Entropy of endomorphisms and relative entropy in finite von Neumann algebras

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Abstract

We show the analogue for the entropy of automorphisms of finite von Neumann algebras of the classical formula $H(T) = H(\bigvee_{i=0}^{\infty} T^{-i}\mathcal{P} \mid \bigvee_{i=1}^{\infty} T^{-i}\mathcal{P})$, where T is a measure preserving transformation of a probability space, and $\mathcal{P}$ is a generator.

1 Introduction

If T is a measure preserving nonsingular transformation on a probability space $(X, \mathcal{B}, \mu)$ one of the basic results on entropy states that if $\mathcal{P}$ is a generator then the entropy of T is given by the relative entropy

$$H(T) = H\left(\bigvee_{0}^{\infty} T^{-i}\mathcal{P} \mid \bigvee_{1}^{\infty} T^{-i}\mathcal{P}\right) . \tag{1.1}$$

In the present paper we shall prove the analogous result for entropy of an automorphism α of a finite von Neumann algebra M. We shall replace the finite partitions $\bigvee_{0}^{n-1} T^{-i}\mathcal{P}$ by an increasing sequence (A_n) of finite dimensional von Neumann subalgebras of M satisfying certain regularity conditions to be specified later. Then we shall show (Theorem 4.1) that if $R = (\bigcup_{n=1}^{\infty} A_n)''$ and if α is considered as an endomorphism of R then

$$H(\alpha) = \tfrac{1}{2}H(R \mid \alpha(R)) + \tfrac{1}{2}\lim_{n\to\infty}\frac{1}{n}H(Z(A_n)) , \tag{1.2}$$

where $H(P|Q)$ denotes the relative entropy in the sense of [C-S] and [P-P] of two von Neumann algebras $P \supset Q$, and $Z(A_n)$ denotes the center of A_n. This formula is a direct generalization of (1.1), because if R is abelian then $A_n = Z(A_n)$, so $H(\alpha) = \lim_{n\to\infty} \frac{1}{n}H(Z(A_n))$. The proof of (1.2) also yields a formula for the index of a subfactor under reasonably general circumstances (Thm. 5.2).

In the special case when α is the so-called canonical endomorphism Γ defined by an inclusion of subfactors, see e.g. [C], formula (1.2) reduces to that found by Hiai [H]. We shall also see how our result fits into the theory of noncommutative Bernoulli shifts, binary shifts, and the shift on Temperley-Lieb algebras arising from the shift on the Jones projections.

The paper is organized as follows. In section 2 we develop the necessary techniques on finite von Neumann algebras needed later on. In section 3 we discuss entropy and how we can reduce our discussion to cases when R is of types II_1 and I_r, $r \in \mathbf{N}$, respectively. We also introduce the regularity conditions we impose on the sequence (A_n). Then formula (1.2) is shown in section 4. Finally in section 5 we discuss the different examples mentioned above.

2 Finite von Neumann algebras

In this section we collect some results on finite von Neumann algebras which will be needed later. Throughout the section R will be a finite von Neumann algebra, τ a faithful normal trace with $\tau(1) = 1$, and α an endomorphism of R, i.e. an injective *-homomorphism $\alpha : R \to R$, such that $\tau \circ \alpha = \tau$.

Lemma 2.1 *Let e_0 (resp. e_n, $n \in \mathbf{N}$) be the central projection in R such that $R_0 = Re_0$ (resp. $R_n = Re_n$) is of type II_1 (resp I_n). Then $\alpha(e_n) = e_n$.*

Proof. Since R_0 is of type II_1, so is $\alpha(R_0)$, hence $\alpha(R_0) \subset R_0$ and therefore $\alpha(e_0) \leq e_0$. Since $\tau(e_0) = \tau(\alpha(e_0))$ and τ is faithful, $e_0 = \alpha(e_0)$.

Let $n \in \mathbf{N}$ and $f_n = \sum_{i>n} e_i$. Then $f_n R$ contains $n+1$ mutually orthogonal abelian projections with the same central carrier, hence $\alpha(f_n R)$ contains the same in $\alpha(R)$. In particular $\alpha(f_n R) \subset f_n R$, and $\alpha(f_n) \leq f_n$. Again by faithfulness $\alpha(f_n) = f_n$. Similarly $\alpha(f_{n-1}) = f_{n-1}$, so that $\alpha(e_n) = \alpha(f_n) - \alpha(f_{n-1}) = f_n - f_{n-1} = e_n$. □

If R is a finite von Neumann algebra of type I then there are central projections $(e_n)_{n\in\mathbf{N}}$ in R such that $e_n R$ is of type I_n, hence is isomorphic to a von Neumann algebra of the form $M_n(\mathbf{C}) \otimes Z$ with Z an abelian von Neumann algebra [D, Ch. 3, §3]. We say R has *maximal type* I_r if $e_r \neq 0$ and $e_n = 0$ for $n > r$.

Throughout the rest of this section R will be a finite von Neumann algebra which is the weak closure of an AF-algebra, i.e. there is an increasing sequence $(A_n)_{n\in\mathbf{N}}$ of finite dimensional von Neumann subalgebras of R with $\bigcup_{n=1}^{\infty} A_n$ weakly dense in R. We denote by $Z(R)$ (resp $Z(A_n)$) the center of R (resp. A_n). The first lemma is well-known.

Lemma 2.2 *If $f_n \in Z(A_n)$ is a projection and $f_n \to f$ strongly, then f is a projection in $Z(R)$.*

Proof. By strong continuity of multiplication on bounded sets, f is a projection. Since $Z(A_n) \subset A'_m$ for $n \geq m$, the projections f_n all commute, and $f_n \in A'_m$ for $n \geq m$, and hence $f = \lim f_n \in A'_m$ for all m. Thus $f \in (\cup A_m)' \cap R = Z(R)$. □

Lemma 2.3 *Suppose each A_n has maximal type I_r. If $p_1, \ldots, p_r$ are nonzero equivalent abelian projections in A_{n_0} then $p = \sum_{i=1}^{r} p_i \in Z(R)$, and pR is of type I_r.*

Proof. If $n \geq n_0$ then $A_n \supset A_{n_0}$ so that $p_1, \ldots, p_r$ are equivalent abelian projections in A_n. Let q be the central support of p in A_n. If $q - p \neq 0$ then there exists an abelian projection p_{r+1} in qA_n orthogonal to p. Let $q' \leq q$ be the central support of p_{r+1} in A_n. Then $q'p_i$,

$i = 1, \dots, r+1$, are nonzero orthogonal abelian projections in A_n with same central support. Hence they are equivalent, [D, Ch. 3, §3], so A_n has maximal type I_k with $k \geq r+1$, contradicting our assumption on A_n. Thus $q = p \in Z(A_n)$. Since n is arbitrary, it follows as in Lemma 2.2 that $p \in Z(R)$, and pR is of type I_r. □

Lemma 2.4 *Suppose R is of type II_1. For each $n \in \mathbf{N}$ let $f_n \in Z(A_n)$ be the projection such that each irreducible representation of f_nA_n (resp. $1 - f_n)A_n$) is a factor of type I_k with $k \leq r$ (resp. $k > r$). Then (f_n) is a decreasing sequence converging strongly to 0.*

Proof. As in the proof of Lemma 2.2 $f_n \in A'_m$ whenever $m \leq n$, and the projections f_n form a commuting family. Furthermore since $m \leq n$ implies $f_nA_m \subset f_nA_n$, each irreducible representation of f_nA_m is of type I_k with $k \leq r$. In particular $f_nA_m = f_nf_mA_m$. Since $1 \in A_m$, $f_n = f_nf_m \leq f_m$, so the sequence (f_n) is decreasing. Let f be its strong limit. By Lemma 2.2 $f \in Z(R)$. Suppose $f \neq 0$. Since $f \leq f_n$ for all n, fA_n has maximal type less than or equal r. Let $k \leq r$ be the maximal type occurring among the algebras fA_n. Then there is n_0 such that fA_n has maximal type I_k for $n \geq n_0$. By Lemma 2.3 there is a nonzero projection $p \in R$, $p \leq f$, such that pR is of type I_k. This contradicts our assumption that R is of type II_1, so that $f = 0$.
□

Lemma 2.5 *Suppose each A_n has maximal type I_r. Then R has maximal type I_r.*

Proof. Let $e_n \in Z(A_n)$ be the projection such that e_nA_n is of type I_r while $(1 - e_n)A_n$ has maximal type strictly less than I_r. Since (A_n) is increasing the sequence (e_n) is increasing, hence converges strongly to a projection $e \in Z(R)$, see Lemma 2.2. If $p_1, \dots, p_r$ are nonzero equivalent abelian projections in A_n for some n, then by Lemma 2.3 $p = \sum_1^r p_i \in Z(R)$, and pR is of type I_r. If $q \leq e$ is a central projection in R then $qe_n \neq 0$ for n sufficiently large, hence qR contains a portion of type I_r. Since this is true for all such q, eR is of type I_r.

Since the maximal type of $(1 - e)A_n$ is strictly less than I_r we can use the same argument to show $(1 - e)R$ has maximal type strictly less than I_r, thus completing the proof. □

From the above proof we immediately obtain

Corollary 2.6 *Suppose R is homogeneous of type I_r. Let $e_n \in Z(A_n)$ be the projection such that e_nA_n is of type I_r while $(1 - e_n)A_n$ has maximal type less than I_r. Then $(e_n)_{n \in \mathbf{N}}$ converges strongly to 1.*

3 Entropy

If R is a finite von Neumann algebra with a faithful normal tracial state τ then the entropy of a τ-invariant automorphism, or endomorphism, was defined and studied in [C-S]. The crucial ingredient was a real function $H(N_1, \dots, N_n)$ defined on the set of finite von Neumann subalgebras of R, which was the analogue of the function

$$H\left(\bigvee_{i=1}^{n} \mathcal{P}_i\right) = H(\mathcal{P}_1, \dots, \mathcal{P}_n)$$

of finite partitions in the classical case. Letting

$$H(N,\alpha) = \lim_n \frac{1}{n} H(N, \alpha(N), \dots, \alpha^{n-1}(N))$$

then the entropy of α was defined to be

$$H(\alpha) = \sup_N H(N,\alpha) \ ,$$

where the sup is taken over all finite dimensional subalgebras N. The relative entropy $H(N|P)$ for two finite dimensional algebras was defined by

$$H(N|P) = \sup_{(x_i)\in S_1} \sum_i (\tau\eta(E_P(x_i)) - \tau\eta(E_N(x_i))) \ ,$$

where $(x_i) \in S_1$ is a finite set of operators $x_i \in R^+$, $\sum x_i = 1$, and η is the function $\eta(0) = 0$, $\eta(t) = -t\log t$ for $t \in (0,1]$, E_P is the τ-invariant conditional expectation of R onto P. If $N \supset P$ this definition is well defined when N and P are infinite dimensional and was studied by Pimsner and Popa in [P-P]. If it is necessary to make reference to the trace τ we write $H_\tau(\alpha), H_\tau(N|P)$ etc. instead of $H(\alpha), H(N|P)$, etc.

We shall find it necessary to study the action of α on each of the portions of R of types II_1 and I_n, $n \in \mathbf{N}$. For this we need the following result.

Lemma 3.1 *Let R be a finite von Neumann algebra with a faithful normal tracial state τ, and suppose α is a τ-invariant endomorphisms. Let $e_1, \dots, e_k$ be nonzero central projections in R with sum 1 such that $\alpha(e_i) = e_i$. Let τ_i be the trace on e_iR given by*

$$\tau_i(x) = \tau(e_i)^{-1}\tau(e_i x) \qquad x \in e_iR \ .$$

Then we have

(i) $H(\alpha) = \sum \tau(e_i) H_{\tau_i}(\alpha|e_iR)$.

If $P \subset N \subset R$ are von Neumann subalgebras we have

(ii) $H(N|P) = \sum \tau(e_i) H_{\tau_i}(e_iN|e_iP)$.

Proof. We have $H(\alpha) = \sup_M H(M,\alpha)$, where the sup is taken over all finite dimensional subalgebras. Since $M \subset N$ implies $H(M,\alpha) \le H(N,\alpha)$ we may consider the sup over all M which contain $e_1, \dots, e_k$. For such M we have by [H-S, Lem. 2]

$$\frac{1}{n} H(M, \alpha(M), \dots, \alpha^{n-1}(M)) = \frac{1}{n} \sum_{i=1}^{k} \tau(e_i) H(e_iM, \dots, e_i\alpha^{n-1}(M)) + \frac{1}{n} \sum_{i=1}^{k} \eta\tau(e_i) \ .$$

Letting $n \to \infty$ we get

$$H(M,\alpha) = \sum_1^k \tau(e_i) H(e_iM, \alpha|e_iR) \ ,$$

which implies (i).

Let $P_i = e_i P$, $N_i = e_i N$. Then for all $x \in R$

$$E_{P_i}(x) = E_{e_iP}(x) = e_i E_P(x) = e_i E_P(e_i x) ,$$

and similarly for N. Thus

$$\begin{aligned} H(N|P) &= \sup_{(x_j)\in S_1} \sum_j \tau\Big(\sum_i e_i \eta E_{P_i}(x_j)\Big) - \tau\Big(\sum_i e_i \eta E_{N_i}(x_j)\Big) \\ &= \sup_{(x_j)\in S_1} \sum_j \sum_i \tau(e_i)[\tau_{e_i}(\eta E_{P_i}(x_j)) - \tau_{e_i}(\eta E_{N_i}(x_j))] \end{aligned}$$

Since $x_j = \sum_i e_i x_j$ for all j the sup adds up as the sum of the sups. Hence

$$\begin{aligned} H(N|P) &= \sum_i \tau(e_i) \sup_{(e_i x_j)} \sum (\tau_{e_i}(\eta E_{P_i}(e_i x_j)) - \tau \eta E_{N_i}(e_i x_j))) \\ &= \sum_i \tau(e_i) H_{\tau_{e_i}}(N_i|P_i) . \end{aligned}$$

□

In section 2 we studied the case when $R = \overline{\bigcup_n A_n}$, where (A_n) is an increasing sequence of finite dimensional von Neumann subalgebras. If N is a mean generator for α in the sense of [G-S] then we shall apply the results to the case when $A_n = \bigvee_0^{n-1} \alpha^i(N)$. However, we do not need A_n to be that restricted.

Definition 3.2 *We say an increasing sequence $(A_n)_{n\in\mathbf{N}}$ of finite dimensional von Neumann subalgebras of R such that $R = \overline{\bigcup_n A_n}$ is a* generating sequence *for a τ-invariant endomorphism α if*

(i) $\alpha(A_n) \subset A_{n+1}, \quad n \in \mathbf{N}$

(ii) $H(\alpha) = \lim_{n\to\infty} \frac{1}{n} H(A_n)$.

(A_n) satisfies the commuting square condition *if (i) holds and*

(iii) $E_{\alpha(A_n)} = E_{\alpha(A_{n+1})} \circ E_{A_{n+1}} \ \forall\, n \in \mathbf{N}$.

Remark 3.3 In [G-S] we modified Voiculescu's definition [V] of the "approximation entropy" $ha_\tau(\alpha)$ to another, smaller approximation entropy $Ha(\alpha)$, and we showed that for the existence of different kinds of generators we have $Ha(\alpha) = H(\alpha)$. Just as for [G-S, Remark 3.5] this can be done when we have the existence of a generating sequence. Hence the tensor product formula $H(\alpha_1 \otimes \alpha_2) = H(\alpha_1) + H(\alpha_2)$ holds under this assumption, see [G-S, Prop. 2.6].

Remark 3.4 If (A_n) satisfies the commuting square condition then

$$\begin{array}{ccc} A_{n+1} & \subset & R \\ \bigcup & & \bigcup \\ \alpha(A_n) & \subset & \alpha(A_{n+1}) \end{array}$$

is a commuting square. In this case, by [P-P, Prop. 3.4]

$$H(R|\alpha(R)) = \lim_{n\to\infty} H(A_{n+1}|\alpha(A_n)) .$$

4 Relative entropy

In this section we prove our main result.

Theorem 4.1 *Let R be a finite von Neumann algebra with a faithful normal tracial state. Suppose α is a τ-invariant endomorphism with entropy $H(\alpha) < \infty$. Suppose $(A_n)_{n\in\mathbf{N}}$ is a generating sequence for α satisfying the commuting square condition. Then we have*

(i) $\lim_{n\to\infty} \frac{1}{n} H(Z(A_n))$ *exists.*

(ii) $H(\alpha) = \frac{1}{2} H(R|\alpha(R)) + \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} H(Z(A_n))$.

Furthermore, if R is of type I then $H(\alpha) = H(R|\alpha(R))$.

The proof will consist of an analysis of the relative entropies $H(A_{n+1}|\alpha(A_n))$ as $n \to \infty$. For this we shall use a formula for relative entropy shown by Pimsner and Popa [P-P, Theorem 6.2]. We follow their notation somewhat closely.

Let $A_n = \bigoplus_{\ell\in K_n} M_\ell^n$, where M_ℓ^n is a factor of type m_ℓ^n. Let e_ℓ^n be the central projection in A_n such that $M_\ell^n = e_\ell^n A_n$. Let $a_{k\ell}^n$ be the multiplicity of $\alpha(M_k^{n-1})$ in M_ℓ^n, i.e. M_ℓ^n contains $a_{k\ell}^n$ copies of $\alpha(M_k^{n-1})$. Then

$$m_\ell^n = \sum_k a_{k\ell}^n m_k^{n-1} \ .$$

Let $b_{k\ell}^n = \tau(e_\ell^n \alpha(e_k^{n-1}))$. Thus

$$b_{k\ell}^n = \frac{a_{k\ell}^n m_k^{n-1} \tau(e_\ell^n)}{m_\ell^n} \ .$$

Proposition 4.2 (Pimsner, Popa) *With the above notation*

$$\begin{aligned} H(A_n|\alpha(A_{n-1})) &= (2H(A_n) - H(Z(A_n))) \\ &\quad -(2H(\alpha(A_{n-1})) - H(Z(\alpha(A_{n-1}))) + \sum_{k,\ell} b_{k\ell}^n \log c_{k\ell}^n \ , \end{aligned}$$

where $c_{k\ell}^n = \min(\frac{m_k^{n-1}}{a_{k\ell}^n}, 1)$.

Since $H(A_{n-1}) = H(\alpha(A_{n-1}))$ and $H(Z(A_{n-1})) = H(\alpha(Z(A_{n-1})))$ the above formula can be rewritten as

$$\begin{aligned} H(A_n|\alpha(A_{n-1})) &= 2(H(A_n) - H(A_n)) \\ &\quad -(H(Z(A_n)) - H(Z(A_{n-1}))) + \sum_{k,\ell} b_{k\ell}^n \log c_{k\ell}^n \ . \end{aligned} \tag{4.1}$$

Lemma 4.3 *With the above notation, if R is homogeneous of type I_r, $r \in \mathbf{N}$, then*

$$\lim_{n\to\infty} \sum_{k\ell} b_{k\ell}^n \log c_{k\ell}^n = 0 \ .$$

Proof. Let $\varepsilon > 0$. By Corollary 2.6 there is $n_0 \in \mathbf{N}$ such that $\tau(e_n) > 1-\varepsilon$ for $n \geq n_0$, where e_n is the central projection in A_n on the type I_r portion of A_n. For each n let $\mathrm{I}_n = \{\ell : e^n_\ell \leq 1-e_n\}$. Then

$$\sum_{k\in \mathrm{I}_{n-1}} \sum_\ell b^n_{k\ell} < \varepsilon \qquad \text{for } n > n_0 \ .$$

If $e^{n-1}_k \leq e_{n-1}$ then $a^n_{k\ell} = 1$, so $\log c^n_{k\ell} = 0$. Since A_n has maximal type I_r, see Corollary 2.6, $c^n_{k\ell} \geq \frac{1}{r}$. Thus when $n > n_0$

$$0 \leq -\sum_{k\ell} b^n_{k\ell} \log c^n_{k\ell} < \varepsilon \log r \ ,$$

proving the lemma. □

Lemma 4.4 *Suppose R is of type II_1 and that*

$$\limsup_N \frac{1}{N} \sum_{n=1}^N \sum_{k\ell} b^n_{k\ell} \log a^n_{k\ell} < \infty \ .$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N \sum_{k\ell} b^n_{k\ell} \log c^n_{k\ell} = 0 \ .$$

Proof. Let $d^n_{k\ell} = (c^n_{k\ell})^{-1} = \max\{\frac{a^n_{k\ell}}{m^{n-1}_k}, 1\}$. Put

$$c = \limsup_N \frac{1}{N} \sum_{n=1}^N \sum_{k\ell} b^n_{k\ell} \log d^n_{k\ell} \ .$$

Put $\mathrm{I}_n = \{(k,\ell) : d^n_{k\ell} > 1\}$. By assumption there is a constant $K > 0$ such that for all N

$$K > \frac{1}{N} \sum_{n=1}^N \sum_{k\ell} b^n_{k\ell} \log a^n_{k\ell} > \frac{1}{N} \sum_{n=1}^N \sum_{(k\ell)\in \mathrm{I}_n} b^n_{k\ell} \log m^{n-1}_k \ .$$

By Lemma 2.4 we can for given $r \in \mathbf{N}$ and $\delta > 0$ find N_0 such that if

$$J_n = \{\ell \in K_n : m^n_\ell \geq r\} \ ,$$

then for $n \geq N_0$

$$\sum_{\substack{k\ell \\ \ell\in K_n\setminus J_n}} b^n_{k\ell} < \delta \ . \tag{4.2}$$

Therefore we have for $N > N_0$

$$\begin{aligned} K &> \frac{1}{N} \sum_{n=1}^N \sum_{k\ell} b_{k\ell} \log a_{k\ell} \\ &> \frac{1}{N} \sum_{n=N_0}^N \sum_{(k,\ell)\in J_n} b^n_{k\ell} \log r \ . \end{aligned}$$

Since this holds for all $r \in \mathbf{N}$ and $N > N_0$, we get in the limit, using (4.2) that

$$\lim \frac{1}{N} \sum_{n=1}^{N} \sum_{(k,\ell)\in \mathrm{I}_n} b_{k\ell}^n = 0 \ . \tag{4.3}$$

For $q \in \mathbf{N}$ put

$$P_q^n = \{(k,\ell) \in \mathrm{I}_n : (q-1)m_k^{n-1} < a_{k\ell}^n \leq q m_k^{n-1}\} \ ,$$

so in particular $d_{k\ell}^n \leq q$ for $(k,\ell) \in P_q^n$, and $\mathrm{I}_n = \bigcup_{q=1}^{\infty} P_q^n$ is a disjoint union. By (4.3) we get for all $q \in \mathbf{N}$,

$$\limsup_N \frac{1}{N} \sum_{n=1}^{N} \sum_{(k,\ell)\in P_q^n} b_{k\ell}^n \log d_{k\ell}^n \leq \lim_N \frac{1}{N} \sum_{n=1}^{N} \sum_{(k,\ell)\in P_q^n} b_{k\ell}^n \log q = 0 \ .$$

Let $f(n,q) = \sum_{(k,\ell)\in P_q^n} b_{k\ell}^n \log d_{k\ell}^n$. Then f is a nonnegative real function on $\mathbf{N} \times \mathbf{N}$. Thus we have

$$\begin{aligned} c &= \limsup_N \frac{1}{N} \sum_{n=1}^{N} \sum_{(k,\ell)\in \mathrm{I}_n} b_{k\ell}^n \log d_{k\ell}^n \\ &= \limsup_N \frac{1}{N} \sum_{n=1}^{N} \sum_{q=1}^{\infty} \sum_{(k,\ell)\in P_q^n} b_{k\ell}^n \log d_{k\ell}^n \\ &= \limsup_N \frac{1}{N} \sum_{n=1}^{N} \sum_{q=1}^{\infty} f(n,q) \\ &= \limsup_N \sum_{q=1}^{\infty} \frac{1}{N} \sum_{n=1}^{N} f(n,q) \\ &\leq \sum_{q=1}^{\infty} \limsup_N \frac{1}{N} \sum_{n=1}^{N} f(n,q) \\ &= 0 \ . \end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 4.1. Let $e_i \in Z(R)$ be the projection such that $e_0 R$ is of type II_1, $e_i R$ is of type I_i, $i \in \mathbf{N}$.

Since $\tau(e_i) \to 0$ as $i \to \infty$, Lemma 3.1 is applicable. If we apply part (ii) of Lemma 3.1 to $P = \mathbf{C}$ we also have

$$H(Z(A_n)) = \sum \tau(e_i) H_{\tau_i}(Z(e_i A_n)) \ .$$

Thus in order to prove the formula in Theorem 4.1 we may consider the algebras $e_i R$ and $\alpha | e_i R$ separately, since by Lemma 2.1 $\alpha(e_i) = e_i$. For each n denote by

$$C_n = \sum_{k,\ell} b_{k\ell}^n \log c_{k\ell}^n \ .$$

Then we have by (4.1), assuming that R is either of type II_1 or homogeneous of type I_r,

$$H(A_n|\alpha(A_{n-1})) = 2(H(A_n) - H(A_{n-1})) - (H(Z(A_n)) - H(Z(A_{n-1}))) + C_n$$

Hence,

$$\frac{1}{N}\sum_{n=1}^{N} H(A_n|\alpha(A_{n-1})) = \tag{4.4}$$
$$= \frac{2}{N}H(A_N) - \frac{2}{N}H(A_0) - \frac{1}{N}H(Z(A_N)) + \frac{1}{N}H(Z(A_0)) + \frac{1}{N}\sum_{1}^{N} C_n \ .$$

By assumption the sequence (A_n) satisfies the commuting square condition, so by Remark 3.4

$$\lim_{n\to\infty} H(A_n|\alpha(A_{n-1})) = H(R|\alpha(R)) \ .$$

Since (A_n) is a generating sequence for α,

$$\lim_{N\to\infty} \frac{1}{N}H(A_N) = H(\alpha) \ .$$

In particular

$$\limsup_{N} \frac{1}{N}H(Z(A_N)) \le \lim_{N} \frac{1}{N}H(A_N) = H(\alpha)$$

We therefore have the existence of $c > 0$ and $N_0 \in \mathbf{N}$ such that if $N \ge N_0$ then

$$\frac{1}{N}\sum_{n=1}^{N}(H(A_n) - H(A_{n-1})) < H(\alpha) + c$$

and

$$\frac{1}{N}\sum_{n=1}^{N}(H(Z(A_n)) - H(Z(A_{n-1}))) < H(\alpha) + c \ .$$

It follows that

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{k\ell} b_{k\ell}^{n}\log a_{k\ell}^{n} \le \frac{1}{N}\sum_{n=1}^{N}\sum_{k\ell} b_{k\ell}^{n}\log\frac{m_\ell^n}{m_k^{n-1}} =$$
$$= \frac{1}{N}\sum_{n=1}^{N}\Big\{(H(A_n) - H(A_{n-1})) - (H(Z(A_n)) - H(Z(A_{n-1})))\Big\} \tag{4.5}$$
$$< 2H(\alpha) + 2c \ .$$

Hence by Lemmas 4.3 and 4.4 $\lim_{N\to\infty} \frac{1}{N}\sum_{1}^{N} C_n = 0$. Since also

$$\lim_{N\to\infty} \frac{2}{N}H(A_0) = 0 \ , \qquad \lim_{N\to\infty} \frac{1}{N}H(Z(A_0)) = 0$$

it follows from (4.4) that $\lim_{N\to\infty} \frac{1}{N} H(Z(A_N))$ exists, hence

$$H(R|\alpha(R)) = 2H(\alpha) - \lim_{N\to\infty} \frac{1}{N} H(Z(A_N)) .$$

Finally if R is of type I_r then since

$$\begin{aligned} H(A_n) &= \sum_\ell \tau(e_\ell^n) \log m_\ell^n + H(Z(A_n)) , \\ H(A_N) &\leq \log r + H(Z(A_N)) , \end{aligned}$$

which shows that

$$\lim_{N\to\infty} \frac{1}{N} H(A_N) = \lim_{N\to\infty} \frac{1}{N} H(Z(A_N)) ,$$

from which we obtain $H(\alpha) = H(R|\alpha(R))$. □

If R is a factor of type II_1 then we can apply a result of Pimsner and Popa [P-P, Theorem 4.4] to obtain a different formula for $H(\alpha)$.

Corollary 4.5 *Let R be the hyperfinite II_1-factor with a τ-invariant endomorphism α with entropy $H(\alpha) < \infty$. Suppose $(A_n)_{n\in\mathbf{N}}$ is a generating sequence for α satisfying the commuting square condition. Then*

(i) $\lim_{n\to\infty} \frac{1}{n} H(Z(A_n))$ *exists.*

(ii) $R \cap \alpha(R)'$ *is atomic with minimal projections f_k, $\sum_k f_k = 1$.*

(iii) $H(\alpha) = H(R\cap\alpha(R)') + \frac{1}{2}\sum_k \tau(f_k)\log[R_{f_k} : \alpha(R)_{f_k}] + \frac{1}{2}\lim_{n\to\infty}\frac{1}{n}H(Z(A_n))$.

Proof. By Theorem 4.1 (i) holds. Since by Theorem 4.1 $H(R|\alpha(R)) < \infty$, $R\cap\alpha(R)'$ is atomic by [P-P, Theorem 4.4]. Thus (iii) is a direct application of [P-P, Theorem 4.4] to $H(R|\alpha(R))$ inserted in Theorem 4.1. □

5 Index of subfactors

An inspection of the proof of Theorem 4.1 shows that we used dynamical entropy only in the assumption that $H(\alpha) = \lim_n \frac{1}{n} H(A_n)$ and therefore that $\lim_n \frac{1}{n} H(Z(A_n))$ existed. We shall in the present section consider a concept closely related to entropy of a matrix algebra, but with the difference that it depends on the dimensions of the irreducible components and not on their ranks. As a consequence we obtain an explicit formula for relative entropy, and for index of subfactors in the irreducible case. We state the definition for finite dimensional C*-algebras, but it is obvious how it extends to other algebras.

Definition 5.1 *Let $M = \bigoplus_{\ell\in K} M_\ell$ where M_ℓ is a I_{m_ℓ}-factor. Let e_ℓ be the central projection in M such that $M_\ell = Me_\ell$, and let τ be a tracial state. Then*

$$D_\tau(M) = \sum_{\ell\in K} \tau(e_\ell) \log \frac{\dim M_\ell}{\tau(e_\ell)} .$$

We shall usually drop the suffix τ and write $D(M)$ for $D_\tau(M)$. A straightforward computation shows that

$$D(M) = 2H(M) - H(Z(M)) \; .$$

As in definition 3.2 we say two increasing sequences $(A_n)_{n\in\mathbf{N}}$ and $(B_n)_{n\in\mathbf{N}}$ of finite dimensional C*-algebras such that $B_n \subset A_n$ satisfy the *commuting square condition* if

$$\begin{array}{ccc} A_n & \subset & A_{n+1} \\ \bigcup & & \bigcup \\ B_n & \subset & B_{n+1} \end{array}$$

is a commuting square for all $n \in \mathbf{N}$. Then the reformulation of Theorem 4.1 becomes.

Theorem 5.2 *Let R be a von Neumann algebra with a faithful normal tracial state τ. Suppose $(A_n)_{n\in\mathbf{N}}$ and $(B_n)_{n\in\mathbf{N}}$ are increasing sequences of finite dimensional C*-subalgebras such that $B_n \subset A_n$ for all $n \in \mathbf{N}$. Let $P = (\bigcup_n A_n)^-$ and $Q = (\bigcup_n B_n)^-$ (weak closures). Assume*

(i) *$(A_n)_{n\in\mathbf{N}}$ and $(B_n)_{n\in\mathbf{N}}$ satisfy the commuting square condition.*

(ii) *$D(A_{n-1}) = D(B_n)$, $n \in \mathbf{N}$.*

(iii) *$\sup_n \frac{1}{n} D(A_n) < \infty$.*

Then the sequence $(\frac{1}{n} D(A_n))_{n\in\mathbf{N}}$ converges, and

$$H(P|Q) = \lim_{n\to\infty} \frac{1}{n} D(A_n)$$

In particular, if P is of type II_1 and $P \cap Q' = \mathbf{C}$ then the index

$$[P{:}\, Q] = \lim_{n\to\infty} \exp\left(\frac{1}{n} D(A_n)\right).$$

Outline of proof. Let notation be as in section 4, so $A_n = \bigoplus_{\ell\in K} M_\ell^n$. Replace $\alpha(A_{n-1})$ by B_n. Then by Proposition 4.2 and assumption (ii) we obtain the analogue of (4.1).

$$H(A_n|B_n) = D(A_n) - D(A_{n-1}) + \sum b_{k\ell}^n \log c_{k\ell}^n \; . \tag{5.1}$$

By (iii) there is $K > 0$ such that $\frac{1}{n} D(A_n) < K$ for all n. Since

$$H(A_n) - H(A_{n-1}) - (H(Z(A_n)) - H(Z(A_{n-1})) \le D(A_n) - D(A_{n-1})$$

it follows from (4.5) that

$$\frac{1}{N} \sum_{n=1}^{N} \sum_{k\ell} b_{k\ell}^n \log a_{k\ell}^n \le K \; ,$$

hence by Lemmas 4.3 and 4.4

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{k\ell} b_{k\ell}^n \log c_{k\ell}^n = 0 \ . \tag{5.2}$$

By assumption (i) and [P-P, Prop. 3.4]

$$H(P|Q) = \lim_{N\to\infty} H(A_N|B_N)$$

Thus by (5.1) and (5.2)

$$H(P|Q) = \lim_{N\to\infty} \frac{1}{N} D(A_N) \ .$$

Finally, if $P \cap Q' = \mathbf{C}$ and P is of type II_1 then by [P-P, Cor. 4.6], see also Cor. 4.5,

$$H(P|Q) = \log[P\colon Q] \ ,$$

From which the last statement of the theorem follows. □

6 Examples

In this section we show how some well-known cases fit into the setup in Theorems 4.1 and 5.2.

6.1 Bernoulli shifts

Noncommutative Bernoulli shifts were constructed in [C-S] as follows. Let $M_i = M_d(\mathbf{C})$. Let $A = \bigotimes_{-\infty}^{\infty} M_i$ be the C*-tensor product. Let φ_0 be a state on M_0 and $\varphi_i = \varphi_0$. Let $\varphi = \bigotimes_{-\infty}^{\infty} \varphi_i$, and let β be the shift on the tensor product. In the GNS-representation π_φ of A defined by φ let $M = \pi_\varphi(A)''$ and let M_φ denote the centralizer of φ in M. Then by [C-S, Theorem 4] M_φ is a II_1-factor, and the extension of β to M restricted to M_φ is the noncommutative Bernoulli shift α defined by φ_0. With the natural embedding of finite tensor products $\bigotimes_{0}^{n-1} M_i$ into M we put

$$A_n = \left(\bigotimes_{i=0}^{n-1} M_i \right) \cap M_\varphi \ .$$

Let $R = (\bigotimes_{i=0}^{\infty} M_i) \cap M_\varphi$, where we consider $\bigotimes_{i=0}^{\infty} M_i$ in its weak closure in M, and let τ be the trace $\varphi|R$. Then $\alpha|R$ is an endomorphism, and (A_n) is a generating sequence for α satisfying the commuting square condition. It was shown in [C-S] that if φ_0 is defined by a positive matrix

$$\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_d \end{pmatrix}$$

with $\sum h_i = 1$, and D_1 is the diagonal matrices in A_1 then

$$H(\alpha) = H_{\varphi_0}(D_1) = H_{\varphi_0}(A_1) = -\sum_1^d h_i \log h_i \ .$$

By definition of A_1 and R it is clear that

$$A_1 \subset R \cap \alpha(R)'$$

Let $f_1, \ldots, f_d$ be the minimal projections in D_1 with sum 1 identified with $f_i \otimes 1$ in $A_1 \otimes \bigotimes_2^\infty M_i$. Then

$$f_i \alpha(R) f_i = f_i \alpha(R) = f_i R f_i \ .$$

Thus by Corollary 4.5

$$\begin{aligned} H(D_1) = H(\alpha) &= H(\alpha(R)' \cap R) + \tfrac{1}{2} \lim_n \frac{1}{n} H(Z(A_n)) \\ &\geq H(D_1) + \tfrac{1}{2} \lim_n \frac{1}{n} H(Z(A_n)) \\ &\geq H(D_1) \ , \end{aligned}$$

hence $\lim_n \frac{1}{n} H(Z(A_n)) = 0$, and D_1 having the same entropy as $\alpha(R)' \cap R$ is a masa in $\alpha(R)' \cap R$, see [H-S, Lemma 4.1].

6.2 The Jones projections

Let $(e_i)_{i \in \mathbf{Z}}$ be a sequence of projections in the hyperfinite II_1-factor satisfying the relations

(i) $e_i e_{i\pm 1} e_i = \lambda e_i$
(ii) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$
(iii) $\lambda \tau(w) = \tau(w e_j)$ if $w \in C^*(e_0, \ldots, e_{j-1})$

Let α_λ be the shift $\alpha(e_i) = e_{i+1}$ on the C*-algebra A generated by the projections e_i. Let R denote the weak closure of $C^*(e_i : i \geq 0)$. Then α_λ is an endomorphism when restricted to R. As remarked in [G-S, Example 3.8] the sequence $(A_n = C^*(e_0, \ldots, e_{n-1}))$ is generating for α_λ on R, and by [GHJ, Example 4.2.9] it satisfies the commuting square condition. It was shown by Pimsner and Popa [P-P] that α_λ is a Bernoulli shift with $d = 2$ defined by the state

$$\varphi_0(x) = \mathrm{Tr}\left(\begin{pmatrix} t & 0 \\ 0 & 1-t \end{pmatrix} x \right) \qquad \text{on } M_2(C), \text{ where } \lambda = t(1-t)$$

when $\lambda \leq \frac{1}{4}$, and if $\frac{1}{4} \leq \lambda < 1$ then

$$H(\alpha_\lambda) = -\tfrac{1}{2} \log \lambda \ .$$

Furthermore it is known, see [P-P], that in this case

$$R \cap \alpha_\lambda(R)' = \mathbf{C} \ .$$

Therefore by Corollary 4.5, if $\lambda \in [\frac{1}{4}, 1)$

$$H(\alpha_\lambda) = \tfrac{1}{2} \log[R : \alpha_\lambda(R)] + \lim_{n\to\infty} \frac{1}{n} H(Z(A_n)) \ .$$

If one shows that $\lim_n \frac{1}{n} H(Z(A_n)) = 0$, as follows from computations in [J], one recovers the result by Jones [J] that $[R : \alpha_\lambda(R)] = \lambda^{-1}$.

6.3 Binary shifts

Let $X \subset \mathbf{N}$ and let $(s_n)_{n\in\mathbf{Z}}$ be a sequence of self-adjoint unitary operators satisfying the commutation relations

$$s_i s_j = \begin{cases} s_j s_i & \text{if } |i-j| \notin X \\ -s_j s_i & \text{if } |i-j| \in X \ . \end{cases}$$

If the set $-X \cup \{0\} \cup X$ is a nonperiodic subset of $\mathbf{Z}$ as we shall assume, the C*-algebra $A(X)$ generated by all the s_n is the CAR-algebra [Po-Pr, V], and the trace τ is 0 on all products $s_{i_1} s_{i_2} \dots s_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. Let α be the shift on $A(X)$ defined by $\alpha(s_i) = s_{i+1}$. Let $A_n = C^*(s_0, s_1, \dots, s_{n-1})$. Then by [Po-Pr]

$$A_n = M_{2^{d_n}} \otimes D_{2^{c_n}} \ , \tag{6.1}$$

where D_k denotes the diagonal in $M_k(\mathbf{C})$. In the GNS-representation of $A(X)$ defined by τ let

$$R = \Big(\bigcup_{n\geq 1} A_n \Big)^-, \qquad \text{weak closure} \ ,$$

where we identify A_n with $\pi_\tau(A_n)$. Then R is the hyperfinite II$_1$-factor, and α is an endomorphism on R. If α has a mean generator in the sense of [G-S] then by [G-S, Propositions 3.3 and 4.8 and Lemmas 4.6 and 4.7] the sequence (A_n) is a generating sequence for α. Now each operator in $\bigcup_{n\geq 1} A_n$ is a sum of products of the form $w = s_{i_1} s_{i_2} \dots s_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. In the Hilbert space structure on $A(X)$ defined by τ we have $w \perp A_n$ if and only if $i_k \geq n$.

Since the conditional expectations E_B, $B \subset A(X)$, can be identified with the orthogonal projections on the subspaces of the Hilbert space corresponding to B, it is immediate that $E_{\alpha(A_{n+1})} \circ E_{A_{n+1}} = E_{\alpha(A_n)}$ for all n, hence (A_n) satisfies the commuting square condition. Since by [G-S, Lemma 4.7] $c_n = 0(n)$ we have

$$\frac{1}{n} H(Z(A_n)) = \frac{1}{n} H(D_{2^{c_n}}) = \frac{1}{n} c_n \log 2 \to 0 \qquad \text{as } n \to \infty \ .$$

This shows that it is in general a quite delicate problem to verify if $\lim \frac{1}{n} H(Z(A_n)) = 0$. For a general binary shift we can compute the index by using Theorem 5.2. Indeed, by (6.1)

$$D(A_n) = \log 2^{2d_n + c_n} = \log 2^n = n \log 2 \ .$$

Since by [Po] $R \cap \alpha(R)' = \mathbf{C}$ Theorem 5.2 implies that

$$[R \colon \alpha(R)] = \exp \log 2 = 2 \ ,$$

a result shown by Powers in [Po].

6.4 Canonical shifts

Let M_1 be a II_1-factor and N a subfactor with finite index. Let

$$\cdots \subset N_2 \subset N_1 \subset N_0 = N \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$$

be the two-sided tower. Put

$$R = \Big(\bigcup_{n \geq 0} M' \cap M_n \Big)''$$

There is an anti-automorphism γ_n of $M' \cap M_{2n}$ given by $\dot{\gamma}_n(x) = J_n x J_n$, where J_n is the canonical involution defined by $M' \cap M_n$. The *canonical shift* Γ on R is the endomorphism defined by

$$\Gamma(x) = \gamma_{n+1} \circ \gamma_n(x) \qquad \text{for } x \in M' \cap M_{2n} .$$

The entropy of Γ has been studied by Choda [C] and Hiai [H]. In [H, Theorem 4.1] Hiai showed that $\lim\limits_{n\to\infty} \frac{1}{n} H(Z(M' \cap M_{2n}))$ exists, and

$$H(\Gamma) = \tfrac{1}{2} H(R|\Gamma(R)) + \tfrac{1}{2} \lim_{n\to\infty} \frac{1}{n} H(Z(M' \cap M_{2n})) . \tag{6.2}$$

This formula is a consequence of Theorem 4.1. Indeed, if we let $A_n = M' \cap M_{2n}$, by [H, Equation 2.2]

$$H(\Gamma) = \lim_n \frac{1}{n} H(A_n) .$$

Furthermore $\Gamma(M'_k \cap M_{2n}) \subset M'_{k+1} \cap M_{n+2}$, [C]. Hence

$$\Gamma(A_n) = \Gamma(M' \cap M_{2n}) \subset M'_1 \cap M_{n+2} \subset M' \cap M_{n+2} = A_{n+1} .$$

Thus (A_n) is a generating sequence for Γ. It follows from [P, Proposition 3.1] that the sequence (A_n) satisfies the commuting square condition. Thus the formula (5.1) of Hiai is nothing but Theorem 4.1 applied to the case $\alpha = \Gamma$.

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