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Approximate second order analysis of unbraced frames reflecting inter-storey interaction in single curvature regions

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ABSTRACT: Present approximate second order methods for the analysis of unbraced multistorey frames may significantly underestimate moments in single curvature regions. To clarify reasons for this that may not be well understood, the mechanics of column interaction in single curvature regions are studied. Suitable tools for sidesway description are derived, including a nearly exact, explicit free-sway effective length expression, that, when high accuracy is required, eliminates the need for cumbersome, iterative solutions of exact effective lengths from the transcendental instability equation. Two reasons for the underestimation are identified. One is related to the local second order $N\delta$ effects, and the other to second order effects causing changes in rotational restraint stiffness at column ends due to vertical, inter-storey column interaction. A modified approximate storey magnifier approach is proposed that accounts for these local second order effects through two separate "flexibility factors". The approach is sufficiently simple to be viable in practical analyses, and predictions are found to compare well with more accurate results.

KEYWORDS: Multistorey sway frames; Columns; Storey magnifier method; Effective lengths; Storey interaction.

Notation

B_s	Sway magnification factor;
EI, EI_b	Cross-sectional stiffness of column and beam;
G_j	Relative rotational restraint flexibility at member end j
H	Applied lateral storey load (sum of column shears and bracing force);
L, L_b	Lengths of considered column and of restraining beam(s);
N	Axial (normal) force;
N_{cr}	Critical load in general $(= \pi^2 EI/(\beta L)^2)$
N_{cb}, N_{cs}	Critical load of columns considered fully braced, and free-to-sway;
N_E	The Euler buckling load of a pinned-end column $(= \pi^2 EI/L^2)$
R_j	Rotational degree of fixity at member end j;
R_m	Mean rotational degree of fixity of the two member ends;
S_0	First order lateral "storey" stiffness;
S_B	Lateral bracing stiffness;
V_0, V	First order and total (first+second order) shear force in a column;
k_j	Rotational restraint stiffness (spring stiffness) at end j
α_{cr}	Member (system) stability index $(= N/N_{cr})$
α_b, α_s	Load index of column considered fully braced, and free-to-sway;
α_{ss}	Storey (system) stability index
α_E	Nominal load index of a column $(= N/N_E)$
β	Effective length factor (from system instability);
β_b, β_s	Effective (buckling) length factor corresponding to N_{cb} and N_{cs}.
γ, γ_n	Flexibility factor in general, and load (N-) dependent flexibility factor;
γ_s, γ_0	Flexibility factor at free-sway, and at zero axial load:
γ_k	Flexibility factor for restraint stiffness correction.
Δ_0, Δ	First order and total lateral displacement;
κ_j	Relative rotational restraint stiffness at end j $(=k_j/(EI/L))$.

1 Introduction

In frame and member analysis, it is often necessary to consider second order load effects on sway, moments and stability caused by axial loads acting on the displacements of the frame and frame members. In frames with sidesway, second order effects affect interconnected members in three ways: (1) in an overall, global sense, due vertical loads acting on the sidesway of the frame system as such ("$N\Delta$" effects), (2) in an individual, local sense, due to axial member loads acting on the deflections away from the chord between member ends and thus causing nonlinear (curved) moment distributions along the members ("$N\delta$" effects), and (3) in a local sense in multi-level columns and frames by changing rotational restraint stiffness at member ends due to vertical, inter-level (inter-storey) column interaction.

Subdivision into global and local second order effects is very common in approximate analyses, such as in the socalled $N - \Delta$ type methods, that consider second-order effects separately following a conventional first-order analysis. A valuable asset of such methods is their transparency with respect to the important variables, and they have been dealt with in a number of studies over the last 40 years or so. Some of these are reviewed in Hellesland [1].

Global effects are well taken care of in such methods, and to some extent also local $N\delta$ effects, through a factor often labelled "flexibility factor" [2], "bending shape factor" [3], or "stiffness reduction factor" [4]. In these and other relevant papers, e.g, [5, 6, 7]), and textbooks, e.g., [8], it is stated or implied incorrectly that the increased column flexibility may be 1 to 1.22 (1.2) times the first order flexibility. This range is acceptable for many practical frames, but may not be adequate for unbraced or partly braced multibay frames that include columns subjected to axial loads far in excess of the free-sway critical load. For such columns, the flexibility factor may be considerably greater than indicated by this range. Extensions to include such cases, which require a load dependent flexibility factor that reflects the transition from sway to braced column response, have been presented and discussed elsewhere [1].

Even for some frames with reasonably low axial column loads, below the free-sway critical loads of the columns considered unbraced, local second order effects may be considerably greater than reflected by a flexibility factor between 1 and 1.22. This is typically the case in single curvature bending regions of multistorey, unbraced frames, where conventional $N - \Delta$ type methods may grossly underestimate moments [9].

In such regions there are two effects that will increase the flexibility beyond what is presently accounted for: (1) Negative restraints will be inflicted at one column end, which in itself may result in flexibility factors outside the mentioned range. This is seemingly not well-known. (2) In addition, there is a strong inter-storey interaction, beyond that reflected in a first order analysis, between columns framing into the same joint. These interaction effects, resulting from changes in the columns' rotational restraint stiffnesses due to the axial column loads (third type second order effects mentioned above), have received little attention.

The emphasis of this paper is on such second order effects in linear elastic two dimensional, unbraced multistorey frames with single curvature regions (typical for "stiff column-flexible beam" frames). Each column has uniform sectional stiffness and axial load along the length. These properties may vary from storey to storey.

The main objective of the study is directed towards deriving a modified approximate storey magnifier approach that may account for the second order effects reviewed above. Towards this goal, (1) the basics of the storey magnifier approach are reviewed, (2) suitable tools for sidesway description are derived (sway magnifier, critical load, and first order storey stiffness expressions), (3) available approximate methods for computing local second order ($N\delta$) effects (flexibility factors) are reviewed, and their accuracy evaluated for a variety of positive and negative end restraint combinations, and (4) the mechanics of restraints and vertical column interaction in single curvature regions are studied to clarify aspects that are not well understood.

2 Second order column analysis

2.1 Unbraced column

In second order analysis, three quantities are normally of interest: the critical load (effective lengths), and the sway and moment magnifier. Reasonably simple, approximate expressions for these can be derived using principles of the socalled $N - \Delta$ analysis approach. The approach is well-known, and only a brief review is given below for the purpose of deriving and defining the quantities of interest for this study.

The isolated, laterally loaded unbraced column in Fig. 1 is considered. It has rotational end restraints, defined by springs with rotational stiffness k_1 and k_2 at the member ends. When these represent the interaction with a larger structure, several bending modes may be possible. The two most common are shown in Fig. 1(b,c). The first, with one inflection point between ends, will result provided the end restraints have positive values. This is typically the case for columns in unbraced frames with stiff (strong) beams. The second, with a negative rotational restraint inflicted at the upper end, and with no inflection point between ends, is typical for columns in lower stories of unbraced frames with flexible (weak) beams.

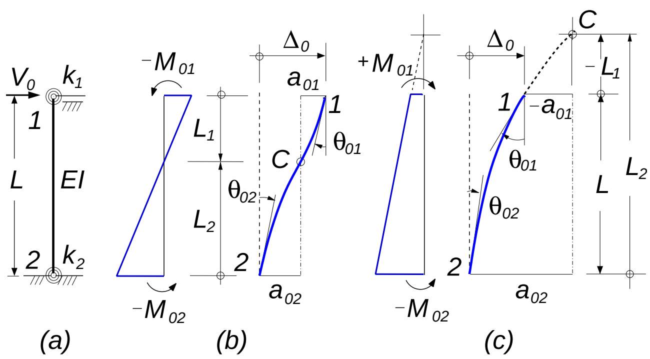

Figure 1: (a) Laterally loaded unbraced column, (b) moments and displacement shapes for given positive/positive end restraints, and (b) for negative/positive end restraints.

Sway magnifier. If the column is subjected to a vertical (axial) load, the relative end displacement will increase from the first order value Δ_0 to

$$\Delta = B_s \Delta_0 \tag{1}$$

where B_s is the sway magnification factor that reflects the second order (global) overturning moment effect, and also the second order member (local) effects through a separate factor, here labelled γ_n. By replacing the overturning moment effect ($N\Delta$) by an equivalent horizontal load ($\gamma_n N\Delta/L$) and scaling up the first order displacement, the final displacement can be written, $\Delta = \Delta_0(1 + (\gamma_n N\Delta/L)/V_0)$. Solving for Δ, it can be written in the form of Eq. (1), with the sway magnifier given by

$$B_s = \frac{\Delta}{\Delta_0} = \frac{1}{1 - \alpha_s} \tag{2}$$

where

$$\alpha_s = \frac{\gamma_n N/L}{S_0} \qquad \text{and} \qquad S_0 = \frac{V_0}{\Delta_0} \tag{3 a, b}$$

are the sidesway "stability index" and the first order lateral stiffness, respectively. This sway magnifier can also be established in other ways, including in an iterative, Vianello type manner (a geometric series).

The γ_n factor reflects an increased flexibility caused by local (member) second order effects. It is strictly axial load-dependent, but is most often taken load-independent. It is discussed in more detail below (Section 4).

Critical load and effective length. The critical load factor, causing infinite displacements, is equal to the inverse of the "stability index" ($\lambda_{cr} = 1/\alpha_s$). The critical free-sway load ($N_{cr} = N_{cs}$) can therefore be expressed by

$$N_{cs} = \frac{N}{\alpha_s} \quad \text{or} \quad N_{cs} = \frac{V_0 L}{\gamma_s \Delta_0} \tag{4 a, b}$$

where γ_n is denoted γ_s at the free-sway condition (i.e., at unbraced buckling of a column considered in isolation).

Alternatively, the critical compression load of an elastic member of length L and uniform cross-sectional bending stiffness EI and axial force along the member may be written in the conventional form as

$$N_{cr} = \frac{\pi^2 EI}{(L_e)^2} \qquad \text{with} \qquad L_e = \beta L \tag{5 a, b}$$

L_e is the effective length (or buckling length), and β is the effective (buckling) length factor of the member. Physically, the buckling length is equal to the distance between inflection points (points of contraflexure) located on the buckled shape, or on the mathematical continuation of the buckled shape.

From Eq. (4 a) and (5), the free-sway effective length ($\beta = \beta_s$) can then be expressed by

$$\beta_s = \sqrt{\frac{N_E}{N}\alpha_s} \qquad \text{with} \qquad N_E = \frac{\pi^2 EI}{L^2} \tag{6}$$

Moment magnifier. End moments due to lateral and axial loads can be expressed by

$$M = B_M M_0 \approx B_s M_0 \tag{7}$$

where B_M is a moment magnification factor that is commonly approximated by the sway magnifier. This is a reasonable approximation for unbraced columns, and for framed columns that contribute to the lateral resistance of the frame, i.e., columns with axial load levels below the free-sway critical loads. It is such cases that are considered here.

2.2 Interacting columns

Laterally interacting columns. For frames with laterally interacting columns with *given* rotational end restraints, the appropriate "storey sway magnifier" and critical load can be obtained by replacing α_s above by a "storey stability index" α_{ss}, obtained from Eq. (3) by replacing the numerator and denominator by the corresponding sums over all the interacting columns plus a possible bracing:

$$\alpha_{ss} = \frac{\sum(\gamma_n N/L)}{S_0} \qquad \text{with} \qquad S_0 = \frac{H}{\Delta_0} = \frac{\sum V_0}{\Delta_0} + S_B \tag{8 a, b}$$

S_0 is now the lateral storey stiffness, H is the total lateral storey load (including possible bracing forces), and S_B is the lateral stiffness of bracings, if present. The shear stiffness may be included in S_0, but is normally neglected.

Details of the derivation of such sway magnifier formulations, recent advances, simplifications and corresponding limitations, and code adaptions, are available elsewhere [1].

Vertically interacting columns. The approach is commonly applied also to individual storeys in multistorey frames. In particular for frames with restraining beams that are sufficiently stiff to cause inflection points (zero moment) within the column lengths, the agreement with more accurate methods is good. However, for storeys with columns in single curvature bending (no inflection points between column ends), the approach may significantly underestimate the storey stability indices and sway magnifiers [9].

An effort is made in this study to clarify the reason for this, and to propose a modified approach that may improve predictions in single curvature regions and that is sufficiently simple to be viable in practical analyses.

3 First order properties

The first order lateral stiffness V_0/Δ_0 in Eq. (3) and (4b) can readily be established by the differential equation, the moment-area theorem, or by other methods, and can be found in the literature. Here, first order properties are derived in forms suitable for use later in the paper.

Reference is made to the laterally loaded column in Fig. 1. At the ends with positive restraints, end moments act in the opposite direction to that of the end rotation and thereby "strengthens" the member by restraining the end rotation. At ends with negative restraints, rotational disturbances are inflicted as reflected by end moments acting in the same direction as the end rotations. End moments and end rotations (θ_0) are taken as positive when they act in the clockwise direction, and segment lengths are defined as positive when they are oriented from the respective column ends toward the other end, as shown in Fig. 1(b). With this definition, a negative segment length implies an inflection point outside the member length such as in Fig. 1(c).

Inflection points, moments, restraints. The location of the inflection point, and a very suitable restraint fixity parameter, can be determined from the slope continuity condition at the common inflection point C of each of the two cantilever segments in the figure. Equal slope can be expressed (for instance using the moment-area theorem) by

$$\theta_{0C} = \frac{(-M_{01})L_1}{2EI} + \theta_{01} = \frac{(-M_{02})L_2}{2EI} + \theta_{02} \tag{9}$$

Substituting $-M_{0j} = V_0 L_j$, which follows from moment equilibrium, and noting that $L_1 + L_2 = L$, Eq. (9) can be expressed in terms of one of the unknown member segment lengths as

$$\frac{L_j}{L} = \frac{-M_{0j}}{V_0 L} = \frac{R_j}{R_1 + R_2} \qquad j = 1, 2 \tag{10}$$

where

$$R_j = \frac{k_j}{k_j + cEI/L} = \frac{1}{1 + c/\kappa_j} \quad \text{with} \quad c = 2 \tag{11}$$

and

$$\kappa_j = \frac{k_j}{(EI/L)} \tag{12}$$

Here, R is a first order **"rotational degree of fixity factor"**, and κ is the non-dimensional rotational restraint stiffness.

The first order fixity factors R are seen to be directly proportional to the end moment, and to the first order inflection point distance from the end at which the factor is computed. It is, consequently, a very useful parameter. Its definition evolves naturally from the mathematics of the problem and is closely related to the physics of the column response. At a rotationally fixed end, R=1, and at a pinned end, R=0 (zero fixity). A negative R at an end implies an inflection point located away from the end, outside the column length. Similar factors defining the approximate inflection point locations of a buckled column, obtained by replacing c=2 in Eq. (11) by c=2.4, is given and discussed in Hellesland [10].

Lateral displacement and stiffness. The total relative lateral displacement can be given by the sum of that of each cantilever segment ($\Delta_0 = a_{01} + a_{02}$, Fig. 1):

$$\Delta_0 = \frac{(-M_{01})L_1^2}{3EI} + \theta_{01}L + \frac{(-M_{02})L_2^2}{3EI} + \theta_{02}L \tag{13}$$

Expressing L_j and $-M_{0j} = V_0 L_j$ in terms of the R-factors (Eq. (10)) and substituting into Eq. (13), the first order relative displacement can be written as

$$\Delta_0 = \left(\frac{6}{R_1 + R_2} - 2 \right) \frac{V_0 L^3}{EI} \tag{14}$$

and the corresponding first order lateral stiffness as

$$\frac{V_0}{\Delta_0} = c_v \frac{EI}{L^3} \quad ; \quad c_v = \frac{12 R_m}{3 - 2 R_m} \tag{15 a, b}$$

where $R_m = 0.5(R_1 + R_2)$ is the mean first order fixity factor. The lateral stiffness is consequently a function of the sum of the fixity factors, and not of the individual components.

Restated in terms of the restraint flexibility parameters G, Eq. (15) becomes

$$c_v = \frac{12(G_1 + G_2 + 6)}{2G_1G_2 + 4(G_1 + G_2) + 6} \tag{16}$$

where

$$G_j = b_o \frac{(EI/L)}{k_i} = \frac{b_o}{\kappa_i} \qquad j = 1, 2 \tag{17}$$

which, in this general form, allows for both positive or negative restraint values [10, 11]. The coefficient b_o is a reference restraint stiffness coefficient, normally taken equal to b_o=6 corresponding to that of a beam bent in antisymmetrical curvature. This reference value is also adopted here. Apart from the generalised G factor definition, the lateral stiffness coefficient Eq. (16) is on a similar form previously derived along different lines by others (e.g., [4]).

4 Flexibility factors for $N\delta$ effects

4.1 Background

An axial force give rise to a nonlinear moment distribution along a column. These local second order ($N\delta$) effects lead in turn to a reduction in lateral column stiffness (or increased flexibility) and to a sideways displacement that is greater than that due to a linear moment distribution. This effect can be accounted for through a factor often denoted γ, and previously (1976) labelled flexibility factor by the author [2]. It reflects, in other words, the reduced lateral column stiffness of an axially loaded member as compared to that of a column with a (first order) linear moment distribution.

This flexibility factor is one of two local second order aspects of importance for correct predictions of column and frame response to lateral loads. It will be considered in some detail, both with regard to alternatives available and accuracy.

The flexibility factor factor is strictly a function of the column axial load (the normal load). A subscript "n" is added, to give γ_n, in order to indicate this dependence in the general case. In its most general form, γ_n is given in Hellesland [1]. Normally, like in this study, simplifications are warranted.

For axial loads approaching zero and the critical free-sway load, γ_n takes on values that for convenience will be labelled γ_0 and γ_s, respectively. Between these two axial load levels, the variation in γ_n is very modest. In this range, which is the one of main interest in this paper, the flexibility factor may, with good accuracy, be taken independent of axial loads, and for instance equal to γ_s or γ_0.

In multibay frames, where some columns may have axial loads far in excess of the free-sway critical load, γ_n for a column may be still be approximated by γ_s, but the accuracy decreases with increasing axial load, in particular as the axial load approaches the critical load of the column considered braced [1].

Common for earlier studies of the flexibility factor [2, 3, 4, 5, 6, 7, 12], is that they deal with positive end restraints only, and that they, with one exception [12], state without any reservations that γ_s is limited to values between 1 and 1.22, or 1 and 1.2 for γ_0. This is a common misconception. As shall be seen below, it is correct for the special case of isolated free-sway columns, for which end restraints always will be positive, but not necessarily correct for other cases, such as framed columns for which also negative end restraints may be inflicted through the interaction with other members in the frame.

4.2 Exact γ_s variation

Evaluated at the free-sway (zero shear) condition, the resulting γ_s can be determined by equating N_{cr} in Eq. (4) to the exact free-sway critical compression load N_{cr} (Eq. (5)) and given by

$$\gamma_s = \frac{(V_0 L/\Delta_0)}{N_{cr}} = \frac{c_v \beta_s^2}{\pi^2} \tag{18}$$

The exact β_s can be found from the transcendental instability equation (Eq. (23)). Results obtained in this manner are shown in Fig. 2 for various positive and negative end restraint combinations. The restraints are conveniently defined in terms of fixity factors R, Eq. (11). For positive restraint stiffness values κ, the R values will always be positive and take on values between 0 (pinned end) and 1.0 (fully fixed end). For large negative κ values (strong negative restraints), the R values will become greater than 1.0, and for small negative κ values (weak negative restraints), the R values will become negative. Results are symmetrical about the the +45 degree diagonal $R_A = R_B$. Therefore, only those above the line are shown.

End restraint combinations giving infinite effective lengths represent outer limits and can be given by [10]

$$\frac{1}{\kappa_1} + \frac{1}{\kappa_2} = -1 \quad \text{or} \quad R_1 + R_2 = 0 \tag{19}$$

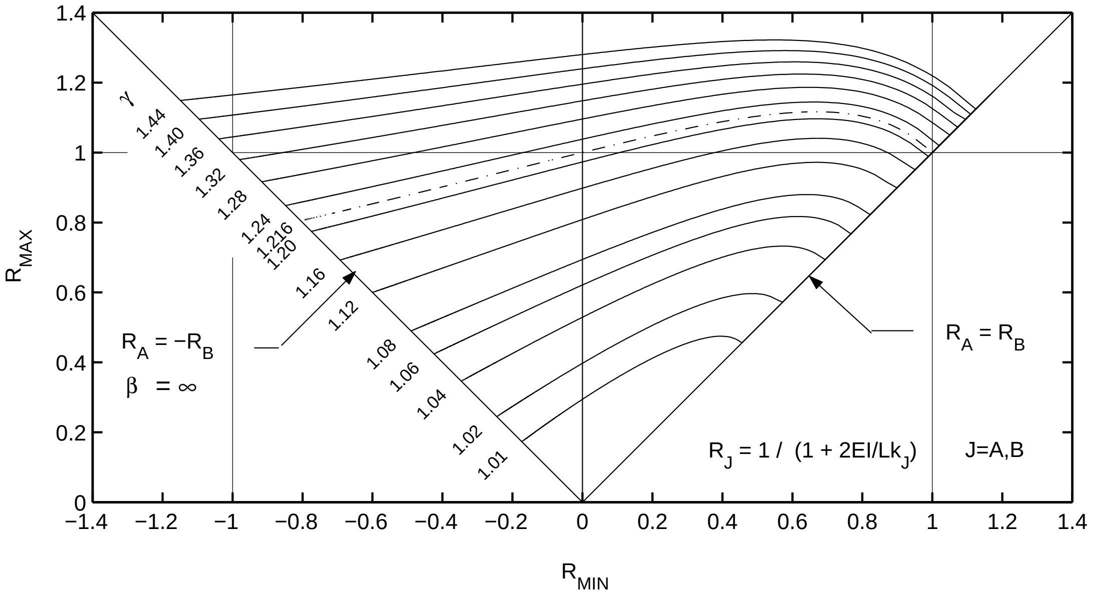

Figure 2: The flexibility factor $\gamma = \gamma_s$ (at the free-sway condition) in terms of first order rotational restraint fixity factors at ends A and B.

in terms of κ and R factors (and $G_1 + G_2 = -6$ in terms of G factors with $b_o = 6$). This restraint combination is shown by the -45 degree diagonal. In the figure, $\gamma = \gamma_s$ values have arbitrarily been terminated at 1.44. This is probably beyond the range of practical interest.

It may be useful for the understanding of these results, to relate them to selected column bending (buckling) shapes presented and discussed in [10], and shown in Fig. 3. Results in the lower right quadrant, for $0 < R_{MAX} < 1$ and $0 < R_{MIN} < 1$, correspond to bending shapes such as illustrated in Fig. 3 (c,d), and those in the lower left quadrant, for $0 < R_{MAX} < 1$ and $-1 < R_{MIN} < 0$, to bending shapes of the type in Fig. 3 (e). Results in these two quadrants represent the most common bending shapes. Results

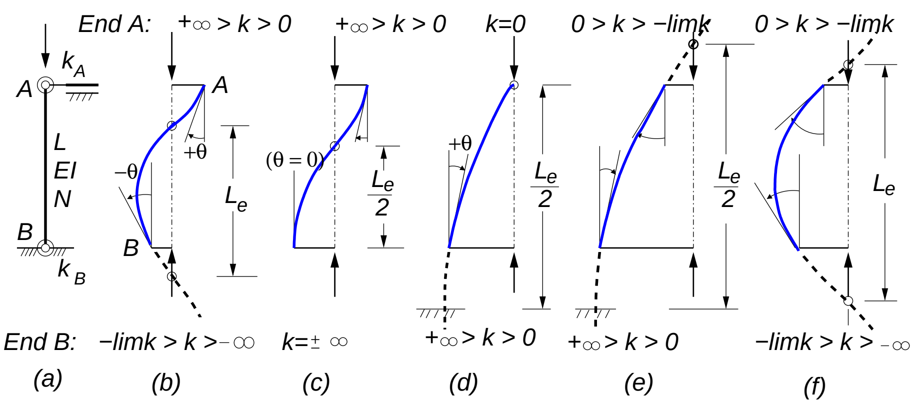

Figure 3: Selected buckling modes of an unbraced compression member.

in upper right quadrant for $R_{MAX} > 1$ and $0 < R_{MIN} < 1$, correspond to buckling shapes of the type in Fig. 3 (b), and those in the upper left quadrant for $R_{MAX} > 1$ and $-1 < R_{MIN} < 0$ to bending shapes of the type in Fig. 3 (f). The label "limk" in the figure refers to the limits given by Eq. (19).

4.3 Approximate γ_s factors

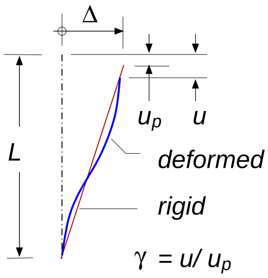

Figure 4: A flexibility factor definition.

Flexibility factor expressions can be derived in various ways, for instance, and simplest, by combining Eq. (18) with approximate free-sway effective length expressions, or alternatively, to establish them directly by other means. The latter approach is considered below.

Based on results obtained using the principle of minimum potential energy, Rubin [3, 13] defined the factor, as illustrated in Fig. 4, as the ratio of the vertical displacements at the top of a flexurally deformed column and a column rotating as a rigid pendulum, $\gamma = u/u_p$. For a column with given end moments and lateral displacement Δ, a deflected shape approximated by that corresponding to a linear moment variation (i.e., a third degree parabola), and neglecting axial deformations, he derived a factor, here denoted γ_0 and expressed by

$$\gamma_0 = 1 + \frac{L^4}{180(EI\Delta)^2}\left[M_1M_2 + 4(M_1 - M_2)^2\right] \tag{20}$$

Also based on energy considerations, Girgin et al. [7] recently derived along similar lines a factor that may be expressed in the exact same form as that above. In the elastic case with positive end restraints, Eq. (20) gives values between 1 and 1.2, which are correct for columns with negligible axial loads for which the third degree parabola assumption is correct. At the free-sway critical load, which is the state of interest here, the correct range for positive restraints is 1 to 1.216 (1.22), as seen in Fig. 2. This minor difference, which is of no practical importance, is adjusted for below by replacing the numeral 180 in Eq. (20) by 167.

In the elastic case, the first order values of Δ, end moments and restraints can be adopted. In this case, the expression can also be rewritten in terms of first order fixity factors R (Eq. (11)) or G factors (Eq. (17)) using the the first order properties, Eq.

(10) and Eq. (15), or these equations in terms of G factors (with $b_o = 6$). The three following alternative forms of the modified factor then result:

$$\gamma_s = 1 + \frac{L^4}{167(EI\Delta_0)^2}\left[M_{01}M_{02} + 4(M_{01} - M_{02})^2\right] \tag{21 a}$$

$$\gamma_s = 1 + 0.216\,\frac{R_1R_2 + 4(R_1 - R_2)^2}{(R_1 + R_2 - 3)^2} \tag{21 b}$$

$$\gamma_s = 1 + 0.216\,\frac{(G_1+3)(G_2+3) + 4(G_1 - G_2)^2}{[(G_1+2)(G_2+2) - 1]^2} \tag{21 c}$$

Values at the zero axial load limit, γ_0, can be obtained by replacing 0.216 above by 0.2. The latter expression with 0.216 replaced by 0.2, has also been presented before in [9],also there based on Eq. (20).

5 Effective length factor expressions

Approximate effective length factors for unbraced columns, $\beta = \beta_s$, can now be obtained from Eq. (6) with the γ_s factors above and with first order lateral stiffness given by V_0/Δ_0, Eq. (15) or (16). They may be given in any of the three following, convenient alternative forms:

$$\beta_s = \left[\frac{\pi^2 EI}{L^2}\cdot\frac{\gamma_s\Delta_0}{V_0 L}\right]^{1/2} \tag{22 a}$$

$$\beta_s = \left[\frac{\gamma_s\pi^2}{12}\left(\frac{3}{R_m} - 2\right)\right]^{1/2} \tag{22 b}$$

$$\beta_s = \left[\frac{\gamma_s\pi^2}{12}\cdot\frac{2G_1G_2 + 4(G_1+G_2) + 6}{G_1 + G_2 + 6}\right]^{1/2} \tag{22 c}$$

The accuracy of the γ_s expressions is investigated implicitly by comparing predictions by Eq. (22) with exact results for an unbraced member. Exact effective length results of a column with uniform section stiffness and axial force along the member, can be obtained from the well-known instability condition (transcendental equation) given by

$$\frac{(\pi/\beta)^2 - \kappa_1\kappa_2}{\kappa_1 + \kappa_2} = \frac{(\pi/\beta)}{\tan(\pi/\beta)} \tag{23}$$

Selected comparisons with exact results given in [10], are presented in Table 1 for combinations of positive/positive, positive/negative and negative/negative rotational restraints. The most relevant results in the table are those for positive/positive end restraint combinations in the lower left quadrant, corresponding to the buckling shapes in Fig. 2 (c,d), and those with positive/negative combinations in the lower right quadrant, corresponding to the buckling shapes in Fig. 2 (e).

For combinations of positive restraints, which are most common in practical cases, results are generally within 0.1% of exact results. This is an extremely good accuracy,

TABLE 1. Unbraced columns – Evaluation of effective length factor formula: Ratios of $\beta_{APPROX,Eq.(22)}/\beta_{EXACT}$.

κ_2 (R_2)	κ_1 (R_1)								
	∞	24	6	1.5	0	-0.3	-0.4	-0.6	-0.75
	(0	0.92	0.75	0.43	1	$-.18$	$-.25$	$-.43$	$-.60$)
-12 (1.20)	1.016	1.015	1.013	1.009	1.008	1.008	1.008	1.008	1.009
-24 (1.09)	1.004	1.003	1.003	1.002	1.002	1.003	1.004	1.005	1.007
∞ (1.00)	1.000	0.999	1.000	0.999	1.000	1.001	1.002	1.004	1.006
24 (0.92)		0.999	0.999	0.998	0.999	1.001	1.001	1.003	1.005
6 (0.75)			1.000	0.999	0.999	1.000	1.001	1.002	1.004
1.5 (0.43)				1.000	1.000	1.001	1.001	1)	
0.75 (0.27)					1.000	1.001	1.001		

- Results that can be obtained by reversing κ_1 and κ_2 are not shown.
- $R_j = 1/(1+(2/\kappa_j))$ • $G_j = 6/\kappa_j$ • 1) Exact eff. length is infinite

and may for most practical cases be considered "exact". Thus, in cases when exact or nearly exact results are required, the tedious iterations required to obtain solutions from the transcendental equation (Eq. (23)), can be avoided. In the lower right quadrant (positive/negative), the accuracy is not quite as good, but still very good and generally exact to two decimals.

This effective length factor will be used later in this study. However, when the high accuracy of this factor is not required, a number of other, simpler approximate effective length factors [10] that are valid for both positive and negative restraints, may be used.

6 Alternative approximate γ_s and γ_0 factors

Prior to learning of the work by Rubin, the author derived a general, but cumbersome γ_s expression based on $\gamma_s = V_0 L/(\Delta_0 N_{cr})$, Eq. (18), with N_{cr} expressed by an approximate expression for cantilever column segments [2]. Written in terms of G factors, it breaks down into the two following simple expressions

$$\gamma_s = 1 + \frac{0.216}{(1+0.5G_2)^2} \quad \text{and} \quad \gamma_s = 1 + \frac{0.216}{(1+G_2)^2} \qquad (24\,\text{a, b})$$

for the special case of a column pinned at one end (κ_1=0, $G_1 = \infty$), and the case with equal end restraints ($G_1 = G_2$), respectively. The same expressions were derived independently in still another way by LeMessurier [4]. It may be noted that Eq. (21 c) breaks down into the exact same expressions for these two cases.

A simple, yet reasonably accurate, expression for γ_s can be given by

$$\gamma_s = 1 + 0.108\,\frac{1 + [1 - (0.5G_{\max})^{\,p}\,]^3}{(1 + 0.5G_{\min})^{\,2}} \tag{25}$$

where $p = 1$ for $|G_{\max}| \leq 2$ and $p = -1$ for $|G_{\max}| > 2$. $G_{\max}$ is the larger and $G_{\min}$ the smaller of the G-factors at the column ends. The absolute signs are included to cover cases with negative end restraints. In such cases, the expression above require the following, rather special rule: $G_{\max}$ should be taken as the G factor with the greater absolute value, but submitted into the expression with its true sign. For instance, in a case with $G_1 = -10$ and G_2=1, one should set $G_{\max} = -10$ and $G_{\min}$=1.

This expression was proposed by the author (during a research stay in 1981 at the University of Alberta, Edmonton), based on observation of the variation of γ_s with changing restraints. For a column pinned at one end ($G_{max} = \infty$), it breaks down into Eq. (24a).

From an effective length factor (and magnification) expression given by Lui [5] in 1992, an expression for local second order effects can be extracted and written in terms of the flexibility factor

$$\gamma_0 = 1 + \frac{H/\Delta_0}{5\eta} \quad \text{with} \quad \eta = \frac{EI}{L^3}\left[3 + 4.8\frac{M_{01}}{M_{02}} + 4.2(\frac{M_{01}}{M_{02}})^2\right] \tag{26}$$

The moment ratio is between the smaller and larger end moment, and is to be taken positive when the moments act in the same direction (giving double curvature bending). For positive end restraints it gives values between 1 and 1.2. For a multibay frame, H is the sum of column shears, and η is replaced by the sum $\sum\eta$ over all columns in the storey. The resulting factor is in this case a sort of mean flexibility factor ($\overline{\gamma}_0$) for all the columns in the summation.

In 1994, Aristizabal-Ochoa [6] presented critical load and effective length expressions for unbraced and partially braced columns from which the following flexibility factor can be extracted:

$$\gamma_s = \frac{12}{\pi^2}\cdot\frac{40 + 8(\rho_1^2 + \rho_2^2) + \rho_1\rho_2(\rho_1 + \rho_2 + 3\rho_1\rho_2 - 34)}{3(4 - \rho_1\rho_2)^2} \tag{27}$$

where

$$\rho_j = \frac{k_j}{k_j + 3EI/L} = \frac{1}{1 + 3/\kappa_j} \qquad j = 1, 2 \tag{28}$$

is a restraint fixity factor. It is similar, but not equal, to the rotational fixity factor defined previously by Eq. (11). The latter is directly related to the first order inflection point location in a laterally loaded column, while that above is not related to any column property as such.

Still another factor, also expressed in terms of the ρ factors above, can be obtained from an approximate elastic lateral column stiffness expression obtained in 2002 by Xu and Liu [12] from a second order Taylor series expansion of the exact lateral stiffness. With symbols used here, a load dependent flexibility factor and its value at zero axial load can be extracted and expressed by

$$\gamma_n = 12(d_1 + d_2\pi^2\alpha_E) \qquad \text{and} \qquad \gamma_0 = 12d_1 \tag{29 a, b}$$

The first is linear in the axial load. The factors d_1 and d_2 (denoted β_1 and β_2 in [12]) are quite cumbersome functions of the end fixity factors ρ. At the free-sway load ($\alpha_E = 1/\beta_s^2$), the resulting values of $\gamma_n (= \gamma_s)$ become close to those obtained with Eq. (27). The γ_0 expression can be written in the same form as Eq. (27), but with 12/10 instead of $12/\pi^2$ in the first fraction. Xu and Liu concluded that the simplified, load independent expression was sufficient for the examples they considered, and recommended the simpler form for use in design practice.

For practical analysis and design, all of the presented flexibility factors (here and above) provide acceptable results. It will be a question of preference which of them to use.

7 Restraint mechanics–Vertical interaction

Another local second order aspect that is of great importance for correct predictions of column and frame response to lateral loads, in particular in single curvature situations, is connected to the vertical inter-column (inter-storey) interaction. A brief review relevant to the present work is considered useful.

The rotational end restraint stiffness of a column at a joint "j" may be defined accurately by

$$k_j = -\frac{M_j}{\theta_j} \tag{30}$$

or

$$k_j = f_j\, k_{bj} \qquad \text{with} \qquad f_j = \frac{M_j}{(\sum M_{col})_j} \tag{31 a, b}$$

In the first definition above, M_j is the end moment and θ_j the rotation, both including second order effects, caused at the considered column end by the other members framing into it. For later discussions in this study, this definition is most useful. Moments and rotations are defined positive when they act in the same direction (here taken as the clockwise direction). The restraint stiffness k_j becomes positive when the end moment ($M_j = -k_j\theta_j$) acts in the opposite direction to the end rotation, and negative otherwise.

In the second definition, k_{bj} is the rotational stiffness (equal to the moment giving a unit rotation) of the restraining beams, tension members, etc., at joint j, and f is the fraction (or multiple) of k_b that is provided to, or "demanded" by, the considered column end. This factor, given by Eq. (31b), is determined from moment equilibrium and rotation compatibility requirements at joint j. The summation in the denominator is over all column (compression member) moments at joint j. The rotational restraint offered by beams etc. at the joint is in other words shared between (or distributed to) the columns meeting at the joint in proportion to their end moments at the joint.

Normally, of course, moments and rotations that include the second order effects are not known, and they are therefore often replaced, such as in storey magnifier approaches, by first order values. Inherent in first order analyses are the restraint stiffnesses obtained from Eq. (30) or (31b) when the total moments and rotations are replaced by the respective first order values ($M_{0,j}$, $M_{0,col}$, θ_{0j}). Eq.(31b) has been given with first order values before for braced frames [14], but it is valid at any frame joint.

Several investigators (e.g., [9]) have documented that the storey magnifier approach, implying first order restraint properties, gives good moment predictions in "flexible column-stiff beam" frames, in which the beams (k_b) are sufficiently stiff to provide double curvature bending of the columns (inflection point between ends). The stiffness definitions of Eq. (30) or (31), are in other words not much affected by the second order axial load effects in such cases.

The same is not the case for "stiff column-flexible beam" frames, in which the beams are not stiff enough to provide double curvature bending. Such a case is illustrated by the four storey frame Fig. 5, where the columns of the three lower storeys are bent in single curvature. The first order stiffness approximation for such cases are studied in more detail below.

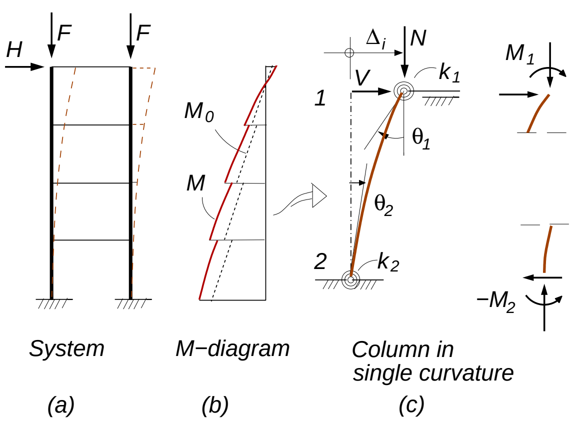

Figure 5: Multistorey "stiff column-flexible beam" frame

8 Frames in single curvature bending

8.1 4-storey unbraced frame

The extreme case of a frame with very flexible beams is obtained by neglecting the beams altogether. The restraint at the bottom of a column in one storey is then required to be provided entirely by the column in the storey below. Such a case is illustrated by the cantilever column shown by the insert in Fig. 6. The column is fixed at the base and can be divided into an arbitrary number of "storeys", or segments. First order and second order analysis solutions for the cantilever column can easily be obtained from rather straightforward hand calculations. Details of the analyses will not be given.

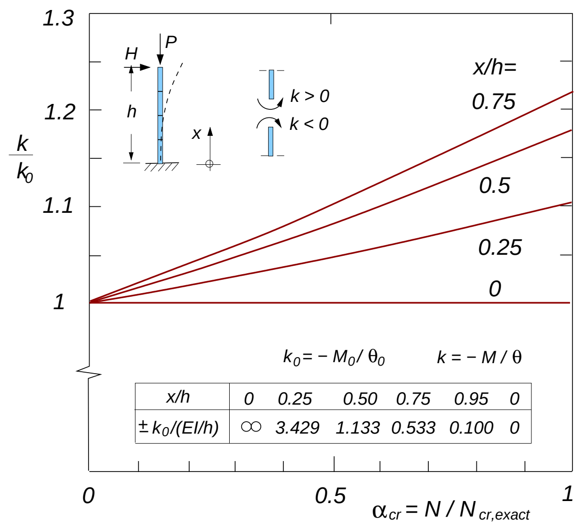

Figure 6: Rotational stiffness of cantilever column at various sections vs. axial load level

Resulting rotational stiffness $k(x)$ at various heights (x) and axial load levels, given as fractions of the exact critical load of the column ($\alpha_{cr} = N/N_{cr,exact}$; $N_{cr,exact} = \pi^2 EI/(4h^2)$) are shown in the figure. They are computed from the corresponding moments and rotations (Eq. (30)), and are given in the figure in terms of the corresponding first order values $k_0(x)$.

Once rotational stiffnesses are known, all quantities of interest can be computed using the previously developed expressions (Section 2). They will become approximate values if the correct stiffnesses, corresponding to the appropriate load level of the situation studied, are not used. This will be illustrated below.

8.2 Effective lengths

Effective lengths will be computed based on the lowest Column (segment) $S1$ of the 4 "storey" frame in Fig. 7(a). At the fixed base, $\kappa_2 = \infty$ and $R_2 = 1$. The rotational restraint stiffness at the upper end are obtained from Fig. 6 and given in Table 2 for three different axial load levels, α_{cr} = 1, 0.5 and 0. The first of these corresponds to buckling, the last to the first order case, and the middle one to a level at which it might be of interest to compute the sway magnifier.

TABLE 2. Effective length computations of Column S1 (Fig. 7) for different end restraints (load levels).

$Col.1$	$\alpha_{cr} = N/N_{cr,exact}$		
	1	0.5	0
κ_1	-0.948	-0.899	-0.857
R_1	-0.901	-0.816	-0.750
γ_s	1.348	1.337	1.329
β_s	8.066	5.804	4.905
L_e/h	2.017	1.451	1.226
$L_e/L_{e,exact}$	1.008	0.725	0.613

• R_2=1 • First order results for $\alpha_{cr} = 0$
• Results at buckling for $\alpha_{cr} = 1$.

Although the restraint stiffness at the upper end 1 does not seem to be too different for the different load levels, it is obvious from the effective length results ($\beta_s = L_e/L$; L_e/h) that they have significant effects. With restraints pertaining to the true buckling condition ($\alpha_{cr} = 1$), the predicted effective length is very good (0.8% above exact result) even though R_1 is strongly negative, and significantly outside the range for which comparisons were made in Table 1. On the other hand, the use of restraint stiffnesses at loads below the critical, α_{cr} = 0.5 and $\alpha_{cr} = 0$ (first order), give poor predictions that are 30 to 40% below exact results.

Otherwise it may be noted that the flexibility factor γ_s (between 1.33 and 1.35) is very little affected by the different load levels. So, the example serves to demonstrate that it is the restraint stiffness that is the most important parameter in isolated analysis of column segments in single curvature like here. The same will be the case for frames with stiff columns-flexible beams where there will still be strong interaction between columns in adjacent stories.

When restraint stiffnesses unlike here are not known, methods that include the stiffness as an unknown could have been employed (e.g., Hellesland [10], Tong et al. [15]). However, except for smaller systems, these are generally cumbersome. An alternative approach will be pursued here.

8.3 Moments for first order restraint stiffness

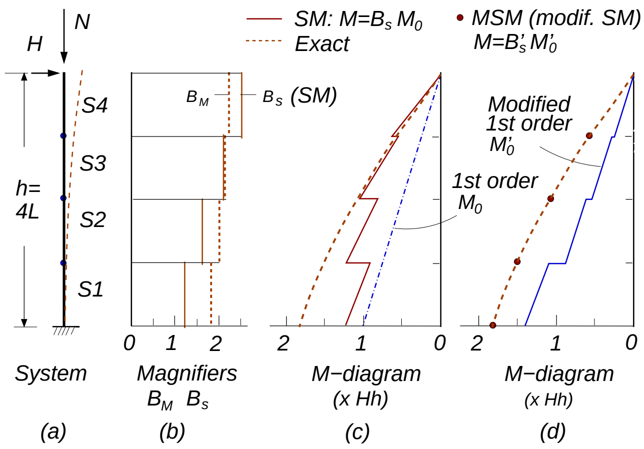

Figure 7: (a) 4-"storey" column; (b) Sway magnifiers; (c) Moments based on 1st order stiffness; and (d) Moments based on exact stiffness. (Exact and 2nd order approximate effects for $N = 0.5N_{cr}$).

In order to study the accuracy of using first order restraints in moment predictions, the same four "storey" column in Fig. 7(a) is considered, but now subjected at the top to a lateral load H and an axial load $N = 0.5N_{cr,exact}$ ($\alpha_{cr} = 0.5$). Magnified moments are first computed using sway magnifiers obtained in a conventional storey analysis based on first order analysis. A sample calculation of the lowest segment is demonstrated below.

Sample computation of S1. Based on the first order rotational stiffnesses $\kappa_{02} = \infty$ at the base, $\kappa_{01} = k_{01}/(EI/L) = -3.429/4 = -0.857$ at the top of $S1$ (from Fig. 6), and γ_n taken equal to γ_s, the following quantities are computed:
(1) Rotational fixity factors, Eq. (11): $R_{02} = 1.0, R_{01} = -0.750$; (2) First order relative displacement between the ends of $S1$, Eq. (14): $\Delta_0 = 1.837\, HL^3/EI$; (3) Flexibility factor, Eq. (21b): $\gamma_s = 1.329$; (4) Sway stability index, Eq. (3): $\alpha_s = 0.188$; (5) Sway magnifier, Eq. (2): $B_s = 1/(1 - 0.188) = 1.231$.

Sway magnifiers obtained in this manner, B_s=1.23, 1.62, 2.11 and 2.53 for $S1$ to $S4$, respectively, are compared to corresponding exact moment magnifiers, B_M=1.82, 2.00, 2.14 and 2.22, in Fig. 7(b). Approximate total moments computed by

$$M = B_s M_0$$

are shown by the stepped line in Fig. 7(c). Also shown are exact moments. The correspondence is good in the upper half ($S3$ and $S4$), but considerably below exact moments in the lower half (-32 and -19% in $S1$ and $S2$, respectively).

This example confirms results of previous studies that the conventional storey magnifier approach is not applicable, or at best very approximate, in single curvature bending regions, or in this particular case, in the lower half of the region.

8.4 Moments for exact restraint stiffness

The inaccuracy in the flexibility factor due to the approximation $\gamma_n(\alpha_{cr} = 0.5) \approx \gamma_s$, is quite small (about 1-1.5% below the exact value). The reason for the large discrepancy in moments above must therefore be due to the first order stiffness properties implied by the first order analysis. To verify this, magnified moment computations are repeated using sway magnifiers obtained based on the exact restraint stiffnesses at the considered axial load level.

Modified first order quantities are now computed in exactly the same manner as above, but for the different restraint stiffnesses. A prime is added to distinguish modified quantities from the conventional first order quantities above.

Sample computation of $S1$: Nondimensional rotational stiffnesses (from Fig. 6) at the actual axial load level α_{cr}=0.5 are $\kappa_{02} = \infty$, $\kappa_{01} = k_{01}/(EI/L) = -1.048 \cdot 3.429/4 = -0.899$. These give the modified quantities:
$R'_2 = 1.0, R'_1 = -0.816$; $\Delta'_0 = 2.554\,HL^3/EI$; $\gamma'_s = 1.337$; $\alpha'_s = 0.263$, and, finally, $B'_s = 1/(1 - 0.263) = 1.358$.

In order to obtain total moment predictions, the modified magnifier (1.358), which is only about 10% greater than that based on first order stiffness (1.231), must be applied to the *modified first order moments*. The latter are given by Eq. (10) with the fixity factors $R'_{1(2)}$ above. Expressed in terms of the total height $h = 4L$, the modified first order moments become $M'_{01} = -(-0.816/(1-0.816))0.25Hh = 1.109Hh$, and $M'_{02} = (-1/(1 - 0.816))0.25Hh = -1.359Hh$. The corresponding approximate, sway magnifier based, total moments

$$M = B'_s M'_0$$

become $M_2 = -1.846\,Hh$ and $M_1 = 1.506\,Hh$.

These, and similar results for the other column segments, are summarized and compared to exact moment results in Table 3. The accuracy is seen to be very good (within 0 to +1.6%). Similarly, total relative displacements, $\Delta = B'_s\Delta'_0$, are summarized in Table 4, and are seen to be within -0.1 and +0.7% of exact displacements.

Modified first order moments and total moment predictions (filled dots) are also shown in Fig. 7(d). The stepped first order moment distribution was to be expected as the slope of the individual portions must be the same since the shear is constant (=H) along the segments.

This last exercise clearly demonstrates, as was to be expected, that the storey magnifier approach as such is applicable provided the correct stiffnesses values are used.

TABLE 3. Moment computations of Column S1-S4 (Fig. 7(a)) for $\alpha_{cr} = 0.5$.

x/h (1)	κ_j (2)	Col. (3)	R'_2/R'_1 (4)	$\overline{\Delta'}_0$ (5)	γ'_s (6)	B'_s (7)	$\overline{M'}_{02}$ (8)	$\overline{M}_{2,exact}$ (9)	$\frac{B'_s M'_{02}}{M_{2,exact}}$ (10)
1.0	0	S4	0.069/0	7.136	1.002	2.237	-0.250	0.556	1.006
0.75	±0.147	S3	0.153/-0.079	6.630	1.005	2.057	-0.520	1.069	1.000
0.5	±0.361	S2	0.301/-0.220	5.402	1.027	1.747	-0.863	1.502	1.004
0.25	±0.899	S1	1/-0.816	2.554	1.337	1.358	-1.359	1.817	1.016
0	∞								

• $\kappa_j = k_j/(EI/L)$ • $\overline{\Delta'} = \Delta'/(HL^3/EI)$ • $\overline{M} = M/Hh$ • $M'_{01} = -M'_{02} - 0.25Hh$

TABLE 4. Relative sway computations of Column S1-S4 (Fig. 5) for $\alpha_{cr} = 0.5$.

x/h (1)	κ_j (2)	Col. (3)	R'_2/R'_1 (4)	$\overline{\Delta'}_0$ (5)	γ'_s (6)	B'_s (7)	$B'_s\overline{\Delta'}_0$ (8)	$\overline{\Delta}_{exact}$ (9)	(8)/(9) (10)
1.0	0	S4	0.069/0	7.136	1.002	2.237	15.955	15.866	1.006
0.75	±0.147	S3	0.153/-0.079	6.630	1.005	2.057	13.645	13.651	0.999
0.5	±0.361	S2	0.301/-0.220	5.402	1.027	1.747	9.440	9.408	1.003
0.25	±0.899	S1	1/-0.816	2.554	1.337	1.358	3.468	3.443	1.007
0	∞								

9 Modified sway magnifier method (MSM)

9.1 General remarks

In order to extend the use of the conventional storey magnifier approach to single curvature regions, means of estimating stiffnesses reasonably accurate at column ends are required, or more approximate procedures must be used in single curvature regions. In this study, the latter alternative is pursued.

In the lower half of the structure in Fig. 7(c) in single curvature, where the conventional storey magnifier approach storey severely underestimated the moments, the change in restraint stiffness caused by the axial loading cause an increase of approximately 10% in the sway magnifier B'_s and 39% in the first order displacement Δ'_0. In the second storey, the increases are smaller (8 and 12%), but together quite significant. The stiffness change manifests itself in increased flexibility.

Based on these observations, a simple approach has been studied whereby a "restraint correction flexibility factor" can be introduced to account for the local effects of axial forces on the rotational restraint stiffnesses k in single curvature bending regions. Indications are that it might be a viable approach, and a proposal on this basis is presented below.

9.2 Proposal

In single curvature regions, it is proposed to replace γ_n in the sway stability index expressions, Eqs. (3) and (8), by a *combined* flexibility factor γ_c such that

$$\alpha_s = \frac{\gamma_c N/L}{S_0} \quad \text{and} \quad \alpha_{ss} = \frac{\sum(\gamma_c N/L)}{S_0} \tag{32 a, b}$$

where

$$\gamma_c = \gamma_n \, \gamma_k \; (\approx \gamma_s \, \gamma_k) \tag{33}$$

and

$$\gamma_k = (c_1 \frac{L_0}{x} + c_2)^g \geq 1 \tag{34}$$

Then, γ_n and γ_k are the flexibility factors accounting for second order effects of axial forces acting on the bended shape ($N\delta$ effects) and on the rotational restraint in single curvature bending regions (vertical interaction effects), respectively. As mentioned previously, γ_n may be approximated by γ_s (or γ_0).

In Eq. (34), L_0 is the distance along the structure from the base to the first order inflection point (at the top of the single curvature region), x is the distance from the base of the structure to the top of the storey considered, c_1, c_2 and the exponent g are constants that will be dependent on several factors including horizontal and axial load distribution and axial load level. Several combinations of the constants have been considered. Tentatively, the values

$$c_1 = 0.11, \; c_2 = 0.89, \; g = 3$$

are suggested, and used below, for computation of sway magnifiers in the practical range of about B_s=1.3-1.5. For storeys above the single curvature region ($x > L_0$), where γ_k=1, predictions are not affected by stiffness changes.

10 Applications to single curvature regions

10.1 4-"storey" column in single curvature

First, the proposal is applied to the 4-"storey" continuous column in Fig. 7(a), with $L_0 = h = 4L$, $x = L, 2L, 3L$ and $4L$ for stories $S1$ to $S4$, respectively, and $\gamma_c = \gamma_s \gamma_k$.

The calculations now follow the same routine described in Section 8.3, and the values of Δ_0 and γ_s are the same used there. Results of the calculations are summarized in Table 5 for two axial load levels of α_{cr}=0.3 and 0.5. These give sway magnifier predictions in the ranges 1.4-1.6 and 1.8-2.5, respectively.

Compared to the exact moment magnifiers in the table, the predictions are seen to be good, and best for the lower load level (α_{cr}=0.3), which is most realistic in practical cases. It should be noted that the predictions for $S4$ is not affected by the γ_k factor.

TABLE 5. Moment computations by approx method of Column S1-S4 (Fig. 7(a)).

				$\alpha_{cr} = 0.5$			$\alpha_{cr} = 0.3$		
Col. (1)	$\overline{\Delta}_0$ (2)	γ_s (3)	γ_k (4)	B_s (5)	$B_{M,exact}$ (6)	(5)/(6) (7)	B_s (8)	$B_{M,exact}$ (9)	(8)/(9) (10)
S4	7.834	1.000	1.0	2.525	2.223	1.136	1.568	1.522	1.031
S3	6.829	1.004	1.114	2.436	2.138	1.139	1.547	1.486	1.041
S2	4.832	1.024	1.368	2.091	2.001	1.045	1.456	1.429	1.019
S1	1.837	1.329	2.353	1.789	1.817	0.984	1.360	1.351	1.007

• $\overline{\Delta} = \Delta/(HL^3/EI)$ • $\alpha_{cr} = N/N_{cr,exact}$ $(N_{cr,exact} = \pi^2 EI/(2h)^2)$

10.2 24-storey frame

The applicability of the method is now demonstrated for a symmetrical 24-storey, 1-bay frame with equal storey heights previously analysed by Lai and MacGregor [9]. The lower 17 storeys are shown in Fig. 8(a). The column stiffnesses are identical within three sets of eight storeys, but significantly different for the three sets. The columns of the lower eight storeys are considerably stiffer than the beams (EI/L of columns are about 15 times EI_b/L_b of beams). Thus, the frame represents an example of a "stiff column-flexible beam" frame. The same vertical loading is applied at all storey levels, and it seems that only a single lateral load is applied at the top of the frame. The first order bending moment diagram shows

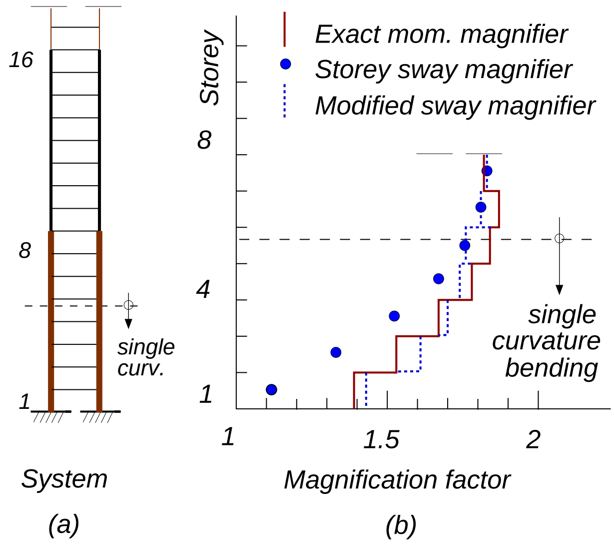

Figure 8: Magnifiers for 24-storey "stiff column-flexible beam" frame

single curvature bending in the bottom four storeys.

Exact moment magnification factors ($B_M = M_2/M_{02}$) at the column end with maximum moment, and corresponding sway magnification factors based on conventional storey stability indices (here denoted $B_{s,lai}$), were computed and presented in diagram form by Lai and MacGregor. Such magnification values, read from an enlarged version of this diagram (considered sufficiently accurate for the present purpose), are replotted in Fig. 8(b) for the lower seven storeys. The estimated inflection point location from the base is $L_0 = 4.7L$, where L is the storey height.

Compared to the exact moment magnifiers (solid, stepped lines), the conventional storey magnifiers (filled circles) are seen to significantly underestimate the moments in the lower stories. This is typical, as also mentioned previously, for regions with single curvature bending.

Modified stability indices are calculated from the results in Lai and MacGregor ("lai"): $\alpha_{s,lai} = 1 - (1/B_{s,lai})$ and $\alpha_{s,mod} = \alpha_{s,lai} \cdot \gamma_s \gamma_k / \gamma_{s,lai} \approx \alpha_{s,lai} \cdot \gamma_k$. The present γ_s values are here for simplicity approximated by the $\gamma_{s,lai}$ values (1.2 in the bottom storey and 1.05 in the others [9]), although a more correct γ_s value for the lower storey probably is somewhat greater than 1.3. For $L_0 = 4.7L$ and L_0/x=4.7, 2.35, 1.57, 1.18 and 0.94 for stories $S1$ to $S5$, respectively, the corresponding γ_k values become 2.79, 1.52, 1.20, 1.06 and 1.0.

Modified storey sway magnifier results, corresponding to B_s (Eq. (2)) defined with $\alpha_{s,mod}$, are also shown in the figure (dashed, stepped lines). They are in good agreement with the exact moment magnifiers (5% above to 2% below).

11 Summary and conclusions

Present approximate second order storey magnifier methods for the analysis of unbraced multistorey frames may significantly underestimate moments in single curvature regions. To provide a better understanding of the reasons for this, the mechanics of column interaction in single curvature regions have been studied. For this purpose, suitable tools for sidesway description were derived, including a nearly exact, explicit free-sway effective length expression, that, when high accuracy is required, eliminates the need for cumbersome, iterative solutions of exact effective lengths from the transcendental instability equation.

Two reasons for the underestimation are identified, and their relative importance have been clarified. One is related to the local second order $N\delta$ effects, which may be greater than commonly assumed, and the other, and most important one, is related to second order effects causing changes in rotational restraint stiffness distributions at column ends due to vertical, inter-storey column interaction.

A modified approximate storey magnifier approach is proposed that accounts for these local second order effects through two separate "flexibility factors", γ_n (γ_s) and γ_k. Both factors increase the flexibility of the columns of the frame beyond their first order values. The first factor may typically vary between 1 and about 1.35. The second ("restraint correction flexibility factor") may vary between much wider limits, such as 1 and 2.8 in the examples considered.

The proposed approach has been found to provide predictions that compare well with more accurate results. It is appealing in that it is sufficiently simple to be viable in practical analyses. Comparisons with accurate analysis results for a wider range of frame parameters are recommended prior to possible inclusion in relevant codes and standards.

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