigma^2\zeta_2^2 A_{22} + \max_{c \geq 0} \left(-cp_1 + \frac{c\gamma}{\gamma} \right)$$

and

$$(3.7) \quad \Lambda_0(A, p, g, \zeta) = -\delta g + \alpha\zeta_2 p_2 + \frac{1}{2}\sigma^2\zeta_2^2 A_{22}, \quad A = [A_{ij}]_{1 \leq i, j \leq 2}.$$

Note that F is not a local operator: The value of F at (A, p, g, ζ) depends on the value of g on the whole space \mathcal{S} .

Also note that

$$(3.8) \quad \bar{F}(A, p, g, \zeta) = \max\{\Lambda(A, p, g, \zeta), (\mathcal{M}g - g)(\zeta)\} \quad \text{for all } \zeta \in \mathcal{S}$$

and that

$$(3.9) \quad \underline{F}(A, p, g, \zeta) = F(A, p, g, \zeta) \quad (\text{i.e. } F \text{ is lsc})$$

Following Barles [B] we now give the definition of viscosity solution of elliptic equations of type (3.4):

Definition 3.5

a) A function $u \in \text{USC}(\mathcal{S})$ is a viscosity subsolution of

$$(3.10) \quad F(D^2u(\zeta), Du(\zeta), u, \zeta) = 0 \quad \text{for all } \zeta = (x, y) \in \mathcal{S}$$

if for every function f which is C^2 in a neighbourhood of \mathcal{S} and for every point $\zeta_0 \in \mathcal{S}$ such that $f \geq u$ on \mathcal{S} and $f(\zeta_0) = u(\zeta_0)$ we have

$$(3.11) \quad \bar{F}(D^2f(\zeta_0), Df(\zeta_0), u, \zeta_0) \geq 0.$$

b) A function $u \in \text{LSC}(\mathcal{S})$ is a viscosity supersolution of (3.10) if for every function f which is C^2 in a neighbourhood of \mathcal{S} and for every point $\zeta_0 \in \mathcal{S}$ such that $f \leq u$ on \mathcal{S} and $f(\zeta_0) = u(\zeta_0)$ we have

$$(3.12) \quad \underline{E}(D^2 f(\zeta_0), Df(\zeta_0), u, \zeta_0) \leq 0 .$$

c) We say that a function $u: \mathcal{S} \rightarrow \mathbf{R}$ is a viscosity solution of (3.10) if u is locally bounded and \bar{u} is a viscosity subsolution and \underline{u} is a viscosity supersolution of (3.10).

An equivalent definition of viscosity solutions which is useful for proving uniqueness results is the following (see [CIL, Section 2]):

Definition 3.6

a) A function $u \in \text{USC}(\mathcal{S})$ is a viscosity subsolution of (3.4) if

$$(3.13) \quad \bar{F}(A, p, u, \zeta) \geq 0 \quad \text{for all } (p, A) \in \bar{J}_{\mathcal{S}}^{2,+} u(\zeta), \zeta \in \mathcal{S}$$

b) A function $u \in \text{LSC}(\mathcal{S})$ is a viscosity supersolution of (3.4) if

$$\underline{F}(A, p, u, \zeta) \leq 0 \quad \text{for all } (p, A) \in \bar{J}_{\mathcal{S}}^{2,-} u(\zeta), \zeta \in \mathcal{S} .$$

Here the second order “superjets” $J_{\mathcal{S}}^{2,+}$, $J_{\mathcal{S}}^{2,-}$ and their “closures” $\bar{J}_{\mathcal{S}}^{2,+}$, $\bar{J}_{\mathcal{S}}^{2,-}$ are defined by

$$(3.14) \quad \begin{aligned} J_{\mathcal{S}}^{2,+} u(\zeta) &= \{(p, A) \in \mathbf{R}^2 \times \mathbf{R}^{2 \times 2} ; \\ &\limsup_{\substack{\eta \rightarrow \zeta \\ \eta \in \mathcal{S}}} \{[u(\eta) - u(\zeta) - p \cdot (\eta - \zeta) - \frac{1}{2}(\eta - \zeta)^T A(\eta - \zeta)]|\eta - \zeta|^{-2}\} \leq 0\} \end{aligned}$$

(where $(\)^T$ denotes matrix transposed)

$$\bar{J}_{\mathcal{S}}^{2,+} u(\zeta) = \{(p, A) \in \mathbf{R}^2 \times \mathbf{R}^{2 \times 2} ; \exists (\zeta_n, p_n, A_n) \in \mathcal{S} \times \mathbf{R}^2 \times \mathbf{R}^{2 \times 2} ,$$

with $(p_n, A_n) \in J_{\mathcal{S}}^{2,+} u(\zeta_n)$ and $(\zeta_n, u(\zeta_n), p_n, A_n)$

$$(3.15) \quad \rightarrow (\zeta, u(\zeta), p, A) \quad \text{when } n \rightarrow \infty\}$$

and

$$(3.16) \quad J_{\mathcal{S}}^{2,-} u = -J_{\mathcal{S}}^{2,+}(-u) , \quad \bar{J}_{\mathcal{S}}^{2,-} u = -\bar{J}_{\mathcal{S}}^{2,+}(-u) .$$

We are now ready for the first main result of this section:

Theorem 3.7 Suppose (2.23) holds. Then the value function Ψ is a viscosity solution of (3.4).

Proof. We first make some useful observations:

Suppose $w = (c, v) \in \mathcal{W}$ is an admissible control with $v = (\tau_1, \tau_2, \dots; \xi_1, \xi_2, \dots)$ where $\tau_1 > 0$ a.s. Then by the *Markov property* we have, with J^w as in (1.14),

$$(3.17) \quad J^w(z) = E^z \left[\int_0^\tau e^{-\delta(s+t)} \frac{c^\gamma(t)}{\gamma} dt + J^w(Z^{(w)}(\tau)) \right]$$

for all stopping times $\tau \leq \tau_1$.

Note that

$$(3.18) \quad \Psi(\zeta) \geq \mathcal{M}\Psi(\zeta) \quad \text{for all } \zeta \in \mathcal{S}.$$

To see this, suppose on the contrary that there exists ζ_1 such that

$$\Psi(\zeta_1) < \mathcal{M}\Psi(\zeta_1).$$

This would mean that we could improve the performance at ζ_1 by making a transaction immediately. This contradicts that $\Psi(\zeta_1)$ is the optimal performance value at ζ_1 .

Also note that since τ_1 is a stopping time, we know that $\{\omega; \tau_1(\omega) = 0\}$ is \mathcal{F}_0 -measurable and hence this event has either probability 1 or 0. So we either have

$$\tau_1(\omega) = 0 \quad \text{a.s.} \quad \text{or} \quad \tau_1(\omega) > 0 \quad \text{a.s.}$$

a) We prove that $\bar{\Psi}$ is a viscosity subsolution. To this end, let f be a C^2 function in a neighbourhood of \mathcal{S} and let $\zeta_0 \in \mathcal{S}$ be such that $f \geq \bar{\Psi}$ on \mathcal{S} and $f(\zeta_0) = \bar{\Psi}(\zeta_0)$. We consider two cases separately:

Case 1. $\bar{\Psi}(\zeta_0) \leq \mathcal{M}\bar{\Psi}(\zeta_0)$.

Then by (3.8) $\bar{F}(D^2f(\zeta_0), Df(\zeta_0), f(\zeta_0), \bar{\Psi}, \zeta_0) \geq (\mathcal{M}\bar{\Psi} - \bar{\Psi})(\zeta_0) = 0$ and hence (3.11) holds at ζ_0 for $u = \bar{\Psi}$.

Case 2. $\bar{\Psi}(\zeta_0) > \mathcal{M}\bar{\Psi}(\zeta_0)$.

It suffices to prove that $\mathcal{L}f(\zeta_0) \geq 0$.

We argue by contradiction: Suppose $\zeta_0 = (x_0, y_0) \in \mathcal{S}$ and $\mathcal{L}f(\zeta_0) < 0$. Then from the definition (3.1) of \mathcal{L} we deduce that $\frac{\partial f}{\partial x}(\zeta_0) > 0$. Hence by continuity, $\frac{\partial f}{\partial x}(\zeta) > 0$ in a neighbourhood G of ζ_0 . But then, with $\zeta = (x, y)$,

$$\mathcal{L}f(\zeta) = -\delta f(\zeta) + (rx - \hat{c}) \frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 f}{\partial y^2} + \frac{\hat{c}^\gamma}{\gamma}$$

with $\hat{c} = \hat{c}(\zeta) = \left(\frac{\partial f}{\partial x}\right)^{\frac{1}{\gamma-1}}$ for all $\zeta \in G \cap \mathcal{S}$.

Hence $\mathcal{L}f(\zeta)$ is continuous on $G \cap \mathcal{S}$ and so there exists a (bounded) neighbourhood G_ρ of ζ_0 such that $G_\rho = \{(x, y); |x - x_0| < \rho \text{ and } |y - y_0| < \rho\}$ for some $\rho > 0$ and

$$(3.19) \quad \mathcal{L}f(\zeta) < \frac{1}{2} \mathcal{L}f(\zeta_0) < 0 \quad \text{for all } \zeta \in G_\rho \cap \mathcal{S}.$$

Now let η be any number such that

$$(3.20) \quad 0 < \eta < (\bar{\Psi} - \mathcal{M}\bar{\Psi})(\zeta_0) .$$

Since $\bar{\Psi}(\zeta_0) > \mathcal{M}\bar{\Psi}(\zeta_0) + \eta$ we can by Corollary 3.4 find a sequence $\{\zeta_n\}_{n=1}^\infty \subset G_\rho \cap \mathcal{S}$ such that $\zeta_n \rightarrow \zeta_0$ and $\Psi(\zeta_n) \rightarrow \bar{\Psi}(\zeta_0)$ as $n \rightarrow \infty$ and

$$(3.21) \quad \mathcal{M}\Psi(\zeta_n) < \Psi(\zeta_n) - \eta \quad \text{for all } n \geq 1 .$$

Choose $\varepsilon \in (0, \eta)$. Since $\bar{\Psi}(\zeta_0) = f(\zeta_0)$ we can choose n_0 such that

$$(3.22) \quad |\Psi(\zeta_n) - f(\zeta_n)| < \varepsilon \quad \text{for all } n \geq n_0 .$$

In the following we fix $n \geq n_0$ and put $\zeta = \zeta_n$.

Let $\tilde{w} = (\tilde{c}, \tilde{v})$ with $\tilde{v} = (\tilde{\tau}_1, \tilde{\tau}_2, \dots; \tilde{\xi}_1, \tilde{\xi}_2, \dots)$ be an ε -optimal control for ζ , in the sense that

$$\Psi(\zeta) \leq J^{\tilde{w}}(0, \zeta) + \varepsilon .$$

If $\tilde{\tau}_1 = 0$ a.s. then $(X^{(\tilde{w})}, Y^{(\tilde{w})})$ makes an immediate jump from ζ to some point $\zeta' \in \mathcal{S}$ and hence by (3.17)

$$J^{\tilde{w}}(0, \zeta) = E^{0, \zeta_0}[J^{\tilde{w}}(0, \zeta')] .$$

But then

$$\Psi(\zeta) \leq J^{\tilde{w}}(0, \zeta) + \varepsilon = E^{0, \zeta}[J^{\tilde{w}}(0, \zeta')] + \varepsilon \leq E^{0, \zeta}[\Psi(\zeta')] + \varepsilon \leq \mathcal{M}\Psi(\zeta) + \varepsilon$$

which contradicts (3.21). We conclude that $\tilde{\tau}_1 > 0$ a.s.

Fix $R < \infty$ and define τ to be the stopping time

$$\tau = \tau(\varepsilon) = \tilde{\tau}_1 \wedge R \wedge \inf\{t > 0; (X^{(\tilde{w})}(t), Y^{(\tilde{w})}(t)) \notin G_\rho\} .$$

Then by the Dynkin formula we have

$$\begin{aligned} E^{0, \zeta}[e^{-\delta\tau} f(X^{(\tilde{w})}(\tau), Y^{(\tilde{w})}(\tau))] &= f(\zeta) + E^{0, \zeta}\left[\int_0^\tau e^{-\delta t} L^{\tilde{c}} f(X^{(\tilde{w})}(t), Y^{(\tilde{w})}(t)) dt\right] \\ &+ E^{0, \zeta}[e^{-\delta\tau} [f(X^{(\tilde{w})}(\tau), Y^{(\tilde{w})}(\tau)) - f(X^{(\tilde{w})}(\tau^-), Y^{(\tilde{w})}(\tau^-))]] \end{aligned}$$

or

$$E^{0, \zeta}[e^{-\delta\tau} f(X^{(\tilde{w})}(\tau^-), Y^{(\tilde{w})}(\tau^-))] = f(\zeta) + E^{0, \zeta}\left[\int_0^\tau e^{-\delta t} L^{\tilde{c}} f(X^{(\tilde{w})}(t), Y^{(\tilde{w})}(t)) dt\right] .$$

Combining this with (3.17) we get, since $\Psi \geq \mathcal{M}\Psi$,

$$\begin{aligned}
\Psi(\zeta) &\leq J^{(\bar{w})}(0, \zeta) + \varepsilon \\
&= E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \frac{\tilde{c}^\gamma(t)}{\gamma} dt + J^{\bar{w}}(Z^{(\bar{w})}(\tau)) \right] + \varepsilon \\
&\leq E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \frac{\tilde{c}^\gamma(t)}{\gamma} dt + e^{-\delta \tau} \Psi(X^{(\bar{w})}(\tau), Y^{(\bar{w})}(\tau)) \right] + \varepsilon \\
&\leq E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \frac{\tilde{c}^\gamma(t)}{\gamma} dt + e^{-\delta \tau} \{ \Psi(X^{(\bar{w})}(\tau^-), Y^{(\bar{w})}(\tau^-)) \cdot \chi_{\tau < \bar{\tau}_1} \right. \\
&\quad \left. + \mathcal{M}\Psi(X^{(\bar{w})}(\tau^-), Y^{(\bar{w})}(\tau^-)) \cdot \chi_{\tau = \bar{\tau}_1} \} \right] + \varepsilon \\
&\leq E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \frac{\tilde{c}^\gamma(t)}{\gamma} dt + e^{-\delta \tau} \Psi(X^{(\bar{w})}(\tau^-), Y^{(\bar{w})}(\tau^-)) \right] + \varepsilon \\
&\leq E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \frac{\tilde{c}^\gamma(t)}{\gamma} dt + e^{-\delta \tau} f(X^{(\bar{w})}(\tau^-), Y^{(\bar{w})}(\tau^-)) \right] + \varepsilon \\
&= f(\zeta) + E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \left(L^{\tilde{c}} f(X^{(\bar{w})}(t), Y^{(\bar{w})}(t)) + \frac{\tilde{c}^\gamma(t)}{\gamma} \right) dt \right] + \varepsilon \\
&\leq \Psi(\zeta) + E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \mathcal{L}f(X^{(\bar{w})}(t), Y^{(\bar{w})}(t)) dt \right] + 2\varepsilon .
\end{aligned}$$

We conclude from this that

$$(3.23) \quad E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \mathcal{L}f(X^{(\bar{w})}(t), Y^{(\bar{w})}(t)) dt \right] \geq -2\varepsilon .$$

On the other hand, from (3.19) we deduce that

$$(3.24) \quad E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \mathcal{L}f(X^{(\bar{w})}(t), Y^{(\bar{w})}(t)) dt \right] \leq \frac{1}{2\delta} \mathcal{L}f(\zeta) (1 - E^{0, \zeta}[e^{-\delta \tau}]) .$$

We claim that

$$(3.25) \quad E^{0, \zeta_n}[e^{-\delta \tau(\varepsilon)}] \quad \text{is bounded away from 1}$$

when $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$.

If this claim is proved, then we see that (3.23) contradicts (3.24) if ε is small enough. This contradiction proves that $\mathcal{L}f(\zeta_0) \leq 0$ and hence (3.11) holds. Therefore, to complete the proof we must verify the claim (3.25).

First note that for $t < \tau$ we have by (1.4)

$$X^{(\bar{w})}(t) = X(0)e^{rt} - e^{rt} \int_0^t e^{-rs} \tilde{c}(s) ds \geq X(0) - \rho$$

and hence, with some constant $C_2 < \infty$,

$$\begin{aligned} \int_0^\tau e^{-\delta t} \frac{\tilde{c}^\gamma(t)}{\gamma} dt &\leq \frac{1}{\gamma} \left[\int_0^\tau e^{-rt} \tilde{c}(t) dt \right]^\gamma \left[\int_0^\tau e^{\frac{r\gamma - \delta}{1-\gamma} t} dt \right]^{1-\gamma} \\ &\leq C_2 \left(X(0)(1 - e^{-r\tau}) + \rho e^{-r\tau} \right)^\gamma, \quad \text{since } r\gamma - \delta < 0 \text{ by (1.2) and (2.23)}. \end{aligned}$$

Combining this with (3.17) we get

$$\begin{aligned} \Psi(\zeta) - \varepsilon &\leq J^{(\bar{w})}(0, \zeta) \\ &\leq E^{0, \zeta} \left[\int_0^\tau e^{-\delta t} \frac{\tilde{c}^\gamma(t)}{\gamma} dt + e^{-\delta\tau} \Psi(X^{(\bar{w})}(\tau), Y^{(\bar{w})}(\tau)) \right] \\ &\leq E^{0, \zeta} [C_2(x - (x - \rho)e^{-r\tau})^\gamma] + E^{0, \zeta} [e^{-\delta\tau} \Psi(X^{(\bar{w})}(\tau^-), Y^{(\bar{w})}(\tau^-)) \cdot \mathcal{X}_{\tilde{\tau}_1 > \tau}] \\ &\quad + E^{0, \zeta} [e^{-\delta\tau} \{ \Psi(X^{(\bar{w})}(\tau), Y^{(\bar{w})}(\tau)) - \Psi(X^{(\bar{w})}(\tau^-), Y^{(\bar{w})}(\tau^-)) \} \cdot \mathcal{X}_{\tilde{\tau}_1 \leq \tau}] \\ &\leq E^{0, \zeta} [C_2(x - (x - \rho)e^{-r\tau})^\gamma] + E^{0, \zeta} [e^{-\delta\tau} \Psi(X^{(\bar{w})}(\tau^-), Y^{(\bar{w})}(\tau^-)) \cdot \mathcal{X}_{\tilde{\tau}_1 > \tau}] \\ &\quad + E^{0, \zeta} [e^{-\delta\tau} \mathcal{M}\Psi(X^{(\bar{w})}(\tau^-), Y^{(\bar{w})}(\tau^-)) \cdot \mathcal{X}_{\tilde{\tau}_1 \leq \tau}] \\ &\leq E^{0, \zeta} [C_2(x - (x - \rho)e^{-r\tau})^\gamma] + E^{0, \zeta} [e^{-\delta\tau} \mathcal{X}_{\tilde{\tau}_1 > \tau}] \cdot \sup\{\Psi(\tilde{\zeta}); \tilde{\zeta} \in G_\rho\} \\ (3.26) \quad &\quad + E^{0, \zeta} [e^{-\delta\tau} \mathcal{X}_{\tilde{\tau}_1 \leq \tau}] \cdot \sup\{\mathcal{M}\Psi(\tilde{\zeta}); \tilde{\zeta} \in G_\rho\}. \end{aligned}$$

Now if there exists a sequence $\varepsilon_k \rightarrow 0$ and a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that

$$E^{0, \zeta_{n_k}} [e^{-\delta\tau(\varepsilon_k)}] \rightarrow 1 \quad \text{when } k \rightarrow \infty,$$

then

$$E^{0, \zeta_{n_k}} [e^{-\delta\tau(\varepsilon_k)} \mathcal{X}_{\tilde{\tau}_1 > \tau}] \rightarrow 0 \quad \text{when } k \rightarrow \infty,$$

so by choosing $\zeta = \zeta_{n_k}$, $\tau = \tau(\varepsilon_k)$ in (3.26) and letting $k \rightarrow \infty$ we obtain

$$\bar{\Psi}(\zeta_0) \leq C_2 \rho^\gamma + \sup\{\mathcal{M}\Psi(\tilde{\zeta}); \tilde{\zeta} \in G_\rho\}.$$

Hence by Lemma 3.3 and (3.20)

$$\begin{aligned} \bar{\Psi}(\zeta_0) &\leq \lim_{\rho \rightarrow 0} (C_2 \rho^\gamma + \sup\{\mathcal{M}\Psi(\tilde{\zeta}); \tilde{\zeta} \in G_\rho\}) \\ &= \overline{\mathcal{M}\Psi}(\zeta_0) \leq \mathcal{M}\bar{\Psi}(\zeta_0) < \Psi(\zeta_0) - \eta. \end{aligned}$$

This contradiction proves the claim (3.25) and completes the proof that $\bar{\Psi}$ is a viscosity subsolution.

b) Next we prove that $\underline{\Psi}$ is a viscosity supersolution. So we let f be a C^2 function in a neighbourhood of \mathcal{S} and we let $\zeta_0 \in \mathcal{S}$ be such that $f \leq \underline{\Psi}$ on \mathcal{S} and $f(\zeta_0) = \underline{\Psi}(\zeta_0)$. We want to show that

$$F(D^2f(\zeta_0), Df(\zeta_0), f(\zeta_0), \underline{\Psi}, \zeta_0) \leq 0.$$

Since by Lemma 3.3 (ii) $\mathcal{M}\underline{\Psi} - \underline{\Psi} \leq 0$ everywhere, we see from (3.5) that this holds for $\zeta_0 \in \ell_1 \cup \ell_2$ and it suffices to show that

$$\mathcal{L}f(\zeta_0) \leq 0 \quad \text{for } \zeta_0 \in \mathcal{S}^0 \cup [0, Q]$$

and

$$\mathcal{L}_0f(\zeta_0) \leq 0 \quad \text{for } \zeta_0 \in [0, P].$$

For $\varepsilon > 0$ let $\hat{w} = \hat{w}_{\varepsilon, c}$ be an admissible control beginning with a constant consumption rate $c \geq 0$ and no transactions up to the first time $\tau = \tau_\varepsilon$ the process $Z^c(t)$ exits from

$$K_\varepsilon = \{(s, x, y); |(s, x, y) - (0, x_0, y_0)| < \varepsilon\} \cap \tilde{\mathcal{S}}$$

where $\zeta_0 = (x_0, y_0)$. Choose $\zeta_n \in K_\varepsilon$ such that $\zeta_n \rightarrow \zeta_0$ and $\Psi(\zeta_n) \rightarrow \underline{\Psi}(\zeta_0)$ as $n \rightarrow \infty$.

Then by combining Dynkin's formula with the *dynamic programming principle* ([Kr, Th. 6, p. 150]) we get for all n

$$\begin{aligned} \Psi(\zeta_n) &\geq E^{0, \zeta_n} \left[\int_0^\tau e^{-\delta t} \frac{c^\gamma}{\gamma} dt + e^{-\delta \tau} \Psi(X^{(\hat{w})}(\tau), Y^{(\hat{w})}(\tau)) \right] \\ &\geq E^{0, \zeta_n} \left[\int_0^\tau e^{-\delta t} \frac{c^\gamma}{\gamma} dt + e^{-\delta \tau} f(X^{(\hat{w})}(\tau), Y^{(\hat{w})}(\tau)) \right] \\ &= f(\zeta_n) + E^{0, \zeta_n} \left[\int_0^\tau e^{-\delta t} \left(L^c f(X^{(\hat{w})}(t), Y^{(\hat{w})}(t)) + \frac{c^\gamma}{\gamma} \right) dt \right]. \end{aligned}$$

We conclude that

$$E^{0, \zeta_n} \left(\int_0^{\tau_\varepsilon} e^{-\delta t} \left\{ L^c f(X^{(\hat{w})}(t), Y^{(\hat{w})}(t)) + \frac{c^\gamma}{\gamma} \right\} dt \right) \leq \Psi(\zeta_n) - f(\zeta_n) \quad \text{for all } n.$$

Taking the limit as $n \rightarrow \infty$ we obtain

$$E^{0, \zeta_0} \left[\int_0^{\tau_\varepsilon} h(t) dt \right] \leq 0.$$

where

$$h(t) = e^{-\delta t} \left(L^c f(X^{(\hat{w})}(t), Y^{(\hat{w})}(t)) + \frac{c^\gamma}{\gamma} \right).$$

By dividing the left hand side by $E^{0,\zeta_0}[\tau_\varepsilon]$ we get

$$\begin{aligned} \frac{E^{0,\zeta_0}[\int_0^{\tau_\varepsilon} h(t)dt]}{E^{0,\zeta_0}[\tau_\varepsilon]} &= \frac{E^{0,\zeta_0}[\int_0^{\tau_\varepsilon} (h(t) - h(0))dt] + h(0)E^{0,\zeta_0}[\tau_\varepsilon]}{E^{0,\zeta_0}[\tau_\varepsilon]} \\ &\longrightarrow h(0) \quad \text{as } \varepsilon \rightarrow 0, \text{ since } h(t) \text{ is continuous at } t = 0. \end{aligned}$$

We conclude that

$$(3.27) \quad h(0) = L^c f(\zeta_0) + \frac{c^\gamma}{\gamma} \leq 0$$

for all $c \geq 0$ such that $\widehat{w}_{\varepsilon,c}$ is admissible for ε small enough. If $\zeta_0 \in \mathcal{S}^0 \cup [0, Q]$ this is clearly the case for all $c \geq 0$ and therefore (3.27) implies that $\mathcal{L}f(\zeta_0) \leq 0$. If $\zeta_0 \in [0, P]$ the only such admissible c is $c = 0$. Therefore we get $\mathcal{L}_0 f(\zeta_0) \leq 0$ in this case, as required. \square

Next we turn to the question of uniqueness. Our second main result in the section is the following:

Theorem 3.8 (Comparison theorem)

a) Suppose u is a viscosity subsolution and v is a viscosity supersolution of (3.4) and that u and v satisfy the estimates

$$(3.28) \quad -C|x + y|^\gamma \leq u(x, y) \quad \text{for all } (x, y) \in \mathcal{S}$$

$$(3.29) \quad v(x, y) \leq C|x + y|^\gamma \quad \text{for all } (x, y) \in \mathcal{S}$$

for some constant $C < \infty$. Then

$$u \leq v \quad \text{in } \mathcal{S}^0.$$

b) Moreover, if in addition

$$(3.30) \quad v(x, y) = \liminf_{\substack{(\xi, \eta) \rightarrow (x, y) \\ (\xi, \eta) \in \mathcal{S}^0}} v(\xi, \eta) \quad \text{for all } (x, y) \in \partial\mathcal{S}$$

then

$$u \leq v \quad \text{in } \mathcal{S}.$$

Corollary 3.9 Suppose u and v are two viscosity solutions of (3.4) satisfying (3.28) and (3.29). Then

$$u = v \quad \text{in } \mathcal{S}^0$$

and u is continuous in \mathcal{S}^0 . In particular, if (2.23) holds then the value function Ψ is continuous on \mathcal{S}^0 .

Proof of Corollary 3.9. Since u is a viscosity solution it follows that \bar{u} is a viscosity subsolution and \underline{u} is a viscosity supersolution and similarly for v . Hence, by Theorem 3.8,

$$\bar{u} \leq \underline{v} \leq \bar{v} \leq \underline{u} \leq \bar{u} \quad \text{in } \mathcal{S}^0 .$$

This implies that

$$\underline{u} = \bar{u} = \underline{v} = \bar{v} \quad \text{in } \mathcal{S}^0 .$$

The last statement of Corollary 3.9 now follows from Theorem 3.7 and Corollary 2.2. \square

Proof of Theorem 3.8. The proof is based on the technique of Ishii (see [B], [CIL] and [IL]) and on the proofs of Lemma 3.12 of [AMS] and Theorem 5.7 in [AST]. Consequently we shall not give here a detailed proof but rather point out the special treatment required to handle the non local intervention operator \mathcal{M} .

Let u and v be as in Theorem 3.8. We first construct a strict supersolution of (3.4) by making a perturbation of v :

Choose $\gamma' \in (\gamma, 1)$ such that (see (1.23))

$$(3.31) \quad \delta > \gamma' \left[r + \frac{(\alpha - r)^2}{2\sigma^2(1 - \gamma')} \right].$$

Put

$$(3.32) \quad g(x, y) = (x + y)^{\gamma'}$$

and choose $\varepsilon > 0$. Then

$$(3.33) \quad \mathcal{M}(v + \varepsilon g) \leq \mathcal{M}v + \varepsilon \mathcal{M}g$$

and hence

$$(3.34) \quad \mathcal{M}(v + \varepsilon g) - (v + \varepsilon g) \leq (\mathcal{M}v - v) + \varepsilon(\mathcal{M}g - g) .$$

Since v is a supersolution we have

$$(3.35) \quad \mathcal{M}v - v \leq 0 .$$

Moreover, with $\zeta = (x, y)$,

$$(3.36) \quad \begin{aligned} (\mathcal{M}g - g)(\zeta) &= \sup_{\xi \neq 0} \{g(x - k - \xi - \lambda|\xi|, y + \xi)\} - g(x, y) \\ &= \sup_{\xi \neq 0} \{(x + y - k - \lambda|\xi|)^{\gamma'}\} - (x + y)^{\gamma'} \\ &\leq (x + y)^{\gamma'} \left[\left(1 - \frac{k}{x + y}\right)^{\gamma'} - 1 \right] . \end{aligned}$$

Therefore, for each compact subset C of $\mathcal{S} \setminus \{0\}$ there exists $\rho_1 > 0$ such that $(\mathcal{M}g - g)(\zeta) \leq -\rho_1$ for all $\zeta \in C$. So from (3.34) and (3.35) we get

$$(3.37) \quad \mathcal{M}(v + \varepsilon g) - (v + \varepsilon g) \leq -\varepsilon\rho_1 \quad \text{in } C .$$

Now if we define the operator L^0 by (see (2.2))

$$(3.38) \quad L^0 = -\delta I + rx \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2}{\partial y^2}$$

where I is the identity operator, then

$$(3.39) \quad \begin{aligned} L^0 g(x, y) &= -\delta(x+y)^{\gamma'} + (rx + \alpha y) \gamma' (x+y)^{\gamma'-1} \\ &\quad + \frac{1}{2} \sigma^2 y^2 \gamma' (\gamma' - 1) (x+y)^{\gamma'-2} \\ &= (x+y)^{\gamma'} \left[-\delta + \gamma' \frac{rx + \alpha y}{x+y} + \frac{1}{2} \sigma^2 \gamma' (\gamma' - 1) \frac{y^2}{(x+y)^2} \right]. \end{aligned}$$

If we put

$$\eta = \frac{y}{x+y} \quad \text{so that} \quad \frac{x}{x+y} = 1 - \eta$$

we get

$$L^0 g(x, y) = (x+y)^{\gamma'} [-\delta + \gamma' r + \gamma' (\alpha - r) \eta + \frac{1}{2} \sigma^2 \gamma' (\gamma' - 1) \eta^2].$$

By (3.31) it follows that

$$L^0 g(x, y) < 0 \quad \text{for all } (x, y) \neq (0, 0).$$

Consequently, on every compact C of $\mathcal{S} \setminus \{0\}$ there exists $\rho_2 > 0$ such that

$$L^0 g(x, y) + \max_{c \geq 0} \left(-c \frac{\partial g}{\partial x} \right) \leq -\rho_2 \quad \text{on } C.$$

Therefore, since v is a supersolution of (3.4) we conclude that on every compact C of $\mathcal{S} \setminus \{0\}$ there exists $\rho > 0$ such that

$$v_\varepsilon := v + \varepsilon g$$

is a viscosity supersolution of

$$F(D^2 v_\varepsilon(\zeta), D v_\varepsilon(\zeta), v_\varepsilon(\zeta), v_\varepsilon, \zeta) = -\varepsilon \rho \quad \text{for } \zeta \in C.$$

Let us now prove the theorem by contradiction. Assume that

$$(3.40) \quad \sup_{\zeta \in \mathcal{S}} \{u(\zeta) - v(\zeta)\} > 0.$$

Choose $\varepsilon > 0$ such that

$$(3.41) \quad \sup_{\zeta \in \mathcal{S}} \{u(\zeta) - v_\varepsilon(\zeta)\} > 0.$$

Define

$$(3.42) \quad h(\zeta) := u(\zeta) - v_\varepsilon(\zeta).$$

Since h is upper semicontinuous and tends to $-\infty$ when $|\zeta| \rightarrow \infty$, the set

$$\text{Argmax } h := \{\bar{\zeta}; h(\bar{\zeta}) = \sup\{h(\zeta); \zeta \in \mathcal{S}\}\}$$

is nonempty and compact in $\mathcal{S} \setminus \{0\}$. Choose an open set $G \subset \mathcal{S} \setminus \{0\}$ containing this compact (G open relative to \mathcal{S}) and with \bar{G} compact. In order to get a contradiction it suffices to prove that

$$u \leq v_\varepsilon \quad \text{in } G.$$

Thus we have reduced the problem to proving a comparison theorem for a strict supersolution v_ε and a subsolution u of (3.4) in an open subset G of $\mathcal{S} \setminus \{0\}$ with compact closure \bar{G} , when the supremum of $u - v_\varepsilon$ is attained in G only. This is proved by using Ishii's technique, adapted as in [B, Theorem 4.6] and in [AMS, Theorem 5.7] for the boundary conditions. We now explain this in more detail:

For $j = 1, 2, \dots$ define, for $(\zeta, \eta) \in \mathcal{S} \times \mathcal{S}$,

$$(3.43) \quad H_j(\zeta, \eta) = u(\zeta) - v_\varepsilon(\eta) - \frac{j}{2}|\zeta - \eta|^2$$

and put

$$(3.44) \quad m_j = \sup\{H_j(\zeta, \eta); (\zeta, \eta) \in \mathcal{S} \times \mathcal{S}\}$$

and

$$(3.45) \quad m = \sup\{h(\zeta); \zeta \in \mathcal{S}\}.$$

Proceeding exactly as in [AMS] we obtain that there exist ζ_j, η_j in \mathcal{S} such that

$$(3.46) \quad m_j = H_j(\zeta_j, \eta_j) < \infty.$$

Moreover,

$$(3.47) \quad j|\zeta_j - \eta_j|^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

and

$$(3.48) \quad m_j \rightarrow m \quad \text{as } j \rightarrow \infty.$$

Suppose

$$(3.49) \quad \text{Argmax } h \text{ is contained in } \mathcal{S}^0$$

and choose $\hat{\zeta} \in \mathcal{S}^0$ such that

$$m = h(\hat{\zeta}).$$

Then we get that

$$(3.50) \quad (\zeta_j, \eta_j) \in \mathcal{S}^0 \times \mathcal{S}^0 \quad \text{for } j \text{ large enough.}$$

By applying [CIL, Theorem 3.2] we now obtain that there exist 2×2 matrices P_j, Q_j such that

$$(p_j, P_j) \in \bar{J}^{2,+}u(\zeta_j) \quad \text{and} \quad (q_j, Q_j) \in \bar{J}^{2,-}v_\varepsilon(\eta_j)$$

and

$$(3.51) \quad \begin{bmatrix} P_j & 0 \\ 0 & -Q_j \end{bmatrix} \leq 3j \begin{bmatrix} I & -I \\ -I & I \end{bmatrix},$$

where

$$p_j = j(\zeta_j - \eta_j) \quad \text{and} \quad q_j = p_j .$$

Since u is a subsolution and v_ε is a strict supersolution we obtain

$$(3.52) \quad \overline{F}(P_j, p_j, u, \zeta_j) \geq 0$$

and

$$(3.53) \quad \underline{F}(Q_j, q_j, v_\varepsilon, \eta_j) \leq -\varepsilon\rho .$$

From (3.53) it follows that

$$\Lambda(Q_j, q_j, v_\varepsilon, \eta_j) < 0$$

and proceeding as in [AMS] we obtain

$$(3.54) \quad \Lambda(P_j, p_j, u, \zeta_j) - \Lambda(Q_j, q_j, v_\varepsilon, \eta_j) < 0 .$$

Consequently $\Lambda(P_j, p_j, u, \zeta_j) < 0$ and from (3.52) we obtain that

$$(3.55) \quad (\mathcal{M}u - u)(\zeta_j) \geq 0 .$$

From (3.53) we have

$$(3.56) \quad (\mathcal{M}v_\varepsilon - v_\varepsilon)(\eta_j) \leq -\varepsilon\rho < 0 .$$

Therefore, combining (3.55) and (3.56) we get

$$(3.57) \quad m_j < u(\zeta_j) - v_\varepsilon(\eta_j) < \mathcal{M}u(\zeta_j) - \mathcal{M}v_\varepsilon(\eta_j) - \varepsilon\rho .$$

Since $\zeta_j, \eta_j \rightarrow \hat{\zeta}$ and u is usc, we obtain

$$(3.58) \quad m < \liminf_{j \rightarrow \infty} [\mathcal{M}u(\zeta_j) - \mathcal{M}v_\varepsilon(\eta_j)] .$$

Since u is usc and v_ε is lsc we see, after some reflections, that

$$(3.59) \quad \limsup_{j \rightarrow \infty} \mathcal{M}u(\zeta_j) \leq \mathcal{M}u(\hat{\zeta})$$

and

$$(3.60) \quad \limsup_{j \rightarrow \infty} (-\mathcal{M}v_\varepsilon(\eta_j)) \leq -\mathcal{M}v_\varepsilon(\hat{\zeta}) .$$

Hence we get the desired contradiction

$$\begin{aligned} m &< \mathcal{M}u(\hat{\zeta}) - \mathcal{M}v_\varepsilon(\hat{\zeta}) \\ &= \sup_{\xi_1 \neq 0} \{u(\hat{x}'(\xi_1), \hat{y}'(\xi_1))\} - \sup_{\xi_2 \neq 0} \{v_\varepsilon(\hat{x}'(\xi_2), \hat{y}'(\xi_2))\} \\ &\leq \sup_{\xi \neq 0} \{(u - v_\varepsilon)(\hat{x}'(\xi), \hat{y}'(\xi))\} \\ &\leq \sup\{(u - v_\varepsilon)(\zeta); \zeta \in \mathcal{S}\} = m , \end{aligned}$$

where $\hat{\zeta} = (\hat{x}, \hat{y})$ and (see (2.3))

$$\hat{x}'(\xi) = \hat{x} - \xi - \lambda|\xi| - k, \quad \hat{y}'(\xi) = \hat{y} + \xi.$$

This contradiction shows that assumption (3.49) cannot hold. Therefore we must have

$$(3.61) \quad \hat{\zeta} \in \text{Argmax } h \cap \partial\mathcal{S}.$$

To treat the boundary points we proceed exactly as in the proof of Theorem 5.7 of [AST], which itself is based on the proof of Theorem 4.6 in [B] (see Appendix, p. 166).

From (3.30) there exists a sequence $\{\zeta_j\} \subset \mathcal{S}^0 \cap G$ converging to $\hat{\zeta}$ such that $v_\varepsilon(\zeta_j) \rightarrow v_\varepsilon(\hat{\zeta})$ when $j \rightarrow \infty$. Let $\varepsilon_j = |\zeta_j - \hat{\zeta}|$. The idea is now to introduce the test function

$$w_j(\zeta, \eta) = u(\zeta) - v_\varepsilon(\eta) - \theta_j(\zeta, \eta); \quad (\zeta, \eta) \in \mathcal{S} \times \mathcal{S}$$

where

$$\theta_j(\zeta, \eta) = \frac{|\zeta - \eta|^2}{2\varepsilon_j} + \frac{1}{4} \left(\frac{d(\eta) - d(\zeta)}{d(\zeta_j)} - 1 \right)^4 + \frac{1}{4} |\zeta - \hat{\zeta}|^4.$$

Here $d(\eta)$ denotes the distance from η to $\partial\mathcal{S}$ and similarly for $d(\zeta), d(\zeta_j)$.

Following exactly the same steps as in [AST] and treating the term $\mathcal{M}u - u$ as we did before, we obtain a contradiction also in the case (3.61). This shows that (3.40) cannot hold and this completes the proof of Theorem 3.8. \square

We summarize the results of this section in the following:

Theorem 3.10 *Let $\Psi(x, y)$ be the value function given by (1.17). Suppose (2.23) holds, i.e.*

$$(3.62) \quad \delta > \gamma\alpha.$$

Then Ψ is continuous on \mathcal{S}^0 and Ψ is the unique viscosity solution of (3.4) with the property that there exists $C < \infty$ such that

$$(3.63) \quad |\Psi(x, y)| \leq C|x + y|^\gamma \quad \text{for all } (x, y) \in \mathcal{S}.$$

4 Numerical results

In this section we present the result of a numerical method used to approximate the viscosity solution of (3.4) in the case when the solvency region is $\mathcal{S}_+ = [0, \infty) \times [0, \infty)$ (see Remark 1.1). This method will be detailed in a forthcoming paper [CØS]. It is based on an iterative method which permits us to obtain the QVHJBI as a limit of variational HJB inequalities. Each variational inequality is approximated by a finite difference scheme and then solved by a Howard algorithm.

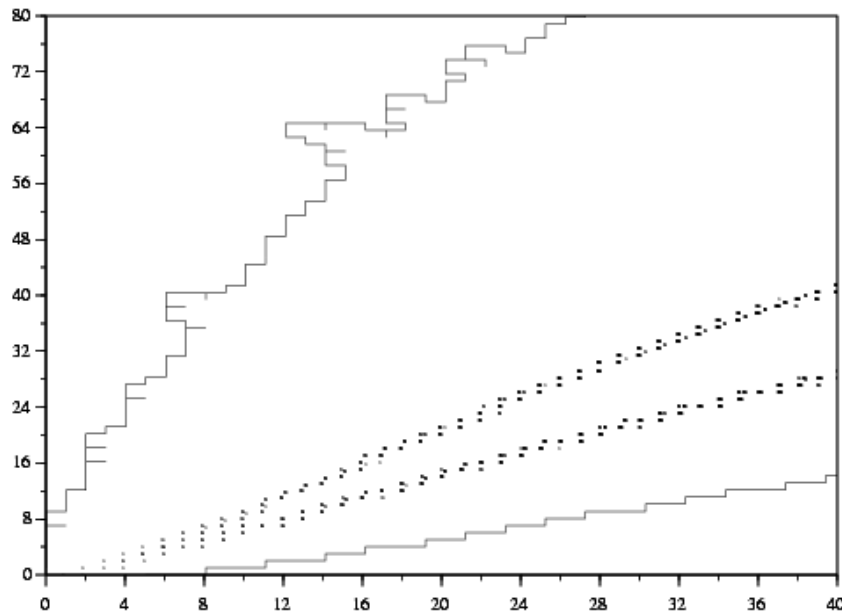


Figure 4: Optimal policy for $k = 0.1, \lambda = 0.01, \sigma = 0.3, r = 0.07, \delta = 0.1, \gamma = 0.3, \alpha = 0.11$

Figure 4 represents the boundary of the no-transaction region for the following values of the parameters: $k = 0.1, \lambda = 0.01, \sigma = 0.3, r = 0.07, \delta = 0.1, \gamma = 0.3, \alpha = 0.11$. The dotted lines (situated inside the no-transaction region) represent the set of positions reached after a transaction.

It is natural to ask if it is possible to obtain more information about the shape of the no-transaction region NT, both for this choice \mathcal{S}_+ of solvency region and for the choice \mathcal{S} given by (1.9). As mentioned in the introduction we know that if $k = 0$, then NT is bounded by two straight lines from the origin [DN] (see Figure 2). If $k > 0$, is NT still bounded by two curves? If so, what can be said about the form of these curves? Can they be given an explicit description?

Remark For results on viscosity solutions of QVIs corresponding to impulse control problems (which, however, do not apply to our situation), see [I], [P] and [TY].

Acknowledgements

We wish to thank Marianne Akian, Guy Barles, Nils Christian Framstad and Kristin Reikvam for very helpful comments and fruitful discussions. This work was partially supported by the French-Norwegian cooperation project Aur 99–050.

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