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ON EXOGENOUSLY RESTRICTED BOUNDED VARIATION CONTROL OF ITÔ DIFFUSIONS

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Abstract. We study bounded variation control of diffusion processes. The controller is allowed to intervene the evolution of the underlying only on the jump times of an observable, independent Poisson process. The control problem is set up as a maximization problem of the expected present value of the total yield for a general underlying diffusion and structure of instantaneous yield. We propose a relatively weak set of assumptions under which we solve the problem. Moreover, we illustrate the main results with an explicit example.

1. Introduction

In this paper, we study bounded variation control of diffusions. In particular, the control problems are set up as maximization problems of the expected present value of the total yield. The underlying dynamics are assumed to follow a fairly general diffusion process. The yield structure is specified by two components. First of these is an integral term representing the accumulation of instantaneous revenue from continuing the process. Moreover, we assume that there is a payoff from the control, which is proportional to the magnitude. Controlling is assumed to be costless. Under this setting, we assume that the class of admissible controls consists of stochastic integrals driven by an independent Poisson process. In other words, at initial time when the underlying is started, we start also an independent, observable Poisson process which drives the admissible controls. In contrast to problems without this type of exogenous constraint, see, e.g., [1], [8], [9], and [19], the underlying dynamics cannot be controlled in continuous time, but only on the jump times of the Poisson process. Consequently, the resulting admissible control policies are temporally discrete impulse controls – for related studies, see, e.g., [2], [10], [13] and [14].

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This type of constraint is familiar from the literature. In [18], the author studies a minimization problem of total cost for underlying Brownian motion with quadratic running cost and proportional cost of control under the same exogenous restriction. An application to modeling liquidity effect is done in [17]. In this study, the authors consider classical investment/consumption optimization á-la Merton, where the asset is available for trade and, consequently, the portfolio can be rebalanced only at the Poissonian jump times. For related studies in utility maximization in the presence of low liquidity, see also [12] and [15]. The restriction was introduced to optimal stopping in [5], where the authors consider a perpetual American call with underlying geometric Brownian motion, which can be exercised only at the jump times of the independent Poisson process. The results of [5] are generalized in [7] to the optimal stopping of a geometric Brownian motion at its maximum, and in [11] to optimal stopping for a more general diffusion and payoff structure.

The reminder of the paper is organized as follows. In Section 2 we set up the stochastic underlying structure and the stochastic control problem. In Section 3 we derive the optimal characteristics of the control problem. In Section 4 we illustrate the main results with an explicit example, and Section 5 concludes the study.

2. The Control Problem

2.1. **The Underlying Dynamics.** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, be a complete filtered probability space satisfying the usual conditions, see [3], p. 2. We assume that the uncontrolled state process X is defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, evolves on $\mathbf{R}_+$, and follows the regular linear diffusion given as the strongly unique solution of the Itô equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x,$$

where the functions μ and $\sigma > 0$ are sufficiently well behaving, cf., [3], p. 45. Here, W is a Wiener process. We denote as $\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ the second order linear differential operator associated to X. For a given $r > 0$, we denote as, respectively, ψ_r and φ_r the increasing and the decreasing solution of the ordinary second-order linear differential equation $(\mathcal{A} - r)f = 0$ defined on the domain of the characteristic operator of X – for the characterization and fundamental properties of the minimal r-excessive functions ψ_r and φ_r, see [3], pp. 18–20. Moreover, we define the scale density S' and speed density m' via the formulæ $S'(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}dy\right)$ and $m'(x) = \frac{2}{\sigma^2(x)S'(x)}$ for all $x \in \mathbf{R}_+$, cf., [3], p. 17. Finally, we assume that the filtration $\mathbb{F}$ is rich enough to carry a Poisson process $N = (N_t, \mathcal{F}_t)_{t\geq 0}$ with intensity λ – we call the process N the signal process, and assume that X and N are independent.

For $r > 0$, we denote as $\mathcal{L}_1^r$ the class of real valued measurable functions f on $\mathbf{R}_+$ satisfying the integrability condition $\mathbf{E}_x\left[\int_0^{\tau_0} e^{-rt}\left|f(X_t)\right| dt\right] < \infty$, where $\tau_0 = \inf\{t \geq 0 : X_t \leq 0\}$ denotes the first exit time of X from $\mathbf{R}_+$. For an arbitrary $f \in \mathcal{L}_1^r$, we define the *resolvent* $R_r f : \mathbf{R}_+ \to \mathbf{R}$ as

$$(R_r f)(x) = \mathbf{E}_x \left[\int_0^{\tau_0} e^{-rs} f(X_s) ds \right], \tag{1}$$

for all $x \in \mathbf{R}_+$. It is worth pointing out that $\psi_r \in L_1^{r+\lambda}$ for all $r, \lambda > 0$. Indeed, since ψ_r is r-harmonic and nonnegative, we find using [11], Lemma 2.1, that

$$\mathbf{E}_x \left[\int_0^{\tau_0} e^{-(r+\lambda)t} |\psi_r(X_t)| dt \right] = (R_{r+\lambda}\psi_r)(x) = \lambda^{-1}\psi_r(x) < \infty,$$

for all $x \in \mathbf{R}_+$. The resolvent R_r and the increasing and decreasing solutions ψ_r and φ_r are connected in a useful way. Indeed, we know from the literature that for a given $f \in \mathcal{L}_1^r$, the resolvent $R_r f$ can be expressed as

$$(R_r f)(x) = B_r^{-1}\varphi_r(x) \int_0^x \psi_r(y) f(y) m'(y) dy + B_r^{-1}\psi_r(x) \int_x^\infty \varphi_r(y) f(y) m'(y) dy, \tag{2}$$

for all $x \in \mathbf{R}_+$, where $B_r = \frac{\psi_r'(x)}{S'(x)}\varphi_r(x) - \frac{\varphi_r'(x)}{S'(x)}\psi_r(x)$ denotes the Wronskian determinant, see [3], pp. 19. We remark that the value of B_r does not depend on the state variable x but on the rate r. In addition, we know from the literature that the family $(R_r)_{r>0}$ is a semigroup, which satisfies the *resolvent equation*

$$R_q - R_r + (q - r) R_q R_r = 0, \tag{3}$$

where $q > r > 0$, cf. [3], p. 4.

2.2. **The Control Problem.** Having the uncontrolled underlying dynamics set up, we formulate now the main stochastic control problem. As was mentioned in the introductory section, we are studying a maximization problem of the expected present value of the total return. The class of admissible controls $\mathcal{Z}$ consists of the non-decreasing processes ζ having the representation

$$\zeta_t = \int_0^{t-} \eta_s dN_s,$$

where N is the signal process and the integrand η is $\mathbb{F}$-predictable. Thus, the admissible interventions are restricted to instantaneous impulse controls taking place at the jump times of the signal process N. The controlled dynamics X^ζ are given by the Itô integral

$$X_t^\zeta = x + \int_0^t \mu(X_s^\zeta) ds + \int_0^t \sigma(X_s^\zeta) dW_s - \zeta_t, \ 0 \le t \le \tau_0^\zeta, \tag{4}$$

where τ_0 is the life-time of the controlled process X^ζ, i.e., $\tau_0^\zeta = \inf\{t \ge 0 : X_t^\zeta \le 0\}$.

The main objective of this study is to consider the following stochastic control problem. First, define the expected present value of the total return as

$$J(x,\zeta) := \mathbf{E}_x\left[\int_0^{\tau_0^\zeta} e^{-rt}\left(\pi(X_t^\zeta)dt + \gamma d\zeta_t\right)\right],$$

where r and γ are exogenously given, positive constants. Here, $\pi : \mathbf{R}_+ \to \mathbf{R}$ is the function measuring the instantaneous revenues from continuing the process which is assumed to be continuous, non-negative and non-decreasing. The optimal control problem is now to find the optimal value function

$$V(x) = \sup_{\zeta \in \mathcal{Z}} J(x,\zeta), \tag{5}$$

and the optimal control ζ^* satisfying $V(x) = J(x,\zeta^*)$ for all $x \in \mathbf{R}_+$.

To set up the framework under which we study the problem (5), define the function $\theta : \mathbf{R}_+ \to \mathbf{R}$ as

$$\theta(x) = \pi(x) + \gamma(\mu(x) - rx). \tag{6}$$

In the economic literature, the function θ is known in as the net convenience yield from holding inventories, cf. [4].

Assumption 2.1. *Throughout the study, we assume that*

- *the upper boundary ∞ is natural and that the lower boundary 0 is either natural, exit or regular for the uncontrolled diffusion X. In the case when the origin is regular, we assume that it is killing,*
- *the functions θ and $\mathrm{id} : x \mapsto x$ are in $\mathcal{L}_1^r$,*
- *there is a unique state $x^* \geq 0$ such that θ is increasing on $(0,x^*)$ and decreasing on (x^*,∞),*
- *the function θ satisfies the limiting conditions $0 \leq \lim_{x\to 0+}\theta(x) < \infty$ and $\lim_{x\to\infty}\theta(x) < 0$.*

In line with most economical and financial applications, we assume that the uncontrolled state variable X cannot become infinitely large in finite time and, therefore, that the process can be killed only at 0 – see [3], pp. 18–20, for a characterization of the boundary behavior of diffusions. From economical point of view, the $\mathcal{L}_1$-condition is natural stating that the expected present value of the total convenience yield must be finite. It is also worth pointing out that in comparison to [2], see also [1], the introduced Poissonian time uncertainty does not impose any severe additional restraints on the solvability of the problem.

For brevity, define the auxiliary function $\pi_\gamma : \mathbf{R}_+ \to \mathbf{R}$ as

$$\pi_\gamma(x) = \pi(x) + \lambda\gamma x. \tag{7}$$

We remark that by Assumption 2.1, the function $\pi_\gamma \in \mathcal{L}_1^r$ and continuous. This function linked to the function θ in a convenient way.

Lemma 2.2. *Let Assumptions 2.1 hold. Then $(R_{r+\lambda}\pi_\gamma)(x) - \gamma x = (R_{r+\lambda}\theta)(x)$, where π_γ is defined in (7) and θ is defined in (6).*

Proof. Define the sequence $n \mapsto \tau_n$ of first exit times as $\tau_n := \inf\{t \geq 0 : X_t \notin (n^{-1}, n)\}$. Applying Dynkin's formula to the identity function $\mathrm{id} : x \mapsto x$ yields

$$\mathbf{E}_x\left[e^{-(r+\lambda)\tau_n} X_{\tau_n}\right] = x + \mathbf{E}_x\left[\int_0^{\tau_n} e^{-(r+\lambda)s}(\mu(X_s) - (r+\lambda)X_s)ds\right],$$

for all $x \in \mathbf{R}_+$. Letting $n \to \infty$, we find by bounded convergence that $x - \lambda(R_{r+\lambda}\,\mathrm{id})(x) = R_{r+\lambda}(r \cdot \mathrm{id} - \mu)(x)$ for all $x \in \mathbf{R}_+$. Given this identity, we readily verify that

$$\begin{aligned}(R_{r+\lambda}\pi_\gamma)(x) - \gamma x &= (R_{r+\lambda}\pi)(x) - \gamma(x - \lambda R_{r+\lambda}\,\mathrm{id})(x) = (R_{r+\lambda}\pi)(x) + \gamma R_{r+\lambda}(\mu - r \cdot \mathrm{id})(x)\\ &= (R_{r+\lambda}\theta)(x),\end{aligned}$$

for all $x \in \mathbf{R}_+$. □

We begin our analysis of Problem (5) by solving first a special case. The following proposition is an analogue of Lemma 2 in [1].

Proposition 2.3. *Assume that $\theta(x) \leq 0$ for all $x \in \mathbf{R}_+$. Then the optimal control is to drive the state variable X to origin at the first jump time T_1, i.e, to set*

$$\zeta_t^* = \begin{cases} 0, & t < T_1, \\ X_{T_1-}, & t \geq T_1. \end{cases}$$

In this case, the value V reads as

$$V(x) = \mathbf{E}_x\left[\int_0^{T_1} e^{-rs}\pi(X_s)ds + \gamma e^{-rT_1}X_{T_1-}\right] = (R_{r+\lambda}\pi_\gamma)(x),$$

for all $x \in \mathbf{R}_+$.

Proof. Let $x \in \mathbf{R}_+$. Define the family of (almost surely finite) stopping times $\{\tau(\rho)\}_{\rho>0}$ as $\tau(\rho) := \tau_0^\zeta \wedge \rho \wedge \tau_\rho^\zeta$, where $\tau_\rho^\zeta = \{t \geq 0 : X_t^\zeta \geq \rho\}$. Since $(\mathcal{A} - r)(R_{r+\lambda}\pi_\gamma)(x) = \lambda(R_{r+\lambda}\pi_\gamma)(x) - \pi_\gamma(x)$, we find by applying the change of variables formula for general semimartingales, cf., e.g., [6], p. 138, to the process $(t, x) \mapsto$

$e^{-rt}(R_{r+\lambda}\pi_\gamma)(X_t^\zeta)$ that

$$(8)\quad \begin{aligned} e^{-r\tau(\rho)}(R_{r+\lambda}\pi_\gamma)(X^\zeta_{\tau(\rho)}) &= (R_{r+\lambda}\pi_\gamma)(x) + \mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}(\lambda(R_{r+\lambda}\pi_\gamma)(X_s^\zeta) - \pi_\gamma(X_s^\zeta))ds\right] \\ &+ \mathbf{E}_x\left[\sum_{T_i\le\tau(\rho)} e^{-rT_i}((R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i}) - (R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i-}))\right]. \end{aligned}$$

To rewrite the right hand side of (8), we observe first that

$$\begin{aligned} &\mathbf{E}_x\left[e^{-rT_i}\left((R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i}) - (R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i-})\right)\right] = \\ &\mathbf{E}_x\left[e^{-rT_{i-1}}\mathbf{E}_{X_{T_{i-1}}}\left[\int_{T_i-}^{T_i} e^{-rs}\pi(X_s^\zeta)ds + \gamma e^{-r(T_i-T_{i-1})}\Delta\zeta_{T_i}\right]\right], \end{aligned}$$

for all $i\ge 1$. Using this and Lemma 2.2, we find after reshuffling the terms of (8) that

$$\begin{aligned} &\mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}(\lambda(R_{r+\lambda}\pi_\gamma)(X_s^\zeta) - \pi_\gamma(X_s^\zeta))ds\right] + \mathbf{E}_x\left[\sum_{T_i\le\tau(\rho)} e^{-rT_i}((R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i}) - (R_{r+\lambda}\pi_\gamma)(X^\zeta_{T_i-}))\right] = \\ &\mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}\lambda(R_{r+\lambda}\theta)(X_s^\zeta)ds\right] - \mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}\pi(X_s^\zeta)ds + \sum_{T_i\le\tau(\rho)} e^{-rT_i}\gamma\Delta\zeta_{T_i}\right]. \end{aligned}$$

By assumption, we find that $(R_{r+\lambda}\theta)(x)\le 0$. Moreover, since $(R_{r+\lambda}\pi_\gamma)(x)\ge 0$, we have the inequality

$$\begin{aligned} &\mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}\pi(X_s^\zeta)ds + \sum_{T_i\le\tau(\rho)} e^{-rT_i}\gamma\Delta\zeta_{T_i}\right] = (R_{r+\lambda}\pi_\gamma)(x) - \mathbf{E}_x\left[e^{-r\tau(\rho)}(R_{r+\lambda}\pi_\gamma)(X^\zeta_{\tau(\rho)})\right] \\ &+\mathbf{E}_x\left[\int_0^{\tau(\rho)} e^{-rs}\lambda(R_{r+\lambda}\theta)(X_s^\zeta)ds\right] \le (R_{r+\lambda}\pi_\gamma)(x). \end{aligned}$$

Now, by letting $\rho\to\infty$, monotone convergence yields

$$(R_{r+\lambda}\pi_\gamma)(x) \ge \mathbf{E}_x\left[\int_0^{\tau_0^\zeta} e^{-rs}\pi(X_s^\zeta)ds + \sum_{T_i\le\tau_0} e^{-rT_i}\gamma\Delta\zeta_{T_i}\right].$$

Since the value $(R_{r+\lambda}\pi_\gamma)(x)$ is attainable with the admissible policy described in the proposition, the claimed result follows. □

Similarly to Lemma 2 in [1], Proposition 2.3 states an intuitively clear result. Indeed, if the net convenience yield θ is non-positive everywhere, the underlying X should be killed at the first possible occasion, i.e., taken instantaneously to origin at instant T_1. Moreover, the optimal control is in this case a threshold stopping rule, where the optimal threshold is origin. The next corollary gives useful bounds for the value function V.

Corollary 2.4. *The value V satisfies*

$$(R_{r+\lambda}\pi_\gamma)(x) \leq V(x) \leq (R_{r+\lambda}\pi_\gamma)(x) + \frac{\lambda}{r} \sup_{x \in \mathbf{R}_+} (R_{r+\lambda}\theta)(x),$$

for all $x \in \mathbf{R}_+$

Proof. Let $x \in \mathbf{R}_+$. Since

$$\mathbf{E}_x \left[\int_0^{\tau_0} e^{-rs} \lambda (R_{r+\lambda}\theta)(X_s^\zeta) ds \right] \leq \frac{\lambda}{r} \sup_{y \in \mathbf{R}_+} (R_{r+\lambda}\theta)(y),$$

the claimed result follows from the proof of proposition 2.3. □

3. The Solution

3.1. **Preliminary analysis.** Before going into the analysis of the control problem (5), we carry out some preliminary analysis. For a given $f \in \mathcal{L}_1^r$, define the function $L_f : \mathbf{R}_+ \to \mathbf{R}$ as

$$L_f(x) = (r+\lambda) \int_x^\infty \varphi_{r+\lambda}(y) f(y) m'(y) dy + \frac{\varphi'_{r+\lambda}(x)}{S'(x)} f(x). \tag{9}$$

The function L_f admits a useful alternate representation proved in the next lemma.

Lemma 3.1. *Let $\lambda > 0$ and $f \in C \cap \mathcal{L}_1^{r+\lambda}$. Then the function L_f can be expressed as*

$$L_f(x) = -\frac{m'(x)}{\lambda} \left[\lambda (R_{r+\lambda}f)''(x) \varphi'_{r+\lambda}(x) - \lambda (R_{r+\lambda}f)'(x) \varphi''_{r+\lambda}(x) \right],$$

for all $x \in \mathbf{R}$. In particular, if f is r-harmonic, then

$$L_f(x) = -\frac{m'(x)}{\lambda} \left[f''(x) \varphi'_{r+\lambda}(x) - f'(x) \varphi''_{r+\lambda}(x) \right],$$

for all $x \in \mathbf{R}_+$.

Proof. Let $x \in \mathbf{R}_+$. Using the definition of $B_{r+\lambda}$ and the representation (2), we readily verify that

$$\begin{aligned} -\lambda S'(x) L_f(x) &= \frac{r+\lambda}{B_{r+\lambda}} \left[\lambda \left(\varphi'_{r+\lambda}(x) \psi_{r+\lambda}(x) - \varphi_{r+\lambda}(x) \psi'_{r+\lambda}(x) \right) \int_x^\infty \varphi(y) f(y) m'(y) dy \right] - \lambda f(x) \varphi'_{r+\lambda}(x) \\ &= (r+\lambda) \left[\lambda (R_{r+\lambda}f)(x) \varphi'_{r+\lambda}(x) - \lambda (R_{r+\lambda}f)'(x) \varphi_{r+\lambda}(x) \right] - \lambda f(x) \varphi'_{r+\lambda}(x). \end{aligned}$$

Since $\varphi_{r+\lambda}$ is $(r+\lambda)$-harmonic and $(r+\lambda)-\mathcal{A}$ is the left inverse of $R_{r+\lambda}$ on $C\cap\mathcal{L}_1^{r+\lambda}$, it is a matter of algebra to show that

$$(r+\lambda)\left[\lambda(R_{r+\lambda}f)(x)\varphi'_{r+\lambda}(x)-\lambda(R_{r+\lambda}f)'(x)\varphi_{r+\lambda}(x)\right]-\lambda f(x)\varphi'_{r+\lambda}(x)=$$
$$\frac{1}{2}\sigma^2(x)\left[\lambda(R_{r+\lambda}g)''(x)\varphi'_{r+\lambda}(x)-\lambda(R_{r+\lambda}g)'(x)\varphi''_{r+\lambda}(x)\right].$$

In particular, if f is r-harmonic, then $f(x)=\lambda(R_{r+\lambda}f)(x)$, cf. [11], Lemma 2.1. Now, the claimed result follows with a direct substitution. □

Define the auxiliary functions $I:\mathbf{R}_+\to\mathbf{R}$ and $J:\mathbf{R}_+\to\mathbf{R}$ as

$$I(x)=\frac{(R_r\pi)'(x)-\gamma}{\psi'_r(x)},\quad J(x)=\frac{(R_{r+\lambda}\pi_\gamma)'(x)-\gamma}{\varphi'_{r+\lambda}(x)}, \tag{10}$$

where π_γ is defined in (7). Next lemma provides us with the needed monotonicity properties of I and J under our standing assumptions.

Lemma 3.2. *Let Assumptions 2.1 hold. Then*

(i) *there exists a unique $\tilde{x}>x^*$ such that $I'(x)\gtreqless 0$ when $x\gtreqless\tilde{x}$,*
(ii) *there exists a unique $\hat{x}_\lambda<x^*$ such that $J'(x)\lesseqgtr 0$ when $x\lesseqgtr\hat{x}_\lambda$.*

Proof. For the proof of the claim on I, see [2], Lemma 3.2. To prove the second claim, we first note that using Lemma 2.2 we can write

$$J'(x)=\frac{d}{dx}\left[\frac{(R_{r+\lambda}\pi_\gamma)'(x)-\gamma}{\varphi'_{r+\lambda}(x)}\right]=\frac{d}{dx}\left[\frac{(R_{r+\lambda}\theta)'(x)}{\varphi'_{r+\lambda}(x)}\right],$$

for all $x\in\mathbf{R}_+$. Consider the expected cumulative present value $(R_{r+\lambda}\theta)(x)$. Using the representation (2), we find that

$$\frac{(R_{r+\lambda}\theta)'(x)}{\varphi'_{r+\lambda}(x)}=B_{r+\lambda}^{-1}\int_0^x\psi_{r+\lambda}(y)\theta(y)m'(y)dy+B_{r+\lambda}^{-1}\frac{\psi'_{r+\lambda}(x)}{\varphi'_{r+\lambda}(x)}\int_x^\infty\varphi_{r+\lambda}(y)\theta(y)m'(y)dy.$$

Since $\varphi''_{r+\lambda}(x)\psi'_{r+\lambda}(x)-\varphi'_{r+\lambda}(x)\psi''_{r+\lambda}(x)=\frac{2(r+\lambda)B_{r+\lambda}S'(x)}{\sigma^2(x)}$, it is a matter of differentiation to show that

$$\frac{d}{dx}\left[\frac{(R_{r+\lambda}\theta)'(x)}{\varphi'_{r+\lambda}(x)}\right]=-\frac{2S'(x)}{\sigma^2(x)\varphi'^2_{r+\lambda}(x)}L_\theta(x),$$

where the function L_θ is defined using (9). Thus in order to prove the claimed result on J, it is sufficient to show that there is a unique $\hat{x}_\lambda$ such that $L_\theta(x)\gtreqless 0$ when $x\lesseqgtr\hat{x}_\lambda$.

First, let $z > x > x^*$. Since the function θ is non-increasing on (x^*, ∞), we find that

$$\begin{aligned}\frac{1}{r+\lambda}(L_\theta(z) - L_\theta(x)) &= -\int_x^z \varphi_{r+\lambda}(y)\theta(y)m'(y)dy + \frac{\theta(z)}{r+\lambda}\frac{\varphi'_{r+\lambda}(z)}{S'(z)} - \frac{\theta(x)}{r+\lambda}\frac{\varphi'_{r+\lambda}(x)}{S'(x)} \\ &> \frac{\theta(x)}{r+\lambda}\left[\frac{\varphi'_{r+\lambda}(x)}{S'(x)} - \frac{\varphi'_{r+\lambda}(z)}{S'(z)}\right] + \frac{\theta(z)}{r+\lambda}\frac{\varphi'_{r+\lambda}(z)}{S'(z)} - \frac{\theta(x)}{r+\lambda}\frac{\varphi'_{r+\lambda}(x)}{S'(x)} \\ &= \frac{\varphi'_{r+\lambda}(z)}{S'(z)}\left[\frac{\theta(z) - \theta(x)}{r+\lambda}\right] \geq 0,\end{aligned}$$

proving that L_θ is increasing on (x^*, ∞). Similarly, we find that when $z < x < x^*$,

$$\begin{aligned}\frac{1}{r+\lambda}(L_\theta(x) - L_\theta(z)) &= -\int_z^x \varphi_{r+\lambda}(y)\theta(y)m'(y)dy + \frac{\theta(x)}{r+\lambda}\frac{\varphi'_{r+\lambda}(x)}{S'(x)} - \frac{\theta(z)}{r+\lambda}\frac{\varphi'_{r+\lambda}(z)}{S'(z)} \\ &< \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\left[\frac{\theta(x) - \theta(z)}{r+\lambda}\right] \leq 0,\end{aligned}$$

proving that L_θ is decreasing on $(0, x^*)$.

Since the upper boundary ∞ is natural for the underlying X, we find that $\lim_{x\to\infty} L_\theta(x) = 0$, and that for all $x \geq x^*$,

$$L_\theta(x) = (r+\lambda)\int_x^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy + \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\theta(x) < \theta(x)\left[\frac{\varphi'_{r+\lambda}(x)}{S'(x)} - \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\right] = 0.$$

On the other hand, mean value theorem implies that for all $x < x^*$,

$$\begin{aligned}L_\theta(x) &= (r+\lambda)\int_x^{x^*} \varphi_{r+\lambda}(y)\theta(y)m'(y)dy + \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\theta(x) + (r+\lambda)\int_{x^*}^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy \\ &= \theta(\xi)\left[\frac{\varphi'_{r+\lambda}(x^*)}{S'(x^*)} - \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\right] + \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\theta(x) + (r+\lambda)\int_{x^*}^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy \\ &= [\theta(x) - \theta(\xi)]\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + \frac{\varphi'_{r+\lambda}(x^*)}{S'(x^*)}\theta(\xi) + (r+\lambda)\int_{x^*}^\infty \varphi_{r+\lambda}(y)\theta(y)m'(y)dy,\end{aligned}$$

for some $\xi \in (x, x^*)$. Since lower boundary 0 is non-entrance, the function $\frac{\varphi'_{r+\lambda}(x)}{S'(x)} \to -\infty$, and, consequently, $L_\theta(x) \to \infty$ as $x \to 0$. This proves the claimed result on J. □

In order to simplify the subsequent notation, define the auxiliary function $g : \mathbf{R}_+ \to \mathbf{R}$ as

(11) $$g(x) = \gamma x - (R_r\pi)(x).$$

Moreover, recall the definition (9). Using these, define the function $Q : \mathbf{R}_+ \to \mathbf{R}$ as the ratio

$$Q(x) = \frac{L_g(x)}{L_{\psi_r}(x)}.$$

We remark that under our assumptions the function Q is well defined. The function Q will be the key quantity when determining the optimal control ζ^*. Next lemma provides us with the needed monotonicity properties of Q under our standing assumptions.

Lemma 3.3. *Let Assumption 2.1 hold. Then there exist a unique* $\hat{x} = \text{argmax}\{Q(x)\} \in (\hat{x}_\lambda, \tilde{x})$ *such that the function* $Q'(x) \gtreqless 0$ *whenever* $x \lesseqgtr \hat{x}$.

Proof. Let $x \in \mathbf{R}_+$. By standard differentiation, we find that

$$\begin{aligned}
L^2_{\psi_r}(x)Q'(x) &= L_{\psi_r}(x)\left[g'(x)\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + g(x)\frac{\varphi''_{r+\lambda}(x)S'(x) - \varphi'_{r+\lambda}(x)S''(x)}{S'^2(x)} - (r+\lambda)\varphi_{r+\lambda}(x)g(x)m'(x)\right] \\
&\quad - L_g(x)\left[\psi'_r(x)\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + \psi_r(x)\frac{\varphi''_{r+\lambda}(x)S'(x) - \varphi'_{r+\lambda}(x)S''(x)}{S'^2(x)} - (r+\lambda)\varphi_{r+\lambda}(x)\psi_r(x)m'(x)\right] \\
&= L_{\psi_r}(x)\left[g'(x)\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + \mathcal{A}\varphi_{r+\lambda}(x)g(x)m'(x) - (r+\lambda)\varphi_{r+\lambda}(x)g(x)m'(x)\right] \\
&\quad - L_g(x)\left[\psi'_r(x)\frac{\varphi'_{r+\lambda}(x)}{S'(x)} + \mathcal{A}\varphi_{r+\lambda}(x)\psi_r(x)m'(x) - (r+\lambda)\varphi_{r+\lambda}(x)\psi_r(x)m'(x)\right] \\
&= \frac{\varphi'_{r+\lambda}(x)}{S'(x)}\left[g'(x)L_{\psi_r}(x) - \psi'_r(x)L_g(x)\right],
\end{aligned}$$

and, consequently, that

$$Q'(x) \lesseqgtr 0 \text{ if and only if } g'(x)L_{\psi_r}(x) \gtreqless \psi'_r(x)L_g(x).$$

Assume that $x > \tilde{x}$. Since $\varphi'_{r+\lambda}(x) < 0$, and $g''(x)\psi'_r(x) < g'(x)\psi''_r(x)$, we find using Lemma 3.1, resolvent equation, and Lemma 3.2 that

$$\begin{aligned}
g'(x)L_{\psi_r}(x) - \psi'_r(x)L_g(x) &> \frac{m'(x)\psi'_r(x)}{\lambda}\left(\varphi'_{r+\lambda}(x)(\lambda(R_{r+\lambda}g)''(x) - g''(x)) - \varphi''_{r+\lambda}(x)(\lambda(R_{r+\lambda}g)'(x) - g'(x))\right) \\
&= \frac{m'(x)\psi'_r(x)\varphi'^2_{r+\lambda}(x)}{\lambda}J'(x) > 0.
\end{aligned}$$

We conclude that the function Q is nondecreasing on $(\tilde{x}, \infty)$. On the other hand, since $g''(x)\psi'_r(x) > g'(x)\psi''_r(x)$ on $(0, x^*)$ and $\hat{x}_\lambda < x^*$, we find using the same argument that

$$g'(x)L_{\psi_r}(x) - \psi'_r(x)L_g(x) < \frac{m'(x)\psi'_r(x)\varphi'^2_{r+\lambda}(x)}{\lambda}J'(x) < 0,$$

and, consequently, that Q is nondecreasing on $(0, \tilde{x}_\lambda)$. By continuity, Q must have a turning point $\hat{x}$ in the interval $(\hat{x}_\lambda, \tilde{x})$. Finally, since $g'(\hat{x})L_{\psi_r}(\hat{x}) = \psi'_r(\hat{x})L_g(\hat{x})$, the uniqueness of $\hat{x}$ follows from Lemma 3.2. $\square$

In Lemma 3.3 we proved that the function $Q : x \mapsto \frac{L_g(x)}{L_{\psi_r}(x)}$ has a unique global maximum $\hat{x}$. We remark the this maximum is characterized by the condition

$$g'(\hat{x})L_{\psi_r}(\hat{x}) = \psi_r'(\hat{x})L_g(\hat{x}). \tag{12}$$

3.2. **Necessary conditions.** Having the necessary auxiliary results at our disposal, we proceed with the study of Problem (5). We start by restricting our attention to a specific subclass of admissible control policies and deriving a unique candidate for the optimal value – denote the candidate as F. Given the infinite time horizon and the constant jump rate of the signal process N, we assume that the optimal value exists and is constituted by a threshold control policy defined as follows: If the state variable X is above some fixed threshold y when the Poisson process N jumps, exert the impulse control to return the state variable to the boundary y and restart the evolution. On the other hand, if $X_{T_i-} < y$ for given i, underlying X is not intervened. Formally, this can be put as follows: if $X_{T_i-} \geq y$ for some $i \geq 0$, invoke the impulse $\Delta\zeta_{T_i} = X_{T_i-} - y$, and start the process anew from y. Now, for a given threshold y, the state space $\mathbf{R}_+$ is partitioned into the *waiting region* $(0, y)$ and the *action region* $[y, \infty)$. At every jump time T_i, the decision maker chooses between two alternatives, she either exerts control or waits. In the continuation region $(0, y)$, the Bellman principle implies that the candidate F should satisfy the balance condition

$$F(x) = \mathbf{E}_x\left[\int_0^U e^{-rs}\pi(X_s)ds + e^{-rU}F(X_U)\right], \tag{13}$$

where U is an independent exponentially distributed random variable with mean λ^{-1}. Since the underlying X is strong Markov, we find that on the waiting region $(0, y)$

$$\mathbf{E}_x\left[\int_0^U e^{-rs}\pi(X_s)ds + e^{-rU}F(X_U)\right] = (R_r\pi)(x) + \lambda(R_{r+\lambda}F)(x) - \mathbf{E}_x\left[\int_U^\infty e^{-rs}\pi(X_s)ds\right] =$$
$$(R_r\pi)(x) + \lambda(R_{r+\lambda}F)(x) - \mathbf{E}_x\left[e^{-rU}(R_r\pi)(X_U)\right] = (R_r\pi)(x) + \lambda(R_{r+\lambda}F)(x) - \lambda(R_{r+\lambda}R_r\pi)(x).$$

By coupling this with (13), Lemma 2.2 of [11] implies that the function $x \mapsto F(x) - (R_r\pi)(x)$ is r-harmonic, i.e., the function F satisfies the ODE

$$(\mathcal{A} - r)F(x) + \pi(x) = 0, \tag{14}$$

for all $x < y$. Since we are looking for a function bounded in origin, we conclude that $G(x) = (R_r\pi)(x) + c\psi_r(x)$ for all $x < y$ for some constant c.

Assume that $x \geq y$. Now, the controller will use the impulse control given that the Poisson process N jumps. In an infinitesimal time dt, the Poisson process jumps with probability λdt. In this case, the controller invokes control which yields a payoff $\gamma(x-y)+F(y)$. On the other hand, the Poisson process does not jump with probability $1-\lambda dt$, in this case the added expected present value is $\pi(x)dt+\mathbf{E}_x[e^{-rdt}F(X_{dt})]$. Now, the Bellman principle coupled with a heuristic usage of Dynkin's formula suggests that

$$\begin{aligned}F(x) &= \lambda dt(\gamma(x-y)+F(y)) + (1-\lambda dt)(\pi(x)dt+\mathbf{E}_x[e^{-rdt}F(X_{dt})])\\ &= \lambda dt(\gamma(x-y)+F(y)) + \pi(x)dt + F(x) + (\mathcal{A}-r)F(x)dt - \lambda F(x)dt,\end{aligned}$$

and, consequently, that the candidate F should satisfy the ODE

$$(\mathcal{A}-(r+\lambda))F(x) = -(\pi(x)+\lambda(\gamma(x-y)+F(y))), \tag{15}$$

for all $x \geq y$. Using representation (2) and partial integration, it is straightforward to establish that a particular solution to (15) can be written as

$$(R_{r+\lambda}\pi_\gamma)(x) + \frac{\lambda}{\lambda+r}(F(y)-\gamma y)\left[1+\delta\varphi_{r+\lambda}(x)\right], \tag{16}$$

where the function π_γ is defined in (7) and

$$\delta = \begin{cases} 0, & \text{when } 0 \text{ is natural,}\\ \frac{\lambda\psi'(0)}{B_{r+\lambda}(\lambda+r)S'(0)}(\gamma y - F(y)), & \text{when } 0 \text{ is exit or regular.}\end{cases}$$

Using Corollary 2.4, we conclude that the candidate F has a representation

$$F(x) = (R_{r+\lambda}\pi_\gamma)(x) + d\varphi_{r+\lambda}(x) + \frac{\lambda}{\lambda+r}(F(y)-\gamma y),$$

for all $x \geq y$, where d is a constant. By substituting $x = y$, solving $F(y)$ and plugging it back to the previous expression, an elementary simplification yields

$$F(x) = \lambda(R_{r+\lambda}\pi_\gamma)(x) + d\varphi_{r+\lambda}(x) + \frac{\lambda}{r}(\lambda(R_{r+\lambda}\pi_\gamma)(y) + d\varphi_{r+\lambda}(y) - \gamma y),$$

for all $x \geq y$. In order to determine the constants c and d, we must impose a pasting principle on the boundary y. We make an *ansatz* that the candidate F is twice continuously differentiable over the boundary y. Moreover, since F is our candidate for the value function, it is reasonable to expect it to be r-superharmonic, i.e., that $(\mathcal{A}-r)F(x)+\pi(x)\leq 0$ for all $x\in\mathbf{R}_+$. To proceed, we first find that

$$(\mathcal{A}-r)F(x)+\pi(x)=\begin{cases}0, & x<y,\\ -\lambda((F(y)-\gamma y)-(F(x)-\gamma x)), & x\geq y.\end{cases}\tag{17}$$

We find from the expression (17) that if the function $x\mapsto F(x)-\gamma x$ has a global maximum, say y^*, then the candidate associated to this boundary is indeed r-superharmonic. We will now assume that the global maximum y^* exists. Since $F\in C^2$, the boundary y^* can be characterized with condition $F'(y^*)=\gamma$. This allows us to determine the constants c and d. Indeed, a simple computation yields $(R_r\pi)'(y^*)+c\psi_r'(y^*)=\gamma=(R_{r+\lambda}\pi_\gamma)'(y^*)+d\varphi_{r+\lambda}'(y^*)$, and, consequently

$$c=\frac{\gamma-(R_r\pi)'(y^*)}{\psi_r'(y^*)},\ d=\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}.$$

Define the function $F:\mathbf{R}_+\to\mathbf{R}$ as

$$F(x)=\begin{cases}(R_{r+\lambda}\pi_\gamma)(x)+\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}\varphi_{r+\lambda}(x)+A(y^*), & x\geq y^*,\\ (R_r\pi)(x)+\frac{\gamma-(R_r\pi)'(y^*)}{\psi_r'(y^*)}\psi_r(x), & x<y^*,\end{cases}\tag{18}$$

where

$$A(y^*)=\frac{\lambda}{r}\left(\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}-\frac{\gamma y-(R_{r+\lambda}\pi_\gamma)(y^*)}{\varphi_{r+\lambda}(y^*)}\right)\varphi_{r+\lambda}(y^*).$$

This function is now our candidate for the optimal value of the problem (5). Since $F\in C^2$, the condition

$$(R_{r+\lambda}\pi_\gamma)''(y^*)+\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}\varphi_{r+\lambda}''(y^*)-(R_r\pi)''(y^*)+\frac{\gamma-(R_r\pi)'(y^*)}{\psi_r'(y^*)}\psi_r''(y^*)=0\tag{19}$$

must be satisfied. Using the definition (11) and the resolvent equation, we find that

$$\begin{aligned}0&=\lambda(R_{r+\lambda}g)''(y^*)+\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi_{r+\lambda}'(y^*)}\varphi_{r+\lambda}''(y^*)-\frac{\gamma-(R_r\pi)'(y^*)}{\psi_r'(y^*)}\psi_r''(y^*)\\ &=\lambda(R_{r+\lambda}g)''(y^*)+\frac{\gamma-(R_r\pi)'(y^*)-\lambda(R_{r+\lambda}g)'(y^*)}{\varphi_{r+\lambda}'(y^*)}\varphi_{r+\lambda}''(y^*)-\frac{\gamma-(R_r\pi)'(y^*)}{\psi_r'(y^*)}\psi_r''(y^*)\\ &=\frac{\lambda(R_{r+\lambda}g)''(y^*)\varphi_{r+\lambda}'(y^*)-\lambda(R_{r+\lambda}g)'(y^*)\varphi_{r+\lambda}''(y^*)}{\varphi_{r+\lambda}'(y^*)}-\frac{g'(y^*)}{\psi_r'(y^*)}\left(\frac{\psi_r''(y^*)\varphi_{r+\lambda}'(y^*)-\psi_r'(y^*)\varphi_{r+\lambda}''(y^*)}{\varphi_{r+\lambda}'(y^*)}\right).\end{aligned}$$

Now, using Lemma 3.1, we find that the C^2-condition (19) can be rewritten as

$$\begin{aligned}&g'(y^*)\left((r+\lambda)\int_{y^*}^{\infty}\varphi_{r+\lambda}(y)\psi_r(y)m'(y)dy+\psi_r(y^*)\frac{\varphi_{r+\lambda}'(y^*)}{S'(y^*)}\right)=\\ &\psi_r'(y^*)\left((r+\lambda)\int_{y^*}^{\infty}\varphi_{r+\lambda}(y)g(y)m'(y)dy+g(y^*)\frac{\varphi_{r+\lambda}'(y^*)}{S'(y^*)}\right).\end{aligned}\tag{20}$$

We established in Lemma 3.3 that under our standing assumptions 2.1, there is a unique threshold $\hat{x}$ satisfying the condition (20) – in the sequel, we will identify y^* as $\hat{x}$. Thus, the function F is twice continuously differentiable, and, by construction, $F'(y^*) = \gamma$. We collect now our findings on the candidate F to the next proposition.

Proposition 3.4. *Let Assumptions 2.1 hold. Then the function F defined* (18)*, where threshold y^* characterized by* (20)*, is the unique solution for the free boundary problem*

$$\begin{cases} F \in C^2, \\ F'(y^*) = \gamma, \\ (\mathcal{A} - r)F(x) + \pi(x) = 0, & x < y^*, \\ (\mathcal{A} - (r + \lambda))F(x) = -(\pi(x) + \lambda(\gamma(x - y) + F(y))), & x \geq y^*. \end{cases}$$

3.3. **Sufficient Conditions.** In Proposition 3.4 we presented our main results on the candidate F to the optimal solution. To prove that the candidate is the optimal solution, we first remark that F satisfies the variational principle

$$(\mathcal{A} - r)F(x) + \pi(x) + \lambda \left[\sup_{y \leq x} (F(y) - \gamma y) - (F(x) - \gamma x) \right] = 0, \tag{21}$$

for all $x \in \mathbf{R}_+$. For brevity, denote

$$\Phi(x) := \sup_{y \leq x} (F(y) - \gamma y) - (F(x) - \gamma x). \tag{22}$$

Let $x < y^*$. Since $y^* < \tilde{x}$, Lemma 3.2 implies that

$$F'(x) - \gamma = \psi_r'(x) \left[\frac{(R_r \pi)'(x) - \gamma}{\psi_r'(x)} - \frac{(R_r \pi)'(y^*) - \gamma}{\psi_r'(y^*)} \right] > 0.$$

On the other hand, when $x \geq y^*$, Lemma 3.2 implies that

$$F'(x) - \gamma = \varphi_{r+\lambda}'(x) \left[\frac{(R_{r+\lambda} \pi_\gamma)'(x) - \gamma}{\varphi_{r+\lambda}'(x)} - \frac{(R_{r+\lambda} \pi_\gamma)'(y^*) - \gamma}{\varphi_{r+\lambda}'(y^*)} \right] \leq 0,$$

since $y^* > \hat{x}_\lambda$. Thus, we conclude that under our standing assumptions 2.1, the function $x \mapsto F(x) - \gamma x$ has unique global maximum at y^*, and we can express Φ as

$$\Phi(x) = \{F(y^*) + \gamma(x - y^*) - F(x)\} \mathbf{1}_{[y^*, \infty)}(x), \tag{23}$$

for all $x \in \mathbf{R}_+$. Using these observations, we prove our main result on Problem (5).

Theorem 3.5. *Let Assumptions 2.1 hold. Then, for all $i \geq 1$, the optimal control policy ζ^* is to take the state variable X^{ζ^*} instantaneously to the state y^* characterized uniquely by (20) whenever $X^{\zeta^*}_{T_i-} > y^*$, i.e., the size of the impulse is $\Delta\zeta^*_{T_i} = (X^{\zeta^*}_{T_i-} - y^*)^+$ for all T_i. Moreover, the value V of the optimal control problem reads as*

$$V(x) = F(x) = \begin{cases} (R_{r+\lambda}\pi_\gamma)(x) + \frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi'_{r+\lambda}(y^*)}\varphi_{r+\lambda}(x) + A(y^*), & x \geq y^*, \\ (R_r\pi)(x) + \frac{\gamma-(R_r\pi)'(y^*)}{\psi'_r(y^*)}\psi_r(x), & x < y^*, \end{cases} \tag{24}$$

where

$$A(y^*) = \frac{\lambda}{r}\left(\frac{\gamma-(R_{r+\lambda}\pi_\gamma)'(y^*)}{\varphi'_{r+\lambda}(y^*)} - \frac{\gamma y-(R_{r+\lambda}\pi_\gamma)(y^*)}{\varphi_{r+\lambda}(y^*)}\right)\varphi_{r+\lambda}(y^*).$$

Proof. Let $x \in \mathbf{R}_+$. We prove first that $F(x) \geq J(x,\zeta)$ for all $\zeta \in \mathcal{Z}$. Recall the definition of the family $\{\tau(\rho)\}_{\rho>0}$ from the proof of Proposition 2.3. Applying the change of variables formula for general semimartingales, cf. [6], p. 138, to the stopped process $(t,x) \mapsto e^{-r(t\wedge\tau(\rho))}F(X^\zeta_{t\wedge\tau(\rho)})$ yields

$$\begin{aligned} e^{-r(t\wedge\tau(\rho))}F(X^\zeta_{t\wedge\tau(\rho)}) = F(x) &+ \int_0^{t\wedge\tau(\rho)} e^{-rs}(\mathcal{A}-r)F(X^\zeta_s)ds + \int_0^{t\wedge\tau(\rho)} e^{-rs}\sigma(X^\zeta_s)F'(X^\zeta_s)dW_s \\ &+ \sum_{s\leq t\wedge\tau(\rho)} e^{-rs}[F(X^\zeta_s) - F(X^\zeta_{s-})], \end{aligned}$$

for all $\rho > 0$. On the other hand, since the control ζ can jump only if the Poisson process N jumps, expression (21) implies that $F(X^\zeta_s) + \gamma(\Delta\zeta_s) - F(X^\zeta_{s-}) \leq \Phi(X^\zeta_{s-})$, where the function Φ is defined in (22). Coupling this with (21) results in

$$\begin{aligned} e^{-r(t\wedge\tau(\rho))}F(X^\zeta_{t\wedge\tau(\rho)}) + \int_0^{t\wedge\tau(\rho)} e^{-rs}\left(\pi(X^\zeta_s)dt + \gamma d\zeta_t\right) &\leq F(x) + \int_0^{t\wedge\tau(\rho)} e^{-rs}\sigma(X^\zeta_s)F'(X^\zeta_s)dW_s \\ -\lambda\int_0^{t\wedge\tau(\rho)} e^{-rs}\Phi(X^\zeta_{s-})ds + \int_0^{t\wedge\tau(\rho)} e^{-rs}\Phi(X^\zeta_{s-})dN_s &= F(x) + M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)}, \end{aligned} \tag{25}$$

where M and Z are local martingales defined as

$$M_t := \int_0^t e^{-rs}\sigma(X^\zeta_s)F'(X^\zeta_s)dW_s,\ Z_t := \int_0^t e^{-rs}\Phi(X^\zeta_{s-})d\tilde{N}_s.$$

Here, $\tilde{N} = (N_t - \lambda t)_{t\geq 0}$ is a compensated Poisson process. Moreover, we observe from expression (25) that the local martingale part $(M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)})$ is bounded uniformly from below by $-F(x)$. Hence $(M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)})$ is a supermartingale, and, in particular, $\mathbf{E}_x[M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)}] \leq 0$ for all $t, \rho > 0$. Taking expectations sidewise in (25), we find that

$$\mathbf{E}_x\left[e^{-r(t\wedge\tau(\rho))}F(X^\zeta_{t\wedge\tau(\rho)})\right] + \mathbf{E}_x\left[\int_0^{t\wedge\tau(\rho)} e^{-rs}\left(\pi(X^\zeta_s)ds + \gamma d\zeta_s\right)\right] \leq F(x),$$

for all $t, \rho > 0$. Letting t and ρ tend to infinity, we obtain $F(x) \geq \lim_{t,\rho\to\infty} \mathbf{E}_x \left[e^{-r(t\wedge\tau(\rho))} F(X^{\zeta}_{t\wedge\tau(\rho)}) \right] + J(x,\zeta)$. Since F is non-negative, we conclude that $F(x) \geq J(x,\zeta)$.

To show that the value F is attainable with the admissible policy ζ^*, it suffices to show that $J(x,\zeta^*) \geq F(x)$. First, since N jumps only upwards and F is non-negative and nondecreasing, we find using (23) that

$$Z_{t\wedge\tau(\rho)} \leq \int_0^{t\wedge\tau(\rho)} e^{-rs}\Phi(X^{\zeta^*}_{s-})dN_s \leq \gamma \int_0^{t\wedge\tau(\rho)} e^{-rs}(X^{\zeta^*}_{s-} - y^*)\mathbf{1}_{[y^*,\infty)}(X^{\zeta^*}_{s-})dN_s \leq \gamma \int_0^{\infty} e^{-rs} d\zeta^*_s,$$

for all $t, \rho > 0$. Thus the process Z is bounded uniformly from above by an integrable random variable and is, consequently, a submartingale. On the other hand, since the functions σ and F' are continuous and the stopped process $X^{\zeta^*}_{\cdot\wedge\tau(\rho)}$ is bounded, we find that the integrand of M is also bounded. This implies that the local martingale M is a martingale and, consequently, that $\mathbf{E}_x[M_{t\wedge\tau(\rho)} + Z_{t\wedge\tau(\rho)}] \geq 0$ for all $t, \rho > 0$. We observe that for the control ζ^* the inequality (25) holds in fact as an equality. Therefore it follows from (25) that

$$\mathbf{E}_x \left[e^{-r(t\wedge\tau(\rho))} F(X^{\zeta^*}_{t\wedge\tau(\rho)}) \right] + \mathbf{E}_x \left[\int_0^{t\wedge\tau(\rho)} e^{-rs} \left(\pi(X^{\zeta^*}_s)ds + \gamma d\zeta^*_s \right) \right] \geq F(x),$$

for all $t, \rho > 0$. Letting t and ρ tend to infinity, we find by bounded convergence that

$$F(x) \leq \liminf_{t,\rho\to\infty} \mathbf{E}_x \left[e^{-r(t\wedge\tau(\rho))} F(X^{\zeta^*}_{t\wedge\tau(\rho)}) \right] + J(x,\zeta^*).$$

Now, recall that y^* is the global maximum of $x \mapsto F(x) - \gamma x$. Thus

$$0 \leq \mathbf{E}_x \left[e^{-r(t\wedge\tau(\rho))} F(X^{\zeta^*}_{t\wedge\tau(\rho)}) \right] \leq \mathbf{E}_x \left[e^{-r(t\wedge\tau(\rho))} (F(y^*) + \gamma(X^{\zeta^*}_{t\wedge\tau(\rho)} - y^*)) \right].$$

Since $\mathrm{id} \in \mathcal{L}_1^r$, we conclude that $\liminf_{t,\rho\to\infty} \mathbf{E}_x \left[e^{-r(t\wedge\tau(\rho))} F(X^{\zeta^*}_{t\wedge\tau(\rho)}) \right] = 0$ and, consequently, that $V(x) = J(x,\zeta^*)$. □

We proved in Theorem 3.5 that under our standing assumptions 2.1 the unique solution of the free boundary problem described in Proposition 3.4 constitutes the optimal solution for Problem (5). It is worth pointing out that we proved in Lemma 3.3 that for all $\lambda > 0$ the optimal control threshold y^* is dominated by the state $\tilde{x} = \mathrm{argmax}\{I(x)\}$, where I is defined in (10). On the other hand, it is proved in [2], Lemma 3.4, that under Assumption 2.1 the state $\tilde{x}$ is the optimal reflection boundary for the version of Problem (5), where the Poissonian time uncertainty N is removed. In this case, the optimal control coincides with the local time of the underlying X at the boundary $\tilde{x}$. Intuitively, this case should correspond to the limit $\lambda \to \infty$ in the present control problem. Indeed, it seems clear, that as the rate λ increases, the opportunities to control appear more frequently in time. Since it is costless to control, the controller should be more inclined to use it. This should result into a higher threshold and on the average smaller but more frequent controls in the neighborhood of this threshold. Unfortunately, a rigorous proof of the property $y^* \to \tilde{x}$ as $\lambda \to \infty$ remains open. However, Lemma 3.3 shows that the introduction of the particular Poissonian time uncertainty studied in this paper unambiguously accelerates the optimal exercise of the control.

4. An Illustration

We illustrate in this section the main results of the paper with an explicit example. To this end, we assume that the uncontrolled underlying dynamics follows a geometric Brownian motion, i.e., the diffusion X given by the Itô equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where $\mu \in \mathbf{R}$, and $\sigma \in \mathbf{R}_+$ are exogenously given constants. The differential operator associated to X reads as $\mathcal{A} = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}$. A straightforward computation yields the scale density $S'(x) = x^{-\frac{2\mu}{\sigma^2}}$ and, consequently, the speed density $m'(x) = \frac{2}{(\sigma x)^2} x^{\frac{2\mu}{\sigma^2}}$ for all $x \in \mathbf{R}_+$.

To set up the control problem, consider the continuous flow of return is constituted by the function

$$\pi(x) = x^\delta, \ 0 < \delta < 1,$$

and fix the constant γ. For the sake of finiteness, we assume that $\mu < r$ and $\mu - \frac{1}{2}\sigma^2 > 0$. This guarantees that we have the optimal exercise thresholds are finite and are attained almost surely in a finite time. It is well known that the minimal excessive functions $\psi_\cdot$ and $\varphi_\cdot$ can now be written as

$$\begin{cases} \psi_r(x) = x^b, \\ \varphi_r(x) = x^a, \end{cases} \quad \begin{cases} \psi_{r+\lambda}(x) = x^\beta, \\ \varphi_{r+\lambda}(x) = x^\alpha, \end{cases}$$

where the constants

$$\begin{cases} b = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \\ a = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0, \end{cases} \quad \begin{cases} \beta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1, \\ \alpha = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0. \end{cases}$$

It is a simple computation to show that the Wronskian $B_{r+\lambda} = 2\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}}$. To check the validity of Assumptions 2.1, we find that the net convenience yield θ reads as $\theta(x) = x^\delta - (r-\mu)x$. We observe that Assumptions 2.1 are satisfied and that

$$x^* = \operatorname{argmax}\{\theta(x)\} = \left(\frac{\delta}{\gamma(r-\mu)}\right)^{\frac{1}{1-\delta}}.$$

Using the representation (2), it is a matter of straightforward integration to show that $(R_r\pi)(x) = \frac{x^\delta}{\iota(\delta)}$, where $\iota(\delta) = r - \delta\mu - \frac{\sigma^2}{2}\delta(\delta-1)$, for all $x \in \mathbf{R}_+$. Using this, we find that

$$\tilde{x} = \left\{ \frac{\delta(b-\delta)}{\gamma\iota(\delta)(r-\mu)} \right\}^{\frac{1}{1-\delta}}.$$

Recall the definition of the operator L_f from (9) and that $g(x) := \gamma x - (R_r\pi)(x) = \gamma x - \frac{x^\delta}{\iota(\delta)}$. To determine the optimal exercise threshold y^*, we need to find the functions L_g and L_{ψ_r} – see (20). For L_g, we find first that

$$\int_x^\infty \varphi_{r+\lambda}(y)g(y)m'(y)dy = \frac{2}{\sigma^2 x^\beta}\left\{ \frac{x^\delta}{\iota(\delta)(\delta-\beta)} - \frac{\gamma x}{1-\beta} \right\},$$

and, consequently, that

$$\begin{aligned} L_g(x) &= \frac{2(r+\lambda)}{\sigma^2 x^\beta}\left\{ \frac{x^\delta}{\iota(\delta)(\delta-\beta)} - \frac{\gamma x}{1-\beta} \right\} + \left\{ \gamma x - \frac{x^\delta}{\iota(\delta)} \right\} \frac{\alpha x^{\alpha-1}}{x^{-\frac{2\mu}{\sigma^2}}} \\ &= \frac{x^{\delta-\beta}}{\iota(\delta)}\left\{ \frac{2(r+\lambda)}{\sigma^2(\delta-\beta)} - \alpha \right\} + \gamma x^{1-\beta}\left\{ \alpha - \frac{2(r+\lambda)}{\sigma^2(1-\beta)} \right\}. \end{aligned} \tag{26}$$

For L_{ψ_r}, we verify readily that

$$\int_x^\infty \varphi_{r+\lambda}(y)\psi_r(y)m'(y)dy = \frac{2}{\sigma^2(\beta-b)}x^{b-\beta},$$

and, consequently, that

$$L_{\psi_r}(x) = \left\{ \frac{2(r+\lambda)}{\sigma^2(\beta-b)} + \alpha \right\} x^{b-\beta}. \tag{27}$$

By inserting the expressions (26) and (27) into the condition (20), we find after a straightforward simplification that

$$y^* = \left\{ \frac{\Lambda(\delta)}{\gamma\iota(\delta)\Lambda(1)} \right\}^{\frac{1}{1-\delta}}, \tag{28}$$

where

$$\Lambda(t) = t\left\{ \alpha - \frac{2(r+\lambda)}{\sigma^2(b-\beta)} \right\} - b\left\{ \alpha - \frac{2(r+\lambda)}{\sigma^2(t-\beta)} \right\}.$$

To illustrate the results numerically, we present in Table 1 numerical values for the optimal exercise thresholds y^* and $\bar{x}$ for different rates λ under the parameter configuration $\mu = 0.02$, $r = 0.07$, $\sigma = 0.15$, $\delta = 0.15$ and $\gamma = 0.8$. For these parameters, the thresholds $x^* = 4.735$ and $\tilde{x} = 6.316$.

λ	0.01	0.1	1	10	100
y^*	3.502	4.486	5.637	6.102	6.249

Table 1. The optimal exercise threshold y^* for various values of λ under the parameter configuration $\mu = 0.02$, $r = 0.07$, $\sigma = 0.15$, $\delta = 0.15$ and $\gamma = 0.8$.

We observe from Table 1 that the numerics are in line with our main results. For this parameter configuration, the optimal exercise threshold y^* is dominated by the optimal reflection threshold $\tilde{x}$. Moreover, the values indicate that y^* tends to $\tilde{x}$ as the rate λ grows.

5. Concluding remarks

In this paper, we studied a class of bounded variation control problems of linear diffusions. This class of problems is a modified generalization of a problem studied in [18], where the problem was formulated as a minimization problem of expected total cost for underlying Brownian motion. In particular, we studied a maximization of the expected present value of the total yield. The main contribution of the study is that the studied class of problems is considerable general in terms of underlying diffusion dynamics X and the instantaneous yield structure π. Moreover, we find in comparison to [2], see also [1], that the introduced Poissonian time uncertainty does not impose additional restraints to the solvability of the problem. We also established that the introduction of the Poissonian time uncertainty unambiguously accelerates the optimal exercise of the control.

This study has at least two interesting generalizations. First, it would be interesting to make the controlling costly. In this case, it seem reasonable to guess that the resulting exercise threshold and the regeneration point are no longer the same. We considered in this paper the case where the rate λ is constant over time. It would be interesting to see if some of the results of this study could generalized to case where λ is given a dynamical structure. These questions are left for future research.

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