The Allure of Infinitesimals: Sergio Albeverio and Nonstandard Analysis

Tom Lindstrøm

Department of Mathematics, University of Oslo
PO Box 1053 Blindern, NO-0316 Oslo, Norway
lindstro@math.uio.no

Abstract. I give a survey of Sergio Albeverio's work in nonstandard analysis, covering applications to operator theory, stochastic analysis, Dirichlet forms, quantum mechanics, and quantum field theory, and making an attempt at putting his contributions into the historical context of what has happened in the field before and since.

Keywords: Nonstandard analysis, stochastic analysis, mathematical physics

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This paper is dedicated to the memory of Raphael Høegh-Krohn (1938-1988) and Jens Erik Fenstad (1935-2020).

Of the more than one thousand items in Sergio Albeverio's bibliography, I have found somewhere between twenty-five and thirty that deal with nonstandard analysis in an essential way. That is not a large percentage, but the picture changes when one realizes that two of these items are books containing substantial amounts of independent research, and it changes even more when one takes the contributions of Sergio's students and collaborators into account.

The purpose of this paper is to trace Sergio's contributions to nonstandard analysis in a broad sense, covering not only his own books and papers, but also those of his students and many associates. As nonstandard analysis is not so much a subject as a method—and a method that can be applied in all areas of mathematics that touches on the infinite—it has been a challenge to find the best structure for the exposition: Should it be organized chronologically or thematically? Having tried both, I have settled for a presentation that is primarily chronological, but where I usually—but not always—allow myself to follow a theme to the end once it is started. As Sergio has often been concentrating on different subjects in different periods, this approach seems to work reasonably well, and it has the advantage of keeping a sense of history without interrupting the thematic developments too much.

I have had to restrict myself. I do not discuss Sergio's many expository articles (such as [1]-[9]) although they are often very instructive, and I have made no attempt to connect the nonstandard papers to the rest of Sergio's production, although there are lots of overlaps, especially in areas such as point interactions, Dirichlet forms, and quantum fields. I have also had to prioritize, and I may clearly be accused of giving preferential treatment to a book [19] that I have myself contributed to, but my defense is that this book laid the foundations for most of what followed later, and that it gives a unique impression of Sergio Albeverio's and Raphael Høegh-Krohn's vision of mathematical physics in the 1980s.

The topics we shall look at span from operator theory to stochastic analysis, with Dirichlet forms as a natural meeting point. And as usual with Sergio, physics is always present – if not center stage, so at least lurking in the wings.

1 Nonstandard Analysis

Although there isn't room in this paper for a systematic introduction to nonstandard analysis, I should say a few words about the subject for those who are not familiar with it ([86] contains more or less what I would have said if I had ten extra pages, and [89] is a full introduction along the same lines).

The basic object of nonstandard analysis is a set of nonstandard reals (or *hyperreals*) ${}^*\mathbb{R}$ which extends the ordinary real line $\mathbb{R}$ by adding infinitely large and infinitely small (*infinitesimal*) numbers.[1] The extension ${}^*\mathbb{R}$ is an ordered field, and hence we can calculate with numbers in ${}^*\mathbb{R}$ and compare their size just the way we are used to from $\mathbb{R}$.

Figure 1 shows the structure of ${}^*\mathbb{R}$: We have infinite negative numbers, finite numbers, and infinite positive numbers. Each finite number x is of the form $x = a + \epsilon$ where $a \in \mathbb{R}$ and ϵ is infinitesimal. We call a the *standard part* of x and write $a = {}^\circ x$ or $a = \mathrm{st}(x)$. We shall also write $x \approx y$ to denote that x and y are two infinitely close hyperreals, i.e. that x and y differ by an infinitesimal.

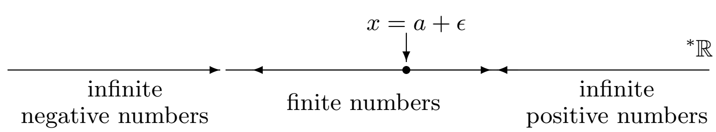

Fig. 1. The hyperfinite line ${}^*\mathbb{R}$.

What separates the hyperreal numbers from most other extensions of $\mathbb{R}$, is that sets and functions also extend: Any set $A \subseteq \mathbb{R}$ has a canonical extension to a subset ${}^*A$ of ${}^*\mathbb{R}$, and every function $f\colon \mathbb{R} \to \mathbb{R}$ has a canonical extension ${}^*f\colon {}^*\mathbb{R} \to {}^*\mathbb{R}$ such that ${}^*f(x) = f(x)$ for all $x \in \mathbb{R}$. These extensions preserve the defining properties of the original objects, but interpreted in a nonstandard context; e.g. will ${}^*(a, b)$ consist of all *nonstandard* numbers between a and b, and the nonstandard extension of the exponential function will satisfy ${}^*\exp(x+y) =$

[1] Actually there isn't just one set of hyperreals, but infinitely many, but for the purpose of this paper it doesn't matter much which one we choose as long as it is sufficiently rich (in technical terminology, it should be $\aleph_1$-saturated).

${}^*\exp(x){}^*\exp(y)$ for all $x, y \in {}^*\mathbb{R}$. As there is usually no danger of confusion, I shall drop the asterisk on the nonstandard extension of a function and write f instead of ${}^*f$.

The nonstandard sets and functions that arise from ordinary sets and functions in this way are (rather confusingly) referred to a *standard* objects in nonstandard parlance. They are what make nonstandard calculus possible, but they are not sufficient for more serious applications of the theory; for this, we need to extend the theory to so-called *internal* sets and functions. It would take me to far afield to give a good description of these sets and functions here; let me only say that they are the sets and functions that can be handled by the theory in a good way, just as the measurable sets and functions are the sets and functions that be handled by measure theory in a good way.

An interesting class of internal sets are the *hyperfinite sets*—these are infinite sets with most of the formal properties of finite sets. To define them, one first observes that the set ${}^*\mathbb{N}$ of nonstandard natural numbers consists of the ordinary natural numbers plus infinitely large elements. If $N \in {}^*\mathbb{N}$ is infinite, the set $A = \{1, 2, 3, \ldots, N\}$ is a hyperfinite set with internal cardinality N, and any other set B for which there is an *internal* bijection $\phi\colon A \to B$, is also a hyperfinite set of internal cardinality N. Hyperfinite sets occur naturally; e.g. is $T = \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$ a hyperfinite set with N elements that can serve as a "hyperdiscrete" timeline.

As already mentioned, hyperfinite sets have many of the combinatorial properties of finite sets. If P is an internal function and Ω is a hyperfinite set, we may, e.g., sum P over Ω—i.e. there is a canonical way to define the sum $\sum_{\omega\in\Omega} P(\omega)$. If this sum equals 1, we may define an internal probability measure on Ω by putting $P(A) = \sum_{\omega\in A} P(\omega)$ for all internal $A \subseteq \Omega$. We can then "standardize" P by taking standard parts: ${}^\circ P(A) = \mathrm{st}(P(A))$.

The internal sets form an algebra, but not a σ-algebra, and hence ${}^\circ P$ is not a measure. It turns out, however, that the conditions of Carathéodory's extension theorem are trivially satisfied, and hence ${}^\circ P$ can be extended to a (complete) measure P_L. This measure is known as the *Loeb measure* of P (the Loeb measure construction was introduced by Peter Loeb in [96] and is much more general than what I have described here).

For a glimpse of what this can be used for, let $T = \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$ be the hyperfinite timeline introduced above, and let Ω be the set of all *internal* functions $\omega\colon T \to \{-1, 1\}$ (think of ω as a sequence of coin tosses, one for each $t \in T$). Then Ω is a hyperfinite set with internal cardinality 2^N, and we let P be the internal counting measure on Ω. Define a process $B\colon \Omega \times T \to {}^*\mathbb{R}$ by

$$B(\omega, t) = \sum_{s<t} \omega(s)\sqrt{\Delta t}, \quad \text{where } \Delta t = \frac{1}{N}\ .$$

Robert M. Anderson [26] showed that the standard part b of B (properly defined) is a Brownian motion on the Loeb space (Ω, P_L), and that stochastic integrals with respect to b can be recovered from hyperfinite sums

$$\sum_{s<t} X(\omega, s)\,\Delta B(\omega, s)\ ,$$

where $\Delta B(\omega, s) = B(\omega, s+\Delta t) - B(\omega, s)$ is the forward increment of B at time s. We shall return to Anderson's random walk again and again throughout this paper.

There is one more use of hyperfinite sets that I need to mention. The notion of finite dimensional vector spaces over $\mathbb{R}$ extends to a notion of hyperfinite dimensional vector spaces over ${}^*\mathbb{R}$. To get one, we may start with a hyperfinite set of basis elements $\{\mathbf{e}_n\}_{n=1}^N$ for an infinitely large $N \in {}^*\mathbb{N}$ and form all hyperfinite sums $\sum_{n=1}^N x_n\mathbf{e}_n$, $x_n \in {}^*\mathbb{R}$. A usual way to obtain such a structure, is to start with a standard Hilbert space with basis $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ and look at its nonstandard extension ${}^*H$ (there are nonstandard extensions of everything, not just ${}^*\mathbb{R}$) and cut off its basis $\{{}^*\mathbf{e}_n\}_{n\in{}^*\mathbb{N}}$ at some infinite N. In this way we get an *outer* approximation of H by something that is formally finite dimensional, and where all the techniques of linear algebra apply.

2 Sergio Albeverio's First Contribution to Nonstandard Analysis

Nonstandard analysis was invented by Abraham Robinson around 1960 [106]. He was not the first to construct an ordered field extension of the reals (see e.g. Levi-Civita [82] and Hahn [61]), but as earlier constructions did not have a way to treat transcendental functions, they could not be used for serious infinitesimal calculus in the spirit of Newton and Leibniz. The first book-length treatment of nonstandard analysis was a set of lecture notes [97] by W.A.J. Luxemburg in 1962, and the first edition of Robinson's own book [107] followed in 1966. The first breakthrough for nonstandard analysis as a research tool also came in 1966 when Bernstein and Robinson [35] solved the invariant subspace problem for polynomially compact operators.

In the summer of 1976, Edward Nelson delivered an AMS address [98] on a new axiomatic approach to nonstandard analysis called Internal Set Theory (IST). At the time, Sergio was in Oslo working with Raphael Høegh-Krohn. They both knew Nelson from Princeton and were interested in his work, but not so much for the new framework as for the applications to probability theory and mathematical physics. There was one particular example that struck their imagination—Nelson's nonstandard treatment of a problem that had previously been discussed by Berezin and Faddeev [34] and Friedman [60]: Describe all Schrödinger operators in $\mathbb{R}^d$ generated by a singular potential of the form $\lambda\delta$, where δ is the Dirac δ-function at the origin. There are no nontrivial examples for $d > 3$, but they exist for $d \leq 3$, and Nelson's thought was to use nonstandard analysis to give a description of these potentials for $d = 3$. The idea (a nonstandard version of Friedman's approach) was simple and natural: Choose an infinitesimal $\epsilon > 0$ and let $V(x) = \frac{3}{4\pi\epsilon^2}\chi_\epsilon(x)$, where χ_ϵ is the indicator function of the ball around the origin with radius ϵ. Consider operators of the form $H(\alpha) = -\Delta + \alpha V$ for standard α. Nelson finds that $H(\alpha)$ is infinitely close to the unperturbed operator $-\Delta$ except when $\alpha = -\frac{\pi^3}{3}(2n+1)^2$ for an integer n. For such α (the standard part of) $H(\alpha)$ is a nontrivial perturbation of $-\Delta$.

Sergio and Raphael soon realized that Nelson's parametrization was too coarse to give all singular perturbations of $-\Delta$. Their heuristic calculations showed that they could get *all* perturbations by instead using the parametrization $H(\lambda) = -\Delta + \lambda\chi_\epsilon$, where λ runs through all of ${}^*\mathbb{R}$, and that the nontrivial perturbations would occur when λ was of the form

$$\lambda(k,\alpha,\epsilon) = -\left(k + \frac{1}{2}\right)^2 \frac{\pi^2}{\epsilon^2} + \frac{2}{\epsilon}\alpha + \beta \ ,$$

where k is a standard integer and α and β are two (standard) real numbers (a quick calculation will show you that Nelson's result corresponds to the situation where $\alpha = \beta = 0$). Moreover, their calculations indicated that which perturbation of $-\Delta$ they got, depended only on α and not on k and β (and hence all Nelson's perturbations are the same as they all correspond to $\alpha = 0$).

The results seemed interesting enough to publish, but Sergio and Raphael needed assistance in turning their heuristic calculations into solid nonstandard analysis, and sought the help of the logician Jens Erik Fenstad. The collaboration was successful and resulted in a joint paper [18], which in addition to treating singular perturbations from two different perspectives also contained a section on singular Sturm-Liouville problems.

The activity in Oslo was seminar–driven, and among the participants in the seminar were Bent Birkeland, Dag Normann, and myself as a beginning graduate student. Bent wrote a paper [31] which treated the singular Sturm-Liouville problem as a problem about hyperfinite difference equations, but after a while the interest of the seminar turned to the new developments in nonstandard probability theory. We studied Anderson's paper on Brownian motion and Itô integration [26], and then turned to the preprint version of Keisler's monograph [74] on infinitesimal stochastic analysis. A natural question at the time was how to extend Anderson's and Keisler's work on diffusions to martingales, and I wrote a thesis [83] on stochastic integration with respect to martingales (many of the results—and more—were discovered independently by Hoover and Perkins [70]).

3 "Nonstandard Methods in Stochastic Analysis and Mathematical Physics"

At some point (I am not quite sure when) Sergio, Raphael, and Jens Erik decided to write a book on nonstandard analysis. Originally, it was meant as a brief introduction with just the basic theory and a few striking applications, but it soon outgrew the original plan. After I had finished my degree and secured a postdoc position with Keisler in Wisconsin, I was invited to join the project.

Over the following years we were often asked when the book would be finished, and we would always answer “before Christmas” and be careful not to specify which Christmas we were talking about. The book [19] eventually appeared in 1986, some six or seven years after the work had started.

Nonstandard Methods in Stochastic Analysis and Mathematical Physics consists of two parts and seven chapters. The first part (called “Basic Theory” and containing the first three chapters) corresponds to some extent to the original plan; it contains the basic theory of nonstandard analysis plus some selected applications: In Chapter 1, the “chasse au canard”, a study of infinitesimal perturbations of dynamical systems due to E. Benoît, J.L. Callot, F. and M. Diener [54], [32], [36], [33]; in Chapter 2, a nonstandard proof of the spectral theorem for compact operators (already treated in Robinson’s book [107]); and in Chapter 3, Anderson’s nonstandard construction of Brownian motion and a few applications of Loeb measures to limit measures and measure extensions, some new, but most taken from [84].

The second part of the book is called “Selected Applications” and as it contains a mixture of original research and reports on (what was then) very recent research, I am going to spend some time on each chapter, especially as I also aim to give an account of subsequent developments where they fit in.

3.1 Chapter 4: Stochastic Analysis

This chapter begins with a quick treatment of Anderson’s version of the Itô integral. One of the big questions at the time was how to extend Anderson’s theory from Brownian motion to martingales, and sections 4.2-4.4 reports on the results obtained by Lindstrøm [83] and Hoover and Perkins [70]. The key to the theory is the close relationship between an internal martingale and its quadratic variation, which can be used to study both path properties and stochastic integrals (see Stroyan and Bayod’s book [112] for another exposition of the theory published about the same time).

Section 4.5 deals with stochastic differential equations, and reports on work by H.J. Keisler [74] and his student S. Kosciuk [81] with some minor simplifications. The basic idea is simple. Translated into a nonstandard setting, the stochastic differential equation

$$\mathrm{d}x(t) = f(t, x(t))\,\mathrm{d}t + g(t, x(t))\,\mathrm{d}b(t) \tag{1}$$

becomes a stochastic difference equation

$$\Delta X(t) = f(t, X(t))\,\Delta t + g(t, B(t))\,\Delta B(t) \tag{2}$$

which, given an initial condition, obviously has an inductively defined solution. The question is when a solution of (2) can be turned into a solution of (1)? As long as the coefficients f and g are continuous in the space variable, it is not very hard to see how this can be done. The result can be extended to jointly measurable coefficients provided $\det g$ is bounded away from zero, but this is more complicated and relies on a deep inequality by Krylov. Kosciuk [81] showed that it is even possible to obtain a solution when the diffusion degenerates as long as the coefficients are continuous on the set of degeneracies, but as later pointed out in [88], the solutions tend to be so nonunique that one loses control.

Section 4.6 deals with stochastic control theory and is based on work by Nigel Cutland (see his two survey papers [39], [41] for more information), although the presentation is modified from a pathwise approach to an approach based on a nonstandard version of Girsanov's theorem. Cutland's fundamental insight was that relaxed (i.e. measure-valued) controls occur naturally as the standard part of wildly oscillating nonstandard controls, and that this can be used to obtain new existence results for optimal controls of partially observed systems.

Section 4.7 takes a quick look at Brownian motion and stochastic integration in Hilbert spaces, based on my paper [85]. The theory starts with a hyperfinite dimensional version of Anderson's random walk, and obtains a standard version of the process by taking standard parts in a norm weaker than the Hilbert space norm. Just as in the finite dimensional theory, stochastic integrals of the standard process can be obtained from hyperfinite sums.

Although infinite dimensional stochastic analysis doesn't play a big part in [19], it has later become a central area of nonstandard research. In a series of papers (neatly summed up in their book [38], see also [37], [49], and [50] for later developments), Marek Capiński and Nigel Cutland developed a nonstandard approach to stochastic fluid dynamics that is formulated in terms of Anderson-like random walks on hyperfinite dimensional spaces. In another direction, Horst Osswald produced a series of papers on a nonstandard approach to Malliavin calculus, culminating in his book [103]. For other papers on nonstandard Malliavin calculus and related topics, see [47], [51], [52], [53], [93], [94], and Chapter 3 of [48].

The last section of Chapter 4 deals with white noise and Lévy Brownian motion, and is based on the Diplomarbeit [109] of Sergio's student Andreas Stoll. As Lévy Brownian motion is a multi-parameter process, we now have to replace our hyperfinite timeline by a hyperfinite lattice in ${}^*\mathbb{R}^d$:

$$\Gamma = \{(k_1\Delta t, k_2\Delta t, \ldots, k_d\Delta t) \ : \ k_i \in {}^*\mathbb{Z} \text{ and } |k_i| \leq N\} ,$$

where Δt is infinitesimal and N is so large that $N\Delta t$ is infinite. The sample space Ω consists of all internal maps $\omega : \Gamma \to \{-1, 1\}$, and the internal probability measure P is simply the normalized counting measure on Ω. If A is an internal subset of Γ, we define

$$\chi(A) = \sum_{a\in A} \omega(a)\, \Delta t^{d/2} .$$

Obviously, χ is our nonstandard representation of white noise. Stoll proved that the standard part of the random field λ given by

$$\lambda(x) - \lambda(y) = k_d \sum_{a\in A} \left(\frac{1}{\|x-a\|^{(d-1)/2}} - \frac{1}{\|y-a\|^{(d-1)/2}} \right) \chi(\{a\})$$

is a Lévy Brownian motion for the right choice of the scaling parameter k_d. He also used the representation to give a Donsker type invariance principle for Lévy Brownian motion.

As was pointed out in [92], Stoll's construction is easily generalized to fractional Brownian fields; just replace Stoll's formula above by

$$\lambda(x) - \lambda(y) = k_{d,p} \sum_{a \in A} \left(\frac{1}{\|x-a\|^{(d-p)/2}} - \frac{1}{\|y-a\|^{(d-p)/2}} \right) \chi(\{a\}) \ ,$$

where $p = 2H$ is twice the Hurst exponent.

3.2 Chapter 5: Hyperfinite Dirichlet Forms and Markov Processes

This chapter consists entirely of original research that has not been published elsewhere. The inspiration for the chapter was twofold—on the one hand the deep study of (standard) Dirichlet forms and their applications that Sergio and Raphael had conducted over the previous decade, and on the other hand the need to build a solid foundation for the nonstandard study of singular perturbations (we shall take a closer look at the latter when we get to Chapter 6).

The first two sections lay the foundations for the rest by constructing a theory for bilinear forms on hyperfinite dimensional inner product spaces and describing their connection to bilinear forms on Hilbert spaces. Starting with a hyperfinite dimensional linear space H with an inner product $\langle \cdot, \cdot \rangle$, we can construct a standard Hilbert space in the following way: Let $\mathrm{Fin}(H)$ consist of the elements in H with finite norm, and define an equivalence relation on $\mathrm{Fin}(H)$ by

$$u \sim v \quad \Longleftrightarrow \quad \|u - v\| \approx 0 \ .$$

If we let $^\circ u$ denote the equivalence class of u, we may introduce an inner product on $^\circ H := \mathrm{Fin}(H)/\sim$ by

$$\langle ^\circ u, ^\circ v \rangle = {}^\circ\langle u, v \rangle.$$

It is not hard to check that $(^\circ H, \langle \cdot, \cdot \rangle)$ is a Hilbert space called the *nonstandard hull* of the hyperfinite space H.

The question we want to look at is this: Given an internal, nonnegative definite, symmetric, bilinear form $\mathcal{E}(\cdot, \cdot)$ on H, can we define a corresponding form E on H? It obviously suffices to define the symmetric terms $E(u, u)$ as we can get the rest by polarization.

If $\mathcal{E}$ is bounded in the sense that there is *standard* number K such that

$$\mathcal{E}(u, u) \leq K \|u\|^2$$

for all $u \in H$, the problem is easy. In this case, $u \approx v$ implies $\mathcal{E}(u, u) \approx \mathcal{E}(v, v)$, and we can just put $E(^\circ u, ^\circ u) = {}^\circ\mathcal{E}(u, u)$ as it doesn't matter which representative u we choose from the equivalence class $^\circ u$.

When $\mathcal{E}$ is unbounded, there are two complications. First of all, the standard part E will now be an unbounded form, and hence only partially defined. This means that we have to determine the domain $D(E)$ of E. The second complication is that since we now may have ${}^\circ\mathcal{E}(u,u) \neq {}^\circ\mathcal{E}(v,v)$ for ${}^\circ u = {}^\circ v$, it is not clear which value to choose for $E({}^\circ u, {}^\circ u)$.

There is a quick fix to these problems: Just define (recall that an element x in ${}^\circ H$ is an equivalence class of elements in H):

$$D(E) = \{x \in {}^\circ H \mid \inf\{{}^\circ\mathcal{E}(u,u) \mid u \in x\} \text{ is finite}\} \tag{3}$$

and

$$E(x,x) = \inf\{{}^\circ\mathcal{E}(u,u) \mid u \in x\}$$

for $x \in D(E)$. I shall refer to E as the *standard part* of $\mathcal{E}$. It turns out that E is always a closed form. This is both convenient and surprising as much of the work in the standard theory goes into showing that forms are closeable.

The problem with the quick fix is that we don't have any control over how the infimum is obtained. To get control, we need to take a closer look at the nonstandard form $\mathcal{E}$. Just as in linear algebra, $\mathcal{E}$ is generated by a symmetric, linear map $A\colon H \to H$ in the sense that

$$\mathcal{E}(u,v) = \langle Au, v\rangle.$$

If $\|A\|$ is the operator norm of A (when $\mathcal{E}$ is unbounded, $\|A\|$ will be an infinitely large number), we choose an infinitesimal time increment Δt so small that $\|A\|\Delta t < 1$, and use

$$T = \{0, \Delta t, 2\Delta t, \ldots\}$$

as our timeline. Put

$$Q^{\Delta t} = I - A\Delta t,$$

where I is the identity operator, and define an internal semigroup by setting $Q^t = (Q^{\Delta t})^k$ for all $t = k\Delta t \in T$. We now define the *domain* $D(\mathcal{E})$ of the nonstandard form $\mathcal{E}$ to consist of those elements $u \in H$ such that

(i) $\mathcal{E}(u,u)$ is finite
(ii) $\mathcal{E}(Q^t u, Q^t u) \approx \mathcal{E}(u,u)$ for all infinitesimal t.

The philosophy (or rationalization) behind (ii) is that Q^t is a smoothing operator, and that the elements in the domain should be so smooth that an infinitesimal amount of smoothing doesn't change them much.

Much of the key to the theory is that the elements in $D(\mathcal{E})$ are exactly the elements in H obtaining the infimum in formula (3), i.e.

$$D(\mathcal{E}) = \{u \in H \mid {}^\circ\|u\| < \infty \text{ and } {}^\circ\mathcal{E}(u,u) = E({}^\circ u, {}^\circ u)\}$$

One may now show that the standard form E can be approximated by less singular, nonstandard objects, e.g., if $G_\alpha = (A-\alpha)^{-1}$ is the resolvent of A, we get:

$$E(x,x) = -\lim_{{}^\circ\alpha \to -\infty} {}^\circ\big(\alpha^2\langle G_\alpha v, v\rangle + \alpha\langle v, v\rangle\big) \tag{4}$$

where v is any element in the equivalence class x. This formula will play a crucial part when we analyze singular perturbations of operators in the next chapter.

The first application of the theory above is in Section 5.3 where it is applied to the theory of hyperfinite Dirichlet forms and their associated Markov processes, including a study of equilibrium potentials and the proof of a nonstandard version of the Feynman-Kac formula. The Markov process generated by a hyperfinite Dirichlet form is a Markov chain with a hyperfinite state space (which may e.g. be a lattice in ${}^*\mathbb{R}^d$ with infinitesimal spacing) and a timeline with infinitesimal increments Δt. The standard part of such a Markov chain is a continuous time (standard) Markov process, and sections 5.4 and 5.5 studies the probabilistic and potential theoretic properties of these processes in finite and infinite dimension. The last section of the chapter sketches some applications to quantum mechanics and stochastic differential equations, but more in terms of illustrations than original research efforts.

Many years later, Sergio, in collaboration with Ruzong Fan and Frederik Herzberg, returned to the theory of hyperfinite Dirichlet forms, but as this resulted in another book [17], it deserves its own section (section 4.3). Except for this book and the papers it builds on, there has unfortunately not been much done with hyperfinite Dirichlet forms. I wrote a quite speculative paper [87] on connections to diffusions on manifolds and fractals, but when I got to write "serious" papers on diffusions on fractals [90], [91], I chose not to use Dirichlet forms, and the same was the case with my student S.O. Nyberg [101], [102]. This is rather ironic as many of the subsequent standard papers used Dirichlet forms.

3.3 Chapter 6: Topics in Differential Operators

This chapter starts and ends with reports of already published results, but the middle three sections consist mainly of original research. As this is a book about Sergio's contributions, I'll concentrate on the middle part, but would like to say a few words about the other two sections first.

Section 6.1 deals with singular Sturm-Liouville problems of the form

$$-Y''(x) + \mu Y'(x) = \lambda Y(x), \qquad 0 \leq x \leq 1 ,$$

where μ is a Borel measure. As mentioned in Section 2, this problem was already treated in Sergio, Jens Erik, and Raphael's first paper on nonstandard analysis [18], but in the book we instead follow the approach by Birkeland [31] who discretized the timeline to get a hyperfinite difference equation that could be treated by linear algebra (plus a lot of estimates).

Section 6.5 reports on Leif Arkeryd's nonstandard approach to the Boltzmann equation (see his own surveys [27]-[29] for more information). Using nonstandard truncation techniques, Arkeryd obtained existence and uniqueness results that was in the forefront of the research at the time.

The final section of the chapter consists of some remarks on the Feynman integral from a nonstandard perspective.

Let us now turn to the central part of the chapter, sections 6.2-6.4, which deals with singular perturbations of operators with applications to point interactions and polymer measures. We have already taken a look at point interactions in connection with [18], but the treatment in Chapter 6 of [19] is much more ambitious and aims to develop a general framework for singular perturbations of operators. The main tool is the theory of standard parts of bilinear forms described above. As this is the heart of the book, I'll go through the arguments in some detail.

To introduce the problem, consider a nonnegative self-adjoint operator A on some L^2-space (most typically $-\Delta$ on $L^2(\mathbb{R}^d, m)$) and the closed, bilinear form E obtained by closing

$$E(f, g) = \langle Af, g\rangle$$

If C is a "small" set, we may wonder whether E (and hence A) has a perturbation supported by C, i.e. a closed form $\tilde{E}$ that is different from E, but agrees with E on all functions vanishing in a neighborhood of C. Formally, it is natural to think of such a form as given by

$$\tilde{E}(f, g) = E(f, g) - \int_C \lambda f g \, \mathrm{d}\tilde{\rho}$$

where $\tilde{\rho}$ is a measure supported on C and λ is a function on C (we could, of course, have incorporated λ in $\tilde{\rho}$, but in many applications $\tilde{\rho}$ is a naturally given measure, and λ is the part we can adjust).

The nonstandard approach starts by replacing the original L^2-space $L^2(X, m)$ by a hyperfinite space $L^2(Y, \mu)$, and the form E by a nonstandard form $\mathcal{E}$ on $L^2(Y, \mu)$ that has E as its standard part (typically, Y is a hyperfinite lattice in ${}^*\mathbb{R}^d$, and $\mathcal{E}$ is the form generated by a hyperdiscrete Laplacian). We also replace C and $\tilde{\rho}$ by nonstandard representations B and ρ in a similar way. The problem is now to figure out when the perturbed, nonstandard form

$$\tilde{\mathcal{E}}(u, v) = \mathcal{E}(u, v) - \sum_{x \in B} \lambda(x) u(x) v(x) \rho(x)$$

has a standard part different from E. For both physical and mathematical reasons, we need the perturbed form to be lower bounded, i.e. $\tilde{\mathcal{E}}(u, u) \geq -K\|u\|^2$ for some finite K.

If L is the operator generating $\mathcal{E}$, the operator H generating $\tilde{\mathcal{E}}$ is given by

$$Hu(x) = Lu(x) - \lambda(x) u(x) \frac{\rho(x)}{\mu(x)}.$$

The best way to control the perturbation seems to be through the resolvents, and if we let $G_\alpha = (L - \alpha)^{-1}$ be the resolvent of L, the resolvent of the perturbed operator H is given by

$$(H - \alpha)^{-1} = G_\alpha \left(I - \frac{\lambda \rho}{\mu} G_\alpha \right)^{-1} = G_\alpha \sum_{l=0}^{\infty} \left(\frac{\lambda \mu}{\rho} G_\alpha \right)^l .$$

Rearranging the terms in the Neumann series and then adding them up again (see [19] for the calculations), we end up with the expression:

$$(H-\alpha)^{-1}f(x) = G_\alpha f(x) + \hat{G}^*_\alpha \left(\frac{1}{\lambda} - G'_\alpha\right)^{-1} \hat{G}_\alpha f(x)\ ,$$

where the operator $\hat{G}\colon L^2(Y,\mu) \to L^2(B,\rho)$ and its adjoint $\hat{G}^*_\alpha\colon L^2(B,\rho) \to L^2(Y,\mu)$ are determined through

$$\hat{G}_\alpha g(x) = \sum_{y\in Y} G_\alpha(x,y)g(y)\mu(y)\ ,$$

and $G'_\alpha : L^2(B,\rho) \to L^2(B,\rho)$ is defined by

$$G'_\alpha g(x) = \sum_{y\in B} G_\alpha(x,y)g(y)\rho(y)\ .$$

This calculation shows that the perturbation is governed by the operator $\frac{1}{\lambda} - G'_\alpha$. If we assume that there is a standard α_0 and a standard $\epsilon > 0$ such that

$$\frac{1}{\lambda(x)} \geq \sum_{y\in B} G_{\alpha_0}(x,y)\rho(y) + \epsilon \tag{5}$$

for all $x \in B$, it follows by a simple calculation that the operator $\frac{1}{\lambda} - G'_{\alpha'}$ is positive for all $\alpha \leq \alpha_0$ and that the perturbed form $\tilde{\mathcal{E}}$ is bounded from below. This means that we can apply the theory from Chapter 5 to find the standard part of $\tilde{E}$ of $\tilde{\mathcal{E}}$. According to formula (4), it is given by

$$\tilde{E}(\tilde{f},\tilde{f}) = -\lim_{^\circ\alpha\to\infty} {}^\circ\big(\alpha^2\langle (H-\alpha)^{-1}f,f\rangle + \alpha\langle f,f\rangle\big)$$

where f is a nonstandard representation of the standard function $\tilde{f}$ (a so-called *lifting*; I admit details are getting a little blurred here!). Using our formulas above, this can be rewritten as

$$\begin{aligned}\tilde{E}(\tilde{f},\tilde{f}) &= -\lim_{^\circ\alpha\to\infty} {}^\circ\left(\alpha^2\langle G_\alpha f,f\rangle + \alpha\langle f,f\rangle + \alpha^2\left\langle \left(\frac{1}{\lambda} - G'_\alpha\right)^{-1}\hat{G}_\alpha f, \hat{G}_\alpha f\right\rangle_{L^2(B,\rho)}\right)\\ &= E(\tilde{f},\tilde{f}) - \lim_{^\circ\alpha\to\infty} {}^\circ\left(\alpha^2\ \left\langle \left(\frac{1}{\lambda} - G'_\alpha\right)^{-1}\hat{G}_\alpha f, \hat{G}_\alpha f\right\rangle_{L^2(B,\rho)}\right)\ .\end{aligned}$$

Let us return to formula (5). The sum $\sum_{y\in B} G_{\alpha_0}(x,y)\rho(y)$ is over a hyperfinite set, and can be both finite and infinite. Let us first assume that it is finite and that we can find a finite function λ satisfying (5). As α goes to $-\infty$ in the limit above, $-\alpha G_\alpha f$ approaches f and $\frac{1}{\lambda} - G'_\alpha$ approaches $\frac{1}{\lambda}$, and we may hope that the whole final term approaches $\sum_{x\in B}\lambda(x)f(x)^2\rho(x)$.

This indeed the case, and the result in standard terms (forgetting all technical conditions) is as follows: Let E be a standard Dirichlet form with resolvent R_α.

Assume that $\tilde{\rho}$ is a Borel measure on a set C and that λ is a (standard) Borel function on C such that for some $\alpha_0 \in \mathbb{R}$

$$\frac{1}{\lambda(x)} \geq \int_C R_{\alpha_0}(x,y)\,\mathrm{d}\tilde{\rho}(y) + \epsilon$$

Then the form

$$\tilde{E}(f,g) = E(f,g) - \int_C \lambda(x) f(x) g(x)\,\mathrm{d}\tilde{\rho}(x)$$

is a closed perturbation of E supported on C. Note that in this situation we have a perturbation that can be described in terms of a measure $\lambda\tilde{\rho}$ on C.

Returning to the nonstandard picture, we may ask what happens if the sum $\sum_{y\in B} G_{\alpha_0}(x,y)\rho(y)$ in formula (5) is infinitely large. By choosing $\lambda(x)$ infinitesimal, it is still possible to keep

$$\frac{1}{\lambda(x)} - \sum_{x\in B} G_{\alpha_0}(x,y)\rho(y)$$

positive and finite. The problem is that since we are interested in the limit as α goes to $-\infty$, we need to keep $\frac{1}{\lambda(x)} - \sum_{x\in B} G_\alpha(x,y)\rho(y)$ finite not only for one value of α, but for *all* finite values. And if $\sum_{x\in B} G_\alpha(x,y)\rho(y)$ is infinite and decaying, this may be seem unlikely.

Let us take a closer look. We need to keep the following quantity finite:

$$\begin{aligned}&\frac{1}{\lambda(x)} - \sum_{x\in B} G_\alpha(x,y)\rho(y)\\ &= \left(\frac{1}{\lambda(x)} - \sum_{x\in B} G_{\alpha_0}(x,y)\rho(y)\right) + \left(\sum_{x\in B} G_{\alpha_0}(x,y)\rho(y) - \sum_{x\in B} G_\alpha(x,y)\rho(y)\right) .\end{aligned}$$

As the first term is finite by assumption, we can concentrate on the second term. By the resolvent equation

$$\sum_{x\in B} G_{\alpha_0}(x,y)\rho(y) - \sum_{x\in B} G_\alpha(x,y)\rho(y) = (\alpha_0 - \alpha)\sum_{y\in B} G_\alpha G_{\alpha_0}(x,y)\rho(y) ,$$

where the kernel $G_\alpha G_{\alpha_0}$ is defined by

$$G_\alpha G_{\alpha_0}(x,y) = \sum_{z\in Y} G_\alpha(x,z) G_{\alpha_0}(z,y)\mu(z) .$$

Now the point is that the kernel $G_\alpha G_{\alpha_0}$ is much less singular than the original kernel $G_\alpha(x,y)$, and hence there is good hope that $\sum_{y\in B} G_\alpha G_{\alpha_0}(x,y)\rho(y)$ is finite even if $\sum_{y\in B} G_{\alpha_0}(x,y)\rho(y)$ is infinite—and if so, our procedure may still lead to a perturbation of the original form.

Translated into standard terms (and again dropping all technical conditions), the final result is: Let E be a standard Dirichlet form with resolvent R_α on a space $L^2(X,m)$. Assume that $\tilde{\rho}$ is a Borel measure on a set C and assume that for some $\alpha_0 \in \mathbb{R}$

$$R_{\alpha_0} R_{\alpha_0}(x,y) \quad \text{is} \quad \tilde{\rho} \times \tilde{\rho} - \text{integrable.}$$

Then the form E has a closed, nontrivial perturbation supported on C. As we now may have to choose λ infinitesimal, the perturbed form can not necessarily be written as

$$\tilde{E}(f,g) = E(f,g) - \int_C \lambda(x) f(x) g(x) \, \mathrm{d}\tilde{\rho}(x)$$

in the standard universe.

To see the difference between the two results, note that if E is the form generated by $-\Delta$, the resolvent kernel $R_\alpha(x,y)$ has a singularity of order $\|x - y\|^{2-d}$ on the diagonal, while the the kernel $R_\alpha R_\alpha(x,y)$ has a singularity of order $\|x-y\|^{4-d}$ (assuming that d is large enough). This means that perturbations of the second kind (corresponding to infinitesimal λ's) usually exist two dimensions higher than perturbations of the first kind.

In Section 6.2, the general theory is applied to point interactions; i.e., the original form E is the closure of $E(f,g) = \langle -\frac{1}{2}\Delta f, g\rangle$ in $L^2(\mathbb{R}^d, m)$, and C is a single point. The result is as expected; perturbations exist for $d \leq 3$, but are only given by a measure for $d = 1$.

Section 6.3 deals with perturbations of the Laplacian along Brownian paths (in the nonstandard setting this means perturbations along the paths of a d-dimensional random walk moving on a lattice with infinitesimal spacing Δx). By a rather straight forward application of the general theory, we show that they exist for $d \leq 5$, but are given by measures only when $d \leq 3$. What is not so straight forward is to show that in dimension 3, we get perturbations of the form

$$E(f,g) - \int_0^1 \lambda(b(t)) f(b(t)) g(b(t)) \, \mathrm{d}t$$

for all bounded functions λ. This requires some hefty estimates involving fifteen dimensional integrals.

There is a close connection between perturbations along Brownian paths and polymer measures. To see why, we apply the nonstandard Feynman-Kac formula proved in Chapter 5 to the semigroup $\tilde{Q}^t$ generated by the perturbed form $\tilde{E}$ (the nonstandard version of the Feynman-Kac formula is strong enough to deal rigorously with extremely singular potentials). The result is

$$\tilde{Q}^t f(x) \approx \tilde{E}_x \left[f(\tilde{B}(t)) \exp\left(\int_0^t \int_0^1 \lambda(\tilde{B}(r)) \tilde{\delta}(B(s) - \tilde{B}(r)) \, \mathrm{d}s \, \mathrm{d}r \right) \right] .$$

Here B is the original Anderson random walk that carries the perturbation, $\tilde{B}$ is a new random walk independent of B and generated by the semigroup, $\tilde{E}_x$ is expectation with respect to the measure of $\tilde{B}$, and $\tilde{\delta}$ is the nonstandard δ-function on the d-dimensional lattice given by

$$\tilde{\delta}(x) = \begin{cases} \Delta x^{-d} & \text{if } x = 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

The formula above shows that our singular perturbation theory gives us a certain control over expressions of the form

$$\exp\left(\int_0^1\int_0^1 \lambda(\tilde{B}(r))\tilde{\delta}(B(s)-\tilde{B}(r))\,\mathrm{d}s\,\mathrm{d}r\right)\ .$$

These expressions also occur in the formal definitions of polymer measures, except that λ is a negative constant and the two Brownian motions are not independent, but the same (the idea is to penalize self-intersections). Self-intersections of the same Brownian path are more singular than intersections of two independent paths, but Westwater [113]–[115] had shown that it is possible (at least in $d = 3$) to use the latter to control the former; the trick is to split the double integral over the square $[0,1]\times[0,1]$ into integrals over smaller squares that just touch the diagonal, and then control the sum of all these contributions. The (open) question is whether it is possible to do something similar in $d = 4$. As we shall see when we get to Chapter 7, this is a question that comes up naturally in quantum field theory.

At the time the book was getting finished, Sergio's student Andreas Stoll was making a more direct, nonstandard attack on self-repellent random walks and polymer measures. He starts his doctoral dissertation [108] (see also the published papers [110], [111]) with a study of local times for Brownian self-intersections of the form

$$L(x,\omega)=\sum_{t=0}^{1/2}\left(\sum_{s=1/2}^{1}\Delta x^{-d}\chi_{\{\omega:B(s,\omega)-B(t,\omega)=x\}}\Delta t\right)\Delta t\ ,$$

where B is a d-dimensional version of Anderson's random walk and χ_A is the indicator function of A. The main tools are the nonstandard version of Kolmogorov's continuity theorem and a hyperdiscrete version of the Fourier inversion formula.

In the second part of the thesis, these results are used to make sense of the heuristic formula

$$\frac{\mathrm{d}\nu(\phi,g)}{\mathrm{d}m}(\omega)=\frac{1}{Z(\phi,g)}\exp\left(-g\int_0^1\int_0^1\phi(\omega(t)-\omega(s))\,\mathrm{d}s\,\mathrm{d}t\right)$$

for the density of a polymer measure against Wiener measure in dimension 2. The nonstandard approach yields among other things a Donsker-type invariance principle for Varadhan's model.

3.4 Chapter 7: Hyperfinite Lattice Models

As the title says, this chapter deals with hyperfinite lattice models for random fields and quantum fields in ${}^*\mathbb{R}^d$. In the first three sections, the lattices have standard spacing (the shortest distance between sites in the lattice is 1), and the only difference between the standard and the nonstandard models is that the nonstandard lattices have additional sites infinitely far out. These sites are important, however, as they allow us to put boundary conditions “at infinity”. The first section of the chapter follows the (nonstandard) papers by Helms and Loeb [63], [64] (see also [62]) in describing how to find the semigroup that governs the evolution of the lattice system—the point here is that if we fix the configuration outside an infinitely large cube in ${}^*\mathbb{R}^d$, the (nonstandard) dynamics inside the cube is easily described, and the standard dynamics can be obtained by "taking standard parts".

Section 2 deals with equilibrium models and thermodynamical properties, and builds partly on earlier nonstandard treatments by Helms and Loeb [63], Hurd [71],[73], and Ostebee, Gambardella, and Dresden [104],[105]. The main idea here is that one can replace the rather cumbersome limit definitions of the standard theory by working directly on a hyperfinite part of the lattice. The section ends with a discussion of how phase transitions can occur in hyperfinite models although they do not occur in finite models—the clue is that a function can be differentiable in a nonstandard sense (with an infinitely large derivative) without being differentiable in the suitable standard sense.

Section 3 deals with the global Markov property of lattice fields. For fields there is a distinction between the local Markov property which deals with the interaction between a bounded set and its exterior, and the global Markov property which also deals with the interaction between two unbounded sets. The intuitive reason is that even if you separate two unbounded sets by a boundary that the interaction does not reach across, one set can still influence the other "through infinity". The nonstandard content of this section is to a large extent based on the work of Sergio's student Christoph Kessler (see [75]-[78], [80]). There is a jungle of conditions leading to the global Markov property, and in his thesis, Kessler helped clarify the relationship between them—in particular, he used nonstandard analysis to construct models satisfying some properties and not others.

The last two sections of the book, 7.4 and 7.5, deal with quantum fields. We are still working with lattices in ${}^*\mathbb{R}^d$, but now the spacing is infinitesimal, i.e. the distance between neighboring sites is $\delta \approx 0$. The contents of these sections were previously unpublished, but some of it must be classified as reworking of standard theory in nonstandard terms.

We first show how the free field on $\mathbb{R}^d$ can be obtained as the standard part of a hyperfinite Gaussian field; one of the advantages of this representation is that the hyperfinite field is defined pointwise and not only in a distributional sense (there are two traditional ways to treat the singularities of quantum field theory: through distributions or through lattice approximations—see Kessler's paper [79] for a nonstandard discussion of the relationship between these two approaches). Interactions are introduced as

$$U_g^\delta = \lambda_\delta \sum \delta^d g(n\delta) u_\delta(\Phi_\delta(n)) ,$$

where Φ_δ is the free field, g is a cut-off function (often taken to be the indicator function of a "large" set), λ_δ is a coupling constant, and the sum is over ${}^*\mathbb{Z}^d$. The function u_δ describes the interaction, and may typically be an exponential $u_\delta(y) = \exp(\alpha y)$ or a polynomial of low order. The associated probability measure is given by

$$d\mu_{g,\Lambda_\delta} = \frac{\exp(-U_g^\delta)}{\int \exp(U_g^\delta)\, d\mu_{0,\Lambda_\delta}}\, d\mu_{0,\Lambda_\delta} ,$$

where $d\mu_{0,\Lambda_\delta}$ is the free measure.

The challenge is twofold: On the one hand to prove that the standard part of the interacting field satisfies the axioms for Euclidean quantum fields, and on the other hand to prove that it is nontrivial, i.e. different from the free field. For exponential interactions in dimension 1 and 2, the situation was well understood through earlier (standard) work by Sergio and Raphael, and this is used a test case for the nonstandard theory. The real challenge is polynomial interactions, especially the famous (or infamous?) Φ_d^4-model of fourth degree interactions.

The final section of the book, 7.5, is called "Fields and polymers" and contains a serious attempt to get a better grip on polynomial interactions. Using Anderson's random walk and a nonstandard representation of Poisson processes, we first construct a"Poisson field of Brownian bridges" in a very concrete way. The second step is to prove that this Poisson field is a probabilistic representation of the square of the free lattice field, and the third step is to use this representation to study interacting scalar fields (representations of this kind were first obtained by Dynkin [57], [58] in a standard context). The famous Φ^4 fields are given by interactions of the form

$$U(\Phi_\delta) = \frac{\lambda}{4}\sum_i \Psi_\delta(i)^2\delta^d + \frac{a}{2}\sum_i \Psi_\delta(i)\delta^d ,$$

where Ψ_δ is the square of the free field, the sums are over the lattice, and λ and a are constants. If we use the representation above to calculate the crucial entity $E[\exp(-\Phi_\delta(g))\exp U]$, we end up with expressions of the type

$$\exp\left(-\int_0^t \int_0^{\tilde{t}} \lambda\delta(b_1(s) - b_2(s))\right)\, d\tilde{s}\, ds , \tag{6}$$

which are exactly the kind of expressions we got acquainted with when we looked at perturbations along Brownian paths. A major problem is that in some of these expressions, b_1 and b_2 are not independent, but the same Brownian motion, and this leads us to the complicated problems of polymer measures that we just touched on at the end of Section 6.4. Westwater managed to tame them when $d = 3$, but $d = 4$ is a much more singular case.

A way to avoid this problem, is to look at two interacting fields Φ_1 and Φ_2 in a $\Phi_1^2\Phi_2^2$-model. The calculations are much the same as before, but as we now have two different fields, we only get the expression in formula (6) for two independent Brownian motions b_1 and b_2. This leads to the questions we analyzed in Section 6.4 on perturbations of the Laplacian along Brownian paths. The main problem in this case is that in the physical dimension $d = 4$, the coupling "constant" in Chapter 6 was positive and allowed to depend on x. In the present situation, we need it to be negative and independent of x. This seems to be a quite difficult problem as we had to work extremely hard to prove that λ can be chosen constant in the much easier three dimensional case (and if it doesn't sound hard to prove that an infinitesimal function can be chosen constant, recall that it is the infinite function $\frac{1}{\lambda(x)}$ that we really need to control).

So the book ends on an open note; we had shown that hyperfinite lattice models were an interesting setting for quantum fields, much closer to intuition than the traditional formalism, but we hadn't been able to obtain the definitive results we were aiming for (but then they have proved to be quite elusive for all kind of approaches!).

4 Later Contributions

After the completion of [19], Sergio has continued to work with nonstandard methods in a variety of subjects and often with different groups of collaborators. Much of this activity can be seen as a natural continuation of ideas and challenges from [19], and I shall try to give an exposition of the main results.

4.1 Nonstandard Constructions of Singular Traces

In the first half of the 1990s, Sergio wrote four papers [20]-[23] on singular traces in collaboration with Daniele Guido, Arcady Ponosov, and Sergio Scarlatti. In spirit these papers are close to Sergio's first nonstandard paper [18] with Fenstad and Høegh-Krohn in the sense that they give concrete, nonstandard descriptions of otherwise rather elusive operators, but the setting of the papers is quite different from [18].

If $\mathcal{R}$ is a von Neumann algebra (you can safely think of the case where $\mathcal{R} = B(H)$ is the algebra of all bounded operators on a Hilbert space H) and $\mathcal{R}^+$ is its cone of positive elements, a *weight* on $\mathcal{R}$ is a linear map

$$\phi\colon \mathcal{R}^+ \to [0,\infty] \ .$$

Using linearity, we can extend ϕ to its natural domain $\mathrm{Span}\{T \in \mathcal{R}^+ : \phi(T) < \infty\}$. A *trace* is a weight τ with the property $\tau(T^*T) = \tau(TT^*)$. We say that τ is *normal* if for all increasing nets $\{T_\alpha \,|\, \alpha \in I\}$ with $T = \sup_{\alpha\in I} T_\alpha$, we have $\phi(T) = \lim_\alpha \phi(T_\alpha)$. A classical result ([56]) tells us that all normal traces are proportional to the usual trace, so the question is how many nonnormal traces are there? If we define a trace τ to be *singular* if it is trivial on all operators of finite rank, it turns out that any trace on the compact operators $K(H)$ can be written uniquely as a sum $\tau = \tau_1 + \tau_2$ of a normal trace τ_1 and a singular trace τ_2, and hence we can concentrate on singular traces.

Dixmier [55] proved that nonnormal traces exist. To get an impression of his construction, we first fix a regular, slowly increasing and divergent sequence α_n of real numbers ($\alpha_n = \log(n+1)$ will do the job, but there are other possibilities). The idea is to use this divergent sequence to speed up the decay of nonsummable sequences of eigenvalues so that they become summable.

Next we choose a state (i.e. a normalized weight) on $l^\infty(\mathbb{N})$. The idea is now to define a trace τ_ϕ on $B(H)^+$ by

$$\tau_\phi(T) = \begin{cases} \phi\left(\left\{\frac{\sigma_n(T)}{\alpha_n}\right\}\right) & \text{if } T \in I(H) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\sigma_n(T) = \sum_{k=1}^{n} \mu_k(T)$, where $\mu_k(T)$ are the eigenvalues of T in decreasing order and counted with multiplicity, and $I(H)$ is the ideal of all compact operators such that the sequence $\{\sigma_n(T)/\alpha_n\}$ is bounded.

The main problem with this construction is that as we in general only have an inequality

$$\sigma_n(T + S) \leq \sigma_n(T) + \sigma_n(S),$$

τ_ϕ will usually not be linear. Dixmier realized that if ϕ is 2-dilation on $l^\infty(\mathbb{N})$, i.e. $\phi(\{a_n\}) = \phi(\{a_{2n}\})$, then we also have the opposite inequality (this needs both the slow growth of α_n and the dilation property), and hence τ_ϕ is linear and a trace. As ϕ vanishes on the set c_0 of sequences converging to 0, τ_ϕ is a nonnormal trace.

So how do we get hold of 2-dilations on $l^\infty(\mathbb{N})$? It is here nonstandard analysis comes in with a very simple and elegant description. If $\{a_n\}_{n\in {}^*\mathbb{N}}$ is the nonstandard extension of a bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, then for any infinite $\omega \in {}^*\mathbb{N}$, we define

$$\phi_\omega(\{a_n\}) = {}^\circ\!\left(\frac{1}{\omega}\sum_{k=1}^{\omega} a_{2^k}\right).$$

As $\phi_\omega(\{a_n\}) - \phi_\omega(\{a_{2_n}\} = \frac{1}{\omega}\left(a_1 - a_{2^{\omega+1}}\right) \approx 0$, we see that ϕ is a 2-dilation, and hence τ_{ϕ_ω} is a nonnormal trace. More generally,

$$\phi_{k,m,n}(\{a_n\}) = {}^\circ\!\left(\frac{1}{n}\sum_{i=k+1}^{k+n} a_{(2m-1)2^{k-1}}\right)$$

is a 2-dilation for all $k, m \in \mathbb{N}$ and all infinite $n \in {}^*\mathbb{N}$. This means that we have a three-parameter family $\tau_{k.m,n}$ of associated traces (with repetitions).

So how general is this construction? It is proved in [20] that any *Dixmier trace* (i.e. any trace coming from a 2-dilation) is in the closure of the convex hull of the traces $\tau_{k,m,n}$. The proof is based on a close study of the extremal dilation invariant states. We cannot go deeper into the arguments here, but would like to say that they exploit the product structure ${}^*(\mathbb{N}\times\mathbb{N}) = {}^*\mathbb{N}\times{}^*\mathbb{N}$ of the nonstandard natural numbers in a way that is not possible in the usual approach through Stone-Čech compactifications as $\overline{\mathbb{N}\times\mathbb{N}} \neq \overline{\mathbb{N}}\times\overline{\mathbb{N}}$.

In [21] the theory is extended to another class of nonnormal traces (called *anti-Dixmier traces* as they are in a sense reflections of the Dixmier traces around the usual trace), but the results and techniques are much the same as in [20]. In [22] the emphasis has shifted. The question now is to classify those compact operators T that admit a singular trace in the sense that there is a singular trace τ with $0 < \tau(T) < \infty$. If we define $\{S_n(T)\}$ to be the sequence such that $S_n(T) - S_{n-1}(T) = \mu_n(T)$ and

$$S_0(T) = \begin{cases} 0 & \text{if } T \notin L^1(H) \\ -\mathrm{tr}(T) & \text{if } T \in L^1(H) \ , \end{cases}$$

we say that T is *generalized eccentric* if 1 is a limit point for the sequence $\{S_{2n}(T)/S_n(T)\}$. The main theorem states that T admits a singular trace if and only if it is generalized eccentric.

The proof of this theorem is entirely standard, but again 2-dilations coming from hyperfinite sums are used to throw light on how these operators occur, and the paper ends with an interesting example of how such sums can be used to calculate a closed formula for a Dixmier trace of a concrete operator. The last paper [23] in the series deals with the same problems as [22], but the main emphasis is now on the nonstandard analysis of 2-dilation invariant states.

4.2 Quantum Fields as Flat Integrals

About ten years after the work on [19] was finished, Sergio returned to hyperfinite models of quantum fields with three papers in collaboration with Jiang-Lun Wu ([14]-[16], see also Wu's later paper [116]). The basic idea was to use nonstandard analysis to make rigorous sense of quantum fields as flat integrals. Intuitively, flat integrals are representations of Gaussian fields as integrals of infinite dimensional Lebesgue measure, and they have been much used as a heuristic tool by both physicists and probabilists. The only problem is that since infinite dimensional Lebesgue measure doesn't exist, flat integrals do not exist—at least not in an immediate sense.

What do exist are nonstandard Lebesgue measures on hyperfinite dimensional spaces, and in a series of papers [40], [43]-[46] Nigel Cutland used these to give nonstandard flat integral representations of a variety of Gaussian fields. In [15] Sergio and Jiang-Lun set out to extend these ideas to the quite singular case of Euclidean quantum fields. Their starting point is that if Λ is a bounded subset of $\mathbb{R}^d$, the free Euclidean field ϕ in Λ with mass m is heuristically given by the flat integral

$$\mathrm{d}\mu(\phi) = \kappa \exp\left\{-\frac{1}{2}\int_\Lambda \left(|\nabla\phi(x)|^2 + m^2\phi(x)\right)\,\mathrm{d}x\right\}\prod_{x\in\Lambda}\mathrm{d}\phi(x) \ ,$$

where $\prod_{x\in\Lambda} d\phi(x)$ is the infinite dimensional Lebesgue measure.

Working on a hyperfinite lattice approximation Λ_δ of Λ, Sergio and Jiang-Lun in [15] obtain a rigorous version of this formula

$$\mathit{\Gamma}(A) = \int_A \kappa \exp\left\{-\frac{1}{2}\sum_{z\in\Lambda_\delta}\left(|\nabla_\delta q_z|^2 + m^2 q_z\right)\,\delta^d\right\}\prod_{z\in\Lambda_d}\mathrm{d}q_z \ ,$$

where ∇_δ is a hyperdiscrete approximation of the appropriate gradient on Λ and $\prod_{z\in\Lambda_d} \mathrm{d}q_z$ is a (well-defined) hyperfinite dimensional Lebesgue integral. Although totally rigorous, this formula only makes sense inside the nonstandard universe, but the authors also derive a standard white noise representation of ϕ as

$$\phi(f,\omega) = \int_\Lambda \left(-\Delta_\Lambda + m^2\right)^{-1/2} f(x)\,\mathrm{d}\xi_x(\omega), \qquad f \in \mathcal{D}(\Lambda), \tag{7}$$

where $\{\xi_x\}$ is an independent family of one-dimensional white noises. This formula is obtained by first defining a nonstandard white noise η on the hyperfinite lattice and showing that the nonstandard lattice field Φ_δ (as defined in Section 7.4 of [19]) can be obtained as an integral of η. Some technical work is needed to show that the standard field is the standard part of Φ_δ in the appropriate Sobolev space. Formula (7) is then used to obtain a Cameron-Martin formula and a Schilder-type large deviation principle for the free Euclidean field on Λ.

The companion paper [14] is written primarily for a nonstandard audience (and not an audience of physicists) and extends the discussion from the free field to fields with exponential interaction. In addition to another discussion of large deviations of quantum fields, the slightly later paper [16] also extends one of Cutland's flat integral representations from l^2 to l^p, $1 \le p < \infty$.

4.3 A Return to Hyperfinite Dirichlet Forms

In 2011, Sergio, in collaboration with Ruzong Fan and Frederik Herzberg, published a book [17] entitled *Hyperfinite Dirichlet Forms and Stochastic Processes*. The work on the project had actually started more than 20 years earlier, and had resulted in a number of papers in the 1990's, mainly by Fan, but in close collaboration with Sergio.

The main difference between the theory in the new book and the one in [19] is that the forms are no longer required to be symmetric, but they do have to be *weakly coercive* in the sense that there is a finite constant C such that

$$\mathcal{E}_1(u,v) \le C\sqrt{\mathcal{E}_1(u,u)}\sqrt{\mathcal{E}_1(v,v)}.$$

This condition works as a replacement for Schwarz' inequality.

The (nonsymmetric) form $\mathcal{E}$ has a *coform* $\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$ that can be used to form the *symmetric part* $\overline{\mathcal{E}}(u,v) = \mathcal{E}(u,v) + \hat{\mathcal{E}}(u,v)$ and the *anti-symmetric part* $\mathring{\mathcal{E}}(u,v) = \mathcal{E}(u,v) - \hat{\mathcal{E}}(u,v)$ of $\mathcal{E}$. All four forms play an important part in the exposition.

One of the differences between the standard and the nonstandard theory of Dirichlet forms is that in the standard theory the domain of the form is usually assumed to be given (at least until one starts looking at examples!), while in the nonstandard theory much of the basic work goes into identifying the domain. Although the theory of weakly coercive forms is in many ways similar to the symmetric theory, there is an important difference in the description of domains: In the symmetric case, the domain is easily described in terms of the semigroup, but in the weakly coercive case it seems necessary to approach the domain via the resolvent. Rather reassuringly, it turns out that the domain of the original form $\mathcal{E}$ coincides with the domain of the much simpler, symmetric form $\overline{\mathcal{E}}$.

After the initial study of domains and standard parts of weakly coercive, hyperfinite forms, the book continues with a detailed study of potential theory and the relationship between a Dirichlet form and its associated Markov process, at the same time generalizing and simplifying the corresponding theory in [19].

The last part of the book is on the theory of hyperfinite Lévy process. As this is the topic of the next subsection, I'll leave the discussion till then.

4.4 Nonstandard Lévy Processes

In 2003, Sergio's student Frederik S. Herzberg wrote a Diplomarbeit on nonstandard Lévy processes. Unaware of Frederik's work, I was at the same time starting my own investigations into the subject. Fortunately, we approached the problem from opposite angles, and when the first papers appeared ([10], [11] by Sergio and Frederik and [94] by me—see also the corrections in [69]), they complemented each other more than they overlapped.

For a quick, intuitive understanding of how the nonstandard theory works, it is convenient to start with the definitions in [94]. Choose a hyperfinite set $A \subseteq {}^*\mathbb{R}^d$, an internal set $\{p_a \mid a \in A\}$ of positive numbers such that $\sum_{a\in A} p_a = 1$, and a positive infinitesimal Δt. Let X be a random walk in ${}^*\mathbb{R}^d$ with timeline $T = \{k\Delta t \mid k \in \mathbb{N}_0\}$ and transition probabilities p_a; i.e. let $X(0) = 0$ and put $P[\Delta X(t) = a \mid X(0), X(1), \ldots, X(t)] = p_a$. We call X a *hyperfinite Lévy process* if almost all paths stay finite for all finite t. This sounds like a silly, totally uncheckable condition, but it turns out that there is an equivalent, easy to verify characterization. Hyperfinite random walks have cadlag standard parts that are Lévy processes, and any Lévy process can be obtained in this way (at least in the sense that every Lévy triple (γ, C, ν) can be realized—see also [100]).

My focus in [94] is very much on the hyperfinite random walks, and the Lévy processes only enter the theory to show that it has achieved what it set out to achieve. In the first two papers [10], [11] by Sergio and Frederik, the Lévy processes are on the contrary the primary objects, and the main focus is to find good, nonstandard representations (*liftings*) preserving the properties of the original process. Typical examples are the lifting results in [10] where the hyperfinite representations live on lattices, and where the time development is divided into sequences of binomial events. Notions from adapted probability logic are used to characterize how close the nonstandard liftings are to the original processes. In [66], Frederik used these hyperfinite representations to construct an intrinsic theory for stochastic integration with respect to Lévy processes.

Lévy processes have been much used to model financial markets, and this is also a clear motivation for Sergio and Frederik. By refining the timeline and modeling the internal processes as sums of binomial increments as in [11], they achieve a model with a unique martingale measure that can be used for hedging (see also [67]). An alternative approach to finance in hyperfinite Lévy markets is presented in [95] where the focus is on minimal martingale measures.

In Chapter 5 of their book [17] with Ruzong Fan, Sergio and Frederik give a review of both approaches to the theory of nonstandard Lévy processes, and Frederik has also given an exposition in another book [68], this time in the framework of Nelson's "radically elementary probability theory" (see [99]). The notion of hyperfinite random walks seems to fit perfectly into Nelson's vision.

4.5 The Power of Loeb Measures

As nonstandard measure and probability theory developed, it soon became clear that Loeb measure spaces have many desirable qualities—e.g., they seem to be *universal* in the sense that anything that can be constructed on some measure space (no matter how exotic), can also be constructed on simple Loeb spaces. This intuitive notion of universality was formalized through several versions of probability logic, mainly developed by Jerry Keisler and his (former) students Douglas N. Hoover and Sergio Fajardo (see [59] for a systematic exposition). Along the way other notions, such as saturation and homogeneity, were added to universality.

Three of Sergio's later papers exploits the richness of Loeb spaces. The first is a joint paper with Yeneng Sun and Jiang-Lun Wu [25] published in 2007. The authors start with two hyperfinite probability spaces I and Ω and study processes $X\colon I \times \Omega \times T \to {}^*\mathbb{R}$, where T is a timeline. If the processes $(\omega, t) \mapsto X(i, \omega, t)$ (with i fixed) are independent martingales, they prove that the "empirical process" $(i, t) \mapsto X(i, \omega, t)$ (with ω fixed) is a martingale for almost all i. Due to measurability problems such questions are hard even to make sense of in a standard setting, but the paper exploits the fact that the Loeb measure of a product of nonstandard measures is richer than the product of the Loeb measures to circumvent these problems. The result extends to sub- and supermartingales.

The other two examples are joint papers [12], [13] with Frederik Herzberg from about the same time. The first of these deals with the optimization of functionals where the main variable is a probability measure P. The functional is evaluated by observing a given process g at a fixed set of points $t_1, t_2, \ldots, t_n$, and then integrating an expression of the type $\phi(g_{t_1}, g_{t_2}, \ldots, g_{t_n})$ against the varying measure P. In the nonstandard setting of an internal, hyperfinite probability space the existence of an optimal measure is almost trivial (it is just a finite dimensional optimization problem), but to transfer this solution to an arbitrary measure space, the full machinery of adapted probability logic is needed.

The third paper in this group [13] is more traditional in its methods, but still uses the power of Loeb measure techniques to the full. The problem is to extend the solution of the classical moment problem from $\mathbb{R}^n$ to Wiener space. Using Anderson's random walk as a representation of Brownian motion, the problem is translated into a hyperfinite dimensional setting where the nonstandard version of the original problem applies. An extra condition on the quadratic variation is needed to pull this nonstandard solution back to the classical Wiener space.

We have reached the end! I hope this little survey has not only given you new insight into the work of Sergio and his school, but also provided you with a better understanding of the power and versatility of nonstandard methods.

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