FINAL REPORT FOR GITEC-TWO
UIO group

G. Pedersen, H.P. Langtangen, H. Johnsgard and B. Gjevik
Department of Mathematics, University of Oslo, Box 1053, 0316 Oslo, Norway

1 Introduction

In the first GITEC project the UiO group performed a series of case studies concerning tsunami events in the Atlantic, the eastern Mediterranean and the Norwegian sea. During the project the focus was slightly shifted towards general model analysis and development. Preliminary Lagrangian run-up models and finite element (FE) techniques for Boussinesq equations were reported. Moreover, tests concerning the convergence and applicability of the standard long wave models were included in the case studies or carried out as separate tasks. Continuing this line of investigation, we have focused mainly on model activities in GITEC-TWO, even though the work on the 1969 tsunami outside Portugal has continued and the study of the Tafjord event (1934) has been renewed. All the model activities rely heavily upon the experience from these and other case studies from the preceding project and elsewhere. Moreover, we have addressed a set of idealized, but challenging test cases, to obtain an improved insight in physics as well as numerics. This will enable us to exploit our new modeling tools in full studies of actual events with better results and control.

A number of test cases have been established, including wave generation and interaction with a shallow seamount, run-up on an idealized headland, slides in fjords, and wave propagation in several two dimensional geometries, usually corresponding to cross sections of actual bathymetries. Some of these problems are also addressed by the LDG group.

The prolonged study of the 1969 tsunami, originating near the Gorringe Bank, is linked to the publication of a joint journal paper [9] with the ICTE and the LDG groups. Due to the complexity of the problem and the diversity of the subtopics involved, the paper has been substantially revised during GITEC-TWO before being finally accepted. This is described further in section 8. In all simulations, concerning real as well as idealized test cases, emphasis has been put on grid refinement tests. Unless otherwise explicitly noted all the presented simulations herein are close to convergence.

It might seem surprising, but some of the properties of the most standard tsunami models are insufficiently documented in the literature. This is alarming since much of the tsunami work world wide still have to rely on the standard methods. Naturally, many of the features will also be inherited by more advanced models. Facing this problem during the preceding project, we have undertaken a study of the optical properties of finite difference (FD) and FE models, as well as on the influence of so called staircase (sawtooth) boundaries on the coastal response to incident waves. The results are summarized in section 2. Together with section 8 these topics form a fairly broad analysis on the shortcomings and accuracy of linear hydrostatic models. Other investigations concerning properties of finite element discretizations, domain decomposition etc. are described in other sections, the references or in manuscripts in progress.

The FE model for the Boussinesq equations has been upgraded, analyzed and fully documented in the journal paper [16]. Evaluation and verification of the method are partly based on the test cases outlined above. More details are given in section 3.

Particular emphasis has been put the on run-up models, that are described in the sections 4 and 6. The FD Lagrangian model for run-up, initiated under the first GITEC project, has been further developed, tested and documented in journal papers [11], [12]. Bore treatment and bottom drag have been included as new features. Careful investigations have revealed both conceptual and practical problems concerning bore run-up (section 6). A related, but more general, FE technique has been implemented and compared to analytical solutions as well as to the pre-existing FD method. Both these results and additional simplified case studies are promising. This is elaborated in section 4. Preliminary model description and test results are published in a conference proceedings [15]. The FE software has been implemented in C++ using object-oriented design techniques and the Diffpack [7] library.

As is usual in tsunami research, our modeling has been based mainly on hydrostatic and dispersive long wave equations. In order to check the validity of such equations, as well to investigate short wave features of tsunamis, we have developed a model that solves the equations of full nonlinear potential theory as described in section 7.

Two essential themes in the tsunami computation of the near future are parallel computing and automatic coupling of models with different numerical and physical characteristics. Both problems reduce to the development of flexible and general software components for implementing domain decomposition methods on top of existing solvers. Our work in this field, within GITEC-TWO, is sketched in section 5.

In the GITEC project we produced a video cassette describing some results concerning the 1969 Portuguese event and the Storegga tsunami, in collaboration with the Computer Center of the University of Oslo. Also in the new project we will produce animations and videos. However, significant upgrading of both hardware and software has been necessary. In combination with vacancy in the Department's position as computer engineer, with particular responsibility for scientific visualization, this has led to substantial delays. Anyway, sufficient resources is now allocated to the visualization task and the work is in steady progress and should be completed within the extended deadline of the GITEC-TWO project.

2 Analysis of long wave models

2.1 Discrete optics

A key question for a researcher in wave theory is to what extent a given numerical procedure defines a virtual medium with properties that are analogous to those of the physical medium. In the present section we pursue this question by developing an optical theory for discrete solutions of finite difference or element methods, with emphasis on amplification and spurious behaviour in shallow water. Particularly, we seek a numerical counterpart to the well known Green's law, which states that the amplitude of a normally incident long wave in shallow water is proportional to $h^{-\frac{1}{4}}$, where h is the depth. There are several textbooks surveying this theory, for instance [17]. We devote most attention to the analysis of the standard long wave model, as is employed in [9], for instance. However, corresponding theories have been established for dispersive equations and element discretizations on regular grids as well. A more complete description of the present topic is found in the manuscript [20] that can be made available at request.

We start from an ansatz

$$\eta_{i,j}^{(n)} = A_{i,j} e^{\mathrm{i}(\xi_{i,j} - \omega n \Delta t)}, \tag{1}$$

where fast variation (on the wavelength scale) is exhibited by the frequency and $\xi_{i,j}$ only, while $A_{i,j}$ and the differences of ξ vary slowly. Substitution of the above expression into the finite difference equations followed by a series of discrete and differential arithmetics leads to transport equations similar to those of geometrical and physical optics for differential equations. Moreover, in the particular case of normally incident waves we obtain closed form solutions for the amplitude. For our standard mid-point scheme for the shallow water equations we obtain the simple expression

$$A = B(h - h_c)^{-\frac{1}{4}}, \quad h_c = \frac{\Delta x^2}{4g} \left(\frac{2}{\Delta t} \sin(\frac{\omega \Delta t}{2}) \right)^2 \tag{2}$$

where B is a constant and h_c is the turning point depth, which is explained below. We note that (2) is a discrete generalization of Green's law and reproduces the latter in the limit $\Delta x \to 0$. When $h \to h_c^+$ the amplitude A becomes infinite and the physical optics collapses. No real solution for k exists for $h < h_c$. At $h = h_c$ we must thus expect a turning point with complete reflection of the incident wave. That reflection *does* occur may be demonstrated by matching the WKBJ solution to a local solution valid in the vicinity of the turning point. However, beyond noting that the local solution has the form of a spatial Nyquist wave, modulated by an Airy function, we omit the details. Naturally, the presence of a total reflection is also verified through direct numerical solution of the difference equations.

The discrete Green's law has been compared to exact discrete solutions, in the sense that they are obtained by solving the difference equations directly. A convincing agreement is found, even for rather steep bottom gradients.

The most important practical results from this study are (1) the existence of a stopping depth close to the shore, with full reflection of the waves, and (2) a discrete Green's law demonstrating that harmonic waves are over-amplified in numerical models. However, it must be noted that numerical dispersion have the opposite effect on a tsunami, which generally inherits a full spectrum.

2.2 Run-up and staircase boundaries

As stated in the preceding subsections the optical theory collapses before the shoreline and cannot describe run-up. Consequently every harmonic retain finite length as well as amplitude. On the other hand, at the shoreline the governing equations inherit a singularity that may be expected to produce errors and artifacts in numerical solutions. The combination of the shoreline singularity with coarse grids, inaccurate digitized bathymetry and staircase boundaries calls for particular caution. In real case studies, as well as more idealized tests, we have observed a strong contamination by unphysical noise at the shore. Naturally, the use of staircase boundaries is a prime suspect concerning the noise production.

In the present section we focus on a simple geometry consisting of an offshore domain of constant depth combined with a plane slope extending to the shoreline. We align a Cartesian coordinate system with σ and γ axes normal and parallel to the shore respectively and obtain a bathymetry independent of γ.

A periodic incident wave is specified by the wavenumber, k, and the angle of incidence, ψ. For normal incidence, $\psi = 0$, the solution can be expressed by the zeroth order Bessel function on the slope. For oblique incidence the form of the solution becomes slightly more complex, now involving the Kummer function. At the slope margin the zeroth and first derivatives of nearshore solutions are patched to the offshore solution, that contains the incident and the reflected harmonic waves. A closer investigation of these solutions, including the discussion of some nontrivial features, is reported in [18].

We now assume that the grid is rotated an angle θ, in the clockwise direction, relative to the bathymetry. Moreover, we align the x, y coordinate system with axes parallel to the axes of the grid. If $\theta \neq 0, \frac{1}{2}\pi$ the shore then has to be represented as a "staircase boundary", consisting of segments being parallel to the x and y axes alternatively. Due to the simplicity of the bathymetry it is natural to assume a grid with regular steps in the boundary, where a single increment in one direction is adjacent to a step in the other direction, counting a fixed number of increments. Without loss of generality we may then assume boundary segments of lengths Δy and $N\Delta x$ respectively, as displayed in figure 1 for the special case $N = 2$ and $\Delta x = \Delta y$. To factorise a discrete solution it is not sufficient that the coefficients of the difference equation are independent of a given coordinate. In addition also the grid, including the representation of the boundary, must be invariant with respect to a shift in the coordinate. Thus we cannot employ separation in neither the γ,σ nor the x,y coordinates. Fortunately, the regularity of the sawtooth boundary enables

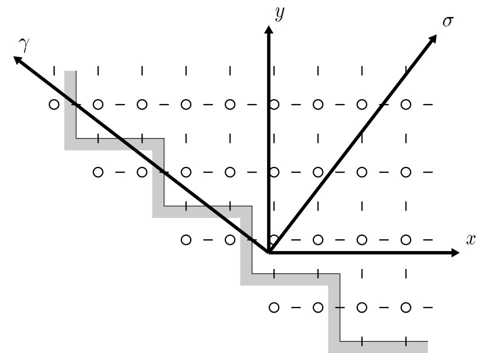


Figure 1:

a separation in the non-orthogonal system spanned by the x and γ axes. Omitting the details, we simply report that elimination of variables and separation yield an ordinary difference equation of order $2N$. The first N of these are modified by the presence of the boundary. At the offshore boundary we implement a combined input/radiation condition N times to close the system, which is then easily inverted by Gaussian elimination. For small N and flat bottom some progress have been made by analytical means as well.

An investigation of the case $N = 1$ (a one by one staircase boundary) reveals that no extra features will be introduced as compared to $N = 0$. The ratio between the maximum surface elevation and the amplitude of the incident wave is denoted by F and depicted in figure 2 for the case $\psi = 35°$, $\Delta x = \frac{1}{25}\lambda$, and slope length equal to the length, λ, of the incident wave. For $N = 1$ we observe good overall convergence in spite of the coarse grid.

For $N \geq 2$ the separated numerical solutions becomes more complex. In addition to the two independent solutions that corresponds to the incident and reflected waves, we now find $N - 1$ decaying (evanescent) and $N - 1$ exponentially growing modes as $x \to \infty$. Naturally, the latter are discarded from the solution, whereas the former give rise to noise adjacent to the shore. In the right panel of 2 we observe a substantial degradation in performance of the numerical method relative to the case $N = 1$.

The main conclusion from this investigation is that a staircase boundary aligned at a general angle relative to the grid might produce significant noise close to the shore and reduce the overall accuracy substantially.

3 The FE Boussinesq solver

We have developed a fairly general finite element simulator for weakly nonlinear and dispersive water waves. The potential advantages of the finite element method, compared to traditional finite difference schemes, are related to more accurate representation of coastal geometries, elimination of staircase boundaries, increased flexibility

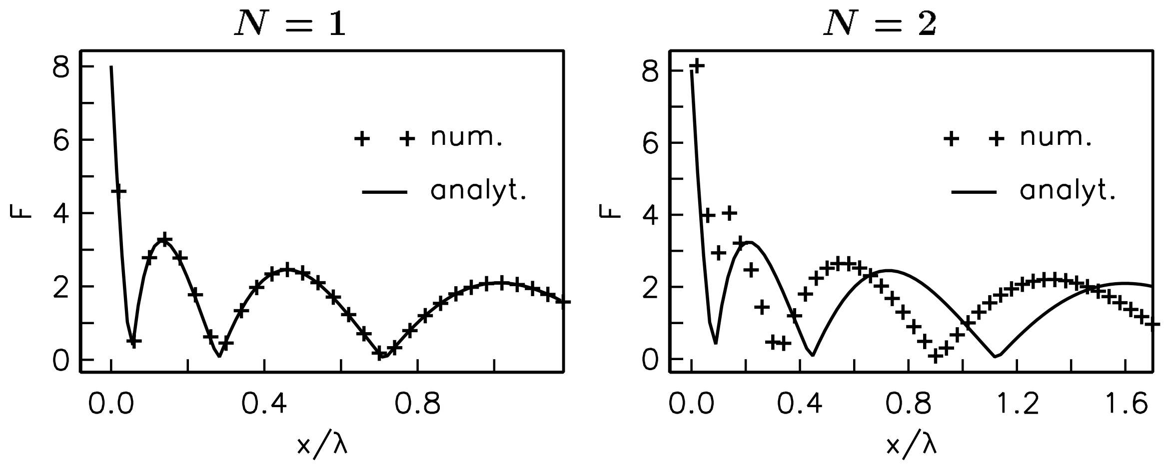


Figure 2:

with respect to adaptive refinements etc., and simple generation of higher order spatial schemes. The disadvantages concern increased CPU-time (mostly due to the finite element assembly process), extra memory requirements, larger computer code, and less obvious means to develop ad hoc improvements of standard schemes.

The finite element simulator solves the Boussinesq equations with the surface elevation and the velocity potential as primary unknowns. For potential flow, this reduces the work by one third compared to the more standard approach where the velocity vector field is used as primary unknown. As in all our finite difference models, the equations are discretized in time by centered differences on a staggered temporal grid. The spatial problems at each time level is then solved by a Galerkin finite element method. Our particular formulation has the no mass flux condition at the coastline as natural boundary condition. The simulator is implemented in an object-oriented, yet efficient, fashion in C++, using the Diffpack library [7].

In some of our FD methods [19], the leading numerical errors have been removed by inclusion of correction terms akin to the physical dispersion terms. Accordingly, as an option we have introduced additional terms for correction of temporal errors in the FE formulation. These (small) terms cancel certain terms in the local truncation error such that the time discretization of the linear hydrostatic equations becomes of fourth order. With a suitable choice of the time step and quadratic elements of size comparable to the depth, the numerical errors will then be of the same order as in the Boussinesq equations themselves.

A comprehensive analysis of the numerical accuracy has been performed by studying the error in the numerical wave velocity as a function of wave length, direction of wave advance, grid increments, grid distortion, consistent vs. lumped mass matrix representations etc. Only linear equations on constant depth are included in the theoretical analysis. We refer to the journal paper [16] for a detailed picture of the performance of various numerical strategies. One important result is that biquadratic elements loose their expected superiority when the elements become significantly distorted.

The finite element method has been investigated further in two idealized, but still challenging, test cases. The first case concerns an incoming plane wave on a bell-shaped beach. A contour plot, showing the reflection of an incident wave, is shown in figure 3, upper panel. It is noteworthy that problems with noise sometimes occur even in this simple bathymetry. Localized noise appear outside the headland, and becomes more intense when the curvature of the bell-shaped coastline is increases. Biquadratic elements seem to be less stable than linear or bilinear elements in this particular case.

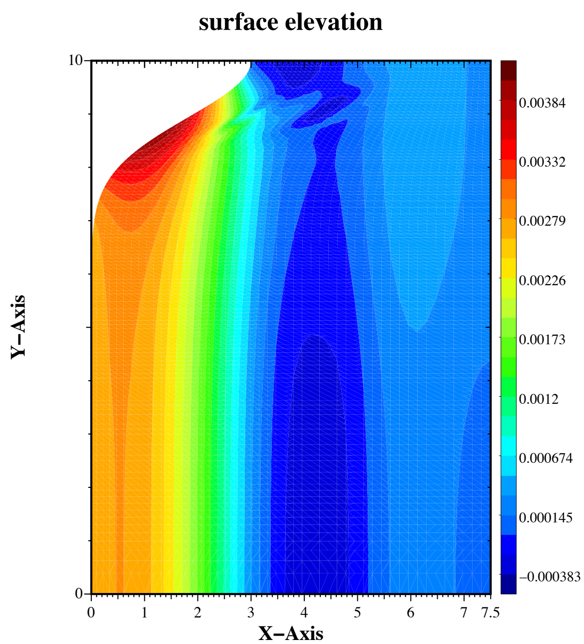
The second application concerns the propagation of waves over a shallow seamount. This case is inspired by tsunamis at the Gorringe bank outside Portugal. Both the depth and the initial surface elevation have the shape of a bell function. Biquadratic elements and grids adapted to the bathymetry are much more efficient than finite difference methods on uniform grids in this case. As the water gap at the summit becomes very small (1 percent of the deep water depth), the superiority of biquadratic elements is somewhat reduced. Results for a plane wave passing over a very shallow seamount are displayed in figure 3, lower panel. The surface elevation is depicted by contours, while the seamount is shown as a wire plot.

In a master's thesis by U. Kolderup [14], initiated by the GITEC-TWO project, some preliminary studies of the finite element model in the Atlantic ocean outside Iberia were carried out. Sponge layers and radiation conditions were included in the finite element model and a first, preliminary, attempt was made towards automatic grid generation from a depth matrix.

Finite element simulation of tsunamis requires the computational domain to be partitioned into quadrilateral or triangular elements. This is in general a complicated task that must be carried out by special grid generation software. However, the usage of such software in tsunami applications is not trivial, because a geometrically complicated polygon, approximating the shoreline, must be specified, with proper marking of boundary segments where different conditions apply. Optionally, the element size should be adapted to the bathymetry to ensure a uniform local Courant number or, equivalently, a uniform distribution of the element crossing times.

As a subproject in GITEC-TWO we have developed a method and associated software that automatically generates the shoreline polygon. We shall refer to this software as the pre-preprocessor. Our main source of shoreline information is a so called "depth matrix", that is, a discrete scalar depth field on a uniform rectangular grid. The pre-preprocessor takes a file with this depth field as input and creates appropriate input files to grid generation programs as output.

The pre-preprocessor is founded on the following method for constructing the


surface elevation
Y−Axis
X−Axis
10
0
0
1
2
3
4
5
6
7
7.5
0.00384
0.00332
0.00279
0.00226
0.00173
0.0012
0.000674
0.000145
−0.000383

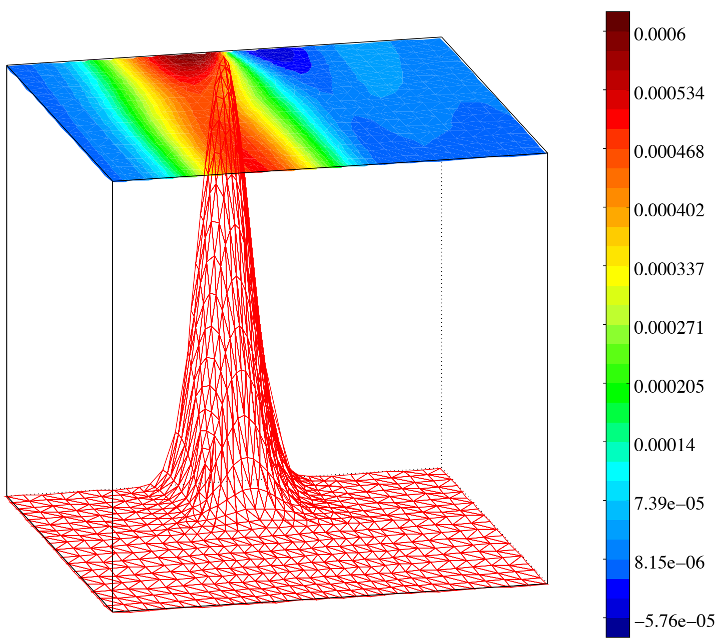

0.0006
0.000534
0.000468
0.000402
0.000337
0.000271
0.000205
0.00014
7.39e−05
8.15e−06
−5.76e−05

Figure 3:

shoreline polygon. First, a contouring algorithm is applied to the depth field to extract the 0-contour (or any other desired contour). This yields a collection of unsorted contour segments. Then the segments are sorted, such that we obtain a closed or open curve. In general, we have a list of n curves, where $n-1$ are closed, representing islands, and one is open or closed, representing the outer boundary. If the outer boundary is open, it must be closed by artificial boundaries in the domain. During the analysis of the curves, it is important to mark all points on the curves that are on the shoreline and all points that are on artificial boundaries, because different boundary conditions may apply to these different types of boundaries.

A basic problem with the algorithm above is that the boundary curves are made of segments whose sizes may differ greatly. Most grid generation software will then create triangles close to the shore with large variation in the element size. Normally, this leads to an unacceptable mesh. To solve the problem, we fit a B-spline curve to the boundary polygons. Having a spline representation of the shoreline, we can sample a new polygon with improved properties for grid generation. A uniform sampling of points has proved successful. Of course, the spline approximation leads to a certain "smoothing" of the true boundary. The amount of smoothing can be controlled by the number of points used in the sampled polygon.

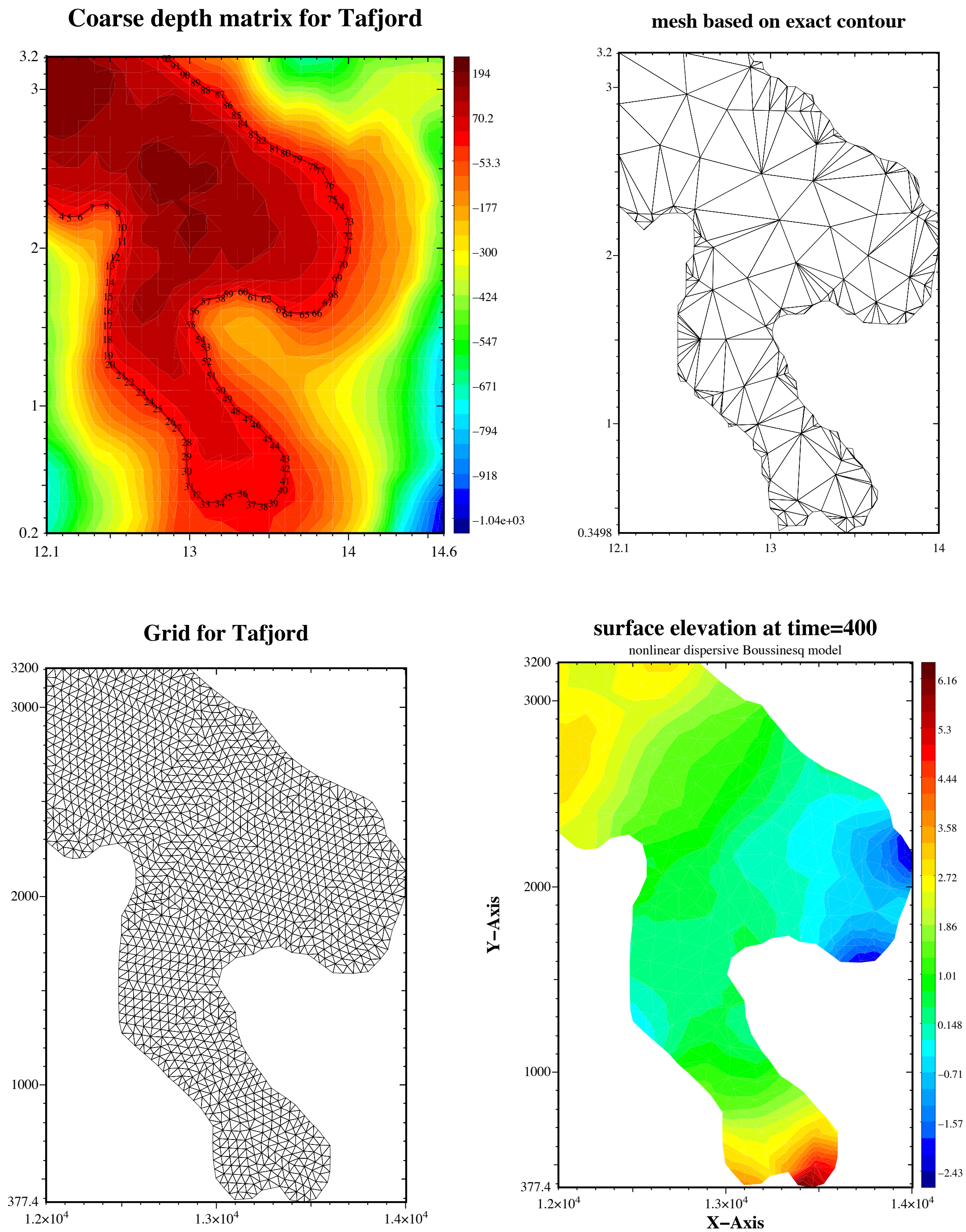
We illustrate our grid generation procedure with some result from the Tafjord case study, as given in figure 4. In the upper left panel we have depicted the original depth data, with the extracted coastline. Direct triangulation now yields the unacceptable grid to the upper right. On the other hand, resampling to obtain a uniform distribution of boundary points results in the mesh in the lower left panel. Using this triangulation we can perform stable simulations of the Boussinesq equations as shown by the preliminary results in the lower right panel.

4 Lagrangian models

The UiO group has developed two sets of Lagrangian models to describe run-up of tsunamis at sloping beaches, based on nonlinear hydrostatic theory. A FD model, in several varieties due to different forms of the momentum equation, has been further developed since the first GITEC project. In addition mixed FE models are in progress. All models have been verified by comparison with two different analytical solutions as well as through intercomparison.

4.1 FD models

In addition to the comparison with analytical solution the FD procedure has been thoroughly tested through a small number of idealized test cases, including run-up on a headland (same geometry as in preceding section) and wave generation by a land slide into a fjord or lake. This work is documented in the journal papers [12] and [11]. In the second part of the GITEC-TWO project the model has been generalized to include formation, propagation and run-up of bores. This is described as a separate



Coarse depth matrix for Tafjord
3.2
3
2
1
0.2
12.1
13
14
14.6
194
70.2
−53.3
−177
−300
−424
−547
−671
−794
−918
−1.04e+03
mesh based on exact contour
3.2
3
2
1
0.3498
12.1
13
14
Grid for Tafjord
3200
3000
2000
1000
377.4
1.2×10^4
1.3×10^4
1.4×10^4
surface elevation at time=400
nonlinear dispersive Boussinesq model
3200
3000
2000
1000
377.4
1.2×10^4
1.3×10^4
1.4×10^4
Y−Axis
X−Axis
6.16
5.3
4.44
3.58
2.72
1.86
1.01
0.148
−0.71
−1.57
−2.43

Figure 4:

theme in section 6.

4.2 FE models

In accordance with the experience from the Eulerian FE models the Lagrangian technique is based on low order elements. Linear or bilinear trial functions defined on triangles and quadrilaterals, respectively, are employed for velocities. The surface elevation is constant over each element, leading to a a mixed FE formulation. The nodes are Lagrangian, in the sense that they move with the fluid velocity, whereas the shape functions are described in Eulerian coordinates. We model the continuity equation simply by requiring mass conservation for each element. A weak formulation for the momentum equation is then designed as to yield a natural boundary condition, corresponding to vanishing fluid depth, at the shoreline. For quadrilaterals we then obtain a representation of the pressure term that is very similar to FD methods based on a conservative formulation of the momentum equation. Employing the standard staggered temporal discretization and lumping the mass matrix we then obtain an explicit scheme. However, a simple analysis reveals that the shoreline modes may display artificial oscillations when the mass is diagonalized. Even though this behaviour is not observed in all tests, consistent mass should be employed close to the shore as a rule. The surface elevation is always found by a purely explicit scheme, while the velocities must be found from an equation system with the mass matrix as coefficient matrix. We remark that keeping a full mass matrix does not reduce the compuational efficiency crucially, as the linear system is well conditioned and an SSOR-preconditioned Conjugate Gradient method converges within a couple of iterations.

In addition to comparison to analytic solutions the FE run-up model have been tested in idealized case studies as well as in preliminary computations on the Tafjord case. More details are given in the conference paper [15].

4.3 Comparison with analytical solutions

A set of particularly simple analytical solutions of the fully non-linear hydrostatic equations, concerning oscillations in parabolic basins, are found in [22]. We have generalized one of these slightly to allow a depth function $h(x,y) = h_0(y) - \alpha x^2$, where h_0 is any smooth function. The analytical solution then yields a linear cross-wise variation of the surface elevation, a spatially uniform velocity field and the eigenfrequency $\omega = \sqrt{2\alpha}$. It can be shown that all our numerical techniques, including FE methods with non uniform grids, reproduce this solution, save for small modifications of the relations between the field variables and the frequency according to $\frac{2}{\Delta t}\sin(\frac{1}{2}\omega\Delta t) = \sqrt{2\alpha}$. This provides a test of the coding rather than on the performance of the method due to the simple spatial distributions. All our run-up code has undergone this test.

In 1958 Carrier and Greenspan [6] published an analytical treatment of the hydrostatic and fully nonlinear run-up on an inclined plane. The assumption of constant

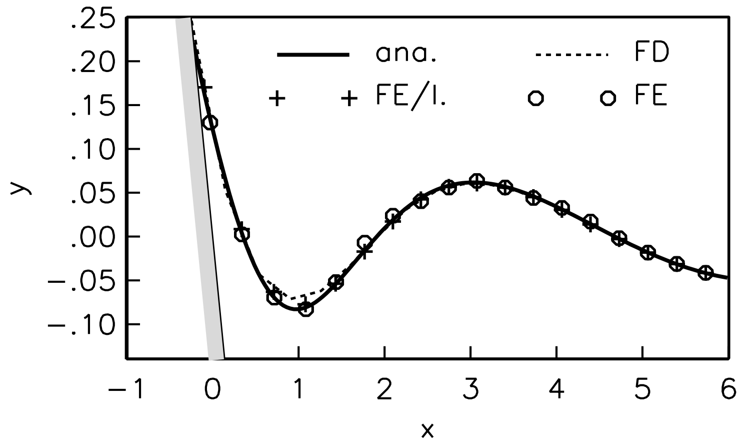


Figure 5:

bottom slope enabled an ingenious transformation, using the Riemann invariants, to a linear problem. Recently the theory has been generalized to include run-up in channels with parabolic cross-sections [21]. We employ one fundamental solution only, namely the standing wave oscillation. This solution is of fundamental importance and is closely related to the linear solutions in section 2. Moreover, it allows us to study the numerical reproduction of waves that are arbitrarily steep at the shoreline, even to the point of breaking. According to our experience this is a more challenging test for Lagrangian models than simulations involving large run-up distances, but small wave steepness. We will not go much into the particulars of the numerical solution nor the outcome of our tests. We refer instead to the mid-term report, the journal papers [11], [15] as well an internal report that can be made available at request. Still to give some idea concerning the tests and the performance of the methods we have depicted the surface at given instant for a case close to breaking (nondimensional frequency equal to 2, $A = 0.95$ as defined in the references). Even for the coarse grid employed (initial grid size 0.33) all methods perform well. We may nevertheless observe that the element techniques (FE, FE/l.) are slightly superior to the difference technique (FD). For other resolutions and larger integration time than in this example, the lumped element method (FE/l.) may be markedly infested by noise at the shoreline.

5 Domain decomposition

Within a single tsunami case study we may generally recognize regions of substantially different characters. The requirements concerning resolution, physical description and numerical methods vary accordingly. Any solution procedure that do not reflect these features will be inefficient or may even fail to produce any complete solution at all. One strategy to deal with this situation is to employ a domain decomposition technique, where solutions on different independent grids are combined to give the full solution. This approach will also allow for coarse-grained parallelization in combination with a suitable message-passing technique.

Our strategy is to encapsulate the existing sequential models to facilitate a simple, effective and robust communication of boundary values etc. The ultimate goal is to integrate general domain decomposition features into the object-oriented Diffpack system and this work will be given high priority in the near future. However, a number of results have already been achieved.

5.1 Parallel models

Explicit FD models are particularly suited for parallel computations based on domain decomposition. In a master's thesis [1], closely linked to the GITEC-TWO project, Elizabeth Acklam has performed a pilot project on the integration of MPI features with Diffpack. This work includes a user-friendly programming environment for implementation of parallel simulation codes using explicit finite difference schemes [3], high-level tools for easy implementation of schemes on staggered grids [4], as well as special techniques for optimizing C++ code [2]. Among the tests in the work with the master's thesis were simulations with the shallow water equations, distributed on 1 – 8 IBM RS6000 workstations. The speed up factor, that is CPU time for a given number of processors divided by the CPU time for the sequential solution on one processor, is given in the table for different sizes of the total grid.

grid	1 CPU	2 CPU	4 CPU	8 CPU
500×500	1	1.92	3.46	6.25
1000×1000	1	1.93	3.72	7.04
2000×2000	1	1.96	3.80	7.23

As seen from the table the actual speed up factor is very close to the theoretical maximum, that equals the number of processors, except from a modest efficiency loss for 8 processor simulations. The shallow water simulations involve little arithmetics per grid point. For a computationally heavier explicit model, like the Lagrangian run-up model, we will expect to get even closer to the theoretical maximum. However, in the general case with FE Boussinesq solvers we have to deal with additional nontrivial problems, like finding an optimal distribution of workload for complex geometries and parallel techniques for solving the implicit equations in the Boussinesq models. The strategy for the latter problem will be discussed briefly below.

5.2 Combination of methods

Fundamental questions concerning combinations of various equations and numerical methods in the different domains have been addressed during the project. This activity mainly falls into two groups: theoretical analysis and extensive numerical experimentation.

Various ways of combining difference and element grids of different resolutions have been analyzed. The results depend slightly on the scheme in question and we will hence omit most of the details. However, for a class of FD/FE Boussinesq solvers we have obtained some results of noteworthy simplicity when the domain decomposition simply consists of two subdomains with differing uniform grid increment, but coinciding boundary nodes. For the linear reflection coefficient of a simple harmonic we find

$$R = \frac{c_{g1} - c_{g2}}{c_{g1} + c_{g2}}$$

where c_{g1} and c_{g2} are the discrete group velocities of the domains. We observe that this relation is a standard law for reflection and that the order of convergenc of the method is not violated. Naturally, the above formula is only valid if a transmitted wave do exist, otherwise we obtain complete reflection.

A topic of particular importance is domain decomposition applied to the implicit Boussinesq solvers. At each time step we then have to solve a discrete Helmholtz equation. One simple approach is to apply an iteration procedure, the additive alternating Schwartz method, where each domain is treated implicitly and boundary values are exchanged after each iteration. However, the Boussinesq equations are nearly hyperbolic which imply a dominant local dependency in the time evolution. For linear equations and constant depth this may be directly observed through a discrete counterpart to the Green's function of the Helmholtz equation. An overlap region of an extent equal to two depths, say, then makes iteration superfluous. Generally, an iterative technique, requiring a small number of iterations, has to be used anyway within each domain and values can be exchanged after each iterations. We have performed numerous tests, including nonlinearity and variable depth, that indicate that an overlap of a few grid points then generally will suffice. Hence, very effective parallel Boussinesq models seem realizable.

Finally, the dynamic coupling of different mathematical descriptions have been investigated. Although this study is far from complete we have tested the combinations of linear/nonlinear and dispersive/nondispersive long wave equations. So far we have not observed any particular numerical problems as long as both descriptions are valid in the overlap zone.

6 Bore propagation and run-up

During the GITEC and GITEC-TWO projects we have developed a Lagrangian FD method for non breaking waves that is documented in the journal paper [11]. Since dangerous tsunamis very frequently arrive at the shores as bores, the next step is to include some sort of wave breaking into the model. Much of this work is described in the report [13], which is available on the Internet. Herein we present a brief discussion of the method, together with a few key results.

In long wave theory bores may be represented as discontinuities, in the velocity and surface elevation, that separates different regions where the assumptions of large wavelength is fulfilled. Details concerning the physical properties of the bore front, as width and velocity distribution, are not inherent in this description, whereas global characteristics like strength and celerity are obtained from preservation of momentum and mass. Energy, on the other hand, is dissipated in the shock. A direct patching of mass and momentum fluxes has been applied in a series of analytical studies of bore formation and propagation. A more sophisticated use of analytical expressions is made in Gudonovs method, where the surface profile is approximated by a piecewise constant function, and the solution is advanced in time by solving a series of Riemann problems. A large family of techniques have been developed from this basis. However, inclusion of such methods has so far not been reported within a Lagrangian description, as we employ in run-up calculation. Hence, we have started with a simpler approach based on inclusion of artificial diffusion. Artificial diffusion may introduced either implicitly by means of the numerical method (like Lax-Wendroff) or explicitly as an additional term in the momentum equation, with a strength governed by a parameter γ. To obtain better control we have preferred the latter option. We note that for refined grids and vanishing artificial diffusion all the methods should converge towards the same limit. It should be noticed that our main observation, namely the slow convergence of bore run-up, are probably much less related to our shock representation than to the physical properties of bores.

In the Lagrangian momentum equations, we include extra terms corresponding to an artificial diffusion, of fairly standard type, and a quadratic bottom drag. For both terms extra caution is required when a shoreline is present in the computational domain. In the finite difference model the diffusion term is implemented by a split-step method. A backward temporal discretization is employed for both terms.

A good reproduction of bores propagating in finite depth is readily obtained, as confirmed through comparison to simple analytical solutions. On the other hand, scrupulous investigations of bore run-up reveal severe problems with convergence concerning both resolution and, even more, vanishing diffusion strength $\gamma \rightarrow 0$. In fact, these problems can be anticipated from the analytical solutions for run-up of a uniform bore, that are available in the literature. According to these, the bore itself collapses at the very shoreline and a thin and smooth "jet" is sent on-shore. Due to the small thickness of the jet the fluid motion is much less affected by the internal pressure distribution than by gravity. Thus, we may consider the run-up event as an ensemble of independent particles, with different velocities, that is released from the collapsing bore during a short period of time. The maximum run-up height is then determined by the beach slope and the maximum velocity at the equilibrium shoreline.

It is very difficult to resolve the bore collapse properly, even in two dimensional computations. Finite bore width due to finite resolution and nonzero γ affects the crucial evolution at the shoreline. In particular, the maximum velocity will not be reproduced, which leads to an underestimation of run-up height. Hence, for a refined grid and a small value of γ we may obtain almost perfect convergence everywhere, apart from the vicinity of the moving shoreline for a times close to the occurrence of maximum run-up. Since tsunami inundations with small fluid depths, maybe even below 0.5 m, can present a mortal danger, this defect may be important. Moreover, similar errors must be expected for every finite difference or element method. On the other hand, the mathematical description of the bore as a singularity may be inappropriate at the shore, where the physical bore length very well may be important.

An improved convergence of run-up can be obtained in two ways. First we may regard a fluid depth under a certain (small) limit as inessential and redefine maximum run-up accordingly. Secondly, we may introduce the physical effect of bottom drag that will yield largest retardation for the fastest moving fluid near the inundation front. Hence, the convergence problems will also be reduced.

In addition to a huge number of tests involving uniform and idealized bores, we have performed a study closer to the reality of European tsunamis. Leaving out all details we sketch the main results of the latter study. As input wave shape in a depth of 20 m, we used the wave shape obtained for the 1969 Portuguese event, by two dimensional Boussinesq simulations and the Okada source [9]. However, the amplitude of the incident wave, close to 2m, is probably more relevant for 1755 event. The wave system then develops over a gentle slope, extending to the shore, of length 5km. Several simulations with different resolutions and drag representations have been performed and some of them are reported in [13]. A bore develops rapidly over the slope and the amplitude *decreases* because dissipation in the bore front dominates over amplification due to shoaling. This may be different for a source of larger extent which leads to a longer bore, in the sense that the length of the wave behind the discontinuity is larger. During run-up we observe the substantial influence of the bottom drag that inhibits the evolution of a long jet as described above. In fact depending on the representation and definition of run-up we find maxima between 3m and 8m. Hence, the run-up is very sensitive to frictional and turbulent effects in the on-shore jet.

Finally, we have also repeated the simplified three dimensional Tafjord simulations in [13] with the new model that includes bores. In three dimensions convergence of bore run-up is hardly attainable. However, the simulations show that the slide generated wave breaks well before half-way across the fjord, while breaking is not observed for the leading crests propagating in the along fjord direction. The amplitudes are fairly consistent with the largest local run-up observations, while no extra problems appear due to run-up of oblique shocks and steep edge waves. A fine grid solution is shown in figure 6 for illustration. The width of the idealized fjord is 1200m, the depth 180m and the duration of slide motion in water 16.6s. Different regions are shown in upper and lower panels, the contour increment is 5m and the arrows indicate the slide position. We note that a bore is nearly developed already in (a), while strongly nonlinear edge waves are present in (b) and (c).

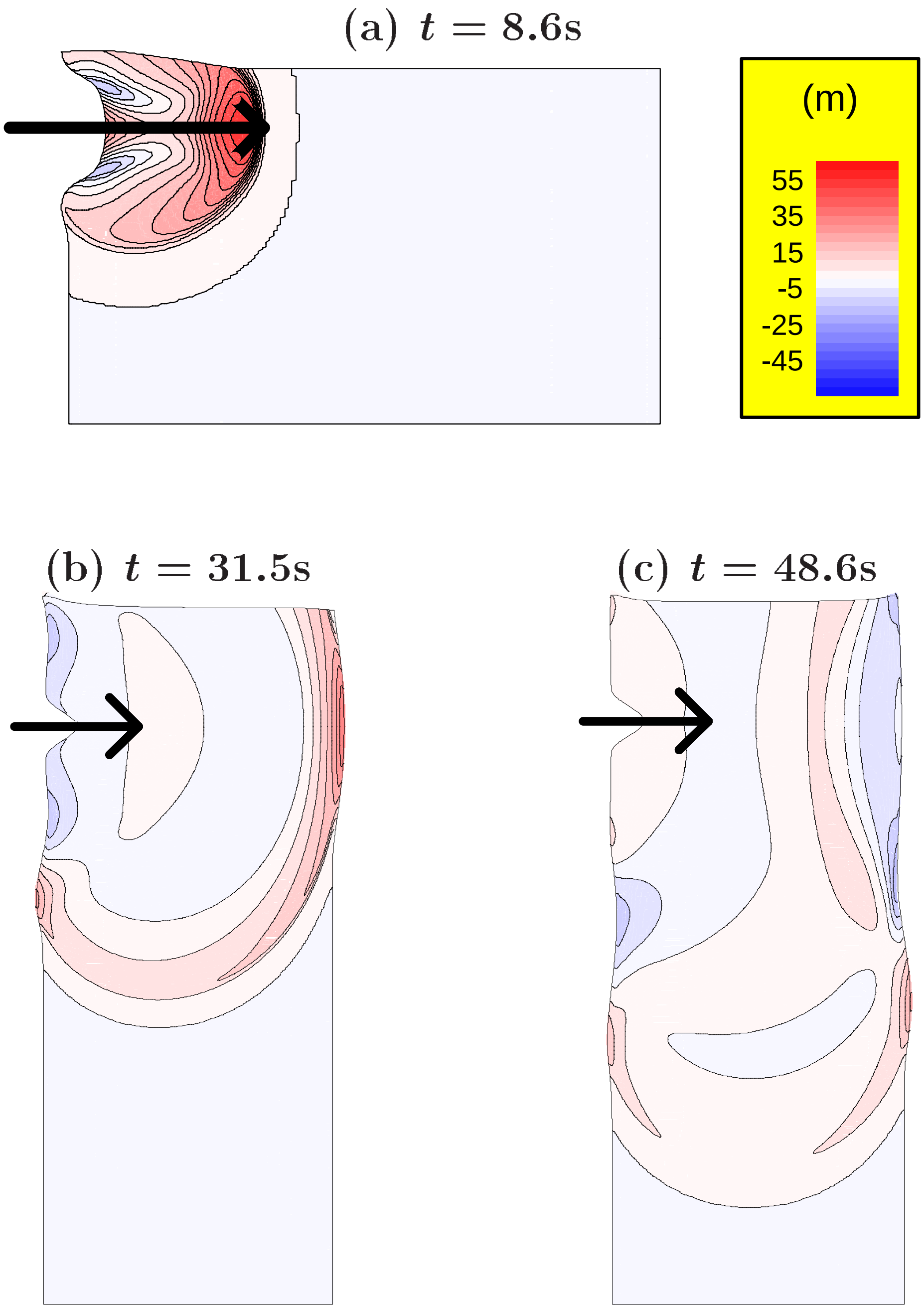


Figure 6:

7 A model based on full potential theory

In the preceding sections we have described modeling activities based on different long wave approximations. However, it is crucial also to integrate fully dispersive techniques in tsunami modeling. This is the motivation for the development of a boundary element method for propagation and run-up of tsunamis. Boundary element methods, and the closely related panel methods, are based on integral equations that only involve boundary values of the field variables. Such formulations exist for potential flow, for instance, where the velocity field anywhere in the fluid is determined by the velocity distribution at the boundaries. Viscous flow, on the other hand, can generally not be treated by boundary integral equations. Fortunately, surface waves may often be approximated very accurately by potential theory.

Compared to volume (FE/FD) methods, based on either potential theory or the NavierStokes equation, the main advantages of boundary element methods (BE) are reduced number of unknowns, flexible treatment of irregular boundaries, simpler adaptive refinement and that higher order methods are more readily invoked. On the other hand, the methods are confined to non-viscous flow, yield full equation sets and their efficiency in 3-D are debatable. On can in fact argue that boundary element methods are asymptotically less efficient than finite element methods in the complete water volume. This observation has led to a work, linked to GITEC-TWO, on an FE model for the equations of full 3D potential theory [5], but the model has not yet been applied to tsunami simulations.

There are a variety of different boundary integral formulations and numerical methods. We have adopted the basic ideas from the method described in [8]. The discretization of the spatial problem is based on Cauchy's theorem for analytical functions, inserted the so called complex velocity. Cauchy's theorem is discretized by collocation at nodes where the field variables are specified. Higher-order polynomials are employed for spatial interpolation. The time evolution, that is invoked through the conditions at the free surface, is accordingly modeled by a higher-order scheme, based on Taylor series expansion and values from previous time steps. We have to make modifications to deal with a shoreline point at a sloping beach. Some extra caution required at the shoreline and the usefulness of higher-order interpolating is doubtful. We have thus attempted piecewise linear interpolants and cubic splines. Both options produce good results for linearized equations while linear interpolation yields too slow convergence in nonlinear simulations. The methods have been tested by comparison to analytic solutions, extensive grid refinement and comparison to long wave computations. Some linear computations are employed in the subsequent section, whereas only preliminary nonlinear simulations have been performed so far.

8 Case studies and physical processes

8.1 The 1969 case study

Even though the joint paper [9] on the 1969 tsunami southwest of Portugal was initiated in the first GITEC project much work has been devoted to this paper also in GITEC-TWO. In addition to a substantial revision of the original manuscript new material has been included, in particular concerning applicability of the long wave assumptions and effects of finite grid resolution. Parts of this study have also served as benchmark problems for verification and comparison of models at the LDG and the UiO. In a more general context this activity has been continued after the completion of the paper. Some results are given in subsequent sections.

8.2 The Tafjord case study

In the present century approximately 170 persons have perished in tsunamis due to slides in fjords and lakes in Norway alone. Such incidents are fairly frequent also in other coastal regions with similar characteristics, as Alaska or Greenland, as well as in mountain lakes world wide. As a well documented and relevant example we have focused on the event in Tafjord that occurred in April 1934 and caused 41 deaths in addition to substantial material damages. The source was a dense rock slide, of volume 1.5 million cubic meters, that plunged into the fjord from heights up to 730m, above sea level, along a slope of more than $30°$ inclination. A study of the waves, based on linear hydrostatic long wave equations, was published in 1993 [10]. In the new study priority has been given to principal investigations of nonlinear effects, dispersion and resolution. Some of this work, concerning bore formation, grid generation and finite element simulations, has been described previously. The study of run up on steep slopes, mentioned below, is also motivated by the incidents like the Tafjord tsunami.

8.3 Dispersive effects

Most of the tsunami computations world wide are still based on the shallow water theory. The computational cost is substantially increased when advancing from hydrostatic equations to, for instance, the Boussinesq equations. This is due partly to the increased number and complexity of terms and partly to the necessity for some degree of implicitness at each time step. Considering the power of present computers fully dispersive theory is still too heavy to be applied to more than idealized or local studies. Hence, it is crucial to establish the limitations of the different long-wave approximations and to point out the features that are lost or misrepresented.

Employing shallow water theory, Boussinesq equations and full potential theory (BE) we have undertaken studies of importance of non-hydrostatic effects during tsunami generation, propagation and run-up. Some of the findings are described in [9], while others will be given in a conference proceeding and also published electronically on the address URL: *http://www.math.uio.no/eprint/appl_math/1998/appl_1998.html*.

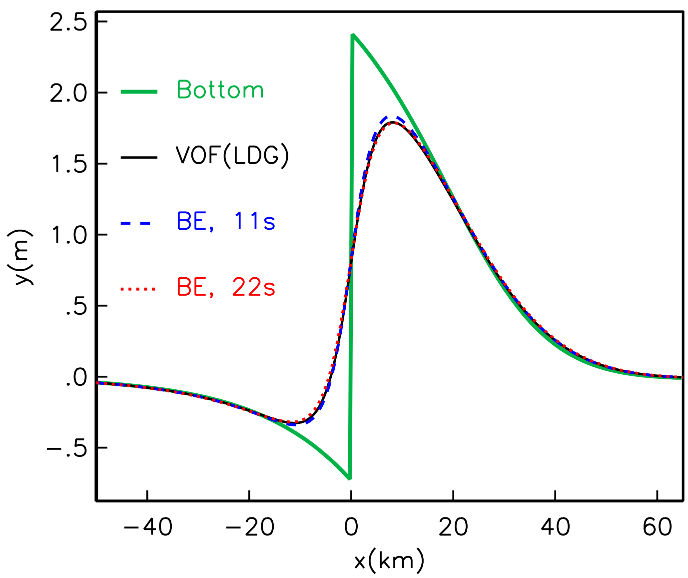


Figure 7:

In two dimensions (the vertical + 1 horizontal) the groups at the LDG and the UiO have produced a fairly complete hydraulic response to the Okada source model by employing a Navier Stokes solver (VOF) and the boundary element method (BE) respectively. The source is specified as a vertical velocity (sink/source distribution) at the bottom, with total periods of duration ranging from 1 to 22 seconds and slightly varying temporal distribution of acceleration. As shown in figure 7 all simulations gave virtually identical surface elevations. Generally the surface elevation is very close to the bottom displacement, except for the discontinuity over the focal line that is replaced by a transition zone of about 18 km. Associated with this zone we still obtain a large content of short waves in the spectrum that will have bearing on the subsequent discussion of dispersion effects and discretization errors. It is also noteworthy that the maximum surface elevation is reduced almost 30% as compared to the bottom deformation, while the region of depression at the surface is substantially reduced.

Using the elevation marked “BE, 11s” in figure 7 as initial condition in depth of 5 km, we have performed a several investigations on dispersion effects in deep water, amplifications due to shoaling and grid effects. This study also rely upon asymptotic expressions for the leading crest of a dispersive wave train, see for instance [17], and geometrical and physical optics. Important features of the asymptotic wave front are attenuation of amplitude and increase in wavelength with time and that the shape is independent of the shape of the initial elevation.

Our main conclusions are that the 1969 tsunami in some respects are strongly influenced by dispersion after a propagation distance of only 150 km, say, and that a negative first arrival can hardly be observed at the coasts. Concerning the performance of the long wave approximations we observe that the Boussinesq solution agrees well with the full theory not only for the front wave, but also for the first few oscillations that follow due to dispersion. While hydrostatic theory reproduces the amplitude fairly well, it yields substantial errors in wave shape, particularly for the system propagating in negative direction. The consequences of this discrepancy are not easily foreseen. The spurious preservation of the steep front in the hydrostatic solution may obviously lead to large errors concerning bore formation and run-up on a sloping beach. On the other hand, it is observed that the response in harbours or other nearly confined coastal basins may be little affected by large deviations in deep water due to the omission of dispersion terms.

When the propagation distance is increased to 400 km the first wave in the dispersive simulations nearly coincide with the asymptotic expression, which imply that information concerning initial shape is lost in the wave front. Consequently, the hydrostatic theory that preserves the initial shape is no longer a meaningful approximation.

We have also performed a number of other simulations, varying the shape and extent of the initial elevation etc., of which some may be reported elsewhere. Moreover, also some 3-D simulations have been performed that indicates that the importance of dispersion is slightly larger than in 2-D. For illustration, the results after 13 minutes are depicted in figure 8.

The study of numerical dispersion naturally fits into the present context. During the project we have performed a huge amount of grid refinement tests as well as studies with focus on coarse grid solutions. It is not convenient to report this large body of simulations herein. However, it is noteworthy that for the 1969 tsunami our investigations indicate that numerical dispersion, owing to a mid-point scheme, *completely corrupts* the wave shape over a propagation distance of 10-20 km in a depth of 50 m, say, for grid increments of order 1 km For a leap-frog method the errors would develop, roughly speaking, four times faster.

In view of the present investigations we anticipate that dispersion often is important for tsunami propagation in the deep sea. Fortunately, the Boussinesq equations seem to approximate the dispersive features of seismic tsunamis quite well.

8.4 Run-up on steep slopes

In many respects tsunamis modeling in fjords or mountain lakes resemble modeling of tsunamis originating in open sea. However, there are also important differences. The applicability of long wave equations for tsunami propagation in fjords are challenged by a large ratio between depth and width and steep bottom slopes rather than huge propagation distances. An analysis of the narrowness in relation to dispersive effects for locally generated tsunamis may be performed in terms of crosswise eigenmodes. It must be noted that the response of a fjord to a long wave incident from open sea, like the Storegga tsunami, often will be dominated by longitudinal modes that

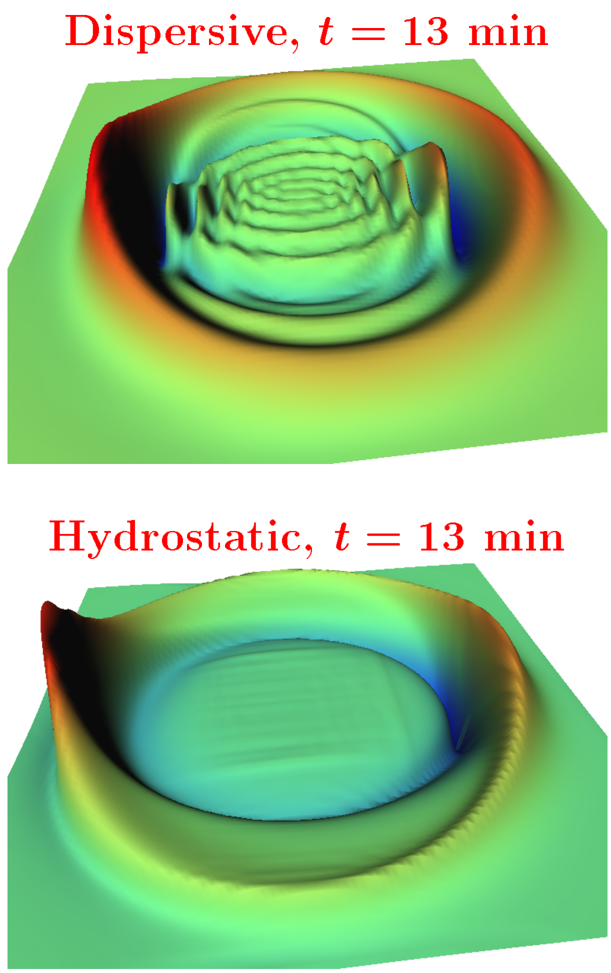


Figure 8:

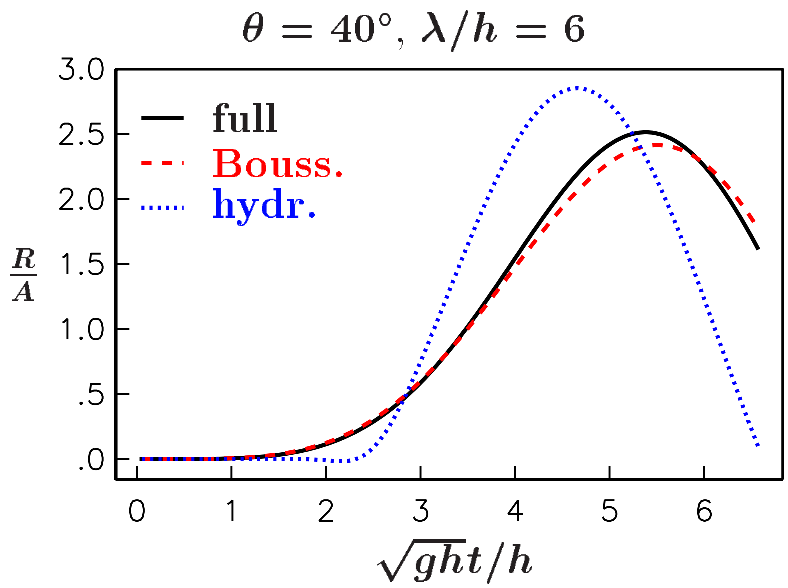


Figure 9:

are virtually unaffected by dispersion. In fact, we have performed some studies of oscillations in fjords, but will instead focus on run-up on steep slopes. This topic is relevant also to run up on breakwaters etc.

In an idealized run-up simulation, where an incident wave is specified in a region of constant depth adjoined by an inclined plane reaching to the shore, we have again compared the different long wave approximations to full potential theory. In figure 9 we have depicted the linear run up relative to the amplitude of the incident wave for a length of incident wave equal to 6 depths and a shoreline slope of 40 degrees. This choice of parameters are relevant for Tafjord. The performance of the Boussinesq equations is remarkably good in view of the slope steepness.

References

[1] Acklam, E. 1997 A Parallel Programming Environment for Finite Difference Methods.
Cand.scient. thesis at Department of Mathematics, Mechanics Division. University of Oslo.

[2] Acklam, E, Jacobsen, A., and Langtangen, H. P. 1998 Optimizing C++ code for explicit finite difference schemes. *Oslo Scientific Computing Archive*, URL: *http://www.math.uio.no/OSCA*.

[3] Acklam, E. and Langtangen, H. P. 1998 Parallelization of explicit finite difference schemes via domain decomposition. *Oslo Scientific Computing Archive*, URL: *http://www.math.uio.no/OSCA*.

[4] Acklam, E. and Langtangen, H.P. 1998 Tools for simplified programming with staggered grids. *Oslo Scientific Computing Archive*, URL: *http://www.math.uio.no/OSCA*.

[5] Cai, X., Langtangen, H.P., Nielsen, B.F., and Tveito, A. 1998 A Finite Element Method for Fully Nonlinear Water Waves. Accepted for publication in *Journal of Computational Physics*.

[6] Carrier, G.F. and Greenspan, H.P. 1958 Water waves of finite amplitude on a sloping beach. *J. Fluid Mech.* **4**, 97-109.

[7] The Diffpack WWW home page. URL: *http://www.nobjects.com/Diffpack*.

[8] Dold, J.W. 1992 An Efficient Surface-Integral Algorithm Applied to Unsteady Gravity Waves. *J. Comp. Phys.***103**, 90–115.

[9] Gjevik, B., Pedersen, G., Dybesland, E., Harbitz, C.B., Miranda P.M.A., Baptista M.A., Mendes-Victor, L., Heinrich, P., Roche, R. and Guesmia, M. 1997 Modeling tsunamis from earthquake sources near Gorringe Bank southwest of Portugal. *J. Geophys. Res.***102, No. C13**, 27–949.

[10] Harbitz, C.B., Pedersen, G., and Gjevik, B. 1993 Numerical simulation of large water waves due to landslides. *J. Hydraulic Enginering*, **119**(12), 1325-1342.

[11] Johnsgard, H. and Pedersen, G. 1997 A numerical model for three-dimensional run-up. *Numerical Methods in Fluids* **24**, 913-931.

[12] Johnsgard, H. and Pedersen, G. 1997 Slide Generated Waves in Near-Shore Regions. A Lagrangian Description. *Phys. Chem. Earth,* **21**-12, 45-49.

[13] Johnsgard H. 1998 A numerical model for run-up of breaking waves. *Preprint Ser. Dept. of Maths., University of Oslo* **4**
URL: *http://www.math.uio.no/eprint/appl_math/1998/04-98.html*. Submitted for publication.

[14] Kolderup, U. M. 1986 (In Norwegian) English title: Solution of the Boussinesq Equations by the Finite Element Method. Cand.scient. thesis at Department of Mathematics, Mechanics Division. University of Oslo.

[15] Langtangen, H.P. and Pedersen, G. 1998 A Lagrangian model for run-up of shallow water waves. In M. Hafez and J. C. Heinrich (editors): *The Proceedings of the Tenth International Conference on Finite Elements in Fluids*, Tuscon, Arizona, USA, January 1998.

[16] Langtangen, H.P. and Pedersen, G. 1998 Computational Models for Weakly Dispersive Nonlinear Water Waves. *In press: Computer Methods in Applied Mechanics and Engineering*

[17] Mei, C.C., 1989 The applied dynamics of ocean surface waves. *Advanced Series on Ocean Engineering* Vol.**I**, 740 pp. World Scientific, London.

[18] Pedersen, G. 1985 Run-up of periodic waves on a straight beach. *Preprint Ser. Dept. of Maths., University of Oslo* **1/85**

[19] Pedersen, G. 1988 On the numerical solution of the Boussinesq equations. *University of Oslo, Research Report in Mechanics* **88–14**

[20] Pedersen, G. 1998 An optical theory for discrete media. *In progress.* URL: *http://www.math.uio.no/eprint/appl_math/1998/appl_1998.html.*

[21] Pelinovsky E., Kozyrev O. and Troshina E. 1995 Tsunami runup in a sloping channel. *Int. Workshop on long wave modelling. Seattle.*

[22] Thacker, W.C. 1981. Some exact solutions to the nonlinear shallow-water wave equations. *J. Fluid Mech.* **107**, 499-508.

[23] Torrey, M.D. et al. 1987 NASA-VOF3D: a three-dimensional computer program for incompressible flows with free surfaces. Los Alamos National Laboratory, **LA1100-9MS**.

[24] Wu, T.Y. 1981 Long waves in ocean and coastal waters. *Proc. ASCE, J. Eng. Mech. Div.* **107**,EM3,501–522