There are no minimal essentially undecidable theories

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Abstract

We show that there is no theory that is minimal with respect to interpretability among recursively enumerable essentially undecidable theories.

Keywords: interpretability, essential undecidability

1 Introduction

For any salient property of recursively enumerable (RE) theories $\mathcal{P}$, one can ask the obvious question *is there a weakest RE theory satisfying* $\mathcal{P}$*?* But what does *weakest* mean here? A traditional answer is to take a theory T to be given by a RE set of axioms $\mathcal{A}$. *Weakest* is then interpreted as T has $\mathcal{P}$ and no theory axiomatized by a proper subset of $\mathcal{A}$ has $\mathcal{P}$.

A paradigmatic example of an answer to our question under this reading, for the case where we take $\mathcal{P}$ to be *essential undecidability*, is the well-known result by Tarski *et al.* that the theory Q with its standard axiomatization is minimally essentially undecidable in the sense that all theories given by a proper subset of the axioms have a decidable extension. See [13, Chapter 2, Theorem 11]. Similar results for theories of concatenation were obtained by Juvenal Murwanashyaka. See [8]. For the theory R, a result in the same spirit is due to Cobham. See [7]. Only here minimality is applied to natural *groups* of axioms rather than to single axioms.

The above results crucially depend on the chosen axiom set. After all, each nontrivial finitely axiomatizable theory is axiomatizable by a single axiom. Suppose pure predicate logic does not satisfy $\mathcal{P}$. If a finitely axiomatized theory has property $\mathcal{P}$, then it is automatically minimal with respect to the single-axiom axiomatization. Along a different line, e.g. in the case of Q, it is easy to

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produce finitely axiomatizable strict sub-theories that are still essentially undecidable, if one allows tampering with the axioms. For example, we may relativize the quantifiers in the axioms for plus and times to the class of x such that $\mathsf{S}x \neq x$. Jones and Shepherdson [7] provide an example of an essentially undecidable sub-theory of R that is strictly below R in the sense that it proves strictly less theorems. However, they do strengthen one axiom group in order to be able to drop another.

In this paper, we zoom in on the property of *essential undecidability of RE theories* and we consider an ordering of RE theories that only depends on the theory-qua-set-of-theorems, to wit *interpretability*. So, we ask whether there is an interpretability-minimal RE essentially undecidable theory. In this paper, we show that there is no such theory. Thus, our main result is the following.

THEOREM 1.1
There is no interpretability-minimal RE essentially undecidable theory.

It is easy to see that, if an RE theory is *minimally* essentially undecidable with respect to interpretability, then it is, *ipso facto*, the *minimum* RE theory, modulo mutual interpretability. This is because essentially undecidable RE theories are closed under finite infima. Thus, it is sufficient to prove the following lemma.

LEMMA 1.2
There is no RE essentially undecidable theory that is the interpretability-minimum.

The paper is structured as follows. In Section 2, we introduce the basic concepts needed for the paper as a whole. This section also contains the reduction of Theorem 1.1 to Lemma 1.2. In Section 3, we give a simple proof that there is no minimal theory with respect to interpretability among *finitely axiomatized theories*. In Section 4, we prove Theorem 1.1. We provide two different proofs. The first proof employs a direct diagonalization argument; the second reduces our question to a general recursion theoretic result.

REMARK 1.3
Our main question can be asked for other notions than essential undecidability. The question whether there is an interpretability minimal recursively inseparable theory is answered negatively in [2]. The question whether there is an interpretability minimal essentially hereditarily undecidable theory is answered negatively in [16].

It seems reasonable to demand that the property under consideration is preserved by the relevant reduction relation. We briefly look at the case of undecidability, which is not preserved under mutual interpretability.

Predicate logic with one binary relation symbol is interpretable in all theories. However, it is mutually interpretable with, e.g. the decidable theory that says that there is precisely one element. So, in a sense, predicate logic with one binary relation symbol is the interpretability minimal undecidable theory, but, we feel, this statement is somewhat misleading.

The more reasonable question would be whether there is an undecidable theory W that is minimal w.r.t. *faithful* interpretability. The answer here is *no*. By a classical theorem of Feferman [3], we have undecidable theories in each RE Turing degree (see Theorem 4.9 of this paper for Shoenfield's strengthening of Feferman's result). If U is faithfully interpretable in V, then the Turing degree of U is lesser or equal to the Turing degree of V. So, W should have the minimum non-recursive RE Turing degree, but there is no such degree.

2 Some Basic Concepts

Theories in the present paper are, by default, one-sorted RE theories of predicate logic in finite signature. We will encounter some theories of infinite signature and some theories that are not RE, but these cases will be clear from the context.

Our results are about theories-qua-sets-of-theorems. However, our methods sometimes demand the intensional perspective where the axiom set is given by a formula or by a recursive index.

There is a whole range of notions of interpretability. We can or cannot have pieces, multidimensionality, parameters, relativization, non-preservation of identity. Our main result is not sensitive to the specific details of the notion of interpretation as long as we consider mixtures of these specific features.

We want to prove that there is no interpretability-minimal RE essentially undecidable theory. The proof can be divided in a preliminary step and a main step.

I. We show that, modulo mutual interpretability, if there is an interpretability-minimal RE essentially undecidable theory, then there is an interpretability-minimum RE essentially undecidable theory. To do this, we show (i) that interpretability is a lower semi-lattice (modulo mutual interpretability), i.e. binary infima exist and (ii) that infima preserve essential undecidability. This gives the reduction of Theorem 1.1 to Lemma 1.2.

II. We prove Lemma 1.2.

Below, we will define one uniform construction on theories defining the binary minimum operation. This construction provides a binary minimum for all notions of interpretability that are given by combinations of the above features. So, (I) works for all such notions.

It is immediate that, if we have (II) for the most inclusive notion, where we have pieces, multidimensionality, parameters, relativization, non-preservation of identity, then we also have it for all weaker notions. So, we only need to prove (II) for our strongest notion of interpretability. Alternatively, one could look at what is used in the proof of (II) and see that all notions satisfy these assumptions.

In this paper, we will use the most inclusive notion, to wit piecewise, multidimensional (with dimensions varying over pieces), relative, non-identity-preserving interpretability with parameters.

We refer the reader for definitions to [15]. Here, we will just fix notations and give some basic facts.

We write $U \rhd V$ for U interprets V and $V \lhd U$ for V is interpretable in U.

Given two theories U and V, we form $W := U \curlyvee V$ in the following way. The signature of W is the disjoint union of the signatures of U and V with an additional fresh zero-ary predicate P. The theory W is axiomatized by the axioms $P \to \phi$ if ϕ is a U-axiom and $\neg P \to \psi$ if ψ is a V-axiom. One can show that $U \curlyvee V$ is the infimum of U and V in the interpretability ordering $\lhd$. This result works for all choices of our notion of interpretation.[1] We note that $\curlyvee$ preserves finite axiomatizability.

A theory is *essentially undecidable* iff all its consistent extensions in the same language are undecidable. Salient examples of essentially undecidable theories are R and Q. See [13]. We have the following basic insights.

[1] It is a bit strange that we use a disjunction-like notation for an infimum. This strangeness is due to the fact that the conventional choice for the interpretability-ordering puts the weakest theory below where in boolean algebras the strongest proposition is the lower one. In our notation, we follow the boolean intuition and view our operation as a kind of disjunction of theories.

THEOREM 2.1
U is essentially undecidable iff every consistent V such that $V \rhd U$ is undecidable.

PROOF. Since any V extending U also interprets U, the 'if' part is trivial. For 'only if' part, we note that if $V \rhd U$ and V is consistent and decidable, then any interpretation $\iota \colon V \rhd U$ gives a decidable consistent extension T of U that is axiomatized by all the sentences φ of the language of U such that V proves the ι-translation of φ. □

THEOREM 2.2
We have

i. if U is essentially undecidable and $V \rhd U$, then V is essentially undecidable;
ii. if U and V are essentially undecidable, then so is $U \curlyvee V$.

PROOF. The first claim follows immediately from Theorem 2.1 and the transitivity of interpretability.

To prove the second claim, we assume for a contradiction that there is a consistent decidable extension T of $U \curlyvee V$. Either T is consistent with P or with $\neg P$. If T is consistent with P, then $T + P$ is a consistent decidable theory that interprets U, contradicting essential undecidability of U. Analogously, if T is consistent with $\neg P$, then $T + \neg P$ is a consistent decidable theory that interprets V, contradicting essential undecidability of V. □

We note that the fact that essentially undecidable theories are closed under binary infima implies that, if there is an interpretability-minimal one, then there is a minimum with respect to interpretability. Thus, we have reduced Theorem 1.1 to Lemma 1.2.

This reduction still works when we restrict ourselves to finitely axiomatized theories, since $\curlyvee$ also preserves finite axiomatizability.

3 Finitely axiomatizable theories

In this section, we prove the non-existence of a minimal essentially undecidable theory with respect to interpretability for the case where we restrict ourselves to finitely axiomatized theories.

Our result is really a triviality as soon as the required machinery is in place. We introduce this machinery in the next subsection.

3.1 Theories of a number

We need the theory TN of a number. This theory is given as follows.

TN1. $\vdash x \not< 0$
TN2. $\vdash (x < y \wedge y < z) \to x < z$
TN3. $\vdash x < y \vee x = y \vee y < x$
TN4. $\vdash x = 0 \vee \exists y\, x = \mathsf{S}y$
TN5. $\vdash \mathsf{S}x \not< x$
TN6. $\vdash x < y \to (x < \mathsf{S}x \wedge y \not< \mathsf{S}x)$
TN7. $\vdash x + 0 = x$
TN8. $\vdash x + \mathsf{S}y = \mathsf{S}(x + y)$
TN9. $\vdash x \cdot 0 = 0$
TN10. $\vdash x \cdot \mathsf{S}y = x \cdot y + x$

We note that TN allows finite models which can be identified with the natural numbers viewed as the finite von Neumann ordinals with the added structure of zero, addition, multiplication and $<$.

Moreover, if a model of TN has a maximal element, the model can be viewed as a (possibly) non-standard number. In this case, we take $\mathsf{S}a = a$ on the maximal element and adapt plus and times accordingly. In [15], the reader may find some further discussion of TN.

A Δ_0-formula is *pure* iff (i) all bounding terms are variables and (ii) all occurrences of terms are in subformulas of the form $\mathsf{S}x = y$, $x + y = z$ and $x \cdot y = z$. A Σ_1-sentence *pure* if it is of the form $\exists \vec{x}\ \sigma_0 \vec{x}$, where σ_0 is a pure Δ_0-formula.

We can transform an arbitrary Σ_1-sentence σ into a pure Σ_1-sentence. See [15] for a sketch of the argument. In Section 3, we will assume that all Σ_1-sentences σ are rewritten in pure form.

Let $\sigma := \exists \vec{y}\ \sigma_0 \vec{y}$, where σ_0 is a pure Δ_0-formula. We define

$$[\sigma] := \mathsf{TN} + \exists x\, \exists \vec{y} < x\, \sigma_0\, \vec{y}.$$

We note that if σ is false, then $[\sigma]$ extends R. Thus, if $[\sigma]$ is, in addition, consistent, we find that $[\sigma]$ is essentially undecidable.

3.2 The main result for finitely axiomatizable theories

We prove our main result for the finitely axiomatized case.

THEOREM 3.1
There is no interpretability minimal essentially undecidable finitely axiomatized theory.

PROOF. Since the finitely axiomatizable essentially undecidable theories are closed under interpretability-infima, it is sufficient to show that there is no minimum theory $A^\star$ among finitely axiomatized essentially undecidable theories.

Suppose there was such an $A^\star$. Consider any Σ^0_1-sentence σ. If σ is true, then $[\sigma]$ has a finite model, so, clearly, $[\sigma] \not\rhd A^\star$. If σ is false and $[\sigma]$ is consistent, we have $[\sigma]$ is essentially undecidable, so $[\sigma] \rhd A^\star$. If $[\sigma]$ is inconsistent, then, trivially, $[\sigma] \rhd A^\star$. *Ergo*, σ is false iff $[\sigma] \rhd A^\star$. However, this is impossible, since the set of σ such that $[\sigma] \rhd A^\star$ is RE. ☐

We note that our proof uses very little about interpretability. So there is a good chance that it will work for even more general notions, like forcing-interpretability. However, we did not explore this.

4 Recursively Enumerable Theories

In this section, we give two proofs of our main result for RE theories. Subsection 4.1 provides the machinery for the first proof, which is given in Subsection 4.2. Subsections 4.1 and 4.3 are preparations for the second proof, which is given in Subsection 4.4.

4.1 Propositional logic-like theories

A central tool in this paper will be the use of a propositional logic like theory. Let us say that a theory T is *propositional-logic-like* or *PLL* iff it has the following property.

- There is a recursive sequence of T-sentences $(\alpha_i)_{i<\omega}$ such that the complete consistent extensions of T are precisely the theories $T_{\mathcal{X}}$, where $\mathcal{X} \subseteq \omega$ and $T_{\mathcal{X}} := T + \{\alpha_i \mid i \in \mathcal{X}\} + \{\neg\alpha_i \mid i \notin \mathcal{X}\}$.

We immediately have the following result.

THEOREM 4.1
Suppose T is PLL as witnessed $(\alpha_i)_{i<\omega}$. We have

i. every ϕ in the T-language is provably equivalent to a boolean combination of the α_i;
ii. T is decidable.

Let $\neg^0 \phi := \phi$ and $\neg^1 \phi := \neg \phi$. Let bseq_n be the set of binary sequences of length n. We identify a binary sequence of length n with a function from $\{0, \ldots, n-1\}$ to $\{0, 1\}$. Suppose $\sigma \in \mathsf{bseq}_n$. We write α_σ for $\bigwedge_{i<n} \neg^{\sigma(i)} \alpha_i$.

PROOF. *Ad (i):* Suppose T is PLL as witnessed $(\alpha_i)_{i<\omega}$. Consider any ϕ. Suppose that, for all n, there is a $\sigma \in \mathsf{bseq}_n$, such that ϕ is independent of $T + \alpha_\sigma$. Then, by König's Lemma, there is an $F : \omega \to \{0, 1\}$, such that ϕ is independent of $T + \{\neg^{F(i)} \alpha_i \mid i \in \omega\}$, contradicting the fact that T is PLL.

So, for some n and for all $\sigma \in \mathsf{bseq}_n$, we have $T + \alpha_\sigma \vdash \phi$ or $T + \alpha_\sigma \vdash \neg \phi$. Let $\mathcal{S} := \{\sigma :\in \mathsf{bseq}_n \mid T + \alpha_\sigma \vdash \phi\}$. It follows that $T \vdash \phi \leftrightarrow \bigvee_{\sigma \in \mathcal{S}} \alpha_\sigma$.

Ad (ii): To see whether ϕ is provable from T or not, we first find an equivalent boolean combination $\beta(\vec{\alpha})$ of the α_i. (By our default assumption, T is RE, so we can effectively find such a combination.) Since the α_i are independent over T, it now suffices to check whether β is a propositional tautology or not. □

REMARK 4.2
Theorem 4.1 really says that T is PLL iff its Lindenbaum algebra, considered as a numbered structure, is recursively boolean isomorphic to the free boolean algebra on infinitely many generators, considered as a numbered structure.[2]

We note that the Lindenbaum algebra of, e.g., Peano Arithmetic is countable and atomless. Hence, it is isomorphic to the free boolean algebra on infinitely many generators. It follows that there is a sequence of arithmetical sentences $(\alpha_i)_{i<\omega}$ such that the complete consistent extensions of PA are precisely the theories $\mathsf{PA}_{\mathcal{X}}$, for $\mathcal{X} \subseteq \omega$. Of course, here, the sequence of the α_i cannot be recursive.

Any PLL theory would be a suitable tool for the development in this paper. In Appendix A.1, we prove a characterization of PLL that suggests that there are many such theories. It is pleasant, however, to employ one specific PLL theory. We choose the theory Jan, that has the further good property that all its extensions are *locally finite*. See below. The theory Jan is the theory of one equivalence relation with the following extra axioms, for each n.

- There is at most one equivalence class of n elements.
- There are at least n equivalence classes of at least n elements.

The theory Jan is an extension of the theory of one equivalence relation studied by Janiczak. See [5].

We define the statement A_n as the sentence expressing that there is at least one equivalence class with precisely $n + 1$ elements.

THEOREM 4.3
Jan is PLL.

[2] This notion is not precisely the same as recursive boolean isomorphism in the sense of the existence of a recursive bijection of sentences that commutes with the propositional connectives and that preserves provability. See [1] and [16] for further discussion of such distinctions.

The theorem follows immediately from the quantifier elimination presented in [5]. For completeness, we present here a simple alternative proof.

PROOF. The countable models of Jan are precisely the models of the following form: (i) for each n, there may or there may not be an equivalence class of size n; and (ii) there are κ countably infinite equivalence classes, where, if there are infinitely many finite equivalence classes, $0 \leq \kappa \leq \aleph_0$ and, if there are finitely many finite equivalence classes, $\kappa = \aleph_0$.

Consider any countable model $\mathcal{M}$ of Jan. We extend the language of Jan with new constants c_i. Let W be the theory of $\mathcal{M}$ in the extended language plus the following axioms:

- c_i is not equivalent to c_j, for $i \neq j$;
- the equivalence class of c_i has at least n elements, for any i and n.

By compactness, W is consistent. So, it has a countable model $\mathcal{N}$ with countably infinitely many countably infinite equivalence classes. In the original language, $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$. Modulo isomorphism, $\mathcal{N}$ is uniquely determined by which finite equivalence classes there are in $\mathcal{M}$.

In the role of the α_i, we take the statements A_i. Thus, given $\mathcal{X} \subseteq \omega$, all models $\mathcal{M}$ of $\mathsf{Jan}_{\mathcal{X}}$ are elementarily equivalent, and so $\mathsf{Jan}_{\mathcal{X}}$ is complete. □

REMARK 4.4
Theorem 4.3, in combination with Theorem 4.1, implies that any extention-in-the-same-language of Jan is *locally finite*. This means that every finite sub-theory has a finite model. It follows that every RE extention-in-the-same-language of Jan is interpretable in the Tarski–Mostowski–Robinson theory R by the main result of [14].

Instead of Jan, we can also employ a certain successor theory Succ°. We treat this theory in Appendix A.2.

4.2 First proof of Theorem 1.1

We prove a theorem that provides, for every essentially undecidable RE theory, a class of theories that do not interpret it. The members of this class can instantiate many desirable recursion theoretic properties. For example, there is a member in any RE degree.

Let $\mathcal{X}$ be a set of numbers. We say that W is a $\mathsf{Jan}, \mathcal{X}$-theory when W is axiomatized over Jan by boolean combinations of sentences A_s for $s \in \mathcal{X}$.

THEOREM 4.5
Consider any essentially undecidable RE theory U. Then, we can effectively find a recursive set $\mathcal{X}$ (from an index of U) such that no consistent $\mathsf{Jan}, \mathcal{X}$-theory interprets U.

PROOF. Let $\mathsf{C}_{n,0}, \ldots, \mathsf{C}_{n,2^n-1}$ be an enumeration of all conjunctions of $\pm\mathsf{A}_i$, for $i < n$. Suppose U is an essentially undecidable RE theory. Let $\upsilon_0, \upsilon_1, \ldots$ be an effective enumeration of the theorems of U. Let $\tau_0, \tau_1, \ldots$ be an effective enumeration of all translations from the U-language into the Jan-language.

Consider n, τ_i and $\mathsf{C}_{n,j}$, for $j < 2^n$. We claim that, for some k, we have $\mathsf{Jan} + \mathsf{C}_{n,j} \nvdash \upsilon_k^{\tau_i}$. If not, then τ_i would carry an interpretation of U in $\mathsf{Jan} + \mathsf{C}_{n,j}$. This contradicts the fact that $\mathsf{Jan} + \mathsf{C}_{n,j}$ is decidable.

Thus, we can effectively find a number $p_{n,i,j}$ as follows. We find the first k such that $\mathsf{Jan} + \mathsf{C}_{n,j} \nvdash \upsilon_k^{\tau_i}$. This can be effectively done since $\mathsf{Jan} + \mathsf{C}_{n,j}$ is decidable. Then, we reduce, over Jan, the sentence $\upsilon_k^{\tau_i}$ to a boolean combination of A_s. Let $p_{n,i,j}$ be the supremum of the $s+1$ such that A_s occurs in this boolean combination.

We define $f(n,i)$ to be the maximum of the $p_{n,i,j}$ and n. Let $F(0) := 0$, and let $F(k+1) := f(F(k)+1,k)$. Clearly F is recursive and strictly increasing. Let $\mathcal{X}$ be the range of F. Clearly, $\mathcal{X}$ is recursive.

Let W be any consistent Jan, $\mathcal{X}$-theory. Suppose we would have $K : W \rhd U$. Let the underlying translation of K be τ_{n^*}.

Clearly, there is a j^* such that $W + \mathsf{C}_{F(n^*)+1,j^*}$ is consistent. (This is a non-constructive step.)

By construction, there is a ϕ with $U \vdash \phi$ and $\mathsf{Jan} + \mathsf{C}_{F(n^*)+1,j^*} \nvdash \phi^K$, such that $\mathsf{Jan} \vdash \phi^K \leftrightarrow \chi$, where χ is a boolean combination of A_s, where $s < F(n^*+1)$. We note that none of the A_s with $F(n^*) < s < F(n^*+1)$ occurs in the axiomatization of W. So, there is a $\mathsf{C}_{F(n^*+1),p}$ that extends $\mathsf{C}_{F(n^*)+1,j^*}$ such that $\mathsf{Jan} + \mathsf{C}_{F(n^*+1),p} \vdash \neg\phi^K$. On the other hand, $\mathsf{C}_{F(n^*+1),p}$ is clearly consistent with W, by the mutual independence of the A_ℓ. A contradiction. □

It now follows:

THEOREM 4.6
For every essentially undecidable RE theory U, there is an essentially undecidable RE theory W such that $W \not\rhd U$. We can find an index for W effectively from an index of U.

PROOF. Suppose U is an essentially undecidable RE theory. Let $\mathcal{X}$ be the recursive set promised for U by Theorem 4.5. Let $\mathcal{Y}, \mathcal{Z}$ be a pair of recursively inseparable RE sets that are subsets of $\mathcal{X}$. We define $W := \mathsf{Jan} + \{\mathsf{A}_n \mid n \in \mathcal{Y}\} + \{\neg\,\mathsf{A}_n \mid n \in \mathcal{Z}\}$. Clearly, W is consistent. Moreover, since any decidable consistent extension of W would give rise to a recursive set separating $\mathcal{Y}$ and $\mathcal{Z}$, we find that W is essentially undecidable. By Theorem 4.5, we have $W \not\rhd U$. □

Clearly, Theorem 4.6 gives us Lemma 1.2. This concludes our first proof of Theorem 1.1.

REMARK 4.7
It is easily seen that we can give the theory W of Theorem 4.5 many extra properties. For example, it can be Turing persistent and at the same time of any given non-zero RE degree. See the next section for these notions.

4.3 A result by Shoenfield

We present some basic ideas from Shoenfield's paper [11]. We will use Shoenfield's result in our second proof of Lemma 1.2 that will be presented in Subsection 4.4.

We first need a purely recursion theoretic result. We write $\leq_{\mathsf{T}}$ for Turing reducibility. Our proof is just a minor variation of Shoenfield's proof.

THEOREM 4.8 (Shoenfield).
Let $\mathcal{A}$ be any RE set with index a. Then, we can effectively find RE-indices b of a set $\mathcal{B}$ and c of a set $\mathcal{C}$ from a, such that we have

i. $\mathcal{B} \leq_{\mathsf{T}} \mathcal{A}$ and $\mathcal{C} \leq_{\mathsf{T}} \mathcal{A}$;
ii. $\mathcal{B} \cap \mathcal{C} = \emptyset$;
iii. suppose $\mathcal{D}$ is an RE set that separates $\mathcal{B}$ and $\mathcal{C}$, i.e. $\mathcal{B} \subseteq \mathcal{D}$ and $\mathcal{C} \cap \mathcal{D} = \emptyset$, then $\mathcal{A} \leq_{\mathsf{T}} \mathcal{D}$.

We represent $x \in \mathcal{A}$ by the formula $\exists y\, \mathsf{T}_1(a,x,y)$. We assume that the computation y is unique when it exists. We write, e.g. $(x)_0 \in \mathcal{A}$ for $\exists y\, \mathsf{T}_1(a,(x)_0,y)$. We treat other indices similarly. The notation $\exists x\, \phi \leq \exists y\, \psi$ means $\exists x\, (\phi \wedge \forall y < x \neg \psi)$ and $\exists x\, \phi < \exists y\, \psi$ means $\exists x\, (\phi \wedge \forall y \leq x \neg \psi)$.

PROOF. We define

- $x \in \mathsf{Z}$ iff $\exists z\, \mathsf{T}_1((x)_1, x, z)$;
- $x \in \mathcal{B}$ iff $((x)_0 \in \mathcal{A}) < (x \in \mathsf{Z})$;
- $x \in \mathcal{B}^{\perp}$ iff $(x \in \mathsf{Z}) \leq ((x)_0 \in \mathcal{A})$;
- $x \in \mathcal{C}$ iff $(x)_0 \in \mathcal{A} \wedge x \in \mathcal{B}^{\perp}$.

Claims (i) and (ii) are trivial.[3]

Consider any $\mathcal{D}$ with index d. We note that $\langle w, d\rangle \in \mathcal{D}$ iff $\langle w, d\rangle \in \mathsf{Z}$. Suppose $\mathcal{D}$ separates $\mathcal{B}$ and $\mathcal{C}$.

We first show $w \in \mathcal{A}$ iff $\langle w, d\rangle \in \mathcal{B}$. From right to left is immediate. Suppose $w \in \mathcal{A}$. Then either $\langle w, d\rangle \in \mathcal{B}$ or $\langle w, d\rangle \in \mathcal{C}$. In case $\langle w, d\rangle \in \mathcal{C}$, we find $\langle w, d\rangle \in \mathsf{Z}$, by the definition of $\mathcal{B}^{\perp}$. Hence, $\langle w, d\rangle \in \mathcal{D}$. *Quod non*, since $\mathcal{C}$ and $\mathcal{D}$ are disjoint. So, $\langle w, d\rangle \in \mathcal{B}$.

In case $\langle w, d\rangle \in \mathcal{D}$, we have $\langle w, d\rangle \in \mathsf{Z}$. So, we can effectively determine whether $\langle w, d\rangle \in \mathcal{B}$ and, thus, whether $w \in \mathcal{A}$. In case $\langle w, d\rangle \notin \mathcal{D}$, we have $\langle w, d\rangle \notin \mathcal{B}$, and, hence, $w \notin \mathcal{A}$. ☐

Let us say that an RE theory U is *Turing-persistent* iff U is consistent and, whenever $U \subseteq V$, where V is RE and consistent, we have $U \leq_{\mathsf{T}} V$. It is easy to see that the Turing-persistence of U implies: if $U \lhd V$, then $U \leq_{\mathsf{T}} V$, whenever V is consistent. We note that, if U is Turing persistent and undecidable, then it is essentially undecidable.

Consider an RE set $\mathcal{A}$ with index a. Let $\mathcal{B}$ and $\mathcal{C}$ be the sets constructed above. Let $\mathsf{sh}(a) := \mathsf{Jan} + \{\mathsf{A}_n \mid n \in \mathcal{B}\} + \{\neg\, \mathsf{A}_m \mid m \in \mathcal{C}\}$. We have the following.

THEOREM 4.9 (Shoenfield).
The theory $\mathsf{sh}(a)$ has the same Turing degree as $\mathcal{A}$. Moreover, the theory is Turing persistent. It follows that, if $\mathcal{A}$ is undecidable, then $\mathsf{sh}(a)$ is essentially undecidable. Thus, there is an essentially undecidable theory in every RE, non-recursive Turing degree.

PROOF. For any consistent RE extension V of $\mathsf{sh}(a)$, the set $\{i \mid V \vdash \mathsf{A}_i\}$ clearly is an RE set separating $\mathcal{B}$ and $\mathcal{C}$ and hence $\mathcal{A}$ is Turing reducible to V. It is obvious that $\mathsf{sh}(a)$ is consistent. Thus, to finish the proof we only need to show that $\mathsf{sh}(a)$ is Turing reducible to $\mathcal{A}$.

We describe an $\mathcal{A}$-recursive procedure of checking whether a given sentence φ of the language of Jan is a theorem of $\mathsf{sh}(a)$. We can effectively find a boolean combination φ^* of A_i that is equivalent over Jan to ϕ. Let the set of indices of A_i occurring in ϕ^* be $\mathcal{I}$. Also, we can effectively find, using $\mathcal{A}$ as an oracle, the conjunction ψ of the A_j with $j \in \mathcal{B} \cap \mathcal{I}$ and the $\neg\, \mathsf{A}_s$ with $s \in \mathcal{C} \cap \mathcal{I}$. Using the mutual independence of the A_n, it is easy to see that $\mathsf{sh}(a) \vdash \phi$ iff $\mathsf{Jan} \vdash \psi \to \phi^*$ and $\mathsf{Jan} \vdash \psi \to \phi^*$ iff $\psi \to \phi^*$ is a propositional tautology. Whether $\psi \to \phi^*$ is a tautology can be checked with a truth table. ☐

REMARK 4.10
The result that there is an essentially undecidable theory in every RE, non-recursive Turing degree is due to Shoenfield. See [11]. This result was improved by Hanf. He proves that there is an essentially undecidable *finitely axiomatized* theory in every RE, non-recursive Turing degree. See [4]. By a

[3]Note that $\mathcal{B}^{\perp}$ need not be Turing reducible to $\mathcal{A}$.

simple adaptation of the argument, Hanf's results imply that there is a Turing persistent finitely axiomatized theory in every RE, non-recursive Turing degree.

EXAMPLE 4.11
We provide an example of an essentially undecidable theory that is not Turing persistent. Consider RE Turing degrees d and e with $0 < d < e$. Let $\mathcal{X}$ be an RE set of degree d, and let $\mathcal{W}$ be an RE set of degree e. We take $\mathcal{Y}$ and $\mathcal{Z}$ to be the recursively inseparable RE sets provided by Shoenfield's result for $\mathcal{X}$. Let

- $U_0 := \mathsf{Jan} + \{\mathsf{A}_{2i} \mid i \in \mathcal{Y}\} + \{\neg\,\mathsf{A}_{2j} \mid j \in \mathcal{Z}\}$;
- $U_1 := U_0 + \{\mathsf{A}_{2k+1} \mid k \in \mathcal{W}\}$;
- $U_2 := U_1 + \{\mathsf{A}_{2k+1} \mid k \in \omega\}$.

Clearly, U_0 and U_2 are Turing persistent and have Turing degree d. The theory U_1 had degree e and, hence, cannot be Turing persistent. Yet, it is clearly essentially undecidable.

So our example shows that there are essentially undecidable theories that are not Turing persistent. Moreover, a Turing persistent theory can have a non-Turing persistent RE extension and a non-Turing persistent recursively inseparable theory can have a Turing persistent RE extension.[4]

OPEN QUESTION 1
A theory is *essentially* Turing-persistent if all its consistent RE extensions are Turing-persistent. Clearly, any Turing-persistent theory in degree $0'$ is essentially Turing-persistent. Are there essentially Turing-persistent theories of other degrees?

4.4 Second proof of Theorem 1.1

In this subsection, we prove our main result for RE theories. We need a basic fact from recursion theory.

Let Rec be the set of indices of recursive sets. We have the following theorem.

THEOREM 4.12 (Rogers, Mostowski).
Rec is complete Σ^0_3.

See [9, Chapter 14, Theorem XVI] or [12, Corollary 4.3.6]. We have the following theorem that is given as exercise 4.3.14 of [12, Page 91]. Let Rsep be the set of pairs of indices of recursively separable RE theories.

THEOREM 4.13
Rsep is complete Σ^0_3.

This follows immediately from Theorem 4.8, since that gives a reduction of Rec to Rsep. We now give to variants of our second proof of Theorem 1.1.

[4] A theory T is *recursively inseparable* if the set of its theorems and the set of its refutable sentences, i.e. $\{\phi \mid T \vdash \neg\phi\}$, are recursively inseparable.

PROOF OF THEOREM 1.1, FIRST VARIANT.It is sufficient to prove Lemma 1.2. Suppose that there is an essentially undecidable RE theory $U^\star$ that is the interpretability minimum. So, we have

$$\begin{aligned} a \notin \mathsf{Rec} \quad &\text{iff} \quad \mathsf{sh}(a) \text{ is essentially undecidable} \\ &\text{iff} \quad \mathsf{sh}(a) \rhd U^\star. \end{aligned}$$

Since, interpretability between RE theories is Σ^0_3, it would follow that Rec is Π^0_3.[5] *Quod non*, by Theorem 4.12. □

We give the second variant of the proof.

PROOF OF THEOREM 1.1, SECOND VARIANT.It is sufficient to prove Lemma 1.2. Suppose that there is an essentially undecidable RE theory $U^\star$ that is the interpretability minimum. Let $\langle a, b\rangle$ be a pair of indices of RE sets, and let $\mathcal{X}$ be the set defined by a and $\mathcal{Y}$ be the set defined by b.

We define

$$\mathsf{so}(a, b) := \mathsf{Jan} + \{\mathsf{A}_i \mid i \in \mathcal{X}\} + \{\neg\, \mathsf{A}_j \mid j \in \mathcal{Y}\}.$$

We have

$$\begin{aligned} \langle a, b\rangle \notin \mathsf{Rsep} \quad &\text{iff} \quad \mathsf{so}(a, b) \text{ is essentially undecidable or inconsistent} \\ &\text{iff} \quad \mathsf{so}(a, b) \rhd U^\star. \end{aligned}$$

Since, interpretability between RE theories is Σ^0_3, it would follow that Rsep is Π^0_3. *Quod non*, by Theorem 4.13. □

We note that our result is insensitive for the precise notion of interpretability used. It could even work for wider notions, like forcing interpretability. However, we did not explore this.

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A More on Propositional-Logic-Like Theories

In this appendix, we provide some additional information about PLL theories.

A.1 Characterisation of Propositional-Logic-Like Theories

A theory U is *f-essentially incomplete* if every consistent *finite* extension of U is incomplete.[6]

LEMMA A.1
Suppose U is f-essentially incomplete and suppose $\phi_0, \ldots, \phi_{n-1}$ are each consistent with U. Then, there is a sentence ψ independent of each of the $U + \phi_i$.

PROOF. If ϕ_i and ϕ_j, for $i \neq j$, are jointly consistent over U, we may replace them both by the single $\phi_i \wedge \phi_j$. Thus, we may assume that all pairs ϕ_i, ϕ_j, for $i \neq j$, are jointly inconsistent over U. Let ψ_i be independent of $U + \phi_i$. We take $\psi := \bigvee_{i<n}(\phi_i \wedge \psi_i)$. □

THEOREM A.2
U is decidable and f-essentially incomplete iff U is PLL.

The theorem is a direct consequence of Theorem 1.1 of [4]. We provide a quick proof.

PROOF. The right-to-left direction is immediate by Theorem 4.1. We prove left-to-right. Suppose U is decidable and f-essentially incomplete. We fix an enumeration of the U-sentences. Suppose we already constructed a sequence $\alpha_0, \ldots, \alpha_{n-1}$, where the α_i are mutually independen (including the case $n = 0$). For $\sigma \in \mathsf{bseq}_n$, we define $\alpha_\sigma := \bigwedge_{i<n} \neg^{\sigma(i)} \alpha_i$. We take α_n to be the first sentence

[6] J.P. Jones, in his paper [6], uses *effectively nonfinitely completable* for an effective version of what we call *f-essentially incomplete*.

in our enumeration that is independent of each of the $U + \alpha_\sigma$. By the decidability of U, we can effectively find α_n.

Clearly, all α_j will be independent of each other. Consider any $F : \omega \to \{0,1\}$. Suppose ϕ were independent of $U + \{\neg^{F(n)} \alpha_j \mid j \in \omega\}$. Then, ϕ would have been added at some stage to the α_j. *A contradiction.* □

A.2 A Successor Theory

Let Succ° be the following theory in the language of arithmetic: zero is not a successor; successor is injective; every number is either zero or a successor; for each $n > 0$, we have an axiom that says: there is at most one cycle of size n.

THEOREM A.3
Succ° is PLL.

PROOF. The countable models of Succ° have precisely the following forms. We have ω. Then, we may have, for each n, one cycle of that size or not. Finally, we can have κ copies of $\mathbb{Z}$, for $0 \leq \kappa \leq \aleph_0$.

Consider any countable model $\mathcal{M}$ of Succ°. We extend the language with new constants c_i. Let W be the theory given by the sentences true in $\mathcal{M}$ plus the following axioms:

- $\vdash \mathsf{c}_i \neq \mathsf{c}_j$, for $i \neq j$.
- $\vdash \exists x_0 \ldots \exists x_{n+1} \, (\bigwedge_{i<n+1} \mathsf{S}x_i = x_{i+1} \wedge x_0 \neq \mathsf{c}_i \wedge x_{n+1} = \mathsf{c}_j)$, for all i, j, n (including the case where $i = j$).

These axioms say that the c_i are in disjoint copies of $\mathbb{Z}$. By compactness, W is consistent. Let $\mathcal{N}$ be a model of W. We see that $\mathcal{N}$ is, w.r.t. the original language, elementarily equivalent to $\mathcal{M}$ and that $\mathcal{N}$ has countably infinitely many copies of $\mathbb{Z}$. This means that $\mathcal{N}$ is uniquely determined by the finite cycles in $\mathcal{M}$.

We take as α_n the statement that there is a cycle of size $n + 1$. For any $\mathcal{X} \subseteq \omega$, we find that all models of $\mathsf{Succ}^\circ_{\mathcal{X}}$ are elementarily equivalent. So, $\mathsf{Succ}^\circ_{\mathcal{X}}$ is complete. □

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