

OPTIMAL STOPPING WITH INFORMATION CONSTRAINT

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ABSTRACT. We study the optimal stopping problem proposed by Dupuis and Wang in [9]. In this maximization problem of expected present value of exercise payoff, the underlying dynamics follow a linear diffusion. The decision maker is not allowed to stop at any time she chooses but rather on the jump times of an independent Poisson process. In [9], the authors solve this problem in the case where the underlying is a geometric Brownian motion and the payoff function is of American call option type. In the current study, we propose a mild set of conditions (covering the setup of [9]) on both the underlying and the payoff and build and use a Markovian apparatus based on the Bellman principle of optimality to solve the problem under these conditions. We also discuss interpretation of this model as optimal timing of an irreversible investment decision under an exogenous information constraint.

1. INTRODUCTION AND THE MAIN RESULT

1.1. **The underlying dynamics.** We assume that the underlying state process X is a regular linear diffusion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ satisfying the usual conditions and evolves on \mathbf{R}_+ with initial state x , see [5]. We denote as \mathbf{P}_x the probability measure \mathbf{P} conditioned on the initial state x and as \mathbf{E}_x the expectation with respect to \mathbf{P}_x . In line with most economical and financial applications, we assume that X does not die inside the state space \mathbf{R}_+ , i.e., that killing of X is possible only at the boundaries 0 and ∞ . Therefore the boundaries 0 and ∞ are either natural, entrance, exit or regular. In the case a boundary is regular, it is assumed to be either killing or reflecting, see [5], pp. 18–20, for a characterization of the boundary behavior of diffusions. Now, the evolution of X is completely determined by its *scale function* S and *speed measure* m inside \mathbf{R}_+ , see [5], pp. 13–14. Furthermore, we assume that the function S and the

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measure m are both absolutely continuous with respect to the Lebesgue measure, have smooth derivatives and that S is twice continuously differentiable. Under these assumptions, we know that the infinitesimal generator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow C_b(\mathbf{R}_+)$ of X can be expressed as $\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ where the functions σ and μ (the *infinitesimal parameters* of X) are related to S and m via the formulæ $m'(x) = \frac{2}{\sigma^2(x)}e^{B(x)}$ and $S'(x) = e^{-B(x)}$ for all $x \in \mathbf{R}_+$, where $B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}dy$, see [5], pp. 17. From these definitions we find that $\sigma^2(x) = \frac{2}{S'(x)m'(x)}$ and $\mu(x) = -\frac{S''(x)}{S'^2(x)m'(x)}$ for all $x \in \mathbf{R}_+$. In what follows, we assume that the functions μ and σ^2 are continuous – a sufficient condition for this is that the derivatives S' and m' do not vanish inside \mathbf{R}_+ . The assumption that the state space is \mathbf{R}_+ is done for reasons of notational convenience. In fact, we could assume that the state space is any interval \mathcal{I} in \mathbf{R} and all our subsequent analysis would hold with obvious modifications. Furthermore, we denote as, respectively, ψ_r and φ_r the increasing and the decreasing solution of the ordinary second-order linear differential equation $\mathcal{A}u = ru$, where $r > 0$, defined on the domain of the characteristic operator of X – for the characterization and fundamental properties of the minimal r -excessive functions ψ_r and φ_r , see [5], pp. 18–20. In addition, we assume that the filtration \mathcal{F}_t is rich enough to carry a Poisson process $N = (N_t, \mathcal{F}_t)$ with intensity λ . We call the process N the *signal process*, and assume that X and N are *independent*.

For $r > 0$, we denote as L_1^r the class of real valued measurable functions f on \mathbf{R}_+ satisfying the condition $\mathbf{E}_x \left[\int_0^\zeta e^{-rt} |f(X_t)| dt \right] < \infty$, where ζ denotes the lifetime, i.e., the first exit time of the state process X from \mathbf{R}_+ . For a function $f \in L_1^r$, the *resolvent* $R_r f : \mathbf{R}_+ \rightarrow \mathbf{R}$ is defined as

$$(1.1) \quad (R_r f)(x) = \mathbf{E}_x \left[\int_0^\zeta e^{-rs} f(X_s) ds \right],$$

for all $x \in \mathbf{R}_+$. The resolvent R_r and the increasing and decreasing solutions ψ_r and φ_r are connected in a computationally very useful way. Indeed, we know from the literature that for given $f \in L_1^r$ the resolvent $R_r f$ can be expressed as

$$(1.2) \quad (R_r f)(x) = B_r^{-1} \varphi_r(x) \int_0^x \psi_r(y) f(y) m'(y) dy + B_r^{-1} \psi_r(x) \int_x^\infty \varphi_r(y) f(y) m'(y) dy,$$

for all $x \in \mathbf{R}_+$, where $B_r = \frac{\psi_r'(x)}{S'(x)} \varphi_r(x) - \frac{\varphi_r'(x)}{S'(x)} \psi_r(x)$ denotes the Wronskian determinant, see [5], pp. 19. We remark that the value of B_r does not depend on the state variable x but on the rate r .

1.2. The optimal stopping problems. Having the underlying dynamics set up, we formulate, following [9], our main optimal stopping problems. In comparison to the classical continuous time case, see, e.g., [2], [7], [14], and [19], see also [16], the key difference is that the decision maker is not allowed to (or cannot) exercise at any time she chooses but rather on the jump times of the independent signal process N . The process N

jumps at times $T_1 < T_2 < \dots < T_n < \dots$, where the intervals $\{T_1, T_2 - T_1, T_3 - T_2, \dots\}$ are exponential IID with mean $\frac{1}{\lambda}$. We remark that by convention $T_0 = 0$ and $T_\infty = \infty$.

In the first optimal stopping problem, the decision maker cannot exercise at the initial time $t = 0$. This means that the first jump time T_1 is the first potentially reasonable moment for her to exercise. In this setting, the class of admissible stopping times reads as

$$(1.3) \quad \mathcal{T} = \{\tau : \text{for all } \omega \in \Omega, \tau(\omega) = T_n(\omega) \text{ for some } n \in 1, 2, \dots, \infty\}.$$

Let $r > 0$ be the constant discount rate and $g : \mathbf{R}_+ \rightarrow \mathbf{R}$ the exercise payoff function, which is assumed to be at least continuous. The optimal stopping problem is now to maximize the expected present value of the exercise payoff under $\{\mathcal{F}_\tau\}_{\tau \in \mathcal{T}}$, i.e. to determine the optimal value function

$$(1.4) \quad V(x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x [e^{-r\tau} g(X_\tau)];$$

following [19], we set $g(X_\tau) = \limsup_{t \rightarrow \infty} g(X_t)$ whenever $\tau(\omega) = \infty$. Moreover, we want to characterize the optimal stopping time τ^* which constitutes this value.

The second optimal stopping problem is otherwise the same as the first but now the decision maker can exercise immediately, i.e., at $t = 0$. Now, the class of admissible stopping times reads as

$$(1.5) \quad \mathcal{T}_0 = \{\tau : \text{for all } \omega \in \Omega, \tau(\omega) = T_n(\omega) \text{ for some } n \in 0, 1, 2, \dots, \infty\}.$$

The corresponding optimal stopping problem reads as

$$(1.6) \quad V_0(x) = \sup_{\tau \in \mathcal{T}_0} \mathbf{E}_x [e^{-r\tau} g(X_\tau)],$$

and the optimal stopping time is denoted as τ_0^* . Analogously to (1.4), we set $g(X_\tau) = \limsup_{t \rightarrow \infty} g(X_t)$ whenever $\tau(\omega) = \infty$. The reason for the simultaneous introduction of these problems is mostly technical, as their analyzes will be intertwined.

1.3. Main result and discussion. In the literature of optimal stopping problems of the form (1.4), the incorporated exogenous Poisson processes, or more general renewal processes, appear in various roles. In principle, this process can affect three different components of the problem, namely the parameters of the underlying dynamics, the payoff structure, and/or the set of admissible exercise times. An example of models where the underlying is affected fall into the class of *regime switching* models, where the changes in the drift and volatility are triggered exogenously, see, e.g., [10], [12], and [13]. The payoff structure is affected, for example, in a real option approach to the *technology adaption* of a value maximizing firm, where the new

technologies emerge according to the jumps of the exogenous innovation process, see, e.g., [3], [4], and [6]. More precisely, the exogenous innovation process affects the firms exit (or entrance) strategy as the adoption of new technologies changes the expected value of the cash flow accrued from the production.

The setup of the study at hand serves as an example of a class of problems where the set of admissible stopping times is affected by the exogenous signal process. This class of problems was first proposed in [9], where the authors solve the special case of perpetual American call with underlying geometric Brownian motion. The same signal process setting was adopted in [11], where the authors generalize the results of [20] for stopping geometric Brownian motion at its maximum. Generally speaking, the process N can be seen as an exogenous constraint on the decision makers ability to exercise. This constraint has different interpretations. In [9], the authors propose, along the lines of [18], that the signal process N reflects liquidity effects, i.e., the process N dictates the times at which the asset is available to trade. Following [11], we remark that the considered optimal stopping problem can also be seen as a valuation problem of a randomized version of a perpetual Bermudan option, where contract allows the holder to exercise only at the jump times of the process N . The process N can also be seen as an information constraint. Now, the holder is able exercise at all times but can observe the return process only at the jump times of N . The holder is forced to make her timing decision based on partial information on X where the signal process N stipulates the exogenous restriction on the information available to her. In this setting, the sample paths observed by the decision maker are pure jump trajectories with jumps at Poissonian times T_i and remaining constant in between, see Figure 1.

By quick inspection, this observed trajectory resembles compound Poisson process. The trajectories of a CPP are pure jump type with IID jumps with exponential arrivals. In the current setting of partially observable diffusion, the jumps occurring at times T_i are independent but *not* identically distributed. In fact, the distribution of a particular jump depends on the time elapsed after the previous jump, i.e., the jump distributions depend on the realization of the signal process N . This suggests that the current setting of partially observable diffusion differs essentially from the compound Poisson case and should be compared to a more general jump dynamic structure than ones with IID jumps.

Our objective is to prove a generalization of the main result in [9]. This generalization, which is new to the authors best knowledge, is formulated in the next theorems.

Theorem 1.1. *Assume that the payoff g is continuous and in L_1^r . Furthermore, assume that there is a unique state \hat{x} which maximizes the function $x \mapsto \frac{g(x)}{\psi_r(x)}$, that this function is nondecreasing on $(0, \hat{x})$ and nonincreasing on (\hat{x}, ∞) , and that it satisfies the limiting conditions $\lim_{x \rightarrow 0^+} \frac{g(x)}{\psi_r(x)} = \lim_{x \rightarrow \infty} \frac{g(x)}{\psi_r(x)} = 0$.*

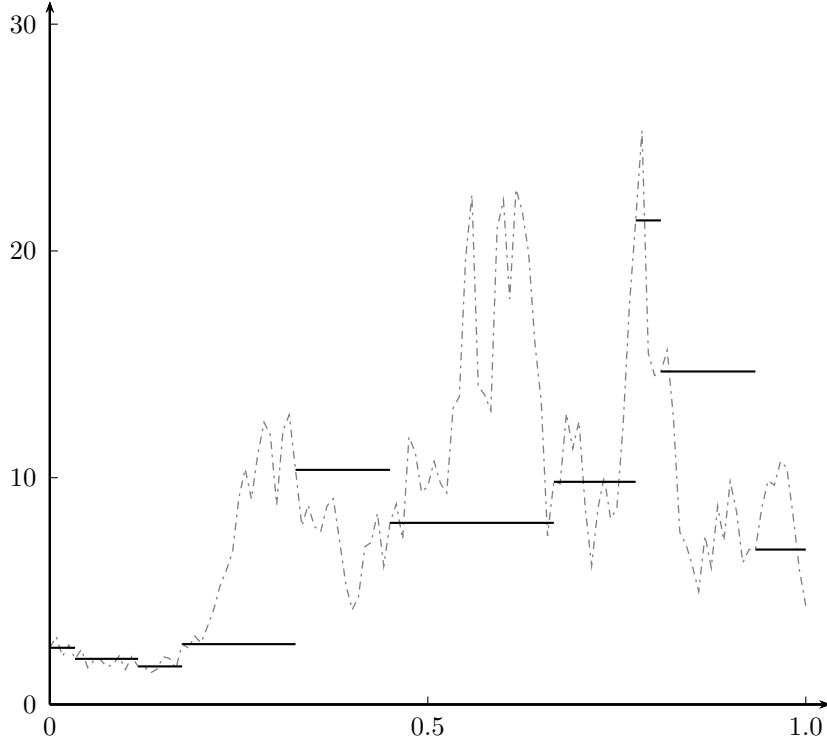


Figure 1. Picture of a possible realization of the underlying diffusion X (grey trajectory) and the pure jump path determined by the exponentially arriving observations of X (black trajectory). In this realization, the high return around $t = 0.6$ is not observed and therefore missed by the investor

Then the threshold $x^* < \hat{x}$ characterized uniquely by the condition

$$\psi_r(x^*) \int_{x^*}^{\infty} \varphi_{r+\lambda}(y) g(y) m'(y) dy = g(x^*) \int_{x^*}^{\infty} \varphi_{r+\lambda}(y) \psi_r(y) m'(y) dy$$

constitutes the optimal exercise rule for the optimal stopping problems (1.4) and (1.6). Moreover, the optimal value functions $V \in C^2(\mathbf{R}_+)$ and $V_0 \in C(\mathbf{R}_+)$ can be written as

$$(1.7) \quad V(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + \frac{g(x^*) - \lambda(R_{r+\lambda}g)(x^*)}{\varphi_{r+\lambda}(x^*)} \varphi_{r+\lambda}(x), & x \geq x^*, \\ \frac{g(x^*)}{\psi_r(x^*)} \psi_r(x), & x < x^*, \end{cases}$$

and

$$(1.8) \quad V_0(x) = \begin{cases} g(x), & x \geq x^*, \\ \frac{g(x^*)}{\psi_r(x^*)} \psi_r(x), & x < x^*. \end{cases}$$

Theorem 1.2. Assume that the payoff g is continuous and in L^r_1 . Furthermore, assume that there is a unique state \tilde{x} which maximizes the function $x \mapsto \frac{g(x)}{\varphi_r(x)}$, that this function is nondecreasing on $(0, \tilde{x})$ and nonincreasing on (\tilde{x}, ∞) , and that it satisfies the limiting conditions $\lim_{x \rightarrow 0+} \frac{g(x)}{\varphi_r(x)} = \lim_{x \rightarrow \infty} \frac{g(x)}{\varphi_r(x)} = 0$.

Then the threshold $x^\dagger > \tilde{x}$ characterized uniquely by the condition

$$\varphi_r(x^\dagger) \int_0^{x^\dagger} \psi_{r+\lambda}(y)g(y)m'(y)dy = g(x^\dagger) \int_0^{x^\dagger} \psi_{r+\lambda}(y)\varphi_r(y)m'(y)dy$$

constitutes the optimal exercise rule for the optimal stopping problems (1.4) and (1.6). Moreover, the optimal value functions $V \in C^2(\mathbf{R}_+)$ and $V_0 \in C(\mathbf{R}_+)$ can be written as

$$(1.9) \quad V(x) = \begin{cases} \frac{g(x^\dagger)}{\varphi_r(x^\dagger)}\varphi_r(x), & x > x^\dagger, \\ \lambda(R_{r+\lambda}g)(x) + \frac{g(x^\dagger) - \lambda(R_{r+\lambda}g)(x^\dagger)}{\psi_{r+\lambda}(x^\dagger)}\psi_{r+\lambda}(x), & x \leq x^\dagger, \end{cases}$$

and

$$(1.10) \quad V_0(x) = \begin{cases} \frac{g(x^\dagger)}{\varphi_r(x^\dagger)}\varphi_r(x), & x > x^\dagger, \\ g(x), & x \leq x^\dagger. \end{cases}$$

We make a few remarks on the assumptions of Theorems 1.1 and 1.2. It is interesting to note that the existence of a unique optimal stopping threshold can be returned essentially to the monotonicity properties of the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ (or $x \mapsto \frac{g(x)}{\varphi_r(x)}$). In comparison to [2], Theorem 3, we make additional assumptions on the limiting behavior of these functions and on integrability of the payoff g . However, these additional assumptions are not very restricting from applications point of view. In this sense, it is interesting to note that the restriction of the admissible stopping times from the entire set of \mathcal{F}_t -stopping times to random times with exponential arrivals does not result into any severe additional restrictions on the underlying X and the payoff g . As was mentioned earlier, the function ψ_r is an increasing solution of the ordinary second order differential equation $(\mathcal{A} - r)\psi_r \equiv 0$ satisfying suitable boundary conditions. Even though it is not possible solve this ODE explicitly except in special cases, there are well developed methods for solving such equations numerically. This makes the numerical verification of the monotonicity and limiting conditions of the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ plausible; the same applies naturally for Theorem 1.2 and the function $x \mapsto \frac{g(x)}{\varphi_r(x)}$.

The remainder of the paper is organized as follows. In section 2, we carry out a proof for Theorem 1.1 by first deriving the candidates for the solutions and then verifying that these candidates are the actual solutions. We remark that the proof of Theorem 1.2 has a completely analogous proof to the one of Theorem 1.1 and will therefore be omitted. In section 2, we also study the asymptotics of the solutions with respect to the parameter λ . In section 3, we illustrate our results with explicit examples by first solving the case of [9] and studying a case of perpetual American put with mean reverting dynamics. Section 4 concludes the study.

2. PROOF OF THE MAIN RESULT

2.1. Some preliminary analysis. We start the preliminary analysis by proving a useful characterization of harmonic functions.

Lemma 2.1. *Assume that the function $f \in C(\mathbf{R}_+)$. Then f is r -harmonic if and only if it satisfies the condition $f(x) = \lambda(R_{r+\lambda}f)(x)$ for all $x \in \mathbf{R}_+$ and $\lambda > 0$.*

Proof. Fix $\lambda > 0$ and $x \in \mathbf{R}_+$. To prove sufficiency, assume that $f(x) = \lambda(R_{r+\lambda}f)(x)$. Now, using the representation (1.2) and the harmonicity properties of functions $\psi_{r+\lambda}$ and $\varphi_{r+\lambda}$, it is a matter of differentiation to show that

$$\begin{aligned} (\mathcal{A} - r)(R_{r+\lambda}f)(x) &= B_{r+\lambda}^{-1}(\mathcal{A} - r)\varphi_{r+\lambda}(x) \int_0^x \psi_{r+\lambda}(y)f(y)m'(y)dy \\ &\quad + B_{r+\lambda}^{-1}(\mathcal{A} - r)\psi_{r+\lambda}(x) \int_x^\infty \varphi_{r+\lambda}(y)f(y)m'(y)dy - f(x) \\ &= B_{r+\lambda}^{-1}\lambda\varphi_{r+\lambda}(x) \int_0^x \psi_{r+\lambda}(y)f(y)m'(y)dy \\ &\quad + B_{r+\lambda}^{-1}\lambda\psi_{r+\lambda}(x) \int_x^\infty \varphi_{r+\lambda}(y)f(y)m'(y)dy - f(x) \\ &= \lambda(R_{r+\lambda}f)(x) - f(x) = 0. \end{aligned}$$

Since $f(x) = \lambda(R_{r+\lambda}f)(x)$, we find that f is r -harmonic.

Assume now that f is r -harmonic. Define the probability measure \mathbf{P}_x^f as

$$\mathbf{P}_x^f(A) = \mathbf{E}_x \left[e^{-rt} \frac{f(X_t)}{f(x)} \mathbf{1}_A \right],$$

for all $A \in \mathcal{F}$ – see [5], p. 34. Using \mathbf{P}_x^f , we find

$$\lambda(R_{r+\lambda}f)(x) = f(x) \int_0^\infty \lambda e^{-\lambda t} \mathbf{E}_x^f \left[\frac{f(X_t)}{f(X_t)} \right] dt = f(x),$$

where \mathbf{E}_x^f denotes the expectation with respect to \mathbf{P}_x^f . □

The assumptions of Theorem 1.1 restraining the choice of the payoff function g and the underlying X are relatively weak and easy to verify, at least numerically. We know from [2], that the ratio function $x \mapsto \frac{g(x)}{\psi_r(x)}$ and its monotonicity properties play a key role in the classical continuous time case. In the current setting, it is not the ratio $x \mapsto \frac{g(x)}{\psi_r(x)}$ but something at least formally quite similar quantity that characterizes the optimal stopping rule. To make a precise statement, define the functions $I : \mathbf{R}_+ \rightarrow \mathbf{R}$ and $J : \mathbf{R}_+ \rightarrow \mathbf{R}$ as

$$(2.1) \quad \begin{aligned} I(x) &= \int_x^\infty \varphi_{r+\lambda}(y)g(y)m'(y)dy, \\ J(x) &= \int_x^\infty \varphi_{r+\lambda}(y)\psi_r(y)m'(y)dy, \end{aligned}$$

for all $x \in \mathbf{R}_+$. We remark that it follows from the proof of Lemma 2.1 that the function J is well-defined. The ratio function $x \mapsto \frac{I(x)}{J(x)}$ will play a key role when proving Theorem 1.1. The next lemma provides us with the required monotonicity properties of this function.

Lemma 2.2. *Let the assumptions of Theorem 1.1 hold. Then there is a unique state $x^* < \hat{x}$ that maximizes the function $x \mapsto \frac{I(x)}{J(x)}$. Moreover, the function $x \mapsto \frac{I(x)}{J(x)}$ is nondecreasing on $(0, x^*)$ and nonincreasing on (x^*, ∞) .*

Proof. First, straightforward differentiation yields the condition

$$(2.2) \quad \frac{d}{dx} \left(\frac{I(x)}{J(x)} \right) \begin{matrix} \geq \\ \leq \end{matrix} 0 \text{ if and only if } \psi_r(x)I(x) \begin{matrix} \geq \\ \leq \end{matrix} g(x)J(x).$$

Let $x \geq \hat{x}$. Since the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ is nonincreasing on (\hat{x}, ∞) , we find that

$$\begin{aligned} \psi_r(x)I(x) - g(x)J(x) &= \psi_r(x) \int_x^\infty \varphi_{r+\lambda}(y) \frac{g(y)}{\psi_r(y)} \psi_r(y) m'(y) dy - g(x)J(x) \\ &< \left(\psi_r(x) \frac{g(x)}{\psi_r(x)} - g(x) \right) J(x) = 0. \end{aligned}$$

Furthermore, since the function $x \mapsto \frac{g(x)}{\psi_r(x)}$ tends to 0 as $x \rightarrow \infty$, we conclude using the condition (2.2) that the function $x \mapsto \frac{I(x)}{J(x)}$ is nonincreasing on (\hat{x}, ∞) and tends to 0 as $x \rightarrow \infty$. On the other hand, since $\lim_{x \rightarrow 0^+} \frac{g(x)}{\psi_r(x)} = 0$ and $\frac{I(\hat{x})}{J(\hat{x})} > 0$, we find using the condition (2.2) that the function $x \mapsto \frac{I(x)}{J(x)}$ must have at least one interior maximum $x^* < \hat{x}$. Finally, since $\frac{g(x^*)}{\psi_r(x^*)} = \frac{I(x^*)}{J(x^*)}$, $x \mapsto \frac{I(x)}{J(x)}$ is continuously differentiable, and $x \mapsto \frac{g(x)}{\psi_r(x)}$ nondecreasing on $(0, \hat{x})$, we conclude, again using (2.2), that the maximum x^* is unique. \square

In Lemma 2.2 we proved that the function $x \mapsto \frac{I(x)}{J(x)}$ has a unique global maximum x^* . We remark that x^* is the unique state satisfying the condition

$$(2.3) \quad \psi_r(x^*) \int_{x^*}^\infty \varphi_{r+\lambda}(y) g(y) m'(y) dy = g(x^*) \int_{x^*}^\infty \varphi_{r+\lambda}(y) \psi_r(y) m'(y) dy.$$

2.2. Necessary conditions. We start the analysis of the optimal stopping problems (1.4) and (1.6) by deriving necessary conditions for the existence of unique optimal solution. As a result we find unique candidates for the optimal values V and V_0 and associated optimal stopping rules. We derive the candidates using two different approaches.

2.2.1. Via the resolvent semigroup. In this section we derive the candidates for optimal characteristics with direct applications of Bellman principle of optimality. We use the HJB variational inequality formulation of Bellman principle, see, e.g., [15]. Furthermore, we exploit the close connection of the resolvent semigroup

and exponentially distributed random times. Denote as G and G_0 the candidates for the optimal values of the problems (1.4) and (1.6), respectively. Given the time homogeneity of the underlying X and the constant jump rate of the signal process N , we make the *ansatz* that the optimal continuation region is an interval $(0, y^*)$ in both problems. The associated candidates for the optimal stopping times are the first exit times $\tau_{y^*} = \inf\{n \geq 1 : X_{T_n} \geq y^*\}$ in (1.4) and $\tau_{y^*}^0 = \inf\{n \geq 0 : X_{T_n} \geq y^*\}$ in (1.6). In the problem (1.6), the decision maker chooses between two actions at every jump time T_i , $i = 0, 1, \dots$: she either exercises or waits. If she chooses to exercise, she gets the payoff $g(x)$. On the other hand, if she waits, the expected discounted value accrued from this choice is determined by the expectation $\mathbf{E}_x [e^{-rU} G_0(X_U)] = \lambda(R_{r+\lambda} G_0)(x)$, where U is an independent, exponentially distributed random time with mean $\frac{1}{\lambda}$. In the problem (1.4) immediate exercise is not allowed, so the decision maker must first wait an exponentially distributed period with mean $\frac{1}{\lambda}$ to make any action. After that she will face the same choice as in the problem (1.6), i.e., the choice of either exercising or postponing the exercise for another exponentially distributed random period. In light of these arguments, we assume that the candidates G and G_0 satisfy the variational inequalities

$$(2.4) \quad \begin{cases} G_0(x) = \max \{g(x), \lambda(R_{r+\lambda} G_0)(x)\}, \\ G(x) = \lambda(R_{r+\lambda} G_0)(x), \end{cases}$$

for all $x \in \mathbf{R}_+$, see also [9], Remark 3, p. 144. To analyze (2.4), we remark that by assumption the candidate G_0 coincides with the payoff g on the exercise region $[y^*, \infty)$ and satisfies the condition $G_0(x) = \lambda(R_{r+\lambda} G_0)(x)$ on the continuation region $(0, y^*)$. Using Lemma 2.1 we find that $G_0(x) = c_1 \psi_r(x) + c_2 \varphi_r(x)$ for all $x \in (0, y^*)$. Since we are looking for a solution that is bounded in origin, we find that $c_2 = 0$. Moreover, since the value function is continuous, we conclude that $G_0(x) = \frac{g(y^*)}{\psi_r(y^*)} \psi_r(x)$ for all $x \in (0, y^*)$.

The next task is to characterize the optimal exercise threshold y^* such that the variational inequalities (2.4) are satisfied. To this end, we find using the balance condition (2.4) and Lemma 2.1 that

$$(2.5) \quad G(x) = \lambda(R_{r+\lambda} G_0)(x) = G_0(x) = \frac{g(y^*)}{\psi_r(y^*)} \psi_r(x) = \lambda \left(R_{r+\lambda} \frac{g(y^*)}{\psi_r(y^*)} \psi_r \right) (x),$$

whenever $x < y^*$. Using the representation (1.2), we find that

$$(2.6) \quad \begin{aligned} \lambda \left(R_{r+\lambda} \frac{g(y^*)}{\psi_r(y^*)} \psi_r \right) (x) &= \lambda(R_{r+\lambda} G_0)(x) \\ &+ \frac{\lambda}{B_{r+\lambda}} \left(\psi_{r+\lambda}(x) \int_{y^*}^{\infty} \varphi_{r+\lambda}(z) \left(\frac{g(y^*)}{\psi_r(y^*)} \psi_r(z) - g(z) \right) m'(z) dz \right), \end{aligned}$$

for all $x < y^*$. By comparing expressions (2.5) and (2.6) to Lemma 2.2 and the expression (2.3), we readily find that in (2.6) the last integral term vanishes and, consequently, the second balance condition in (2.4) is

satisfied on the continuation region if and only if $y^* = x^*$, where x^* is defined in (2.3). By choosing $y^* = x^*$, we can write

$$G(x) = \begin{cases} \lambda(R_{r+\lambda}G_0)(x), & x \geq x^*, \\ \frac{g(x^*)}{\psi_r(x^*)}\psi_r(x), & x < x^*. \end{cases}$$

Let $x \geq x^*$. Then using Lemma 2.1 and representation (1.2), we find that

$$\begin{aligned} G(x) &= \lambda(R_{r+\lambda}g)(x) + \frac{\lambda}{B_{r+\lambda}} \left(\int_0^{x^*} \psi_{r+\lambda}(z) \left(\frac{g(x^*)}{\psi_r(x^*)}\psi_r(z) - g(z) \right) m'(z) dz \right) \frac{\varphi_{r+\lambda}(x)}{\varphi_{r+\lambda}(x^*)} \\ &= \lambda(R_{r+\lambda}g)(x) + \frac{g(x^*) - \lambda(R_{r+\lambda}g)(x^*)}{\varphi_{r+\lambda}(x^*)} \varphi_{r+\lambda}(x), \end{aligned}$$

and, consequently, that the candidate G can be written as

$$(2.7) \quad G(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + \frac{g(x^*) - \lambda(R_{r+\lambda}g)(x^*)}{\varphi_{r+\lambda}(x^*)} \varphi_{r+\lambda}(x), & x \geq x^*, \\ \frac{g(x^*)}{\psi_r(x^*)}\psi_r(x), & x < x^*. \end{cases}$$

Finally, by setting $y^* = x^*$, the candidate G_0 can be expressed as

$$(2.8) \quad G_0(x) = \begin{cases} g(x), & x \geq x^*, \\ \frac{g(x^*)}{\psi_r(x^*)}\psi_r(x), & x < x^*. \end{cases}$$

We have now derived unique candidates (G, x^*) given by (2.7) and (2.3), and (G_0, x^*) given by (2.8) and (2.3) for the optimal characteristics of the problems (1.4) and (1.6), respectively, under the assumptions of Theorem 1.1. Furthermore, we remark that since G and G_0 satisfy the condition (2.4), the candidate G is twice continuously differentiable over the boundary x^* . Also, since $x^* < \hat{x} = \operatorname{argmax}\{\frac{g}{\psi_r}\}$, we conclude that the candidate G_0 is only continuous over the boundary x^* , cf. [2].

2.2.2. Via a free boundary problem. In the previous section we derived the candidates (G, x^*) and (G_0, x^*) for the optimal characteristics of the problems (1.4) and (1.6) using the resolvent operator. These candidates can also be derived using the free boundary approach of [9]. To do this, we investigate the problem (1.4) and, similarly to Section 2.2.1, make the *ansatz* that the optimal exercise rule is a one-sided threshold rule constituted by the first exit time from the continuation region $(0, y^*)$. According to the Bellman principle, we expect the candidate G to be r -harmonic in $(0, y^*)$. On the other hand, on the exercise region $[y^*, \infty)$ the decision maker cannot exercise unless the signal process N jumps. In an infinitesimal time interval dt , the signal process N has probability λdt of making a jump. This means that in time dt , the jump and, consequently, exercise with payoff $g(x)$ has probability λdt . On the other hand, the absence of jump forces

the decision maker to wait with probability $1 - \lambda dt$. Formally, this suggests with a heuristic use of Dynkin's theorem, see, e.g., [15], that

$$\begin{aligned} G(x) &= g(x)\lambda dt + (1 - \lambda dt)\mathbf{E}_x[e^{-rdt}G(X_{dt})] \\ &= \lambda g(x)dt + (1 - \lambda dt)[G(x) + (\mathcal{A} - r)G(x)dt] \\ &= G(x) + (\mathcal{A} - r)G(x)dt + \lambda(g(x) - G(x))dt, \end{aligned}$$

for all $x > y^*$ under the intuition $dt^2 = 0$. Finally, this yields the condition

$$(2.9) \quad (\mathcal{A} - (r + \lambda))G(x) = -\lambda g(x),$$

for all $x > y^*$. Moreover, we can expect that $g(x) < G(x)$ on $(0, y^*)$ and due to the possibility that N doesn't jump when $X \geq y^*$ that $G(x) < g(x)$ on (y^*, ∞) . To complete the free boundary problem, we must pose a boundary condition at y^* . Following [9], we require the *smooth pasting principle* to hold, i.e., the candidate G to be continuously differentiable over the boundary y^* . Under this condition it is elementary to check that $G(y^*) = g(y^*)$. Now we are in position to pose the free boundary problem: *Determine the unique solution (G, y^*) for the problem*

$$(2.10) \quad \begin{cases} G(0+) \geq 0, \\ G(y^*) = g(y^*), \\ (\mathcal{A} - r)G(x) = 0, \text{ and } G(x) > g(x), & x < y^*, \\ (\mathcal{A} - (r + \lambda))G(x) = -\lambda g(x) \text{ and } G(x) < g(x), & x > y^*. \end{cases}$$

Assume now that the unique solution (G, y^*) exists and that $x < y^*$. The condition $(\mathcal{A} - r)G(x) = 0$ implies that G can be expressed as $G(x) = c_1\psi_r(x) + c_2\varphi_r(x)$, where $c_i \geq 0$. Since we are looking for a solution that is bounded in origin, we find that $c_2 = 0$. Now, let $x \geq y^*$. A particular solution to the fourth condition of the free boundary problem (2.10) is the resolvent $\lambda(R_{r+\lambda}g)(x)$ and, consequently, the general solution can be written as $G(x) = \lambda(R_{r+\lambda}g)(x) + d_1\psi_{r+\lambda}(x) + d_2\varphi_{r+\lambda}(x)$. We observe that the assumptions of Theorem 1.1 imply that $d_1 = 0$. Now, the second condition in (2.10) implies that $g(y^*) = c_1\psi_r(y^*) = \lambda(R_{r+\lambda}g)(y^*) + d_2\varphi_{r+\lambda}(y^*)$. This in turn implies that

$$c_1 = \frac{g(y^*)}{\psi_r(y^*)}, \quad d_2 = \frac{g(y^*) - \lambda(R_{r+\lambda}g)(y^*)}{\varphi_{r+\lambda}(y^*)},$$

and, consequently, that the candidate G can be expressed as

$$(2.11) \quad G(x) = \begin{cases} \lambda(R_{r+\lambda}g)(x) + \frac{g(y^*) - \lambda(R_{r+\lambda}g)(y^*)}{\varphi_{r+\lambda}(y^*)} \varphi_{r+\lambda}(x), & x \geq y^*, \\ \frac{g(y^*)}{\psi_r(y^*)} \psi_r(x), & x < y^*. \end{cases}$$

To identify the candidate for the optimal stopping threshold, we use the smooth pasting principle. Indeed, since the candidate G is assumed to be continuously differentiable over the boundary y^* , we observe that the condition

$$(2.12) \quad g(y^*) \frac{\psi_r'(y^*)}{\psi_r(y^*)} - \lambda(R_{r+\lambda}g)'(y^*) - \frac{g(y^*) - \lambda(R_{r+\lambda}g)(y^*)}{\varphi_{r+\lambda}(y^*)} \varphi_{r+\lambda}'(y^*) = 0$$

must be satisfied. This can be rewritten as

$$g(y^*) \left(\frac{\psi_r'(y^*)}{\psi_r(y^*)} - \frac{\varphi_{r+\lambda}'(y^*)}{\varphi_{r+\lambda}(y^*)} \right) = \lambda(R_{r+\lambda}g)'(y^*) - \frac{\varphi_{r+\lambda}'(y^*)}{\varphi_{r+\lambda}(y^*)} \lambda(R_{r+\lambda}g)(y^*).$$

By invoking the representation (1.2) and straightforward differentiation, we find that the righthand-side can be expressed as

$$\lambda(R_{r+\lambda}g)'(y^*) - \frac{\varphi_{r+\lambda}'(y^*)}{\varphi_{r+\lambda}(y^*)} \lambda(R_{r+\lambda}g)(y^*) = \lambda \frac{S'(y^*)}{\varphi_{r+\lambda}(y^*)} \int_{y^*}^{\infty} \varphi_{r+\lambda}(y) g(y) m'(y) dy.$$

Consequently, the optimality condition (2.12) can be rewritten as

$$\lambda \psi_r(y^*) \int_{y^*}^{\infty} \varphi_{r+\lambda}(y) g(y) m'(y) dy = g(y^*) \left(\frac{\psi_r'(y^*)}{S'(y^*)} \varphi_{r+\lambda}(y^*) - \frac{\varphi_{r+\lambda}'(y^*)}{S'(y^*)} \psi_r(y^*) \right).$$

Denote as $w(x) = \frac{\psi_r'(x)}{S'(x)} \varphi_{r+\lambda}(x) - \frac{\varphi_{r+\lambda}'(x)}{S'(x)} \psi_r(x)$. It is a straightforward application of the harmonicity properties of ψ_r and $\varphi_{r+\lambda}$ to establish that $w'(x) = -\lambda \varphi_{r+\lambda}(x) \psi_r(x) m'(x)$ for all $x \in \mathbf{R}_+$. Now, Fundamental Theorem of Calculus implies that

$$w(y^*) = \lambda \int_{y^*}^{\infty} \varphi_{r+\lambda}(y) \psi_r(y) m'(y) dy,$$

and, consequently, that the optimality condition (2.12) can be expressed as

$$(2.13) \quad \psi_r(y^*) \int_{y^*}^{\infty} \varphi_{r+\lambda}(y) g(y) m'(y) dy = g(y^*) \int_{y^*}^{\infty} \varphi_{r+\lambda}(y) \psi_r(y) m'(y) dy.$$

Under the assumptions of Theorem 1.1, we know from Lemma 2.2 that this equation has a unique solution denoted as x^* . By combining expressions (2.11) and (2.13), we have the same candidate for the value of the problem (1.4) and, consequently, of the problem (1.6) as we did in Section 2.2.1. However, we had to make a

priori assumptions on the differentiability of the candidate G over the optimal stopping boundary in setting up and solving the free boundary problem (2.10). This is in contrast to Section 2.2.1, where we formulated the variational inequalities (2.4) in terms of the resolvent operator and used its properties to identify the boundary x^* and compute the candidates G and G_0 directly without such assumptions.

2.3. The verification phase. In the previous subsection we derived the candidates G and G_0 for the solutions of the valuation problems (1.4) and (1.6), respectively. From the point of view of the verification, the continuous time formulations (1.4) and (1.6) are not that handy. In order to remedy this, define the filtration \mathcal{G} as $\mathcal{G}_n := \mathcal{F}_{T_n}$ for all $n \geq 0$, where T_i is the i th jump time of the signal process N , and, the \mathcal{G} -adapted process Z as $Z_n := (T_n, X_{T_n})$. Moreover, define the sets \mathcal{N} and \mathcal{N}_0 as

$$\mathcal{N} = \{N \geq 1 : N \text{ is a } \mathcal{G}\text{-stopping time}\},$$

$$\mathcal{N}_0 = \{N \geq 0 : N \text{ is a } \mathcal{G}\text{-stopping time}\}.$$

Then Lemma 1 of [9] implies that the optimal stopping problems (1.4) and (1.6) can be formulated alternatively as

$$(2.14) \quad \begin{aligned} V(x) &= \sup_{N \in \mathcal{N}} \mathbf{E} [\tilde{g}(Z_N) | Z_0 = (0, x)], \\ V_0(x) &= \sup_{N \in \mathcal{N}_0} \mathbf{E} [\tilde{g}(Z_N) | Z_0 = (0, x)], \end{aligned}$$

for all $x \in \mathbf{R}$ where $\tilde{g}(Z_N) := e^{-rT_n} g(X_{T_n})$. Formulations (2.14) allow a straightforward usage of martingale techniques in the verification phase, as we will shortly see. We recall that the candidates G and G_0 were connected via the condition $G(x) = \lambda(R_{r+\lambda}G_0)(x)$ for all $x \in \mathbf{R}_+$. Using this, we are in position to prove the following.

Lemma 2.3. *Let the assumptions of Theorem 1.1 hold. Then the process*

$$S := (e^{-rT_n} G_0(X_{T_n}); \mathcal{G}_n)_{n \geq 0}$$

is a non-negative uniformly integrable supermartingale for all initial states $X_0 = x \in \mathbf{R}_+$.

Proof. Let U be an exponentially distributed random time with mean $\frac{1}{\lambda}$ and independent of X . Then $G_0(x) \geq G(x) = \lambda(R_{r+\lambda}G_0)(x) = \mathbf{E}_x [e^{-rU} G_0(X_U)]$ for all $x \in \mathbf{R}_+$. Thus the process S is a non-negative supermartingale. In order to prove uniform integrability, it suffices to show that $\sup_n \mathbf{E}_x [S_n] < \infty$ and $\sup_n \mathbf{E}_x [S_n \mathbf{1}_A] \rightarrow 0$ as $\mathbf{P}(A) \rightarrow 0$; then uniform integrability follows from [21], p. 190, Lemma 2.

Let $x \in \mathbf{R}_+$. Define the process $L := \left(e^{-rT_n} \frac{\psi_r(X_{T_n})}{\psi_r(x)}; \mathcal{G}_n \right)_{n \geq 0}$. First, we find using Lemma 2.1 that $\mathbf{E}_x[L_1] = \frac{\lambda(R_r + \lambda \psi_r)(x)}{\psi_r(x)} = 1$. Thus, the strong Markov property of the underlying X implies that L satisfies $\mathbf{E}_x[L_n] = 1$ for all $n \geq 0$. Now, define the measure \mathbf{P}_x^* on (Ω, \mathcal{F}) as

$$\mathbf{P}_x^*(A) = \mathbf{E}_x[L_n \mathbf{1}_A],$$

see [5], p. 34. Let $A \in \mathcal{F}$ and $n \geq 0$. By substituting G_0 into S , we find that

$$(2.15) \quad \begin{aligned} \frac{\mathbf{E}_x[S_n \mathbf{1}_A]}{\psi_r(x)} &= \mathbf{E}_x \left[\frac{G_0(X_{T_n})}{\psi_r(X_{T_n})} \mathbf{1}_A L_n \right] \\ &= \mathbf{E}_x \left[\frac{g(x^*)}{\psi_r(x^*)} \mathbf{1}_A \mathbf{1}_{\{X_{T_n} < x^*\}} L_n \right] + \mathbf{E}_x \left[\frac{g(X_{T_n})}{\psi_r(X_{T_n})} \mathbf{1}_A \mathbf{1}_{\{X_{T_n} \geq x^*\}} L_n \right]. \end{aligned}$$

Since \hat{x} is the global maximum of the function $x \mapsto \frac{g(x)}{\psi_r(x)}$, expression (2.15) yields

$$(2.16) \quad 0 \leq \frac{\mathbf{E}_x[S_n \mathbf{1}_A]}{\psi_r(x)} = \frac{g(\hat{x})}{\psi_r(\hat{x})} \{ \mathbf{E}_x [\mathbf{1}_A \mathbf{1}_{\{X_{T_n} < x^*\}} L_n] + \mathbf{E}_x [\mathbf{1}_A \mathbf{1}_{\{X_{T_n} \geq x^*\}} L_n] \} \leq \frac{g(\hat{x})}{\psi_r(\hat{x})} \mathbf{P}_x^*(A).$$

First, let $A = \Omega$ in the inequality (2.16). Since $\mathbf{E}_x[S_n] < \frac{g(\hat{x})}{\psi_r(\hat{x})} \psi_r(x)$, we find that $\sup_n \mathbf{E}_x[S_n] \leq \frac{g(\hat{x})}{\psi_r(\hat{x})} \psi_r(x) < \infty$. On the other hand, it is evident from the definition of \mathbf{P}_x^* that $\mathbf{P}_x^*(A) \rightarrow 0$ whenever $\mathbf{P}_x(A) \rightarrow 0$. Thus, we conclude using the inequality (2.16) that $\mathbf{E}_x[S_n \mathbf{1}_A] \rightarrow 0$ and, consequently, $\sup_n \mathbf{E}_x[S_n \mathbf{1}_A] \rightarrow 0$ as $\mathbf{P}_x(A) \rightarrow 0$. \square

In Lemma 2.3 we show that under the assumptions of Theorem 1.1 the process $n \mapsto e^{-rT_n} g(X_{T_n})$ is not only a non-negative \mathcal{G} -supermartingale but also uniformly integrable. Uniform integrability will be needed in the proof of next lemma, where we use optional stopping with a stopping time which is not almost surely bounded.

Lemma 2.4. *Let the assumptions of Theorem 1.1 hold. Let $\tau_0^* = T_{N_0^*}$ where $N_0^* = \inf\{n \geq 0 : X_{T_n} \geq x^*\}$.*

Then

$$G_0(x) = \mathbf{E}_x \left[e^{-r\tau_0^*} g(X_{\tau_0^*}) \right] = V_0(x),$$

for all $x \in \mathbf{R}_+$.

Proof. Coupled with Lemma 2.3, the optional sampling theorem now implies that $G_0(x) \geq \mathbf{E}_x [e^{-rT_N} G_0(T_N)] \geq \mathbf{E}_x [e^{-rT_N} g(T_N)]$ for all \mathcal{G} -stopping times N . Hence, $G_0(x) \geq V_0(x)$ for all $x \in \mathbf{R}_+$. To prove that this inequality holds as an equality, i.e., that the function G_0 can be attained by the admissible stopping rule "stop at time τ_0^* ", it suffices to show that the stopped process

$$Q = \left(e^{-rT_{N_0^* \wedge n}} G_0(X_{T_{N_0^* \wedge n}}); \mathcal{G}_n \right)_{n \geq 0}$$

is a martingale. We recall the definition of the process S from Lemma 2.3. Now for each $n \geq 1$, we find that

$$(2.17) \quad \begin{aligned} \mathbf{E}[Q_n | \mathcal{G}_{n-1}] &= \mathbf{E}[S_n \mathbf{1}_{\{N_0^* \geq n\}} | \mathcal{G}_{n-1}] + \sum_{i=0}^{n-1} \mathbf{E}[S_i \mathbf{1}_{\{N_0^* = i\}} | \mathcal{G}_{n-1}] \\ &= \mathbf{E}[S_n \mathbf{1}_{\{N_0^* \geq n\}} | \mathcal{G}_{n-1}] + \sum_{i=0}^{n-1} S_i \mathbf{1}_{\{N_0^* = i\}}. \end{aligned}$$

Denote as U an independent exponentially distributed random time with mean $\frac{1}{\lambda}$. Using the strong Markov property and the property $G(x) = \lambda(R_{r+\lambda}G_0)(x)$, we find that the first term on the right hand side of (2.17) can be written as

$$(2.18) \quad \begin{aligned} \mathbf{E}[S_n \mathbf{1}_{\{N_0^* \geq n\}} | \mathcal{G}_{n-1}] &= e^{-rT_{n-1}} \mathbf{E}_{X_{T_{n-1}}} [e^{-rU} G_0(X_U)] \mathbf{1}_{\{N_0^* \geq n\}} \\ &= e^{-rT_{n-1}} G(X_{T_{n-1}}) \mathbf{1}_{\{N_0^* \geq n\}}. \end{aligned}$$

Now, since $G_0(x) = G(x)$ when $x \leq x^*$, the expressions (2.17) and (2.18) imply that

$$\mathbf{E}[Q_n | \mathcal{G}_{n-1}] = S_{n-1} \mathbf{1}_{\{N_0^* \geq n\}} + \sum_{i=0}^{n-1} S_i \mathbf{1}_{\{N_0^* = i\}} = Q_{n-1}.$$

Finally, since Q is also uniformly integrable, the result follows by optional sampling, i.e.,

$$G_0(x) = \mathbf{E}_x [Q_{N_0^*}] = \mathbf{E}_x [e^{-r\tau_0^*} G_0(X_{\tau_0^*})] = \mathbf{E}_x [e^{-r\tau_0^*} g(X_{\tau_0^*})],$$

for all $x \in \mathbf{R}_+$. □

We proved in Lemma 2.4 that our candidates G_0 and $\tau_{y^*}^0$ are the optimal characteristics of the problem (1.6). We now turn now back to the problem (1.4) and use Lemmas 2.3-2.4 to prove that the candidates G and τ_{x^*} are the optimal characteristics of the problem (1.4).

Lemma 2.5. *Let the assumptions of Theorem 1.1 hold. Let $\tau^* = T_{N^*}$ where $N^* = \inf\{n > 0 : X_{T_n} \geq x^*\}$.*

Then

$$G(x) = \mathbf{E}_x [e^{-r\tau^*} g(X_{\tau^*})] = V(x),$$

for all $x \in \mathbf{R}_+$.

Proof. Since the process S from Lemma 2.3 is a non-negative supermartingale, we find that

$$\mathbf{E}_x [e^{-rT_n} g(X_{T_n})] \leq \mathbf{E}_x [e^{-rT_n} G_0(X_{T_n})] \leq \mathbf{E}_x [e^{-rT_1} G_0(X_{T_1})] = \lambda(R_{r+\lambda}G_0)(x) = G(x),$$

for all \mathcal{G} -stopping times $N \geq 1$ and $x \in \mathbf{R}_+$. By taking supremum over all such N , we obtain the inequality $V(x) \leq G(x)$ for all $x \in \mathbf{R}_+$. To prove that this inequality hold as an equality, it suffices to show that the

value G is attained by the admissible stopping rule "stop at time τ^* ". By conditioning on the first jump time T_1 , we find by using the strong Markov property, Lemma 2.4, and the condition $G(x) = \lambda(R_{r+\lambda}G_0)(x)$ that

$$\mathbf{E}_x \left[e^{-r\tau^*} g(X_{\tau^*}) \right] = \mathbf{E}_x \int_0^\infty e^{-rt} \mathbf{E}_{X_t} \left[e^{-r\tau_0^*} g(X_{\tau_0^*}) \right] \lambda e^{-\lambda t} dt = G(x),$$

for all $x \in \mathbf{R}_+$. \square

2.4. A note on the asymptotics. We study the asymptotics of the optimal characteristics x^* , V and V_0 as $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$. To this end, we remark that the thresholds \hat{x} and \tilde{x} defined in Theorem 1.1 and 1.2 are the optimal exercise thresholds for the classical continuous time stopping problems corresponding Theorems 1.1 and 1.2 and, consequently, satisfy (uniquely) the smooth-pasting conditions $g(\hat{x})\psi_r'(\hat{x}) = \psi_r(\hat{x})g'(\hat{x})$ and $g(\tilde{x})\varphi_r'(\tilde{x}) = \varphi_r(\tilde{x})g'(\tilde{x})$, cf. [2]. Moreover, the value functions \hat{V} and \tilde{V} corresponding to \hat{x} and \tilde{x} read as

$$(2.19) \quad \hat{V}(x) = \begin{cases} g(x), & x \geq \hat{x}, \\ \frac{g(\hat{x})}{\psi_r(\hat{x})}\psi_r(x), & x < \hat{x}, \end{cases} \quad \tilde{V}(x) = \begin{cases} \frac{g(\tilde{x})}{\varphi_r(\tilde{x})}\varphi_r(x), & x > \tilde{x}, \\ g(x), & x \leq \tilde{x}. \end{cases}$$

cf. [2]. Using this notation, we prove the following result.

Proposition 2.6. *Let x^* , V and V_0 be given by Theorem 1.1. Then*

- (1) x^* is an increasing function of λ
- (2) $x^* \rightarrow \hat{x}$, $V(x) \rightarrow \hat{V}(x)$ and $V_0(x) \rightarrow \tilde{V}(x)$ as $\lambda \rightarrow \infty$
- (3) $V(x) = 0$ and $V_0(x) = g(x)$ when $\lambda = 0$,

for all $x \in \mathbf{R}_+$.

Proof. First, we notice that on the limit $\lambda = 0$ the signal process jumps only at $T_0 = 0$ and $T_\infty = \infty$ implying that $V(x) = 0$ and $V_0(x) = g(x)$ for all $x \in \mathbf{R}_+$. Now, let $x \geq \hat{x}$. Since diffusions are Feller processes, we have that $\lambda(R_{r+\lambda}g) \rightarrow g$ as $\lambda \rightarrow \infty$ in sup-norm, see [17], pp. 235. By coupling this with the representation

$$V(x) = \lambda(R_{r+\lambda}g)(x) + (g(x^*) - \lambda(R_{r+\lambda}g)(x^*)) \mathbf{E}_x \left[e^{-(r+\lambda)\tau_{x^*}} \right]$$

(see (1.7)), we deduce that $V(x) \rightarrow g(x)-$ as $\lambda \rightarrow \infty$. Monotonicity of this convergence and continuity of V across the boundary x^* imply that x^* increases as λ increases and, consequently, that $x^* \rightarrow \hat{x}$ as $\lambda \rightarrow \infty$. Finally we conclude that $V(x) \rightarrow \hat{V}(x)$ and $V_0(x) \rightarrow \tilde{V}(x)$ for all $x \in \mathbf{R}_+$ as $\lambda \rightarrow \infty$. \square

Finally, the following proposition can be proved completely analogously under the assumptions of Theorem 1.2.

Proposition 2.7. *Let x^\dagger , V and V_0 be given by Theorem 1.2. Then*

- (1) x^\dagger is an decreasing function of λ
- (2) $x^\dagger \rightarrow \tilde{x}$, $V(x) \rightarrow \tilde{V}(x)$ and $V_0(x) \rightarrow \tilde{V}(x)$ as $\lambda \rightarrow \infty$
- (3) $V(x) = 0$ and $V_0(x) = g(x)$ when $\lambda = 0$,

for all $x \in \mathbf{R}_+$.

The results of Propositions 2.6 and 2.7 are intuitively plausible. In fact, Propositions 2.6 and 2.7 establish unambiguously that the optimal exercise threshold of full information case dominates the optimal exercise threshold under constrained information. This is a reasonable result and reflects the phenomenon that the decision maker will settle for less return when facing uncertainty on the length of the waiting time before the next information update. Moreover, due to the partial information on the underlying X , profitable moments can be missed and therefore decision maker has an incentive to lower her return requirement. Propositions 2.6 and 2.7 establish also that increased information on the underlying X (in the sense of increased λ) postpones the exercise. This again makes sense, since increased λ results into shorter expected gap between the observations. This means it is less likely that decision maker misses a profitable moment and has therefore an incentive to increase her return requirement.

3. ILLUSTRATIONS

3.1. Geometric Brownian motion and perpetual American call. In this subsection we consider the problem studied in [9], namely the perpetual American call option with underlying geometric Brownian motion. Let X be the regular linear diffusion with infinitesimal generator

$$\mathcal{A} = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx},$$

where $\mu \in \mathbf{R}$ and $\sigma > 0$. The scale density S' reads as $S'(x) = x^{-\frac{2\mu}{\sigma^2}}$ and the speed density m' reads as $m'(x) = \frac{2}{(\sigma x)^2} x^{\frac{2\mu}{\sigma^2}}$. It is well known that the increasing and decreasing solutions ψ . and φ . can be expressed as

$$\begin{cases} \psi_r(x) = x^b, & \psi_{r+\lambda}(x) = x^\beta, \\ \varphi_r(x) = x^a, & \varphi_{r+\lambda}(x) = x^\alpha, \end{cases}$$

where the constants

$$\begin{cases} b = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, & \beta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1, \\ a = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0, & \alpha = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0. \end{cases}$$

It is a simple computation to show that the Wronskian $B_{r+\lambda} = 2\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$.

The optimal stopping problem is now written as

$$(3.1) \quad V(x) = \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau} (X_{\tau} - K)^+ \right],$$

where $r > 0$ is the constant discount factor and K is an exogenously given constant. For the sake of finiteness, we assume that $\mu < r$ and $\mu - \frac{1}{2}\sigma^2 > 0$. This guarantees that we have the optimal exercise thresholds are finite and are attained almost surely in a finite time. Since the payoff $g(x) = (x - K)^+ = 0$ when $x \leq K$, we find after straightforward integration that the resolvent $\lambda(R_{r+\lambda}g)$ can be written as

$$(3.2) \quad \lambda(R_{r+\lambda}g)(x) = \begin{cases} \frac{\lambda}{r+\lambda-\mu}x - \frac{\lambda}{r+\lambda}K - \frac{2\lambda K^{1-\alpha}}{\sigma^2 B_{r+\lambda}\alpha(1-\alpha)}x^{\alpha}, & x > K, \\ \frac{2\lambda K^{1-\beta}}{\sigma^2 B_{r+\lambda}(\beta-1)\beta}x^{\beta}, & x \leq K. \end{cases}$$

We use now Theorem 1.1 to determine of the optimal exercise threshold x^* and the optimal value function V . First, elementary integration implies that

$$J(x) = \frac{2}{\sigma^2 \kappa} x^{-\kappa},$$

for all $x \in \mathbf{R}_+$, where $\kappa = \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0$. Similarly we find that

$$I(x) = \begin{cases} \frac{2}{\sigma^2} x^{-\beta} \left(\frac{x}{\beta-1} - \frac{K}{\beta} \right), & x > K, \\ \frac{2K^{-(\beta-1)}}{\sigma^2 \beta(\beta-1)}, & x < K. \end{cases}$$

Let $x > K$. It is an elementary computation to see that $\frac{I(x)}{J(x)} = \frac{\kappa x^{-b}}{\beta(\beta-1)} (\beta x - K(\beta-1))$ and, consequently, that

$$\frac{d}{dx} \left(\frac{I(x)}{J(x)} \right) \underset{\geq}{\leq} 0, \quad \text{when } x \underset{\geq}{\leq} x^* := \frac{b(\beta-1)}{\beta(b-1)} K.$$

We remark that it is a straightforward computation to verify that

$$\frac{b(\beta-1)}{\beta(b-1)} = \frac{b - \frac{r}{r+\lambda}a}{b - \frac{(r-\mu)a-\lambda}{r+\lambda-\mu}};$$

see [9], p. 147, expression (15). Finally, using the expressions (3.2) and (1.7) we obtain the representation

$$V(x) = \begin{cases} \frac{\lambda}{r+\lambda-\mu}x - \frac{\lambda}{r+\lambda}K + \frac{\frac{r-\mu}{r+\lambda-\mu}x^* - \frac{r}{r+\lambda}K}{\varphi_{r+\lambda}(x^*)} \varphi_{r+\lambda}(x), & x \geq x^*, \\ \frac{x^*-K}{\psi_r(x^*)} \psi_r(x), & x < x^*, \end{cases}$$

for the optimal value V ; see [9], pp. 146–147, expressions (13), (14) and (16). Thus we have derived the results on x^* and V by Dupuis and Wang from ours. A straightforward differentiation yields

$$\frac{dx^*}{d\lambda} = \hat{x} \frac{1}{\beta^2} \frac{d\beta}{d\lambda} = \hat{x} \left(\beta^2 \sigma^2 \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} \right)^{-1} > 0,$$

this observation is in line with Part (1) of Proposition 2.6. Moreover, since $\beta \rightarrow \infty$ as $\lambda \rightarrow \infty$, we see immediately from the representation of x^* that $x^* \rightarrow \hat{x} := \frac{bK}{b-1}$ as $\lambda \rightarrow \infty$. Finally, since $\frac{\varphi_{r+\lambda}(x)}{\varphi_{r+\lambda}(x^*)} < 1$ whenever $x > x^*$, we find after elementary manipulations that

$$\frac{\lambda}{r+\lambda-\mu}x - \frac{\lambda}{r+\lambda}K + \frac{\frac{r-\mu}{r+\lambda-\mu}x^* - \frac{r}{r+\lambda}K}{\varphi_{r+\lambda}(x^*)} \varphi_{r+\lambda}(x) \rightarrow x - K$$

for all $x > x^*$ and, consequently, that both $V(x)$ and $V_0(x)$ tend to

$$\hat{V}(x) = \begin{cases} x - K, & x \geq \hat{x}, \\ \frac{\hat{x}-K}{\psi_r(\hat{x})} \psi_r(x), & x < \hat{x}, \end{cases}$$

as $\lambda \rightarrow \infty$.

To end the subsection, we illustrate graphically in Figure 2 the value functions V , V_0 and \hat{V} under the parameter configuration $\mu = 0.01$, $r = 0.05$, $\sigma^2 = 0.1$, $\lambda = 0.1$ and $K = 1.2$.

3.2. Logistic diffusion and perpetual American put. As a generalization of the geometric Brownian setting and an illustration of Theorem 1.2, we consider the case of perpetual American put with a mean reverting underlying X . More precisely, let X follow the regular linear diffusion with infinitesimal generator

$$\mathcal{A} = \frac{1}{2} \sigma^2 x^2 \frac{d^2}{dx^2} + \mu x(1 - \gamma x) \frac{d}{dx},$$

where exogenous constants $\mu, \gamma, \sigma \in \mathbf{R}_+$. This process is called the *logistic diffusion* (or the *geometric Ornstein-Uhlenbeck process* [14] or the *radial Ornstein-Uhlenbeck process* [5]) and was made famous in literature of real options at the latest by [8]. As above, a straightforward computation yields the scale density $S'(x) = x^{-\frac{2\mu}{\sigma^2}} e^{\frac{2\gamma\mu}{\sigma^2}x}$ and, consequently, the speed density $m'(x) = \frac{2}{(\sigma x)^2} x^{\frac{2\mu}{\sigma^2}} e^{-\frac{2\gamma\mu}{\sigma^2}x}$ for all $x \in \mathbf{R}_+$. The optimal stopping problem is now formulated as

$$(3.3) \quad V(x) = \sup_{\tau} \mathbf{E}_x [e^{-r\tau} (K - X_{\tau})^+],$$

with $\mu < r$ and $K > 0$.

We use Theorem 1.2 study the optimal exercise threshold x^\dagger the optimal value functions V and V_0 . We know from the literature, see, [14], Lemma 3.4.3 or [7], Section 6.5, that the decreasing solution φ_r and the increasing solution $\psi_{r+\lambda}$ can be expressed as

$$\begin{cases} \varphi_r(x) = x^b U(b, 2b + \frac{2\mu}{\sigma^2}, \frac{2\mu\gamma}{\sigma^2}x), \\ \psi_{r+\lambda}(x) = x^\beta M(\beta, 2\beta + \frac{2\mu}{\sigma^2}, \frac{2\mu\gamma}{\sigma^2}x), \end{cases}$$

where $\beta = (\frac{1}{2} - \frac{\mu}{\sigma^2}) + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}}$ and the functions $M : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ and $U : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ are the two linearly independent solutions of the Kummer's equation, i.e., the so-called *confluent hypergeometric functions of first and second kind*, cf. [1], pp. 504. Due to the analytically demanding nature of the functions φ_r and $\psi_{r+\lambda}$, we will now fix a parameter setting and illustrate our results numerically and graphically. In Table 1 we present the optimal stopping thresholds for different information rates λ under the parameter configuration $\mu = 0.01$, $r = 0.05$, $\sigma^2 = 0.1$, $\gamma = 0.5$, and $K = 2.4$.

λ	0.005	0.1	0.5	1	5	10	50	250	∞
x^\dagger	1.837	1.626	1.248	1.150	1.023	0.994	0.956	0.939	0.926

Table 1. The optimal stopping threshold x^\dagger for various information rates λ and the smooth-pasting boundary \tilde{x} ($\lambda = \infty$) of ordinary the ordinary case under the parameter configuration $\mu = 0.01$, $r = 0.05$, $\sigma^2 = 0.1$, $\gamma = 0.5$, and $K = 2.4$.

The numerical results reported in Table 1 are in line with our main results. In particular, these numerics indicate that the optimal exercise threshold x^\dagger is a decreasing function of the intensity λ and that these thresholds tend to the smooth-pasting threshold \tilde{x} of the ordinary case as λ increases. To end the subsection, we illustrate graphically in Figure 3 the value functions V , V_0 and \tilde{V} under the parameter configuration $\mu = 0.01$, $r = 0.05$, $\sigma^2 = 0.1$, $\gamma = 0.5$, $\lambda = 0.1$ and $K = 2.4$.

4. CONCLUDING COMMENTS

We studied in this paper the optimal stopping problems (1.4) and (1.6) proposed originally by Dupuis and Wang in [9]. In [9], the authors solve these problems in the case of perpetual American call with underlying geometric Brownian motion. As our main result, we proposed a mild set of conditions on the underlying and the payoff and solved the problems under these conditions. These results are formulated in Theorems 1.1 and 1.2. After proving the necessary auxiliary results, we proposed in Section 2.2.1 the HJB variational equalities (2.4) and solved them directly using the Markovian theory of linear diffusions. As a result, we produced candidates (2.7) and (2.8) for the optimal solutions. We also derived these candidates using the free boundary approach of [9] and established that the approaches are consistent. The verification

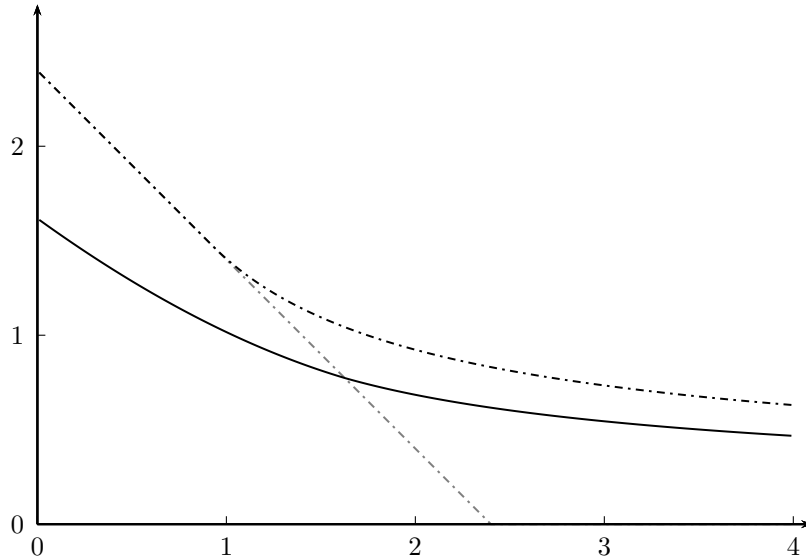


Figure 2. The value functions \tilde{V} under the complete information (black dashed curve) and V under the information rate $\lambda = 0.1$ (black solid curve). The grey dashed line is the payoff $g : x \mapsto (K - x)^+$ and the value function V_0 can be recovered from the figure by first following the payoff g and then after the intersection the curve V . The corresponding optimal stopping thresholds are $\tilde{x} = 0.926$ and $x^\dagger = 1.626$.

phase was carried out in Section 3 in the spirit of [9]. In [9], the authors interpret the signal process N as an exogenous liquidity constraint. In this paper, we proposed and discussed an alternate interpretation of N as an exogenous information constraint.

The main contribution of this paper is that it generalizes considerably the results of [9] with respect to the underlying and payoff structure. In comparison to [2], Theorem 3, we made additional assumptions on the limiting behavior of these functions and on integrability of the payoff g . However, these additional assumptions are not very restricting from applications point of view. In this sense, this study shows that the introduction of an independent Poissonian signal process N to the problem lowers the degree of solvability of the problem only slightly. Moreover, we avoided making any *a priori* differentiability assumptions in deriving the candidates in Section 2.2.1. In fact, we saw that the smoothness properties of the values can be seen as a consequence of the HJB variational formulation connecting the values.

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