Curves of genus 2 on rational normal scrolls and scrollar syzygies

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Contents

Chapter 1

Introduction

Given a variety $X \subseteq \mathbf{P}^n$, one interesting aspect is to analyze the syzygy modules of its homogeneous ideal I_X, e.g. via the minimal free resolution of I_X.

The first interesting problem that arises is to look for a decomposition of the ideal I_X. Since for any two projective varieties X and Y such that $X \subseteq Y$, we have the reverse inclusion of the ideals, i.e. $I_Y \subseteq I_X$, in order to describe the structure of I_X, it is natural to look for varieties $Y_1, \ldots, Y_m$ that contain X and such that the union of the ideals $I_{Y_1}, \ldots, I_{Y_m}$ generates I_X. The first engaging question that now arises is whether we can find varieties $Y_1, \ldots, Y_m$ that contain X and such that I_X is generated by the union of the ideals $I_{Y_1}, \ldots, I_{Y_m}$. In addition, we naturally want m to be as small as possible.
Continuing in this direction, we may pose the same question for the higher syzygy modules of I_X. Denoting by the 0th syzygy module the ideal I_X and by r the length of the minimal free resolution of I_X, one motivating question is the following:

(∗) For all i in the range $0 \leq i \leq r$, can we find varieties $Y_1, \ldots, Y_m$ such that the ith syzygy module of I_X is generated by the union of the ith syzygy modules of the ideals $I_{Y_1}, \ldots, I_{Y_m}$?

This question is for instance considered for elliptic normal curves and their secant varieties and bielliptic canonical curves in [HvB04] and for canonical curves in [vB07].

In all these cases that have been studied, rational normal scrolls are natural candidates for the varieties $Y_1, \ldots, Y_m$. One nice property that turns rational normal scrolls into good candidates for these varieties $Y_1, \ldots, Y_m$ is the fact that for a curve C lying inside a rational normal scroll Y generated by some g^1_k on C, the resolution of I_C is obtained via a mapping cone construction using the minimal free resolution of I_Y in $\mathbf{P}^n$, and thus for $0 \leq i \leq \operatorname{codim}(Y)$, there is a natural inclusion of the ith syzygy modules, $\operatorname{Syz}_i(Y) \subseteq \operatorname{Syz}_i(C)$.

In this thesis we study smooth curves C of genus 2, embedded in $\mathbf{P}^{d-2}$ by a complete linear system of degree $d \geq 5$.

We use the notation $g^1_k(C)$ to denote a g^1_k on C.

We are interested in rational normal scrolls defined by linear systems $g^1_2(C)$ and $g^1_3(C)$'s.

A simple use of the Riemann-Roch Theorem for curves shows that there exists exactly one g^1_2 on C, and that this is equal to the canonical system $|K_C|$ on C, and that the family of g^1_3's on C is two-dimensional.

The unique $g^1_2(C)$ gives rise to one scroll S of dimension 2:

$$S = \overline{\bigcup_{E \in g^1_2(C)} \operatorname{span}(E)}.$$

We will denote a $g^1_3(C)$ by $|D|$. Each $g^1_3(C)$ gives rise to a scroll $V = V_{|D|}$ of dimension 3:

$$V_{|D|} = \overline{\bigcup_{D' \in |D|} \operatorname{span}(D')}.$$

This gives a two-dimensional family of three-dimensional rational normal scrolls that contain the curve C.
For $d \geq 6$ the ideal I_C is generated by quadrics, for $d = 5$ the ideal I_C is generated by one quadric and two cubics. Since the ideal of a rational normal scroll is generated by quadrics as well, we will mostly only consider the case $d \geq 6$ and mention the case $d = 5$ occasionally as some exceptional example.
It will be shown, from the minimal free resolution of I_C in Chapter 4, that the ith syzygy module of I_C can be generated by linear syzygies for $1 \leq i \leq d-5$. Moreover, it is a well-known fact that, for $i \geq 1$, the ith syzygy-module of I_X, where X is a rational normal scroll, can be generated by linear syzygies. We will thus focus on the linear syzygies and for any variety Z denote by $\operatorname{Syz}_i(I_Z)$ the vector space of linear ith syzygies of I_Z. In our cases, all varieties are arithmetically Cohen-Macaulay, and thus the length of the minimal free resolution is equal to the codimension, which is equal to $d-5$ for a $g^1_3(C)$-scroll.
Now, Question $(*)$ becomes in this case our following motivating question:

$(**)$ Let S be the $g^1_2(C)$-scroll, and let V run through the two-dimensional family of $g^1_3(C)$-scrolls. For fixed i such that $0 \leq i \leq d-5$, is the space of the ith linear syzygies of C spanned by the ith linear syzygies of I_S and the ith linear syzygies of *all* I_V?

We will give a positive answer to this question in the case $i = 0$, i.e. for the ideal of C. More precisely, we will show that I_C is generated by the union of I_S and the ideal of *one* $g^1_3(C)$-scroll V, that obviously does not contain S.
Considering the higher syzygies of I_C we restrict ourselves to the first syzygies and the case when the degree of C is equal to $d = 7$, the first interesting case.
We provide examples of smooth curves C and two $g^1_3(C)$-scrolls V_1 and V_2 such that the first szyygies of I_C are generated by the first syzygies of I_S and the first syyzgies of I_{V_1} and I_{V_2}.

The thesis is organized as follows:

In **Chapter 2** we introduce rational normal scrolls, in particular rational normal scrolls of dimension 2 and 3 that contain the curve C and give a description of rolling factor coordinates on a scroll, which will be useful in Chapter 3.
Moreover, we give a connection between $|H|$, the complete linear system that embeds C into $\mathbf{P}^{d-2}$, and the scroll types of the $g^1_2(C)$-scroll S and a $g^1_3(C)$-scroll $V = V_{|D|}$, for a given $g^1_3(C)$ $|D|$.

In **Chapter 3** we let $d \geq 6$. We use rolling factor coordinates on the $g^1_2(C)$-scroll S to give quadrics that together with I_S generate the ideal of C.

Chapter 4 deals with the minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module. We will give the Betti diagram of $\mathcal{O}_C$. The description of the quadrics in Chapter 3 that together with I_S generate I_C is useful here in order to describe the differentials in the resolution explicitly.

Chapter 5 contains a proof of the following main result in this chapter:

Theorem 5.1. *Let C be a curve of genus 2, linearly normal embedded in $\mathbf{P}^{d-2}$ by a complete linear system $|H|$ of degree $d \geq 6$. Then*

$$I_S + I_V = I_C$$

for a $g^1_3(C)$-scroll V that does not contain the $g^1_2(C)$-scroll S.

In **Chapter 6** we first study quadrics of rank 3 and 4 containing a curve C of degree $d = 6$ and discover a connection to a quartic Kummer surface in $\mathbf{P}^3_C = \mathbf{P}(H^0(\mathcal{I}_C(2)))$. We will give examples of smooth, singular and reducible curves of degree 6 using the computer algebra system Macaulay 2 ([GS]).

In **Chapter 7** the motivating problem is to find, for a given smooth curve $C \subseteq \mathbf{P}^5$ of genus 2 and degree 7, $g^1_3(C)$-scrolls $V_1, \ldots, V_m$ that do not contain the $g^1_2(C)$-scroll S such that the space of first syzygies of I_C is spanned by the first syzygies of I_S and the first syzygies of $I_{V_1}, \ldots, I_{V_m}$.
We will find examples of smooth curves C and two $g^1_3(C)$-scrolls V_1 and V_2 such that the space of the first syzygies of I_C is spanned by the first syzygies of I_S, the first syzygies of I_{V_1} and the first syzygies of I_{V_2} and consequently prove by semi-continuity the following theorem:

Theorem 7.4. *For a general curve $C \in \mathcal{M}_2$ and a general $\mathcal{O}_C(H) \in \mathrm{Pic}^7(C)$ such that the complete linear system $|H|$ embeds C into $\mathbf{P}^5$ as a smooth curve, there exist two $g^1_3(C)$-scrolls V_1 and V_2 such that the space of first syzygies of I_C is generated by the first syzygies of I_S, the first syzygies of I_{V_1} and the first syzygies of I_{V_2}.*

Moreover, we give families of singular and reducible curves C and three-dimensional rational normal scrolls V_i containing C such that the space of first syzygies of I_C is spanned by the first syzygies of I_S and the first syzygies of all I_{V_i}. In most of the cases two three-dimensional scrolls V_1 and V_2 are enough, in one case we give three three-dimensional scrolls V_1, V_2 and V_3.

In **Chapter 8** we use the description of the third secant variety of C, $\mathrm{Sec}_3(C)$, as the union of all $g^1_3(C)$-scrolls and thus give, for $d \geq 8$, another proof of the formula of the degree of $\mathrm{Sec}_3(C)$, which is also known as Berzolari's formula, namely that the number of trisecant lines to a smooth curve of genus 2 and degree d in $\mathbf{P}^4$ is equal to

$$\binom{d-2}{3} - 2(d-4).$$

In **Appendix A** we list the matrices that give the maps in the resolutions we found in Chapter 4 for curves of degree $d = 7$ and $d = 8$ such that the $g^1_2(C)$-scroll is maximally balanced.

In this thesis we will use basic results in algebraic geometry as in [Har77], [Ful98] and [ACGH85], sometimes without further reference.

Chapter 2

Scrolls containing C

2.1 Preliminaries

In this thesis, if not mentioned otherwise, by a curve we will always mean a non-singular and irreducible curve of genus 2.
For a positive integer m we denote by a $g^1_m(C)$ a linear system of projective dimension 1 and degree m on the curve C. In this thesis we are only interested in the cases $m = 2$ and $m = 3$ and denote by $G^1_3(C)$ the family of $g^1_3(C)$'s.

Proposition 2.1. *There exists exactly one $g^1_2(C)$, and this is equal to the canonical system $|K_C|$. The family $G^1_3(C) := \{g^1_3(C)'s\}$ is two-dimensional.*

Proof. We use the Riemann-Roch Theorem for curves (see e.g. [Har77], Thm. 1.3 in Chapter IV.1):
If D is a divisor of degree 2, then

$$h^0(\mathcal{O}_C(D)) = 1 + h^0(\mathcal{O}_C(K_C - D)) = \begin{cases} 1 & \text{if} \quad D \notin |K_C|, \\ 2 & \text{if} \quad D \in |K_C|. \end{cases}$$

Hence we can conclude that the linear system $|D|$ is a $g^1_2(C)$ if and only if $|D| = |K_C|$.
If D is a divisor of degree 3 on C, then $h^0(\mathcal{O}_C(D)) = 2$, i.e. each linear system $|D|$ of degree 3 is a $g^1_3(C)$. The set of all effective divisors of degree 3 on C is given by $C_3 := (C \times C \times C)/S_3$, where S_3 denotes the symmetric group on 3 letters. The dimension of this family is equal to 3, and since each linear system $|D|$ of degree 3 has dimension 1, as shown above, the family of $g^1_3(C)$'s has to be two-dimensional. □

Let now $|H|$ be a complete linear system on C of degree $d \geq 5$. Since $d \geq 2g + 1$, $|H|$ is very ample, and thus $|H|$ embeds the curve into projective space. By the Riemann-Roch Theorem for curves we obtain $h^0(\mathcal{O}_C(H)) = d - 1$, and this yields an embedding

$$C \overset{\phi_{|H|}}{\hookrightarrow} \mathbf{P}^{d-2}.$$

Since $|H|$ is complete, the embedded curve $C \subseteq \mathbf{P}^{d-2}$ is linearly normal.

In this thesis we will work with rational normal scrolls of dimension 2 and 3 that contain the curve C. There are several different presentations of a rational normal scroll. We will use the following two (cf. [Sch86], [Ste02]):

A first definition of rational normal scrolls

Let $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_k)$ be a locally free sheaf of rank k on $\mathbf{P}^1$, and let $\pi : \mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$ be the corresponding $\mathbf{P}^{k-1}$-bundle. Moreover, let $e_1 \geq e_2 \geq \cdots \geq e_k \geq 0$ and $e_1 + e_2 + \cdots + e_k \geq 2$.

Definition 2.2. *A rational normal scroll X is the image of $\iota : \mathbf{P}(\mathcal{E}) \hookrightarrow \mathbf{P}^N := \mathbf{P}H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))$.*
We define the scroll type of X to be equal to $(e_1, e_2, \cdots, e_k)$.
We say that a scroll X of scroll type $(e_1, e_2, \ldots, e_k)$ is maximally balanced if $e_1 - e_k \leq 1$.

Remark 2.3. *The dimension of X is equal to k, and the degree of X is equal to the degree of $\mathcal{E}$ which is equal to $f := \sum_{i=1}^k e_i$. Moreover, by the Riemann-Roch Theorem for vector bundles, $h^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = h^0(\mathbf{P}^1, \mathcal{E}) = \mathrm{rk}(\mathcal{E}) + \deg(\mathcal{E}) = k + \sum_{i=1}^k e_i$, i.e. the dimension of the ambient projective space is equal to $N = k + \sum_{i=1}^k e_i - 1$. Thus for a rational normal scroll X we obtain $\dim(X) + \deg(X) = k + \sum_{i=1}^k e_i = N + 1$, and consequently a rational normal scroll $X \subseteq \mathbf{P}^N$ is a non-degenerate irreducible variety of minimal degree $f = \mathrm{codim}(X) + 1$.*

Remark 2.4. *The scroll X is smooth if and only if all e_i, $i = 1, \ldots, k$, are positive. If this is the case, then $\iota : \mathbf{P}(\mathcal{E}) \to X \subseteq \mathbf{P}^N$ is an isomorphism. If X is singular, then $\iota : \mathbf{P}(\mathcal{E}) \to X \subseteq \mathbf{P}^N$ is a resolution of singularities.*

Proposition 2.5. *Each linearly normal scroll X over $\mathbf{P}^1$ is a rational normal scroll.*

Proof. If X is a linearly normal scroll over $\mathbf{P}^1$, then $X = \iota(\mathbf{P}(\mathcal{E}))$, where $\mathcal{E} = \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is a vector bundle over $\mathbf{P}^1$ and $\iota : \mathbf{P}(\mathcal{E}) \hookrightarrow \mathbf{P}(H^0(\mathcal{E}))$. By Grothendieck's splitting Theorem (cf. [HM82]) every vector bundle over $\mathbf{P}^1$ splits, i.e. $\mathcal{E}$ is of the form $\mathcal{E} = \oplus_i \mathcal{O}_{\mathbf{P}^1}(e_i)$. □

From now on by a scroll we will always mean a rational normal scroll.

An alternative description of rational normal scrolls

Now we will come to a more geometric description of a rational normal scroll X: With the e_i as above, let for $i = 1, \ldots, k$, $\phi_i : \mathbf{P}^1 \to C_i \subseteq \mathbf{P}^{e_i} \subseteq \mathbf{P}^N$, where $N = \sum_{i=1}^k e_i - k - 1$, parametrize a rational normal curve of degree e_i, $i = 1, \ldots, k$ such that $\mathbf{P}^{e_1}, \ldots, \mathbf{P}^{e_k}$ span the whole $\mathbf{P}^N$. Then

$$X = \overline{\bigcup_{P \in \mathbf{P}^1} \langle \phi_1(P), \ldots, \phi_k(P) \rangle}$$

is a rational normal scroll of dimension k, degree $e_1 + \cdots + e_k$ and scroll type $(e_1, \ldots, e_k)$. In other words, each fiber of X is spanned by k points where each of these lies on a different rational normal curve. We call these k rational normal curves C_i directrix curves of the scroll.

The Picard group of rational normal scrolls

Let $H = [\iota^* \mathcal{O}_{\mathbf{P}^N}(1)]$ denote the hyperplane class and $F = [\pi^* \mathcal{O}_{\mathbf{P}^1}(1)]$ be the class of a fiber of $\mathbf{P}(\mathcal{E})$. In the following we will use H and F to denote both the classes and divisors in the respective classes.

The *Picard group* of $\mathbf{P}(\mathcal{E})$ is generated by H and F:

$$\mathrm{Pic}(\mathbf{P}(\mathcal{E})) = \mathbf{Z}[H] \oplus \mathbf{Z}[F].$$

We have the following intersection products:

$$H^k = f, \quad H^{k-1}.F = 1, \quad F^2 = 0.$$

A minimal section of $\mathbf{P}(\mathcal{E})$ is given by $[B_0] = H - rF$ where $r \in \mathbf{N}$ is maximal such that $H - rF$ is effective, in other words, $[B_0] = H - e_1 F$.

2.2 Rolling factor coordinates

We can describe coordinates on the scroll $\mathbf{P}(\mathcal{E})$ via coordinates on $\mathbf{P}^1$ and coordinates in each fiber of $\mathbf{P}(\mathcal{E})$. These are called *rolling factor coordinates*. We will now describe how we can choose coordinates in $\mathbf{P}^N$ in such a way that they restrict to these rolling factor coordinates on $\mathbf{P}(\mathcal{E})$:
Let $(s : t)$ be the homogeneous coordinates in $\mathbf{P}^1$ and $(z_1 : z_2 : \ldots : z_k)$ be the homogeneous coordinates on a fiber of $\mathbf{P}(\mathcal{E})$. Then we can choose coordinates x_{ij}, $i = 1, \ldots, k$, $j = 0, \ldots, e_i$ on $\mathbf{P}^N$ such that

$$x_{ij}|_{\mathbf{P}(\mathcal{E})} = s^{e_i - j} t^j z_i.$$

We assign the degree $-e_i$ to z_i, i.e. the coordinates all have weighted degree 0.
It is straightforward to check that the (2×2)-minors of the matrix

$$\begin{pmatrix} s^{e_1} z_1 & s^{e_1-1} t z_1 & \cdots & s t^{e_1-1} z_1 & \cdots & s^{e_k} z_k & s^{e_k-1} t z_k & \cdots & s t^{e_k-1} z_k \\ s^{e_1-1} t z_1 & s^{e_1-2} t^2 z_1 & \cdots & t^{e_1} z_1 & \cdots & s^{e_k-1} t z_k & s^{e_k-2} t^2 z_k & \cdots & t^{e_k} z_k \end{pmatrix}$$

are equal to 0. In fact, the ideal of $\mathbf{P}(\mathcal{E})$ in $\mathbf{P}^N$ is generated by the (2×2)-minors of the following matrix:

$$M = \begin{pmatrix} x_{10} & x_{11} & \cdots & x_{1,e_1-1} & \cdots & x_{k0} & x_{k1} & \cdots & x_{k,e_k-1} \\ x_{11} & x_{12} & \cdots & x_{1,e_1} & \cdots & x_{k1} & x_{k2} & \cdots & x_{k,e_k} \end{pmatrix}.$$

Proposition 2.6. *Let $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_k)$ with all $e_i \geq 0$. For all $a \geq 0$ there is an isomorphism*

$$H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(aH + bF)) \cong H^0(\mathbf{P}^1, \mathcal{S}\mathrm{ym}^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^1}(b)).$$

Proof. By the rolling factor coordinate construction each divisor in $H^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(aH+bF))$ can be represented by a bihomogeneous polynomial of degree a in the z_i and total degree b. □

Corollary 2.7. *Set $f := \sum_{i=1}^k e_i$. For all $a \geq 0$, $b \geq -1$*

$$h^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(aH + bF)) = f \binom{a+k-1}{k} + (b+1) \binom{a+k-1}{k-1},$$

in particular, this dimension does not depend on the scroll type of X, only the degree and the dimension of X.

Proof. We will use the isomorphism in Proposition 2.6 and consequently compute the dimension of

$$H^0(\mathbf{P}^1, \mathcal{S}\mathrm{ym}^a(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_k)) \otimes \mathcal{O}_{\mathbf{P}^1}(b)).$$

Since

$$\begin{aligned} & \mathcal{S}\mathrm{ym}^a(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_k)) \otimes \mathcal{O}_{\mathbf{P}^1}(b) \\ = \; & \oplus_{i=1}^k \mathcal{O}_{\mathbf{P}^1}(ae_i + b) \oplus \oplus_{i=1}^k \oplus_{j \neq i} \mathcal{O}_{\mathbf{P}^1}((a-1)e_i + e_j + b) \\ & \oplus \oplus_{i=1}^k \oplus_{j,l \neq i} \mathcal{O}_{\mathbf{P}^1}((a-2)e_i + e_j + e_l + b) \oplus \cdots \end{aligned}$$

we obtain

$$\begin{aligned} & h^0(\mathbf{P}^1, \mathcal{S}\mathrm{ym}^a(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_k)) \otimes \mathcal{O}_{\mathbf{P}^1}(b)) \\ = \; & f \sum_{i=0}^{a-1} (a-i) \binom{k-2+i}{i} + (b+1) \sum_{i=0}^{a} \sum_{l=0}^{i} \binom{k-3+l}{l} \\ = \; & f \left(a \binom{a+k-2}{k-1} - \sum_{i=1}^{a-1} i \binom{k-2+i}{k-2} \right) + (b+1) \binom{k-1+a}{a} \\ = \; & f \sum_{i=0}^{a-1} \binom{k-1+i}{k-1} + (b+1) \binom{k-1+a}{k-1} \\ = \; & f \binom{a+k-1}{k} + (b+1) \binom{a+k-1}{k-1}. \end{aligned}$$

□

Proposition 2.8. *A rational normal scroll X is projectively normal.*

Proof. In order to show that

$$H^0(\mathcal{O}_{\mathbf{P}^N}(mH)) \to H^0(\mathcal{O}_X(mH))$$

is surjective for all integers $m \geq 1$, we will use rolling factor coordinates on X.
By Proposition 2.6 there is an isomorphism

$$H^0(X, \mathcal{O}_X(mH)) \;\cong\; H^0(\mathbf{P}^1, \mathcal{S}\mathrm{ym}^m(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_k))),$$

and we find that a section in $H^0(X, \mathcal{O}_X(mH))$ can be identified with a polynomial of degree m in the z_i's and of degree $r_1e_1 + \cdots + r_ke_k$ with $r_1 + \cdots + r_k = m$ in (s,t). More precisely, the sections of the form $s^{r_1e_1+r_2e_2+\cdots+r_ke_k-l}t^l z_1^{r_1} z_2^{r_2} \cdots z_k^{r_k}$, where $r_1 + \cdots + r_k = m$ and $l = 0, \ldots, r_1e_1 + r_2e_2 + \cdots + r_ke_k$, form a basis of $H^0(X, \mathcal{O}_X(mH))$. Obviously, all these sections can be realized as polynomials of degree m in the restrictions of the x_{ij}. □

2.3 $g_2^1(C)$-scrolls and $g_3^1(C)$-scrolls

Now we consider our curve $C \subseteq \mathbf{P}^{d-2}$ of genus 2 and degree $d \geq 6$. From the $g_2^1(C)$ and the $g_3^1(C)$'s we construct rational normal scrolls that contain the curve C in a natural way:

Set

$$S = \overline{\bigcup_{E \in g^1_2(C)} \operatorname{span}(E)},$$

where by $\operatorname{span}(E)$ we denote the line between the two points in E.
For each $|D| \in G^1_3(C)$ we set

$$V_{|D|} = \overline{\bigcup_{D' \in |D|} \operatorname{span}(D')},$$

where by $\operatorname{span}(D')$ we denote the plane spanned by the three points in D'.

Proposition 2.9. *Let $C \subseteq \mathbf{P}^{d-2}$ be a linearly normal curve of degree d and genus 2, let S be the $g^1_2(C)$-scroll, and for a $g^1_3(C)$ $|D|$ let $V_{|D|}$ be the $g^1_3(C)$-scroll associated to $|D|$. The scrolls S and $V_{|D|}$ are rational normal scrolls.*

Proof. The rationality of S and each $V_{|D|}$ is obvious. For the rest notice that if a scroll X contains a linearly normal curve C, then also X has to be linearly normal: If X was the image of a non-degenerate variety in higher-dimensional projective space under some projection, then C had to be as well. We conclude that since C is linearly normal, all $V_{|D|}$ and S are linearly normal. By Proposition 2.5 we can conclude that S and all $V_{|D|}$ are rational normal scrolls. □

Note that the dimension of S is equal to $\dim |K_C| + \dim \operatorname{span}(E) = 2$ and that the dimension of $V_{|D|}$ is equal to $\dim |D| + \dim \operatorname{span}(D') = 3$. By Proposition 2.9 S and $V_{|D|}$ are rational normal scrolls which implies by the observations in Remark 2.3 that we obtain the following degrees:

$$\deg S = d - 3, \qquad \deg V_{|D|} = d - 4. \tag{2.1}$$

The next proposition will be useful in Chapter 4, when we study the resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module via the resolution of $\mathcal{O}_C$ as $\mathcal{O}_S$-module and the resolution of $\mathcal{O}_S$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module:

Proposition 2.10. *If $X \subseteq \mathbf{P}^N$ is a rational normal scroll of degree f, then the minimal free resolution of $\mathcal{O}_X$ as $\mathcal{O}_{\mathbf{P}^N}$-module is linear.*

Proof. The minimal free resolution of $\mathcal{O}_X$ as $\mathcal{O}_{\mathbf{P}^N}$-module is given by the Eagon-Northcott complex associated to the map

$$\Phi : \mathcal{O}^f_{\mathbf{P}^{d-2}}(-1) \to \mathcal{O}^2_{\mathbf{P}^{d-2}},$$

which is given by multiplication with the matrix M as described above, which (2×2)-minors generate the ideal I_X . □

Proposition 2.11. *Let C be a curve of genus 2, and let $\mathcal{E}$ be a $\mathbf{P}^1$-bundle such that the image of $\iota : \mathbf{P}(\mathcal{E}) \to \mathbf{P}^{d-2}$ is the $g^1_2(C)$-scroll S.*

(i) The class of C on $\mathbf{P}(\mathcal{E})$ is equal to $[C] = 2H - (d-6)F$.

(ii) The class of the canonical divisor $K_{\mathbf{P}(\mathcal{E})}$ on $\mathbf{P}(\mathcal{E})$ is given by $[K_{\mathbf{P}(\mathcal{E})}] = -2H + (d-5)F$.

Proof. (i) Write $[C] = aH + bF$ with $a, b \in \mathbf{Z}$. Since $[C].F = 2$, we obtain $a = 2$, and $[C].H = d$ implies that $d = 2(d-3) + b$, i.e. $b = 6 - d$.

(ii) Write $[K_{\mathbf{P}(\mathcal{E})}] = aH + bF$ with $a, b \in \mathbf{Z}$ and use the adjunction formula: $-2 = ([K_{\mathbf{P}(\mathcal{E})}] + F).F = a$ and $-2 = ([K_{\mathbf{P}(\mathcal{E})}] + H).H = -(d-3) + b$, i.e. $b = d - 5$. □

Proposition 2.12. *Let C be a curve of genus 2, and let $\mathcal{E}$ be a $\mathbf{P}^1$-bundle such that the image of $\iota : \mathbf{P}(\mathcal{E}) \to \mathbf{P}^{d-2}$ is a $g_3^1(C)$-scroll V such that C does not pass through the (possibly empty) singular locus of V.*
The class of C on $\mathbf{P}(\mathcal{E})$ is equal to $[C] = 3H^2 - 2(d-6)H.F$.

Proof. Since C is of codimension 2 on $\mathbf{P}(\mathcal{E})$, we can write the class of C on $\mathbf{P}(\mathcal{E})$ as $[C] = aH^2 + bH.F$ with $a, b \in \mathbf{Z}$. Since $[C].F = 3$, we obtain $a = 3$ and $[C].H = d$ implies that $d = 3(d-4) + b$, i.e. $b = 2(6-d)$. □

2.4 Possible scroll types

Proposition 2.13. *For the scroll type (e_1, e_2) of S we have $e_1 - e_2 \leq 3$.*

Proof. Let $[C_0] = H - e_1F$ as described as $[B_0]$ in general in Section 2.1.
Since C and C_0 are effective and C is smooth, so $C_0 \not\subseteq C$, we have $[C].[C_0] \geq 0$, which means by Proposition 2.11 that $(2H - (d-6)F).(H - e_1F) \geq 0$, consequently $2e_1 + 2e_2 - 2e_1 - (d-6) \geq 0$. Since $d = e_1 + e_2 + 3$ the result follows. □

Proposition 2.14. *If V is a $g_3^1(C)$-scroll such that the curve C does not intersect the (possibly empty) singular locus of V, then for its scroll type (e_1, e_2, e_3) we have $2e_1 - e_2 - e_3 \leq 4$.*

Proof. Since $h^0(\mathcal{O}_V(H - B_0)) = h^0(\mathcal{O}_V(e_1F)) = e_1 + 1 \geq 1$, B_0 is contained in at least one hyperplane, consequently B_0 does not span all of $\mathbf{P}^{d-2}$. Since C spans all of $\mathbf{P}^{d-2}$, B_0 cannot contain C, thus we have that $[C].[B_0] \geq 0$, i.e. by Proposition 2.12 $(3H^2 - 2(d-6)HF).(H - e_1F) \geq 0$, which means that $3e_1 + 3e_2 + 3e_3 - 3e_1 - 2(d-6) \geq 0$. The result follows from $d = e_1 + e_2 + e_3 + 4$. □

Proposition 2.15. *If $V = V_{|D|}$ is a singular scroll of scroll type $(e_1, e_2, 0)$ such that the curve C intersects the singular locus of V, then e_1 and e_2 satisfy the following: $e_1 - e_2 \leq 3$.*

Proof. If $V = V_{|D|}$ is a singular $g_3^1(C)$-scroll of type $(e_1, e_2, 0)$ such that C intersects its singular locus, then a point $P \in \mathrm{sing}(V) \cap C$ is a basepoint of $|D|$, i.e. $|D| = |K_C + P|$. The projection from P maps C to a curve C' of degree $d-1$ in $\mathbf{P}^{d-3}$ and it maps $V_{|D|}$ to the $g_2^1(C')$-scroll of type (e_1, e_2). By Proposition 2.13 we obtain $e_1 - e_2 \leq 3$. □

We will come to the converse of Proposition 2.15, namely the existence part, after we have described a method to find the scroll type of the $g_2^1(C)$-scroll S and a $g_3^1(C)$-scroll $V = V_{|D|}$, given a class $|H|$ of degree d on C that embeds the curve into $\mathbf{P}^{d-2}$. This method we found in [Sch86], p. 114.

(1) The scroll type of S:
Set

$$\begin{aligned}
d_0 &= h^0(\mathcal{O}_C(H)) - h^0(\mathcal{O}_C(H-K_C)),\\
d_1 &= h^0(\mathcal{O}_C(H-K_C)) - h^0(\mathcal{O}_C(H-2K_C)),\\
d_2 &= h^0(\mathcal{O}_C(H-2K_C)) - h^0(\mathcal{O}_C(H-3K_C)),\\
&\vdots\\
d_{\lfloor\frac{d}{2}\rfloor} &= h^0(\mathcal{O}_C(H-\lfloor\frac{d}{2}\rfloor K_C)) - \underbrace{h^0(\mathcal{O}_C(H-(\lfloor\frac{d}{2}\rfloor+1)K_C))}_{=0}.
\end{aligned}$$

Then the scroll type (e_1, e_2) of S is given by

$$\begin{aligned}
e_1 &= \#\{j|d_j \geq 1\} - 1,\\
e_2 &= \#\{j|d_j \geq 2\} - 1.
\end{aligned}$$

(2) The scroll type of $V = V_{|D|}$:
Here we will distinguish between two cases, namely whether $|D|$ is basepoint-free or has one basepoint:

(a) If $|D|$ is basepoint-free, then we set

$$\begin{aligned}
d_0 &= h^0(\mathcal{O}_C(H)) - h^0(\mathcal{O}_C(H-D)),\\
d_1 &= h^0(\mathcal{O}_C(H-D)) - h^0(\mathcal{O}_C(H-2D)),\\
d_2 &= h^0(\mathcal{O}_C(H-2D)) - h^0(\mathcal{O}_C(H-3D)),\\
&\vdots\\
d_{\lfloor\frac{d}{3}\rfloor} &= h^0(\mathcal{O}_C(H-\lfloor\frac{d}{3}\rfloor D)) - \underbrace{h^0(\mathcal{O}_C(H-(\lfloor\frac{d}{3}\rfloor+1)D))}_{=0}.
\end{aligned}$$

Then the scroll type (e_1, e_2, e_3) of V is given by

$$\begin{aligned}
e_1 &= \#\{j|d_j \geq 1\} - 1,\\
e_2 &= \#\{j|d_j \geq 2\} - 1,\\
e_3 &= \#\{j|d_j \geq 3\} - 1.
\end{aligned}$$

(b) If the $g^1_3(C)$ $|D|$ has one basepoint, i.e. $|D| = |K_C + P|$, then we set

$$
\begin{aligned}
d_0 &= h^0(\mathcal{O}_C(H)) - h^0(\mathcal{O}_C(H-D)),\\
d_1 &= h^0(\mathcal{O}_C(H-D)) - h^0(\mathcal{O}_C(H-P-2K_C)),\\
d_2 &= h^0(\mathcal{O}_C(H-P-2K_C)) - h^0(\mathcal{O}_C(H-P-3K_C)),\\
&\vdots\\
d_{\lfloor\frac{d-1}{2}\rfloor} &= h^0(\mathcal{O}_C(H-P-\lfloor\frac{d-1}{2}\rfloor K_C))\\
&- \underbrace{h^0(\mathcal{O}_C(H-P-\lfloor\frac{d+1}{2}\rfloor K_C))}_{=0}.
\end{aligned}
$$

The scroll type (e_1, e_2, e_3) of V is given by

$$
\begin{aligned}
e_1 &= \#\{j|d_j \geq 1\} - 1,\\
e_2 &= \#\{j|d_j \geq 2\} - 1,\\
e_3 &= \#\{j|d_j \geq 3\} - 1.
\end{aligned}
$$

We will now discuss the connection between the scroll type and $|H|$ and thus give alternative proofs of Propositions 2.13 and 2.14.
Let us first take a look at the scroll type (e_1, e_2) of S:

(1) Let us first consider the case when d is even:
Since every linear system of degree 2 is non-empty by the Riemann-Roch Theorem for curves we can always write $|H| = |\frac{d-2}{2}K_C + P + Q|$, and there are exactly two possibilities for $P+Q$, namely it is either a divisor in $|K_C|$ or it is not.

(a) If $|H| = |\frac{d}{2}K_C|$, then the d_i are as follows:

$$
\begin{aligned}
d_0 &= d-1-(d-3) = 2,\\
&\vdots\\
d_{\frac{d}{2}-3} &= h^0(\mathcal{O}_C(3K_C)) - h^0(\mathcal{O}_C(2K_C)) = 2,\\
d_{\frac{d}{2}-2} &= h^0(\mathcal{O}_C(2K_C)) - h^0(\mathcal{O}_C(K_C)) = 1,\\
d_{\frac{d}{2}-1} &= h^0(\mathcal{O}_C(K_C)) - h^0(\mathcal{O}_C) = 1,\\
d_{\frac{d}{2}} &= h^0(\mathcal{O}_C) = 1.
\end{aligned}
$$

Thus, $e_1 = \frac{d}{2}$ and $e_2 = \frac{d}{2} - 3$.

(b) In the case $|H| = |\frac{d-2}{2}K_C + P + Q|$ with $P+Q \notin |K_C|$ we have

$$
\begin{aligned}
d_0 &= d-1-(d-3) = 2,\\
&\vdots\\
d_{\frac{d}{2}-2} &= h^0(\mathcal{O}_C(K_C+P+Q)) - h^0(\mathcal{O}_C(P+Q)) = 2,\\
d_{\frac{d}{2}-1} &= h^0(\mathcal{O}_C(P+Q)) = 1,\\
d_{\frac{d}{2}} &= 0.
\end{aligned}
$$

Thus, $e_1 = \frac{d}{2} - 1$ and $e_2 = \frac{d}{2} - 2$.

(2) If d is odd, then there are also two possibilities for $|H|$:
Since every linear system of degree 3 is non-empty by the Riemann-Roch Theorem for curves, we can write $|H| = |\frac{d-3}{2}K_C + P + Q + R|$, where either one of $P+Q$, $P+R$ or $Q+R$ is a divisor in $|K_C|$ or none of those three sums is a divisor in $|K_C|$.

(a) If $|H| = |\frac{d-1}{2}K_C + P|$, then we obtain

$$\begin{aligned} d_0 &= d-1-(d-3) = 2, \\ &\vdots \\ d_{\frac{d-1}{2}-2} &= h^0(\mathcal{O}_C(2K_C+P)) - h^0(\mathcal{O}_C(K_C+P)) = 2, \\ d_{\frac{d-1}{2}-1} &= h^0(\mathcal{O}_C(K_C+P)) - h^0(\mathcal{O}_C(P)) = 1, \\ d_{\frac{d-1}{2}} &= h^0(\mathcal{O}_C(P)) = 1. \end{aligned}$$

Hence, $e_1 = \frac{d-1}{2}$ and $e_2 = \frac{d-1}{2} - 2$.

(b) If $|H| = |\frac{d-3}{2}K_C + P + Q + R|$, where none of the divisors $P+Q$, $P+R$, $Q+R$ lies in $|K_C|$, then we have

$$\begin{aligned} d_0 &= d-1-(d-3) = 2, \\ &\vdots \\ d_{\frac{d-1}{2}-2} &= h^0(\mathcal{O}_C(K_C+P+Q+R)) - h^0(\mathcal{O}_C(P+Q+R)) = 2, \\ d_{\frac{d-1}{2}-1} &= h^0(\mathcal{O}_C(P+Q+R)) = 2, \\ d_{\frac{d-1}{2}} &= 0. \end{aligned}$$

Consequently, $e_1 = e_2 = \frac{d-3}{2}$.

Now we will study the scroll type (e_1, e_2, e_3) of $V = V_{|D|}$ in the case when $|D|$ is basepoint-free:

(1) Let us first consider the case $d \equiv_{(3)} 0$:
Since every linear system of degree 3 is non-empty by the Riemann-Roch Theorem for curves, we can write $|H| = |\frac{d-3}{3}D + P + Q + R|$, where either $P+Q+R$ is a divisor in $|D|$ or it is not. If it is not a divisor in $|D|$, then since $h^0(\mathcal{O}_C(P+Q+R)) = 2$, $P+Q+R$ is a divisor in some other pencil $|D'|$.

(a) If $|H| = |\frac{d}{3}D|$, then

$$\begin{aligned} d_0 &= d-1-(d-4)=3,\\ &\vdots\\ d_{\frac{d}{3}-2} &= h^0(\mathcal{O}_C(2D))-h^0(\mathcal{O}_C(D))=3,\\ d_{\frac{d}{3}-1} &= h^0(\mathcal{O}_C(D))-h^0(\mathcal{O}_C)=1,\\ d_{\frac{d}{3}} &= h^0(\mathcal{O}_C)=1. \end{aligned}$$

Consequently, $e_1=\frac{d}{3}$, $e_2=e_3=\frac{d}{3}-2$.

(b) If $|H|=|\frac{d-3}{3}D+D'|$ with $|D'|\neq|D|$, then:

$$\begin{aligned} d_0 &= d-1-(d-4)=3,\\ &\vdots\\ d_{\frac{d}{3}-2} &= h^0(\mathcal{O}_C(D+D'))-h^0(\mathcal{O}_C(D'))=3,\\ d_{\frac{d}{3}-1} &= h^0(\mathcal{O}_C(D'))=2,\\ d_{\frac{d}{3}} &= 0. \end{aligned}$$

Thus, $e_1=e_2=\frac{d}{3}-1$ and $e_3=\frac{d}{3}-2$.

(2) The next case to consider is the case when $d\equiv_{(3)}1$: By similar arguments as in (1) we can write $|H|=|\frac{d-4}{3}D+P_1+P_2+P_3+P_4|$ where either one combination $P_i+P_j+P_k$ is a divisor in $|D|$ or else some combination $P_i+P_j+P_k$ is a divisor in a suitable linear system $|D'|$.

(a) If $|H|=|\frac{d-1}{3}D+P|$, then

$$\begin{aligned} d_0 &= 3,\\ &\vdots\\ d_{\frac{d-1}{3}-2} &= h^0(\mathcal{O}_C(2D+P))-h^0(\mathcal{O}_C(D+P))=3,\\ d_{\frac{d-1}{3}-1} &= h^0(\mathcal{O}_C(D+P))-h^0(\mathcal{O}_C(P))=2,\\ d_{\frac{d-1}{3}} &= h^0(\mathcal{O}_C(P))=1. \end{aligned}$$

Thus, $e_1=\frac{d-1}{3}$, $e_2=\frac{d-1}{3}-1$, $e_3=\frac{d-1}{3}-2$.

(b) If $|H|=|\frac{d-4}{3}D+D'+P|$, then

$$\begin{aligned} d_0 &= 3,\\ &\vdots\\ d_{\frac{d-1}{3}-2} &= h^0(\mathcal{O}_C(D+D'+P))-h^0(\mathcal{O}_C(D'+P))=3,\\ d_{\frac{d-1}{3}-1} &= h^0(\mathcal{O}_C(D'+P))=3,\\ d_{\frac{d-1}{3}} &= 0. \end{aligned}$$

Thus, $e_1 = e_2 = e_3 = \frac{d-1}{3} - 1$.

(3) The last case to consider is the case when $d \equiv_{(3)} 2$:
Since every linear system of degree 2 is non-empty by the Riemann-Roch Theorem for curves, we have two possibilities for $|H|$:
Either $|H| = |\frac{d-2}{3}D + K_C|$ or $|H| = |\frac{d-2}{3}D + P + Q|$, $P + Q \notin |K_C|$.

(a) If $|H| = |\frac{d-2}{3}D + K_C|$, then the d_i's are of the following form:

$$\begin{aligned} d_0 &= 3, \\ &\vdots \\ d_{\frac{d-2}{3}-2} &= h^0(\mathcal{O}_C(2D + K_C)) - h^0(\mathcal{O}_C(D + K_C)) = 3, \\ d_{\frac{d-2}{3}-1} &= h^0(\mathcal{O}_C(D + K_C)) - h^0(\mathcal{O}_C(K_C)) = 2, \\ d_{\frac{d-2}{3}} &= h^0(\mathcal{O}_C(K_C)) = 2. \end{aligned}$$

Consequently, $e_1 = e_2 = \frac{d-2}{3}$ and $e_3 = \frac{d-2}{3} - 2$.

(b) If $|H| = |\frac{d-2}{3}D + P + Q|$, $P + Q \notin |K_C|$, then we have

$$\begin{aligned} d_0 &= 3, \\ &\vdots \\ d_{\frac{d-2}{3}-2} &= h^0(\mathcal{O}_C(2D + P + Q)) - h^0(\mathcal{O}_C(D + P + Q)) = 3, \\ d_{\frac{d-2}{3}-1} &= h^0(\mathcal{O}_C(D + P + Q)) - h^0(\mathcal{O}_C(P + Q)) = 3, \\ d_{\frac{d-2}{3}} &= h^0(\mathcal{O}_C(P + Q)) = 1. \end{aligned}$$

Hence, $e_1 = \frac{d-2}{3}$ and $e_2 = e_3 = \frac{d-2}{3} - 1$.

Finally, we will study the connection between $|H|$ and the scroll type of $V_{|D|}$ in the case when the linear system $|D|$ has one basepoint, i.e. $|D| = |K_C + P|$. Since a $g^1_3(C)$-scroll in the case $d = 5$ is all of $\mathbf{P}^3$, the interesting cases here are given by $d \geq 6$. Notice that if $|D|$ has one basepoint, then the scroll $V_{|D|}$ is necessarily singular, the curve C passes through the singular locus. Thus we already know that $e_3 = 0$.
We divide again into the cases $d \equiv_{(3)} 0$, $d \equiv_{(3)} 1$ and $d \equiv_{(3)} 2$.
There is the following connection between $|H|$ and the scroll type of $V_{|D|}$:

(1) $d \equiv_{(3)} 0$:

$\lvert H\rvert$	scroll type of $V_{\lvert D\rvert}$
$\lvert\frac{d}{3}D\rvert = \lvert\frac{d}{3}K_C + \frac{d}{3}P\rvert$	$(2,0,0)$ if $d=6$ $(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$ if $d \geq 9, 2P \notin \lvert K_C\rvert$ $(\lfloor\frac{d-1}{2}\rfloor, \lfloor\frac{d-6}{2}\rfloor, 0)$ if $d \geq 9, 2P \in \lvert K_C\rvert$
$\lvert\frac{d-3}{d}D + D'\rvert$ $= \lvert\frac{d-3}{3}K_C + \frac{d-3}{3}P + D'\rvert$, $\lvert D'\rvert$ basepoint-free	$(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$
$\lvert\frac{d-3}{3}D + D'\rvert$ $= \lvert\frac{d}{3}K_C + \frac{d-3}{3}P + Q\rvert$, $\lvert D'\rvert$ with one basepoint Q, $P + Q \notin \lvert K_C\rvert$	$(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$ if $\lvert H - P - \lfloor\frac{d-1}{2}\rfloor K_C\rvert = \emptyset$ $(\lfloor\frac{d-1}{2}\rfloor, \lfloor\frac{d-6}{2}\rfloor, 0)$ if $\lvert H - P - \lfloor\frac{d-1}{2}\rfloor K_C\rvert \neq \emptyset$
$\lvert\frac{d-3}{3}D + D'\rvert$ $= \lvert\frac{d+3}{3}K_C + \frac{d-6}{3}P\rvert$, $\lvert D'\rvert$ with one basepoint Q, $P + Q \in \lvert K_C\rvert$	$(4,1,0)$ if $d=9$ $(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$ if $d \geq 12, P \neq Q$ $(\lfloor\frac{d-1}{2}\rfloor, \lfloor\frac{d-6}{2}\rfloor, 0)$ if $d \geq 12, P = Q$

(2) $d \equiv_{(3)} 1$:

$\lvert H\rvert$	scroll type of $V_{\lvert D\rvert}$
$\lvert\frac{d-1}{3}D + Q\rvert$ $= \lvert\frac{d-1}{3}K_C + \frac{d-1}{3}P + Q\rvert$, $P + Q \notin \lvert K_C\rvert$	$(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$ if $\lvert H - P - \lfloor\frac{d-1}{2}\rfloor K_C\rvert = \emptyset$ $(\lfloor\frac{d-1}{2}\rfloor, \lfloor\frac{d-6}{2}\rfloor, 0)$ if $\lvert H - P - \lfloor\frac{d-1}{2}\rfloor K_C\rvert \neq \emptyset$
$\lvert\frac{d-1}{3}D + Q\rvert$ $= \lvert\frac{d-1}{3}K_C + \frac{d-1}{3}P + Q\rvert$ $= \lvert\frac{d+2}{3}K_C + \frac{d-4}{3}P\rvert$ $= \lvert\frac{d-4}{3}D + 2K_C\rvert$, $P + Q \in \lvert K_C\rvert$	$(3,0,0)$ if $d=7$ $(4,2,0)$ if $d=10$ $(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$ if $d \geq 13, P \neq Q$ $(\lfloor\frac{d-1}{2}\rfloor, \lfloor\frac{d-6}{2}\rfloor, 0)$ if $d \geq 13, P = Q$
$\lvert\frac{d-4}{3}D + \sum_{i=1}^{4} Q_i\rvert$ $= \lvert\frac{d-1}{3}K_C + \frac{d-4}{3}P + R + R'\rvert$, $\lvert\sum_{i=1}^{4} Q_i\rvert = \lvert K_C + R + R'\rvert$, $R + R' \notin \lvert K_C\rvert$	$(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$ if $\lvert H - P - \lfloor\frac{d-1}{2}\rfloor K_C\rvert = \emptyset$ $(\lfloor\frac{d-1}{2}\rfloor, \lfloor\frac{d-6}{2}\rfloor, 0)$ if $\lvert H - P - \lfloor\frac{d-1}{2}\rfloor K_C\rvert \neq \emptyset$

(3) $d \equiv_{(3)} 2$:

$\lvert H\rvert$	Scroll type of $V_{\lvert D\rvert}$
$\lvert \frac{d-2}{3}D + K_C\rvert$ $= \lvert \frac{d+1}{3}K_C + \frac{d-2}{3}P\rvert$	$(3,1,0)$ if $d=8$ $(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$ if $d \geq 11, 2P \notin \lvert K_C\rvert$ $(\lfloor\frac{d-1}{2}\rfloor, \lfloor\frac{d-6}{2}\rfloor, 0)$ if $d \geq 11, 2P \in \lvert K_C\rvert$
$\lvert \frac{d-2}{3}D + Q_1 + Q_2\rvert$ $= \lvert \frac{d-2}{3}K_C + \frac{d-2}{3}P + Q_1 + Q_2\rvert$, $Q_1+Q_2 \notin \lvert K_C\rvert$	$(\lfloor\frac{d-3}{2}\rfloor, \lfloor\frac{d-4}{2}\rfloor, 0)$ if $\lvert H - P - \lfloor\frac{d-1}{2}\rfloor K_C\rvert = \emptyset$ $(\lfloor\frac{d-1}{2}\rfloor, \lfloor\frac{d-6}{2}\rfloor, 0)$ if $\lvert H - P - \lfloor\frac{d-1}{2}\rfloor K_C\rvert \neq \emptyset$

As an example we do the computations in the case $d \equiv_{(6)} 0$:
Since every linear system of degree 3 is non-empty by the Riemann-Roch Theorem for curves, we can write $|H| = |\frac{d-3}{3}D+Q_1+Q_2+Q_3|$, where either $Q_1+Q_2+Q_3$ is a divisor in $|D|$ or it is not. If it is not a divisor in $|D|$, then since $h^0(\mathcal{O}_C(Q_1+Q_2+Q_3)) = 2$, $Q_1+Q_2+Q_3$ is a divisor in some other pencil $|D'|$, and $|D'|$ is basepoint-free or it has one basepoint, say Q_1, i.e. $|D'| = |K_C + Q_1|$.

(a) If $|H| = |\frac{d}{3}D| = |\frac{d}{3}K_C + \frac{d}{3}P|$, then

$$
\begin{aligned}
d_0 &= h^0(\mathcal{O}_C(H)) - h^0(\mathcal{O}_C(H-D)) = 3,\\
d_1 &= h^0(\mathcal{O}_C(H-K_C-P)) - h^0(\mathcal{O}_C(H-2K_C-P))\\
&= \begin{cases} 1 & \text{if } d=6,\\ 2 & \text{if } d \geq 12,\end{cases}\\
&\;\;\vdots\\
d_{\frac{d-6}{2}} &= \begin{cases} 3 & \text{if } d=6,\\ h^0\left(\mathcal{O}_C\left(\frac{d-3}{3}P + \frac{18-d}{6}K_C\right)\right) & \\ -h^0\left(\mathcal{O}_C\left(\frac{d-3}{3}P + \frac{12-d}{6}K_C\right)\right) = 2 & \text{if } d \geq 12,\end{cases}\\
d_{\frac{d-4}{2}} &= \begin{cases} h^0(\mathcal{O}_C(P+K_C)) - h^0(\mathcal{O}_C(P)) = 1 & \text{if } d=6,\\ h^0\left(\mathcal{O}_C\left(\frac{d-3}{3}P + \frac{12-d}{6}K_C\right)\right) & \\ -h^0\left(\mathcal{O}_C\left(\frac{d-3}{3}P + \frac{6-d}{6}K_C\right)\right) = 1 & \text{if } d \geq 12 \text{ and } 2P \in |K_C|,\\ h^0\left(\mathcal{O}_C\left(\frac{d-3}{3}P + \frac{12-d}{6}K_C\right)\right) & \\ -h^0\left(\mathcal{O}_C\left(\frac{d-3}{3}P + \frac{6-d}{6}K_C\right)\right) = 2 & \text{if } d \geq 12 \text{ and } 2P \notin |K_C|,\end{cases}\\
d_{\frac{d-2}{2}} &= \begin{cases} h^0(\mathcal{O}_C(P)) = 1 & \text{if } d=6,\\ h^0\left(\mathcal{O}_C\left(\frac{d-3}{3}P + \frac{6-d}{6}K_C\right)\right) = 1 & \text{if } d \geq 12 \text{ and } 2P \in |K_C|,\\ h^0\left(\mathcal{O}_C\left(\frac{d-3}{3}P + \frac{6-d}{6}K_C\right)\right) = 0 & \text{if } d \geq 12 \text{ and } 2P \notin |K_C|.\end{cases}
\end{aligned}
$$

We obtain the following conclusion:

If $d = 6$, then $(e_1, e_2, e_3) = (2, 0, 0)$.

If $d \geq 12$ and $2P \in |K_C|$, then

$$\boxed{e_1 = \tfrac{d-2}{2},\ e_2 = \tfrac{d-6}{2},\ e_3 = 0.}$$

If $d \geq 12$ and $2P \notin |K_C|$, then

$$\boxed{e_1 = \tfrac{d-4}{2},\ e_2 = \tfrac{d-4}{2},\ e_3 = 0.}$$

(b) If $|H| = |\frac{d}{3}D| = |\frac{d-3}{3}D + D'|$ such that $|D'|$ has one basepoint Q_1, i.e. $|H| = |\frac{d}{3}K_C + \frac{d-3}{3}P + Q_1|$, then

$$\begin{aligned}
d_0 &= h^0(\mathcal{O}_C(H)) - h^0(\mathcal{O}_C(H-D)) = 3,\\
d_1 &= h^0(\mathcal{O}_C(H-K_C-P)) - h^0(\mathcal{O}_C(H-2K_C-P))\\
&= \begin{cases} h^0(\mathcal{O}_C(K_C+Q_1)) - h^0(\mathcal{O}_C(Q_1)) = 1 & \text{if } d=6,\\ 2 & \text{if } d \geq 12,\end{cases}\\
&\vdots\\
d_{\frac{d-6}{2}} &= \begin{cases} 3 & \text{if } d=6,\\ h^0(\mathcal{O}_C(\frac{d-3}{3}P + \frac{18-d}{6}K_C)) & \\ -h^0(\mathcal{O}_C(\frac{d-3}{3}P + \frac{12-d}{6}K_C)) = 2 & \text{if } d \geq 12,\end{cases}\\
d_{\frac{d-4}{2}} &= \begin{cases} h^0(\mathcal{O}_C(P+K_C)) - h^0(\mathcal{O}_C(P)) = 1 & \text{if } d=6,\\ h^0(\mathcal{O}_C(\frac{d-3}{3}P + \frac{12-d}{6}K_C)) & \\ -h^0(\mathcal{O}_C(\frac{d-3}{3}P + \frac{6-d}{6}K_C)) = 1 & \text{if } d \geq 12 \text{ and } 2P \in |K_C|,\\ h^0(\mathcal{O}_C(\frac{d-3}{3}P + \frac{12-d}{6}K_C)) & \\ -h^0(\mathcal{O}_C(\frac{d-3}{3}P + \frac{6-d}{6}K_C)) = 2 & \text{if } d \geq 12 \text{ and } 2P \notin |K_C|,\end{cases}\\
d_{\frac{d-2}{2}} &= \begin{cases} h^0(\mathcal{O}_C(Q_1)) = 1 & \text{if } d=6,\\ h^0(\mathcal{O}_C(\frac{d-6}{3}P + \frac{6-d}{6}K_C + Q_1)) = 1 & \text{if } d \geq 12 \text{ and } 2P \in |K_C|,\\ h^0(\mathcal{O}_C(\frac{d-3}{3}P + \frac{6-d}{6}K_C)) = 0, & \text{if } d \geq 12 \text{ and } 2P \notin |K_C|.\end{cases}
\end{aligned}$$

(c) If $|H| = |\frac{d-3}{3}D + D'| = |\frac{d-3}{3}K_C + \frac{d-3}{3}P + D'|$ with $|D'|$ basepoint-free, then:

$$\begin{aligned}
d_0 &= h^0(\mathcal{O}_C(H)) - h^0(\mathcal{O}_C(H-D)) = 3,\\
d_1 &= h^0(\mathcal{O}_C(H-K_C-P)) - h^0(\mathcal{O}_C(H-2K_C-P)) = 2,\\
&\vdots\\
d_{\frac{d-4}{2}} &= h^0(\mathcal{O}_C(\frac{6-d}{6}K_C + \frac{d-6}{3}P + D'))\\
&- h^0(\mathcal{O}_C(\frac{-d}{6}K_C + \frac{d-6}{3}P + D')) = 2.
\end{aligned}$$

Thus, $e_1 = e_2 = \frac{d-4}{2}$ and $e_3 = 0$.

We will now come to the converse of Proposition 2.15, i.e. the existence part:

Proposition 2.16. *If e_1 and e_2 are integers with $e_1 \geq e_2 \geq 0$, $e_1 - e_2 \leq 3$ and $e_1 + e_2 = d - 4$ with $d \geq 6$, then there exists a curve C of genus 2 and a divisor class $|H|$ on C of degree d that embeds C into $\mathbf{P}^{d-2}$ such that there exists a $g_3^1(C)$-scroll of type $(e_1, e_2, 0)$ such that its singular locus intersects the curve C.*

Proof. Let $e_1 \geq e_2 \geq 0$ be integers with $e_1 - e_2 \leq 3$ and $e_1 + e_2 = d - 4$. By the method just described above there exists a curve C of genus 2, embedded with a system $|H'|$ of degree $d-1$ into $\mathbf{P}^{d-3}$ such that its $g_2^1(C)$-scroll is of type (e_1, e_2). Take a point P on C and reembed the curve C with the linear system $|H| := |H' + P|$ into $\mathbf{P}^{d-2}$. The cone over the $g_2^1(C)$-scroll in $\mathbf{P}^{d-3}$ with P as vertex is a $g_3^1(C)$-scroll in $\mathbf{P}^{d-2}$ of type $(e_1, e_2, 0)$, and the point P lies in the intersection of its singular locus and the curve C. □

As suggested in Proposition 2.16, given $|H|$ and $|D| = |K_C + P|$, we can also find the scroll type of $V_{|D|}$ by projecting from the point P and using the analysis of the scroll type of the $g^1_2(C')$-scroll, where C' is the image of C under the projection. Let in this situation P' always denote the point in $|K_C - P|$.

(1) If d is even, then, since each linear system of degree 2 is non-empty, we can write $|H| = |\frac{d}{2}K_C| = |\frac{d-2}{2}K_C + P + P'|$ or $|H| = |\frac{d-2}{2}K_C + Q_1 + Q_2| = |\frac{d-4}{2}K_C + Q_1 + Q_2 + P + P'|$, s.t. $Q_1 + Q_2 \notin |K_C|$. Projecting from P yields a curve C' which is embedded in $\mathbf{P}^{d-3}$ by the linear system $|H'| := |H - P|$. Under this projection the scroll $V_{|D|}$ maps to the $g^1_2(C')$-scroll S'. The scroll $V_{|D|}$ is thus the cone over S' with P as vertex, so if S' is of scroll type (e_1, e_2), then $V_{|D|}$ is of scroll type $(e_1, e_2, 0)$.
Either $|H'| = |\frac{d-2}{2}K_C + P'|$ or $|H'| = |\frac{d-4}{2}K_C + Q_1 + Q_2 + P'|$. From our analysis above we obtain the following:

$\lvert H\rvert$	Conditions	Scroll type of $V_{\lvert D\rvert}$
$\lvert\frac{d-2}{2}K_C + P + P'\rvert$		$\left(\frac{d-2}{2}, \frac{d-6}{2}, 0\right)$
$\lvert\frac{d-4}{2}K_C + Q_1 + Q_2 + P + P'\rvert$	$P' + Q_1 \notin \lvert K_C\rvert$, $P' + Q_2 \notin \lvert K_C\rvert$	$\left(\frac{d-4}{2}, \frac{d-4}{2}, 0\right)$
$\lvert\frac{d-4}{2}K_C + Q_1 + Q_2 + P + P'\rvert$	$P' + Q_i \in \lvert K_C\rvert$ for at least one i	$\left(\frac{d-2}{2}, \frac{d-6}{2}, 0\right)$

(2) If d is odd, then we can write $|H| = |\frac{d-1}{2}K_C + Q| = |\frac{d-3}{2}K_C + Q + P + P'|$ or $|H| = |\frac{d-3}{2}K_C + \sum_{i=1}^3 Q_i| = |\frac{d-5}{2}K_C + \sum_{i=1}^3 Q_i + P + P'|$, s.t. $Q_i + Q_j \notin |K_C|$ for $i, j \in \{1, 2, 3\}$, $i \neq j$.

$\lvert H\rvert$	Conditions	Scroll type of $V_{\lvert D\rvert}$
$\lvert\frac{d-3}{2}K_C + Q + P + P'\rvert$	$P' + Q \notin \lvert K_C\rvert$	$\left(\frac{d-3}{2}, \frac{d-5}{2}, 0\right)$
$\lvert\frac{d-3}{2}K_C + Q + P + P'\rvert$	$P' + Q \in \lvert K_C\rvert$	$\left(\frac{d-1}{2}, \frac{d-7}{2}, 0\right)$
$\lvert\frac{d-5}{2}K_C + \sum_{i=1}^3 Q_i + P + P'\rvert$ $= \lvert\frac{d-3}{2}K_C + P + R + R'\rvert$	$R + R' \notin \lvert K_C\rvert$	$\left(\frac{d-3}{2}, \frac{d-5}{2}, 0\right)$

Another approach to the connection between $|H|$ and scroll types

In this section we want to analyze the connection between $|H|$ and scroll types of the $g^1_2(C)$-scroll S and the $g^1_3(C)$-scrolls V. We start with the cases $d = 6$ and $d = 7$ and end with a description of the connection between $|H|$, expressed with respect to $|K_C|$, and the scroll type of the $g^1_2(C)$-scroll S for arbitrary $d \geq 6$.

$d = 6$

(1) Here the degree of S is equal to 3, we have two possible scroll types: $(2, 1)$ and $(3, 0)$.

   (a) S has scrolltype $(2, 1)$: A general hyperplane section of the scroll that contains the directrix line is the union of this line and two fibers of S.
   In this case $|H| = |2K_C + P + Q|$ for two points P, Q on C such that $P + Q$ is not a divisor in the system $|K_C|$.

(b) S has scrolltype $(3,0)$: A general hyperplane section of the scroll that contains the singular point of the scroll is the union of three fibers of S which all meet in this singular point.
In this case $|H| = |3K_C|$.

(2) The degree of each $V = V_{|D|}$ is equal to 2, there are two possible scroll types: $(1,1,0)$ and $(2,0,0)$.

(a) A scroll of type $(1,1,0)$ is the cone over a smooth quadric in $\mathbf{P}^3$ with one vertex point P, and thus it is a quadric of rank 4 in $\mathbf{P}^4$. A general hyperplane section of such a scroll which contains the singular point and a directrix line of the scroll decomposes into two planes, A_1 and A_2, where A_1 is spanned by P and a line in one family of lines on $\mathbf{P}^1 \times \mathbf{P}^1$, and A_2 is spanned by P and a line in the other family of lines on $\mathbf{P}^1 \times \mathbf{P}^1$. These two planes A_1 and A_2 intersect in a line.

(b) A scroll of type $(2,0,0)$ is the cone over a singular quadric in $\mathbf{P}^3$ with a line as vertex and thus a quadric of rank 3 in $\mathbf{P}^4$.
A general hyperplane section of such a scroll that contains the vertex line is the union of two planes intersecting in this vertex line.

The $g_3^1(C)$-scroll $V_{|D|}$ is a quadric of rank 3 if $|H - D| = |D|$. Since $|D|$ and $|H - D|$ can be basepoint-free or have one basepoint, there are the following possibilities:

$\|H\|$	Conditions on basepoint loci	Scrolltype of $V_{\|D\|}$ or $V_{\|D_i\|}$, $i \in \{1,2\}$
$\|D_1 + D_2\|$, $\|D_1\| \neq \|D_2\|$	$\|D_1\|$ and $\|D_2\|$ basepoint-free	$(1,1,0)$
$\|D_1 + D_2\|$	$\|D_1\|$ basepointfree, $\|D_2\|$ with one basepoint	$(1,1,0)$
$\|2D\|$	$\|D\|$ basepoint-free	$(2,0,0)$
$\|2D\|$	$\|D\|$ with one basepoint	$(2,0,0)$
$\|D_1 + D_2\|$	$\|D_1\|$ with one basepoint P_1, $\|D_2\|$ with one basepoint P_2, $P_1 \neq P_2$	$(2,0,0)$

$d = 7$

(1) Here the degree of S is equal to 4, we have two possible scroll types: $(2,2)$ and $(3,1)$.

(a) S has scroll type $(2,2)$:
A general hyperplane section of the scroll that contains a directrix conic decomposes into this conic and two fibers of S.
In this situation $|H| = |2K_C + P + Q + R|$, P, Q, R points on the curve with all $P + Q$, $P + R$ and $Q + R$ not divisors in $|K_C|$.

(b) S has scroll type $(3,1)$:
A general hyperplane section of the scroll that contains the directrix line consists of this line and three fibers of S. In this case $|H| = |3K_C + P|$.

(2) The degree of $V = V_{|D|}$ is equal to 3, there are three possible scroll types: $(1,1,1)$, $(2,1,0)$ and $(3,0,0)$.

(a) A scroll of type $(1,1,1)$ is smooth, which means that the $g^1_3(C)$ $|D|$ is basepoint-free. Since two fibers in the scroll do not intersect and two fibers span a $\mathbf{P}^5$, we have $|H-2D| = \emptyset$.

(b) A scroll of type $(2,1,0)$ is the cone over the smooth scrollar surface in $\mathbf{P}^4$. Since two fibers in the scroll meet at one point, namely the singular point of the scroll, $|H-2D| \neq \emptyset$.

(c) Two fibers in a scroll of type $(3,0,0)$ intersect in a line, i.e. the span of the union of two fibers is a $\mathbf{P}^3$. The $g^1_3(C)$ $|D|$ has a basepoint, which means a point on the line along which the scroll is singular. In this case $|H| = |3K_C + P|$.

As conclusion, we give three descriptions of the connection between $|H|$ and the scroll type of $V_{|D|}$:

$\|H\|$	Conditions on basepoint locus of $\|D\|$	Scroll type of $V_{\|D\|}$
$\|H-2D\| = \emptyset$	No basepoints	$(1,1,1)$
$\|2D+P\|$	No basepoints	$(2,1,0)$
$\|2K_C+P+Q+R\|$, $P+Q, P+R, Q+R \notin \|K_C\|$	One basepoint P	$(2,1,0)$
$\|3K_C+P\|$	One basepoint P	$(3,0,0)$

The system $|H-3K_C|$ is of degree 1, and thus it is either empty or consists of one point.

$\|H-3K_C\|$	Conditions on basepoint locus of $\|D\|$	Scroll type of $V_{\|D\|}$
$\emptyset$	No basepoints	$(1,1,1)$
$\emptyset$	One basepoint	$(2,1,0)$
P	No basepoints	$(2,1,0)$
P	One basepoint P	$(3,0,0)$

Since $\deg(H-D-K_C) = 2$, the linear system $|H-D-K_C|$ is non-empty by the Riemann-Roch Theorem for curves. There are exactly two possibilities for $|H-D-K_C|$, namely $|H-D-K_C| = |K_C|$ and $|H-D-K_C| = |P+Q|$, where $P+Q$ is not a divisor in $|K_C|$. Hence we obtain the following third description of the connection between $|H|$ and the scroll type of $V_{|D|}$:

$\|H\|$	Conditions on basepoint locus of $\|D\|$	Scroll type of $V_{\|D\|}$
$\|D+2K_C\|$	No basepoints	$(1,1,1)$
$\|D+K_C+P+Q\|$, $P+Q \notin \|K_C\|$	No basepoints	$(2,1,0)$
$\|2K_C+P+Q+R\|$, $P+Q \notin \|K_C\|$	One basepoint R	$(2,1,0)$
$\|3K_C+R\|$	One basepoint R	$(3,0,0)$

More generally, we can study the connection between $|H|$ with respect to $|K_C|$ and the scroll type of S in the following way:

(1) d even:
The scroll S has scroll type $\left(\frac{d-2}{2}, \frac{d-4}{2}\right)$ or $\left(\frac{d}{2}, \frac{d-6}{2}\right)$. We describe $|H|$ in these two cases:

(a) If the scroll type of S is equal to $\left(\frac{d-2}{2}, \frac{d-4}{2}\right)$, then a minimal section C_0 is of degree $\frac{d-4}{2}$, and a general hyperplane section of S containing C_0 consists of C_0 and $\frac{d-2}{2}$ fibers of S. Consequently, $|H| = |\frac{d-2}{2}K_C + P + Q|$, where P and Q are points on $C_0 \cap C$ and $P + Q \notin |K_C|$.

(b) In the case when S is of scroll type $\left(\frac{d}{2}, \frac{d-6}{2}\right)$, a general hyperplane section of S that contains a minimal section C_0, which is of degree $\frac{d-6}{2}$, decomposes into C_0 and $\frac{d}{2}$ fibers of S. Hence $|H| = |\frac{d}{2}K_C|$.

(2) d odd:
The scroll S is of scroll type $\left(\frac{d-3}{2}, \frac{d-3}{2}\right)$ or $\left(\frac{d-1}{2}, \frac{d-5}{2}\right)$.

(a) If S is of scroll type $\left(\frac{d-3}{2}, \frac{d-3}{2}\right)$, then a general hyperplane section of S containing C_0 is equal to the union of C_0 and $\frac{d-3}{2}$ fibers of S. We obtain that $|H| = |\frac{d-3}{2}K_C + P + Q + R|$, where P, Q, R are points lying on $C_0 \cap C$ and none of $P + Q$, $P + R$ or $Q + R$ is a divisor in $|K_C|$.

(b) If the scroll type of S is equal to $\left(\frac{d-1}{2}, \frac{d-5}{2}\right)$, then a general hyperplane section of S that contains C_0 decomposes into C_0 and $\frac{d-1}{2}$ fibers of S. Consequently, $|H| = |\frac{d-1}{2}K_C + P|$ where P is a point on $C_0 \cap C$.

In Chapters 5 and 7 we will only be interested in $g^1_3(C)$-scrolls $V_{|D|}$ that do not contain the $g^1_2(C)$-scroll S. For this purpose we will now give a criterion for when a given $g^1_3(C)$-scroll $V_{|D|}$ does not contain the $g^1_2(C)$-scroll S. We will distinguish between the cases $d = 6$, $d = 7$ and $d \geq 8$.

Proposition 2.17. *Let $C \subseteq \mathbf{P}^{d-2}$ be a curve of genus 2 and degree $d \geq 6$, embedded with the system $|H|$, and let S be the $g^1_2(C)$-scroll. A $g^1_3(C)$-scroll $V = V_{|D|}$ contains S if and only if at least one of the following holds:*

- *$|D|$ has a basepoint,*
- *$d = 6$ and $|H - D|$ has a basepoint or*
- *$d = 7$ and $|H| = |D + 2K_C|$.*

Proof. If $|D|$ has a basepoint, then $|D| = |K_C + P|$, hence each fiber of S is contained in a fiber of $V_{|D|}$, and consequently $V_{|D|}$ contains S.
Conversely, if $S \subseteq V_{|D|}$ and $|D|$ is basepoint-free, then each fiber of $V_{|D|}$ intersects each fiber of S in one point, since if it did not, then each fiber of S had to be contained in a fiber of V which meant that $|D|$ had a basepoint. This implies that each fiber of $V_{|D|}$, which is a plane, intersects the scroll S in a directrix curve of S. This curve is a smooth rational planar curve, consequently the degree of this curve is equal to 1 or 2. This means that, since the degree of C is greater or equal to 6, the scroll type of S is equal to $(2,1)$ or $(2,2)$, i.e. $d = 6$ and $|H| = |D + K_C + P|$, or $d = 7$ and $|H| = |D + 2K_C|$. □

Chapter 3

The ideal of C

Let $C \subseteq \mathbf{P}^{d-2}$ be a smooth curve of genus 2 and degree $d \geq 6$.

In this chapter we will first prove that the ideal of C is generated by quadrics and then describe quadrics which together with the quadrics in I_S generate I_C.

As we have shown in Proposition 2.11, $[C] = 2H-(d-6)F$ on S, so each curve of genus 2 and degree $d \geq 6$ can be seen as a section in $H^0(S, \mathcal{O}_S(2H-(d-6)F)$, and via the rolling factor coordinates $s^{e_i-j}t^j x$, $s^{e_i-j}t^j y$ on S, each such curve can be represented by an equation f_C in $H^0(\mathbf{P}^1, S^2\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}(d-6))$, where $\mathbf{P}(\mathcal{E}) \to S \subseteq \mathbf{P}^{d-2}$. Moreover, the intersection of a quadric with the scroll S consists of a curve C of genus 2 and degree d plus $d-6$ lines on the scroll. So in order to describe the ideal of a curve of genus 2 and degree d on S, we will give $d-5$ quadrics that together cut out the curve on S.

We will now focus on the lines L_1 and L_2 on S, where L_1 is given by $s=0$ and L_2 is given by $t=0$. We want to give $d-5$ quadrics $q_1, \ldots, q_{d-5}$ in such a way that $q_1 \cap S = C \cup (d-6)L_1$, $q_2 \cap S = C \cup (d-7)L_1 \cup L_2$, ..., $q_{d-5} \cap S = C \cup (d-6)L_2$ for one and the same curve C of genus 2.

Theorem 3.1. *Let C be a curve of genus 2 embedded with a complete linear system $|H|$ of degree $d \geq 6$ in $\mathbf{P}^{d-2}$. The ideal of C is generated by $\binom{d-3}{2} + d - 5$ quadrics.*

Proof. Theorem (4.a.1) in [Gre84] shows that I_C is generated by quadrics for all $d \geq 6$. For another proof of this fact we use the rolling factor coordinates on S. Since $\mathcal{I}_{C,S} = \mathcal{O}_S(-C)$, showing that the ideal $I_{C,S}$ of C on S is generated by quadrics is equivalent to showing that the map

$$H^0(S, \mathcal{O}_S(2H-C)) \otimes H^0(S, \mathcal{O}_S((n-2)H)) \to H^0(S, \mathcal{O}_S(nH-C)),$$

$$q \otimes h \mapsto qh,$$

is surjective for all $n \geq 2$.

Since $[C] = 2H-(d-6)F$ on S by Proposition 2.11 and $H^0(S, \mathcal{O}_S(aH+bF)) \cong H^0(\mathbf{P}^1, \mathcal{S}\mathrm{ym}^a(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2)) \otimes \mathcal{O}_{\mathbf{P}^1}(b))$ with $e_1 + e_2 = d-3$ by Proposition 2.6 this is equivalent to showing that the map

$$\begin{aligned}\psi : &H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d-6)) \otimes H^0(\mathbf{P}^1, \oplus_{j=0}^{n-2}\mathcal{O}_{\mathbf{P}^1}((n-2-j)e_1 + je_2)) \\ &\rightarrow H^0(\mathbf{P}^1, \oplus_{l=0}^{n-2}\mathcal{O}_{\mathbf{P}^1}((n-1-l)e_1 + (l+1)e_2 - 3)),\end{aligned}$$

$$\begin{aligned}\psi(&s^{d-6-i}t^i \otimes s^{(n-2-j)e_1+je_2-k}t^k x^{n-2-j}y^j) \\ &:= s^{d-6-i}t^i \cdot s^{(n-2-j)e_1+je_2-k}t^k x^{n-2-j}y^j \\ &= s^{(n-1-j)e_1+(j+1)e_2-3-(i+k)}t^{i+k}x^{n-2-j}y^j,\end{aligned}$$

is surjective. This is straightforward, since a basis of

$$H^0(\mathbf{P}^1, \mathcal{S}\mathrm{ym}^{n-2}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2)) \otimes \mathcal{O}_{\mathbf{P}^1}(d-6))$$

is exactly given by

$$\left\{ s^{(n-1-l)e_1+(l+1)e_2-3-m}t^m x^{n-2-l}y^l \;\middle|\; \begin{array}{l} l = 0, \ldots, n-2, \\ m = 0, \ldots, (n-1-l)e_1 + (l+1)e_2 - 3 \end{array} \right\}.$$

It remains to show that $h^0(\mathcal{I}_C(2)) = \frac{d^2}{2} - \frac{5d}{2} + 1$:
We use the following exact sequence of ideal sheaves:

$$0 \rightarrow \mathcal{I}_S(2) \rightarrow \mathcal{I}_C(2) \rightarrow \mathcal{I}_{C,S}(2) \rightarrow 0$$

and the associated exact sequence in cohomology:

$$0 \rightarrow H^0(\mathcal{I}_S(2)) \rightarrow H^0(\mathcal{I}_C(2)) \rightarrow H^0(\mathcal{I}_{C,S}(2)) \rightarrow 0.$$

Here we know that $H^1(\mathcal{I}_S(2)) = 0$ since S is projectively normal by Proposition 2.9. From the last exact sequence we obtain

$$\begin{aligned} h^0(\mathcal{I}_C(2)) &= h^0(\mathcal{I}_S(2)) + h^0(\mathcal{I}_{C,S}(2)) \\ &= \binom{d-3}{2} + d - 5. \end{aligned}$$

□

Corollary 3.2. *Let $C \subseteq \mathbf{P}^{d-2}$ be a curve of genus 2 embedded as a smooth and irreducible curve with a complete linear system of degree $d \geq 6$. Then C has no trisecant lines.*

Proof. Since the ideal of C is generated by quadrics, we can write $C = Q_1 \cap \ldots \cap Q_r$, where $r = h^0(\mathcal{I}_C(2))$ and the Q_i are quadrics. Any line that intersects C in three points intersects each Q_i in at least three points, consequently it is contained in each Q_i, hence in the intersection of all Q_i's which is equal to C. □

Remark 3.3. *The condition that the ideal I_C is generated by quadrics is important for Corollary 3.2. For instance, the ideal of a curve $C \subseteq \mathbf{P}^3$ of genus 2 and degree 5 is generated by one quadric and two cubics. The curve C is of type $(2,3)$ on a quadric which is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and thus it has infinitely many trisecants.*

Let (e_1, e_2) with $e_1 \geq e_2 \geq 0$ be the scroll type of S, and set $e := e_1 - e_2$. In Proposition 2.13 we proved that $e \in \{0, 1, 2, 3\}$. In Sections 3.1, 3.2, 3.3, 3.4 we will give a set of quadrics that together with I_S generate I_C in each of the cases $e = 0$, $e = 1$, $e = 2$ and $e = 3$.
Our aim is to find $d - 5$ quadrics that together with I_S generate $H^0(\mathcal{I}_C(2))$ using rolling factor coordinates on S. We will use rolling factor coordinates in order to find a basis for $H^0(S, \mathcal{O}_S(2H + (d-6)F))$ and thus the equation of C on S and then afterwards find $d - 5$ quadrics in $\mathbf{P}^{d-2}$ that cut out the curve on S. The dimension of $H^0(S, \mathcal{O}_S(2H + (d-6)F))$ is independent of d and is equal to 12, a fact that also can be verified with the Riemann-Roch Theorem for surfaces.

3.1 $d \geq 7$ odd, $e = 0$, i.e. S of scroll type $(\frac{d-3}{2}, \frac{d-3}{2})$

In this section we will consider curves C of odd degree $d \geq 7$ such that the $g^1_2(C)$-scroll S is maximally balanced. In this case $S \cong \mathbf{P}(\mathcal{E}_d)$ where

$$\mathcal{E}_d = \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-3}{2}\right) \oplus \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-3}{2}\right).$$

Motivating examples

(1) $d = 7$:
In this case the $g^1_2(C)$-scroll S has scroll type $(2, 2)$.
After possibly a coordinate change, the ideal of the scroll S is generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix}.$$

The rolling factor coordinates take the following form:

$$\begin{aligned} x_0|_S &= s^2x, \\ x_1|_S &= stx, \\ x_2|_S &= t^2x, \\ x_3|_S &= s^2y, \\ x_4|_S &= sty, \\ x_5|_S &= t^2y. \end{aligned}$$

The curve C can be identified with a section in $H^0(S, \mathcal{O}_S(2H-F))$, and by Proposition 2.6 the curve C is represented by a polynomial f_C in $H^0(\mathbf{P}^1, (S^2\mathcal{E}_7)(-1))$ via the rolling factor coordinates, where

$$\mathcal{E}_7 = \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(2),$$

so

$$(S^2\mathcal{E}_7)(-1) = \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3).$$

A basis for $H^0(\mathbf{P}^1, (S^2\mathcal{E}_7)(-1))$ is given by

$$\{s^3x^2, s^2tx^2, st^2x^2, t^3x^2, s^3xy, s^2txy, st^2xy, t^3xy, s^3y^2, s^2ty^2, st^2y^2, t^3y^2\},$$

i.e. the equation of C on S is given by

$$\begin{aligned} f_C &= a_1 s^3 x^2 + a_2 s^2 t x^2 + a_3 s t^2 x^2 + a_4 t^3 x^2 + a_5 s^3 xy + a_6 s^2 txy \\ &\quad + a_7 s t^2 xy + a_8 t^3 xy + a_9 s^3 y^2 + a_{10} s^2 t y^2 + a_{11} s t^2 y^2 + a_{12} t^3 y^2. \end{aligned}$$

Now we want to find two quadrics q_1 and q_2 such that $q_1 = s f_C$ and $q_2 = t f_C$ on S. The result is the following:

$$\begin{aligned} q_1 &= a_1 x_0^2 + a_2 x_0 x_1 + a_3 x_0 x_2 + a_4 x_1 x_2 + a_5 x_0 x_3 + a_6 x_0 x_4 \\ &\quad + a_7 x_0 x_5 + a_8 x_1 x_5 + a_9 x_3^2 + a_{10} x_3 x_4 + a_{11} x_3 x_5 + a_{12} x_4 x_5, \\ q_2 &= a_1 x_0 x_1 + a_2 x_1^2 + a_3 x_1 x_2 + a_4 x_2^2 + a_5 x_1 x_3 + a_6 x_1 x_4 \\ &\quad + a_7 x_1 x_5 + a_8 x_2 x_5 + a_9 x_3 x_4 + a_{10} x_3 x_5 + a_{11} x_4 x_5 + a_{12} x_5^2. \end{aligned}$$

(2) $d = 9$:
Here the $g_2^1(C)$-scroll S has type $(3,3)$. After possibly a coordinate change the ideal I_S is generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 & x_5 & x_6 \\ x_1 & x_2 & x_3 & x_5 & x_6 & x_7 \end{pmatrix}.$$

The rolling factor coordinates take the following form:

$$\begin{aligned} x_0|_S &= s^3 x, \\ x_1|_S &= s^2 t x, \\ x_2|_S &= s t^2 x, \\ x_3|_S &= t^3 x, \\ x_4|_S &= s^3 y, \\ x_5|_S &= s^2 t y, \\ x_6|_S &= s t^2 y, \\ x_7|_S &= t^3 y. \end{aligned}$$

By Proposition 2.6 the curve C is represented by a polynomial f_C in $H^0(\mathbf{P}^1, (S^2\mathcal{E}_9)(-3))$ via the rolling factor coordinates, where

$$\mathcal{E}_9 = \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3),$$

so

$$(S^2\mathcal{E}_9)(-3) = \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3).$$

In fact, we will see later that $(S^2\mathcal{E}_d)(6-d)$ is independent of d, so from now we will only write $\mathcal{E}$ instead $\mathcal{E}_d$, and it will be clear from the context which degree we consider.

Consequently, we obtain the same equation f_C for the curve C on S as in the case $d = 7$.

Here we want to find four quadrics q_1, q_2, q_3 and q_4 such that $q_1 = s^3 f_C$, $q_2 = s^2 t f_C$, $q_3 = st^2 f_C$ and $q_4 = t^3 f_C$ on S. The result is the following:

$$\begin{aligned}
q_1 &= a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + a_4x_0x_3 + a_5x_0x_4 + a_6x_0x_5 \\
&\quad +a_7x_0x_6 + a_8x_0x_7 + a_9x_4^2 + a_{10}x_4x_5 + a_{11}x_4x_6 + a_{12}x_4x_7, \\
q_2 &= a_1x_0x_1 + a_2x_1^2 + a_3x_1x_2 + a_4x_1x_3 + a_5x_1x_4 + a_6x_1x_5 \\
&\quad +a_7x_1x_6 + a_8x_1x_7 + a_9x_4x_5 + a_{10}x_5^2 + a_{11}x_5x_6 + a_{12}x_5x_7, \\
q_3 &= a_1x_0x_2 + a_2x_1x_2 + a_3x_2^2 + a_4x_2x_3 + a_5x_2x_4 + a_6x_2x_5 \\
&\quad +a_7x_2x_6 + a_8x_2x_7 + a_9x_4x_6 + a_{10}x_5x_6 + a_{11}x_6^2 + a_{12}x_6x_7, \\
q_4 &= a_1x_0x_3 + a_2x_1x_3 + a_3x_2x_3 + a_4x_3^2 + a_5x_3x_4 + a_6x_3x_5 \\
&\quad +a_7x_3x_6 + a_8x_3x_7 + a_9x_4x_7 + a_{10}x_5x_7 + a_{11}x_6x_7 + a_{12}x_7^2.
\end{aligned}$$

(3) The main pattern will be visible when we consider the next example, $d = 11$: In this case the $g_2^1(C)$-scroll S has type $(4, 4)$. After possibly a coordinate change the ideal I_S is generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_5 & x_6 & x_7 & x_8 \\ x_1 & x_2 & x_3 & x_4 & x_6 & x_7 & x_8 & x_9 \end{pmatrix}.$$

The rolling factor coordinates take the following form:

$$\begin{aligned}
x_0|_S &= s^4x, \\
x_1|_S &= s^3tx, \\
x_2|_S &= s^2t^2x, \\
x_3|_S &= st^3x, \\
x_4|_S &= t^4x, \\
x_5|_S &= s^4y, \\
x_6|_S &= s^3ty, \\
x_7|_S &= s^2t^2y, \\
x_8|_S &= st^3y, \\
x_9|_S &= t^4y.
\end{aligned}$$

By Proposition 2.6 the curve C is represented by a polynomial f_C in $H^0(\mathbf{P}^1, (S^2\mathcal{E})(-5))$ where $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(4) \oplus \mathcal{O}_{\mathbf{P}^1}(4)$. As mentioned above, we will see later that $S^2(\mathcal{E}_d)(6-d)$ is independent of d, so we obtain the same equation f_C for C on S as in the two previous examples.
Now we want to find six quadrics $q_1, \ldots, q_6$ such that $q_1 = s^5 f_C$, $q_2 = s^4 t f_C$, $q_3 = s^3t^2 f_C$, $q_4 = s^2t^3 f_C$, $q_5 = st^4 f_C$ and $q_6 = t^5 f_C$ on S.
The result is the following:

$$
\begin{aligned}
q_1 &= a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + a_4x_0x_3 \\
&+ a_5x_0x_5 + a_6x_0x_6 + a_7x_0x_7 + a_8x_0x_8 \\
&+ a_9x_5^2 + a_{10}x_5x_6 + a_{11}x_5x_7 + a_{12}x_5x_8, \\
q_2 &= a_1x_0x_1 + a_2x_1^2 + a_3x_1x_2 + a_4x_1x_3 \\
&+ a_5x_1x_5 + a_6x_1x_6 + a_7x_1x_7 + a_8x_1x_8 \\
&+ a_9x_5x_6 + a_{10}x_6^2 + a_{11}x_6x_7 + a_{12}x_6x_8, \\
q_3 &= a_1x_0x_2 + a_2x_1x_2 + a_3x_2^2 + a_4x_2x_3 \\
&+ a_5x_2x_5 + a_6x_2x_6 + a_7x_2x_7 + a_8x_2x_8 \\
&+ a_9x_5x_7 + a_6x_6x_7 + a_7x_7^2 + a_8x_7x_8, \\
q_4 &= a_1x_0x_3 + a_2x_1x_3 + a_3x_2x_3 + a_4x_3^2 \\
&+ a_5x_3x_5 + a_6x_3x_6 + a_7x_3x_7 + x_8x_3x_8 \\
&+ a_9x_5x_8 + a_{10}x_6x_8 + a_{11}x_7x_8 + a_{12}x_8^2, \\
q_5 &= a_1x_0x_4 + a_2x_1x_4 + a_3x_2x_4 + a_4x_3x_4 \\
&+ a_5x_4x_5 + a_6x_4x_6 + a_7x_4x_7 + a_8x_4x_8 \\
&+ a_9x_5x_9 + a_{10}x_6x_9 + a_{11}x_7x_9 + a_{12}x_8x_9, \\
q_6 &= a_1x_1x_4 + a_2x_2x_4 + a_3x_3x_4 + a_4x_4^2 \\
&+ a_5x_1x_9 + a_6x_2x_9 + a_7x_3x_9 + a_8x_4x_9 \\
&+ a_9x_6x_9 + a_{10}x_7x_9 + a_{11}x_8x_9 + a_{12}x_9^2.
\end{aligned}
$$

General $d \geq 9$

Let now $d \geq 9$ be arbitrary. The scroll type of the $g_2^1(C)$-scroll S is equal to $(\frac{d-3}{2}, \frac{d-3}{2})$, and the ideal I_S is, possibly after a coordinate change, generated by the (2×2)-minors of the following matrix:

$$
\begin{pmatrix}
x_0 & x_1 & \cdots & x_{\frac{d-3}{2}-1} & x_{\frac{d-3}{2}+1} & \cdots & x_{d-3} \\
x_1 & x_2 & \cdots & x_{\frac{d-3}{2}} & x_{\frac{d-3}{2}+2} & \cdots & x_{d-2}
\end{pmatrix}.
$$

In terms of rolling factor coordinates this corresponds to:

$$
\begin{pmatrix}
s^{\frac{d-3}{2}}x & s^{\frac{d-3}{2}-1}tx & \cdots & st^{\frac{d-3}{2}-1}x & s^{\frac{d-3}{2}}y & s^{\frac{d-3}{2}-1}ty & \cdots & st^{\frac{d-3}{2}-1} \\
s^{\frac{d-3}{2}-1}tx & s^{\frac{d-3}{2}-2}t^2x & \cdots & t^{\frac{d-3}{2}}x & s^{\frac{d-3}{2}-1}ty & s^{\frac{d-3}{2}-2}t^2y & \cdots & t^{\frac{d-3}{2}}y
\end{pmatrix}.
$$

We have seen that for the class of C on S we can write $[C] = 2H - (d-6)F$. Moreover, $S = \iota(\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(\frac{d-3}{2}) \oplus \mathcal{O}_{\mathbf{P}^1}(\frac{d-3}{2})))$, where ι is the map as in Definition 2.2, $\iota : \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(\frac{d-3}{2}) \oplus \mathcal{O}_{\mathbf{P}^1}(\frac{d-3}{2})) \to \mathbf{P}^{d-2}$. Then by Proposition 2.6 C is represented by a polynomial in

$$
\begin{aligned}
&H^0\left(\mathbf{P}^1, S^2\left(\mathcal{O}_{\mathbf{P}^1}\left(\frac{d-3}{2}\right) \oplus \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-3}{2}\right)\right) \otimes \mathcal{O}_{\mathbf{P}^1}(6-d)\right) \\
&\cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3)).
\end{aligned}
$$

A basis for the vector space $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3))$ is given by

$$
\{x^2s^3, x^2s^2t, x^2st^2, x^2t^3, xys^3, xys^2t, xyst^2, xyt^3, y^2s^3, y^2s^2t, y^2st^2, y^2t^3\}.
$$

We obtain thus the following equation of the curve C on S (i.e. $I_{C,S} = (f_C)$):

$$\begin{aligned} f_C &= a_1x^2s^3 + a_2x^2s^2t + a_3x^2st^2 + a_4x^2t^3 + a_5xys^3 + a_6xys^2t \\ &\quad + a_7xyst^2 + a_8xyt^3 + a_9y^2s^3 + a_{10}y^2s^2t + a_{11}y^2st^2 + a_{12}y^2t^3, \end{aligned}$$

for $a_1, \ldots, a_{12} \in k$.
Here we immediately see that $h^0(S, \mathcal{O}_S(2H - (d-6)F)) = 12$, a fact that also can be verified by the Riemann-Roch Theorem for surfaces (cf. also Corollary 2.7).
Varying the coefficients $a_1, \ldots, a_{12}$ produces all members in $|2H - (d-6)F|$.
The adjunction formula

$$2p_a(C) - 2 = (2H - (d-6)F).F = 2$$

yields that the arithmetic genus of C is equal to $p_a(C) = 2$. By the Bertini Theorem (cf. [Har77], Theorem II.8.18) a general curve in the system $|2H - (d-6)F|$ on S is smooth, so its geometric genus is equal to 2.
Now we are able to state and prove the following result:

Theorem 3.4. *Let C be a curve on S, and let its ideal on S be given by $I_{C,S} = (f_C)$, where*

$$\begin{aligned} f_C &= a_1x^2s^3 + a_2x^2s^2t + a_3x^2st^2 + a_4x^2t^3 + a_5xys^3 + a_6xys^2t \\ &\quad + a_7xyst^2 + a_8xyt^3 + a_9y^2s^3 + a_{10}y^2s^2t + a_{11}y^2st^2 + a_{12}y^2t^3, \end{aligned}$$

with $a_1, \ldots, a_{12} \in k$ and where (s,t) are the homogeneous coordinates on $\mathbf{P}^1$ and (x,y) are the homogeneous coordinates on a fiber $F \cong \mathbf{P}^1$ of the scroll S.
For $d \geq 9$ the quadrics $q_1, \ldots, q_{d-5}$ given by the following formula cut out the curve C in the linear system $|2H - (d-6)F|$ on S, more precisely, the restrictions $q_1|_S, \ldots, q_{d-5}|_S$ form a basis for the vector space $H^0(\mathcal{I}_{C,S}(2))$:

$$q_i = \begin{cases} a_1x_0x_{i-1} + a_2x_1x_{i-1} + a_3x_2x_{i-1} + a_4x_3x_{i-1} \\ + a_5x_{i-1}x_{\frac{d-1}{2}} + a_6x_{i-1}x_{\frac{d+1}{2}} + a_7x_{i-1}x_{\frac{d+3}{2}} \\ + a_8x_{i-1}x_{\frac{d+5}{2}} + a_9x_{\frac{d-1}{2}}x_{\frac{d-1}{2}+i-1} \\ + a_{10}x_{\frac{d+1}{2}}x_{\frac{d-1}{2}+i-1} + a_{11}x_{\frac{d+3}{2}}x_{\frac{d-1}{2}+i-1} \\ + a_{12}x_{\frac{d+5}{2}}x_{\frac{d-1}{2}+i-1}, & \text{for} \quad 1 \leq i \leq \frac{d-1}{2}, \\ \\ a_1x_{\frac{d-9}{2}}x_{i-\frac{d-7}{2}} + a_2x_{\frac{d-7}{2}}x_{i-\frac{d-7}{2}} \\ + a_3x_{\frac{d-5}{2}}x_{i-\frac{d-7}{2}} + a_4x_{\frac{d-3}{2}}x_{i-\frac{d-7}{2}} \\ + a_5x_{\frac{d-9}{2}}x_{i+3} + a_6x_{\frac{d-7}{2}}x_{i+3} \\ + a_7x_{\frac{d-5}{2}}x_{i+3} + a_8x_{\frac{d-3}{2}}x_{i+3} \\ + a_9x_{d-5}x_{i+3} + a_{10}x_{d-4}x_{i+3} \\ + a_{11}x_{d-3}x_{i+3} + a_{12}x_{d-2}x_{i+3}, & \text{for} \quad \frac{d+1}{2} \leq i \leq d-5. \end{cases}$$

Proof. With the equation f_C of C on S as above we have the following for $1 \leq i \leq \frac{d-1}{2}$:

$$\begin{aligned} q_i|_S &= a_1s^{d-2-i}t^{i-1}x^2 + a_2s^{d-3-i}t^ix^2 + a_3s^{d-4-i}t^{i+1}x^2 + a_4s^{d-5-i}t^{i+2}x^2 \\ &\quad + a_5s^{d-2-i}t^{i-1}xy + a_6s^{d-3-i}t^ixy + a_7s^{d-4-i}t^{i+1}xy + a_8s^{d-5-i}t^{i+2}xy \\ &\quad + a_9s^{d-2-i}t^{i-1}y^2 + a_{10}s^{d-3-i}t^iy^2 + a_{11}s^{d-4-i}t^{i+1}y^2 + a_{12}s^{d-5-i}t^{i+2}y^2 \\ &= s^{d-5-i}t^{i-1}f_C. \end{aligned}$$

For $\frac{d+1}{2} \leq i \leq d-5$ we obtain the same:

$$q_i|_S \quad = \quad s^{d-5-i}t^{i-1}f_C.$$

Consequently, $q_1, \ldots, q_{d-5}$ cut out a curve C in $H^0(S, 2H-(d-6)F)$ which equation on S is given by the polynomial f_C. □

Our system of choosing the quadrics $q_1, \ldots, q_{d-5}$ continues in Sections 3.2, 3.3 and 3.4.

3.2 $d \geq 6$ even, $e = 1$, i.e. S of scroll type $(\frac{d-2}{2}, \frac{d-4}{2})$

After possibly a coordinate change the ideal I_S is generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{\frac{d-2}{2}-1} & x_{\frac{d-2}{2}+1} & \cdots & x_{d-3} \\ x_1 & x_2 & \cdots & x_{\frac{d-2}{2}} & x_{\frac{d-2}{2}+2} & \cdots & x_{d-2} \end{pmatrix}.$$

In terms of rolling factor coordinates the above matrix looks like:

$$\begin{pmatrix} s^{\frac{d-2}{2}}x & s^{\frac{d-2}{2}-1}tx & \cdots & st^{\frac{d-2}{2}-1}x & s^{\frac{d-4}{2}}y & s^{\frac{d-4}{2}-1}ty & \cdots & st^{\frac{d-4}{2}-1}y \\ s^{\frac{d-4}{2}-1}tx & s^{\frac{d-4}{2}-2}t^2x & \cdots & t^{\frac{d-2}{2}}x & s^{\frac{d-4}{2}-1}ty & s^{\frac{d-4}{2}-2}t^2y & \cdots & t^{\frac{d-4}{2}}y \end{pmatrix}.$$

By Proposition 2.6 we have

$$\begin{aligned} & H^0\left(\mathbf{P}\left(\mathcal{O}_{\mathbf{P}^1}\left(\frac{d-2}{2}\right) \oplus \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-4}{2}\right)\right), 2H+(6-d)F\right) \\ \cong\; & H^0\left(\mathbf{P}^1, S^2\left(\mathcal{O}_{\mathbf{P}^1}\left(\frac{d-2}{2}\right) \oplus \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-4}{2}\right)\right) \otimes \mathcal{O}_{\mathbf{P}^1}(6-d)\right) \end{aligned}$$

which in turn is isomorphic to

$$H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(2))$$

with a basis given by

$$\{s^4x^2, s^3tx^2, s^2t^2x^2, st^3x^2, t^4x^2, s^3xy, s^2txy, st^2xy, t^3xy, s^2y^2, sty^2, t^2y^2\}.$$

Consequently, the equation of C on S is of the following form:

$$\begin{aligned} f_C \quad = \quad & a_1s^4x^2 + a_2s^3tx^2 + a_3s^2t^2x^2 + a_4st^3x^2 + a_5t^4x^2 + a_6s^3xy \\ & + a_7s^2txy + a_8st^2xy + a_9t^3xy + a_{10}s^2y^2 + a_{11}sty^2 + a_{12}t^2y^2 \end{aligned}$$

with $a_1, \ldots, a_{12} \in k$.

Theorem 3.5. *Let*

$$\begin{aligned} f_C \quad = \quad & a_1s^4x^2 + a_2s^3tx^2 + a_3s^2t^2x^2 + a_4st^3x^2 + a_5t^4x^2 + a_6s^3xy \\ & + a_7s^2txy + a_8st^2xy + a_9t^3xy + a_{10}s^2y^2 + a_{11}sty^2 + a_{12}t^2y^2, \end{aligned}$$

where $a_1, \ldots, a_{12} \in k$, be the equation of a curve C on S.

For $d \geq 10$ the quadrics $q_1, \ldots, q_{d-5}$ cut out the curve C on S:

$$q_i = \begin{cases} a_1x_0x_{i-1} + a_2x_1x_{i-1} + a_3x_2x_{i-1} + a_4x_3x_{i-1} \\ +a_5x_4x_{i-1} + a_6x_{i-1}x_{\frac{d}{2}} + a_7x_{i-1}x_{\frac{d+2}{2}} \\ +a_8x_{i-1}x_{\frac{d+4}{2}} + a_9x_{i-1}x_{\frac{d+6}{2}} \\ +a_{10}x_{\frac{d}{2}}x_{\frac{d}{2}+i-1} + a_{11}x_{\frac{d+2}{2}}x_{\frac{d}{2}+i-1} \\ +a_{12}x_{\frac{d+4}{2}}x_{\frac{d}{2}+i-1}, & \text{for} \quad 1 \leq i \leq \frac{d-2}{2}, \\ \\ a_1x_{\frac{d-10}{2}}x_{i-\frac{d-8}{2}} + a_2x_{\frac{d-8}{2}}x_{i-\frac{d-8}{2}} \\ +a_3x_{\frac{d-6}{2}}x_{i-\frac{d-8}{2}} + a_4x_{\frac{d-4}{2}}x_{i-\frac{d-8}{2}} \\ +a_5x_{\frac{d-2}{2}}x_{i-\frac{d-8}{2}} + a_6x_{\frac{d-8}{2}}x_{i+3} \\ +a_7x_{\frac{d-6}{2}}x_{i+3} + a_8x_{\frac{d-4}{2}}x_{i+3} \\ +a_9x_{\frac{d-2}{2}}x_{i+3} + a_{10}x_{d-4}x_{i+3} \\ +a_{11}x_{d-3}x_{i+3} + a_{12}x_{d-2}x_{i+3}, & \text{for} \quad \frac{d}{2} \leq i \leq d-5. \end{cases}$$

Proof. The proof is analogous to the proof of Theorem 3.4. □

Remark 3.6. *In the statement of Theorem 3.5 we have to exclude the lowest values for d. However, as in Section 3.4, we can still find the desired quadrics $q_1, \ldots, q_{d-5}$ for these values for d, i.e. for $d = 6$ and $d = 8$:*

(1) *The case $d = 6$ is probably the easiest to consider: Here we know that $I_C = I_S + (Q)$ for a general quadric Q that does not lie in I_S.*
The ideal of S is probably after a coordinate change generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix}.$$

The rolling factor coordinates are given as follows:

$$\begin{aligned} x_0|_S &= s^2x, \\ x_1|_S &= stx, \\ x_2|_S &= t^2x, \\ x_3|_S &= sy, \\ x_4|_S &= ty. \end{aligned}$$

We have that $h^0(\mathcal{O}_S(2H)) = 12$ which is exactly equal to the dimension of the vector space of quadrics in $\mathbf{P}^4$ minus $h^0(\mathcal{I}_S(2))$. Notice that if we set

$$\begin{aligned} Q &= a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + a_4x_1x_2 + a_5x_2^2 + a_6x_0x_3 \\ &\quad +a_7x_0x_4 + a_8x_1x_4 + a_9x_2x_4 + a_{10}x_3^2 + a_{11}x_3x_4 + a_{12}x_4^2, \end{aligned}$$

then $Q|_S = f_C$.

(2) *In the case* $d = 8$ *we obtain the following:*
After possibly a coordinate change the ideal I_S *is generated by the* (2×2)*-minors of the following matrix:*

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_5 & x_6 \end{pmatrix}.$$

The rolling factor coordinates are given as follows:

$$\begin{aligned} x_0|_S &= s^3x, \\ x_1|_S &= s^2tx, \\ x_2|_S &= st^2x, \\ x_3|_S &= t^3x, \\ x_4|_S &= s^2y, \\ x_5|_S &= sty, \\ x_6|_S &= t^2x. \end{aligned}$$

As above, the equation of C *on* S *is given by*

$$\begin{aligned} f_C &= a_1s^4x^2 + a_2s^3tx^2 + a_3s^2t^2x^2 + a_4st^3x^2 + a_5t^4x^2 + a_6s^3xy \\ &+ a_7s^2txy + a_8st^2xy + a_9t^3xy + a_{10}s^2y^2 + a_{11}sty^2 + a_{12}t^2y^2 \end{aligned}$$

with $a_1, \ldots, a_{12} \in k$.
The aim is now to find three quadrics q_1, q_2 *and* q_3 *such that* $q_1 = s^2 f_C$, $q_2 = stf_C$ *and* $q_3 = t^2 f_C$ *on* S.
The result is the following:

$$\begin{aligned} q_1 &= a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + a_4x_0x_3 + a_5x_1x_3 + a_6x_0x_4 \\ &+ a_7x_0x_5 + a_8x_0x_6 + a_9x_1x_6 + a_{10}x_4^2 + a_{11}x_4x_5 + a_{12}x_4x_6, \\ q_2 &= a_1x_0x_1 + a_2x_1^2 + a_3x_1x_2 + a_4x_1x_3 + a_5x_2x_3 + a_6x_1x_4 \\ &+ a_7x_1x_5 + a_8x_1x_6 + a_9x_2x_6 + a_{10}x_4x_5 + a_{11}x_5^2 + a_{12}x_5x_6, \\ q_3 &= a_1x_0x_2 + a_2x_1x_2 + a_3x_2^2 + a_4x_2x_3 + a_5x_3^2 + a_6x_2x_4 \\ &+ a_7x_2x_5 + a_8x_2x_6 + a_9x_3x_6 + a_{10}x_4x_6 + a_{11}x_5x_6 + a_{12}x_6^2. \end{aligned}$$

3.3 $d \geq 7$ odd, $e = 2$, i.e. S of scroll type $(\frac{d-1}{2}, \frac{d-5}{2})$

After possibly a coordinate change the ideal of the scroll S is generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{\frac{d-1}{2}-1} & x_{\frac{d-1}{2}+1} & \cdots & x_{d-3} \\ x_1 & x_2 & \cdots & x_{\frac{d-1}{2}} & x_{\frac{d-1}{2}+2} & \cdots & x_{d-2} \end{pmatrix}.$$

In terms of rolling factor coordinates the above matrix becomes

$$\begin{pmatrix} s^{\frac{d-1}{2}}x & s^{\frac{d-1}{2}-1}tx & \cdots & st^{\frac{d-1}{2}-1}x & s^{\frac{d-5}{2}}y & s^{\frac{d-5}{2}-1}ty & \cdots & st^{\frac{d-5}{2}-1}y \\ s^{\frac{d-1}{2}-1}tx & s^{\frac{d-1}{2}-2}t^2x & \cdots & t^{\frac{d-1}{2}}x & s^{\frac{d-5}{2}-1}ty & s^{\frac{d-5}{2}-2}t^2y & \cdots & t^{\frac{d-5}{2}}y \end{pmatrix}.$$

By Proposition 2.6 we obtain

$$\begin{aligned} & H^0\left(\mathbf{P}\left(\mathcal{O}_{\mathbf{P}^1}\left(\frac{d-1}{2}\oplus \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-5}{2}\right)\right)\right), 2H+(6-d)F\right) \\ \cong\ & H^0\left(\mathbf{P}^1, S^2\left(\mathcal{O}_{\mathbf{P}^1}\left(\frac{d-1}{2}\right)\oplus \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-5}{2}\right)\right)\otimes \mathcal{O}_{\mathbf{P}^1}(6-d)\right) \\ \cong\ & H^0(\mathcal{O}_{\mathbf{P}^1}(5)\oplus \mathcal{O}_{\mathbf{P}^1}(3)\oplus \mathcal{O}_{\mathbf{P}^1}(1)) \end{aligned}$$

with a basis given by

$$\{s^5x^2, s^4tx^2, s^3t^2x^2, s^2t^3x^2, st^4x^2, t^5x^2, s^3xy, s^2txy, st^2xy, t^3xy, sy^2, ty^2\}.$$

As before, we observe thus that $I_{C,S} = (f_C)$ where the equation f_C of C on S is of the following form:

$$\begin{aligned} f_C \ =\ & a_1s^5x^2 + a_2s^4tx^2 + a_3s^3t^2x^2 + a_4s^2t^3x^2 + a_5st^4x^2 + a_6t^5x^2 \\ & + a_7s^3xy + a_8s^2txy + a_9st^2xy + a_{10}t^3xy + a_{11}sy^2 + a_{12}ty^2 \end{aligned}$$

with $a_1, \ldots, a_{12} \in k$.

Theorem 3.7. *Let*

$$\begin{aligned} f_C \ =\ & a_1s^5x^2 + a_2s^4tx^2 + a_3s^3t^2x^2 + a_4s^2t^3x^2 + a_5st^4x^2 + a_6t^5x^2 \\ & + a_7s^3xy + a_8s^2txy + a_9st^2xy + a_{10}t^3xy + a_{11}sy^2 + a_{12}ty^2 \end{aligned}$$

with $a_1, \ldots, a_{12} \in k$ be the equation of a curve C on S.
For $d \geq 11$ the quadrics $q_1, \ldots, q_{d-5}$ cut out the curve C on S:

$$q_i = \begin{cases} a_1x_0x_{i-1} + a_2x_1x_{i-1} + a_3x_2x_{i-1} + a_4x_3x_{i-1} \\ +a_5x_4x_{i-1} + a_6x_5x_{i-1} + a_7x_{i-1}x_{\frac{d+1}{2}} \\ +a_8x_{i-1}x_{\frac{d+3}{2}} + a_9x_{i-1}x_{\frac{d+5}{2}} \\ +a_{10}x_{i-1}x_{\frac{d+7}{2}} + a_{11}x_{\frac{d+1}{2}}x_{\frac{d+1}{2}+i-1} \\ +a_{12}x_{\frac{d+3}{2}}x_{\frac{d+1}{2}+i-1}, & \text{for} \quad 1 \leq i \leq \frac{d-3}{2}, \\ \\ a_1x_{\frac{d-11}{2}}x_{i-\frac{d-9}{2}} + a_2x_{\frac{d-9}{2}}x_{i-\frac{d-9}{2}} \\ +a_3x_{\frac{d-7}{2}}x_{i-\frac{d-9}{2}} + a_4x_{\frac{d-5}{2}}x_{i-\frac{d-9}{2}} \\ +a_5x_{\frac{d-3}{2}}x_{i-\frac{d-9}{2}} + a_6x_{\frac{d-1}{2}}x_{i-\frac{d-9}{2}} \\ +a_7x_{\frac{d-7}{2}}x_{i+3} + a_8x_{\frac{d-5}{2}}x_{i+3} \\ +a_9x_{\frac{d-3}{2}}x_{i+3} + a_{10}x_{\frac{d-1}{2}}x_{i+3} \\ +a_{11}x_{d-3}x_{i+3} + a_{12}x_{d-2}x_{i+3}, & \text{for} \quad \frac{d-1}{2} \leq i \leq d-5. \end{cases}$$

Proof. The proof is analogous to the proof of Theorem 3.4. ☐

Remark 3.8. *Again, when we state Theorem 3.7 we have to exclude the lowest values for d. We give the quadrics $q_1, \dots, q_{d-5}$ in the cases when $d = 7$ and $d = 9$:*

(1) $d = 7$*: The $g^1_2(C)$-scroll S has type $(3,1)$, its ideal I_S is after possibly a coordinate change generated by the (2×2)-minors of the following matrix:*

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 \\ x_1 & x_2 & x_3 & x_5 \end{pmatrix}.$$

The rolling factor coordinates are of the following form:

$$\begin{aligned} x_0|_S &= s^3x, \\ x_1|_S &= s^2tx, \\ x_2|_S &= st^2x, \\ x_3|_S &= t^3x, \\ x_4|_S &= sy, \\ x_5|_S &= ty. \end{aligned}$$

Since also in the case $e = 2$ we obtained that the equation f_C for C on S is independent of d, we want to find two quadrics q_1 and q_2 such that $q_1 = sf_C$ and $q_2 = tf_C$ on S.
The result is the following:

$$\begin{aligned} q_1 &= a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + a_4x_0x_3 + a_5x_1x_3 + a_6x_2x_3 \\ &+ a_7x_0x_4 + a_8x_0x_5 + a_9x_1x_5 + a_{10}x_2x_5 + a_{11}x_4^2 + a_{12}x_4x_5, \\ q_2 &= a_1x_0x_1 + a_2x_0x_2 + a_3x_0x_3 + a_4x_1x_3 + a_5x_2x_3 + a_6x_3^2 \\ &+ a_7x_0x_5 + a_8x_1x_5 + a_9x_2x_5 + a_{10}x_3x_5 + a_{11}x_4x_5 + a_{12}x_5^2. \end{aligned}$$

(2) $d = 9$*: In this case the $g^1_2(C)$-scroll has type $(4,2)$ and the ideal I_S is generated by the (2×2)-minors of the following matrix:*

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_5 & x_6 \\ x_1 & x_2 & x_3 & x_4 & x_6 & x_7 \end{pmatrix}.$$

The rolling factor coordinates take the following form:

$$\begin{aligned} x_0|_S &= s^4x, \\ x_1|_S &= s^3tx, \\ x_2|_S &= s^2t^2x, \\ x_3|_S &= st^3x, \\ x_4|_S &= t^4x, \\ x_5|_S &= s^2y, \\ x_6|_S &= sty, \\ x_7|_S &= t^2y. \end{aligned}$$

Now we want to find four quadrics q_1, q_2, q_3 and q_4 such that $q_1 = s^3 f_C$, $q_2 = s^2 t f_C$, $q_3 = st^2 f_C$ and $q_4 = t^3 f_C$ on S. The result is the following:

$$\begin{array}{rcl} q_1 & = & a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + a_4x_0x_3 + a_5x_0x_4 + a_6x_1x_4, \\ & + & a_7x_0x_5 + a_8x_0x_6 + a_9x_0x_7 + a_{10}x_1x_7 + a_{11}x_5^2 + a_{12}x_5x_6, \\ q_2 & = & a_1x_0x_1 + a_2x_1^2 + a_3x_1x_2 + a_4x_1x_3 + a_5x_1x_4 + a_6x_2x_4 \\ & + & a_7x_1x_5 + a_8x_1x_6 + a_9x_1x_7 + a_{10}x_2x_7 + a_{11}x_5x_6 + a_{12}x_6^2, \\ q_3 & = & a_1x_0x_2 + a_2x_1x_2 + a_3x_2^2 + a_4x_2x_3 + a_5x_2x_4 + a_6x_3x_4 \\ & + & a_7x_2x_5 + a_8x_2x_6 + a_9x_2x_7 + a_{10}x_3x_7 + a_{11}x_5x_7 + a_{12}x_6x_7, \\ q_4 & = & a_1x_0x_3 + a_2x_1x_3 + a_3x_2x_3 + a_4x_3^2 + a_5x_3x_4 + a_6x_4^2 \\ & + & a_7x_3x_5 + a_8x_3x_6 + a_9x_3x_7 + a_{10}x_4x_7 + a_{11}x_6x_7 + a_{12}x_7^2. \end{array}$$

3.4 $d \geq 6$ even, $e = 3$, i.e. S of scroll type $(\frac{d}{2}, \frac{d-6}{2})$

In this case I_S is after possibly a coordinate change generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{\frac{d}{2}-1} & x_{\frac{d}{2}+1} & \cdots & x_{d-3} \\ x_1 & x_2 & \cdots & x_{\frac{d}{2}} & x_{\frac{d}{2}+2} & \cdots & x_{d-2} \end{pmatrix}.$$

In terms of rolling factor coordinates this corresponds to:

$$\begin{pmatrix} s^{\frac{d}{2}}x & s^{\frac{d}{2}-1}tx & \cdots & st^{\frac{d}{2}-1}x & s^{\frac{d-6}{2}}y & s^{\frac{d-6}{2}-1}ty & \cdots & st^{\frac{d-6}{2}-1} \\ s^{\frac{d}{2}-1}tx & s^{\frac{d}{2}-2}t^2x & \cdots & t^{\frac{d}{2}}x & s^{\frac{d-6}{2}-1}ty & s^{\frac{d-6}{2}-2}t^2y & \cdots & t^{\frac{d-6}{2}}y \end{pmatrix}.$$

By Proposition 2.6 we obtain

$$\begin{aligned} & H^0\left(\mathbf{P}\left(\mathcal{O}_{\mathbf{P}^1}\left(\frac{d}{2}\right) \oplus \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-6}{2}\right)\right), 2H + (6-d)F\right) \\ & \cong H^0\left(\mathbf{P}^1, S^2\left(\mathcal{O}_{\mathbf{P}^1}\left(\frac{d}{2}\right) \oplus \mathcal{O}_{\mathbf{P}^1}\left(\frac{d-6}{2}\right)\right) \otimes \mathcal{O}_{\mathbf{P}^1}(6-d)\right) \\ & \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(6) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}) \end{aligned}$$

with a basis given by

$$\{s^6x^2, s^5tx^2, s^4t^2x^2, s^3t^3x^2, s^2t^4x^2, st^5x^2, t^6x^2, s^3xy, s^2txy, st^2xy, t^3xy, y^2\}.$$

Consequently, we see that the equation of C on S is of the following form:

$$\begin{array}{rcl} f_C & = & a_1s^6x^2 + a_2s^5tx^2 + a_3s^4t^2x^2 + a_4s^3t^3x^2 + a_5s^2t^4x^2 + a_6st^5x^2 \\ & & +a_7t^6x^2 + a_8s^3xy + a_9s^2txy + a_{10}st^2xy + a_{11}t^3xy + a_{12}y^2, \end{array}$$

with $a_1, \ldots, a_{12} \in k$.

Theorem 3.9. *Let*

$$\begin{aligned} f_C &= a_1s^6x^2 + a_2s^5tx^2 + a_3s^4t^2x^2 + a_4s^3t^3x^2 + a_5s^2t^4x^2 + a_6st^5x^2 \\ &\quad + a_7t^6x^2 + a_8s^3xy + a_9s^2txy + a_{10}st^2xy + a_{11}t^3xy + a_{12}y^2, \end{aligned}$$

with $a_1, \dots, a_{12} \in k$, be the equation of a curve C on S.
For $d \geq 12$ the quadrics $q_1, \dots, q_{d-5}$ given by the following formula cut out the curve C on S:

$$q_i = \begin{cases} a_1x_0x_{i-1} + a_2x_1x_{i-1} + a_3x_2x_{i-1} + a_4x_3x_{i-1} \\ +a_5x_4x_{i-1} + a_6x_5x_{i-1} + a_7x_6x_{i-1} \\ +a_8x_{i-1}x_{\frac{d+2}{2}} + a_9x_{i-1}x_{\frac{d+4}{2}} \\ +a_{10}x_{i-1}x_{\frac{d+6}{2}} + a_{11}x_{i-1}x_{\frac{d+8}{2}} \\ +a_{12}x_{\frac{d+2}{2}}x_{\frac{d+2}{2}+i-1}, & \text{for} \quad 1 \leq i \leq \frac{d-4}{2}, \\ \\ a_1x_{\frac{d-12}{2}}x_{i-\frac{d-10}{2}} + a_2x_{\frac{d-10}{2}}x_{i-\frac{d-10}{2}} \\ +a_3x_{\frac{d-8}{2}}x_{i-\frac{d-10}{2}} + a_4x_{\frac{d-6}{2}}x_{i-\frac{d-10}{2}} \\ +a_5x_{\frac{d-4}{2}}x_{i-\frac{d-10}{2}} + a_6x_{\frac{d-2}{2}}x_{i-\frac{d-10}{2}} \\ +a_7x_{\frac{d}{2}}x_{i-\frac{d-10}{2}} + a_8x_{\frac{d-6}{2}}x_{i+3} \\ +a_9x_{\frac{d-4}{2}}x_{i+3} + a_{10}x_{\frac{d-2}{2}}x_{i+3} \\ +a_{11}x_{\frac{d}{2}}x_{i+3} + a_{12}x_{d-2}x_{i+3}, & \text{for} \quad \frac{d-2}{2} \leq i \leq d-5. \end{cases}$$

Proof. The proof is analogous to the proof of Theorem 3.4. □

Remark 3.10. *As in the previous sections we have to exclude the lowest values for d when stating Theorem 3.9. We will now give the quadrics $q_1, \dots, q_{d-5}$ when $d = 6$, $d = 8$, $d = 10$:*

(1) $d = 6$: In this case the scroll type of the $g^1_2(C)$-scroll S is equal to $(3, 0)$, the ideal I_S is after possibly a coordinate change generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

The rolling factor coordinates are given as follows:

$$\begin{aligned} x_0|_S &= s^3x, \\ x_1|_S &= s^2tx, \\ x_2|_S &= st^2x, \\ x_3|_S &= t^3x, \\ x_4|_S &= y. \end{aligned}$$

Note again that $I_C = I_S + (Q)$ for a general quadric Q in $\mathbf{P}^4$ that is not contained in I_S.

If we set

$$\begin{aligned} Q &= a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + a_4x_0x_3 + a_5x_1x_3 + a_6x_2x_3 \\ &+ a_7x_3^2 + a_8x_0x_4 + a_9x_1x_4 + a_{10}x_2x_4 + a_{11}x_3x_4 + a_{12}x_4^2, \end{aligned}$$

which is a general quadric in $\mathbf{P}^4$ *modulo* I_S*, then* $Q|_S = f_C$*.*

(2) $d = 8$*: Here the* $g_2^1(C)$*-scroll* S *is of type* $(4, 1)$*, the ideal* I_S *is, possibly after a coordinate change, generated by the* (2×2)*-minors of the following matrix:*

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_5 \\ x_1 & x_2 & x_3 & x_4 & x_6 \end{pmatrix}.$$

The rolling factor coordinates take the following form:

$$\begin{aligned} x_0|_S &= s^4x, \\ x_1|_S &= s^3tx, \\ x_2|_S &= s^2t^2x, \\ x_3|_S &= st^3x, \\ x_4|_S &= t^4x, \\ x_5|_S &= sy, \\ x_6|_S &= ty. \end{aligned}$$

We want to find three quadrics q_1*,* q_2 *and* q_3 *such that* $q_1 = s^2f_C$*,* $q_2 = stf_C$ *and* $q_3 = t^2f_C$*. The result is the following:*

$$\begin{aligned} q_1 &= a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + a_4x_0x_3 + a_5x_0x_4 + a_6x_1x_4 \\ &+ a_7x_2x_4 + a_8x_0x_5 + a_9x_0x_6 + a_{10}x_1x_6 + a_{11}x_2x_6 + a_{12}x_5^2, \\ q_2 &= a_1x_0x_1 + a_2x_1^2 + a_3x_1x_2 + a_4x_1x_3 + a_5x_1x_4 + a_6x_2x_4 \\ &+ a_7x_3x_4 + a_8x_1x_5 + a_9x_1x_6 + a_{10}x_2x_6 + a_{11}x_3x_6 + a_{12}x_5x_6, \\ q_2 &= a_1x_0x_2 + a_2x_1x_2 + a_3x_2^2 + a_4x_2x_3 + a_5x_2x_4 + a_6x_3x_4 \\ &+ a_7x_4^2 + a_8x_2x_5 + a_9x_2x_6 + a_{10}x_3x_6 + a_{11}x_4x_6 + a_{12}x_6^2. \end{aligned}$$

(3) $d = 10$*: Here the scroll type of the* $g_2^1(C)$*-scroll* S *is equal to* $(5, 2)$*, the ideal* I_S *is, possibly after a coordinate change, generated by the* (2×2)*-minors of the following matrix:*

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_6 & x_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_7 & x_8 \end{pmatrix}.$$

The rolling factor coordinates are given as follows:

$$
\begin{aligned}
x_0|_S &= s^5x, \\
x_1|_S &= s^4tx, \\
x_2|_S &= s^3t^2x, \\
x_3|_S &= s^2t^3x, \\
x_4|_S &= st^4x, \\
x_5|_S &= t^5x, \\
x_6|_S &= s^2y, \\
x_7|_S &= sty, \\
x_8|_S &= t^2y.
\end{aligned}
$$

Here we want to find five quadrics q_1, q_2, q_3, q_4 and q_5 such that $q_1 = s^4f_C$, $q_2 = s^3tf_C$, $q_3 = s^2t^2f_C$, $q_4 = st^3f_C$ and $q_5 = t^4f_C$. The result is the following:

$$
\begin{aligned}
q_1 &= a_1x_0^2 + a_2x_0x_1 + a_3x_0x_2 + x_4x_0x_3 + a_5x_0x_4 + a_6x_0x_5 \\
&+ a_7x_1x_5 + a_8x_0x_6 + a_9x_0x_7 + a_{10}x_0x_8 + a_{11}x_1x_8 + a_{12}x_6^2, \\
q_2 &= a_1x_0x_1 + a_2x_1^2 + a_3x_1x_2 + x_4x_1x_3 + a_5x_1x_4 + a_6x_1x_5 \\
&+ a_7x_2x_5 + a_8x_1x_6 + a_9x_1x_7 + a_{10}x_1x_8 + a_{11}x_2x_8 + a_{12}x_6x_7, \\
q_3 &= a_1x_0x_2 + a_2x_1x_2 + a_3x_2^2 + x_4x_2x_3 + a_5x_2x_4 + a_6x_2x_5 \\
&+ a_7x_3x_5 + a_8x_2x_6 + a_9x_2x_7 + a_{10}x_2x_8 + a_{11}x_3x_8 + a_{12}x_7^2, \\
q_4 &= a_1x_0x_3 + a_2x_1x_3 + a_3x_2x_3 + x_4x_3^2 + a_5x_3x_4 + a_6x_3x_5 \\
&+ a_7x_4x_5 + a_8x_3x_6 + a_9x_3x_7 + a_{10}x_3x_8 + a_{11}x_4x_8 + a_{12}x_7x_8, \\
q_5 &= a_1x_0x_4 + a_2x_1x_4 + a_3x_2x_4 + x_4x_3x_4 + a_5x_4^2 + a_6x_4x_5 \\
&+ a_7x_5^2 + a_8x_4x_6 + a_9x_4x_7 + a_{10}x_4x_8 + a_{11}x_5x_8 + a_{12}x_8^2.
\end{aligned}
$$

3.5 An alternative presentation when $7 \leq d \leq 12$

Inspired by Theorems 3.4, 3.5, 3.7 and 3.9 we will now list sets of quadrics that together with I_S generate I_C in the cases where the degree d of C satisfies $7 \leq d \leq 12$. In order to give a compact form which will be more practical in Chapter 4 where we look at resolutions of I_C, we will instead of using the coefficients $a_1, \ldots, a_{12} \in k$ use general linear forms $l_i \in k[x_0, \ldots, x_{d-2}]$. Modulo I_S this gives the same result. For $d = 10, 11, 12$ we cannot use this notation with the linear forms l_i but have to use a mixed version with l_i and a_i's. For $d \geq 13$ we have to entirely go back to the notation with the a_i's as in Theorems 3.4, 3.5, 3.7, 3.9, since e.g. we would need that the term lx_0 in terms of rolling factor coordinates contains s^{d-6}. But if $l \in k[x_0, \ldots, x_{d-2}]$ is a general linear form, then the monomial x_0x_{d-2} appears in lx_0, and in terms of rolling factor coordinates we have $x_0x_{d-2} = s^{e_1}t^{e_2}xy$, where (e_1, e_2) is the scroll type of S. But in all cases we have $e_1 \leq \frac{d}{2}$, so $e_1 < d - 6$ for all $d \geq 13$. For the cases $d = 11$ and $d = 12$ we have two versions depending on the scroll type.

Table 3.1: Generators

$\deg(C)$	e	Generating matrix for I_S	Generators of I_C modulo I_S
7	0	$\begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix}$	$q_1 = l_1x_0 + l_2x_1 + l_3x_3 + l_4x_4,$ $q_2 = l_1x_1 + l_2x_2 + l_3x_4 + l_4x_5$
7	2	$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 \\ x_1 & x_2 & x_3 & x_5 \end{pmatrix}$	$q_1 = l_1x_0 + l_2x_1 + l_3x_2 + l_4x_4,$ $q_2 = l_1x_1 + l_2x_2 + l_3x_3 + l_4x_5$
8	1	$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_5 & x_6 \end{pmatrix}$	$q_1 = l_1x_0 + l_2x_1 + l_3x_4,$ $q_2 = l_1x_1 + l_2x_2 + l_3x_5,$ $q_3 = l_1x_2 + l_2x_3 + l_3x_6$
8	3	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_5 \\ x_1 & x_2 & x_3 & x_4 & x_6 \end{pmatrix}$	$q_1 = l_1x_0 + l_2x_1 + l_3x_2,$ $q_2 = l_1x_1 + l_2x_2 + l_3x_3,$ $q_3 = l_1x_2 + l_2x_3 + l_3x_4$
9	0	$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 & x_5 & x_6 \\ x_1 & x_2 & x_3 & x_5 & x_6 & x_7 \end{pmatrix}$	$q_1 = l_1x_0 + l_2x_4,$ $q_2 = l_1x_1 + l_2x_5,$ $q_3 = l_1x_2 + l_2x_6,$ $q_4 = l_1x_3 + l_2x_7$
9	2	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_5 & x_6 \\ x_1 & x_2 & x_3 & x_4 & x_6 & x_7 \end{pmatrix}$	$q_1 = l_1x_0 + l_2x_1,$ $q_2 = l_1x_1 + l_2x_2,$ $q_3 = l_1x_2 + l_2x_3,$ $q_4 = l_1x_3 + l_2x_4$
10	1	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_5 & x_6 & x_7 \\ x_1 & x_2 & x_3 & x_4 & x_6 & x_7 & x_8 \end{pmatrix}$	$q_1 = l_1x_0 + a_{10}x_5^2 + a_{11}x_5x_6 + a_{12}x_5x_7,$ $q_2 = l_1x_1 + a_{10}x_5x_6 + a_{11}x_6^2 + a_{12}x_6x_7,$ $q_3 = l_1x_2 + a_{10}x_5x_7 + a_{11}x_6x_7 + a_{12}x_7^2,$ $q_4 = l_1x_3 + a_{10}x_5x_8 + a_{11}x_6x_8 + a_{12}x_7x_8,$ $q_5 = l_1x_4 + a_{10}x_6x_8 + a_{11}x_7x_8 + a_{12}x_8^2$
10	3	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_6 & x_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_7 & x_8 \end{pmatrix}$	$q_1 = l_1x_0 + a_7x_1x_5 + a_{11}x_1x_8 + a_{12}x_6^2,$ $q_2 = l_1x_1 + a_7x_2x_5 + a_{11}x_2x_8 + a_{12}x_6x_7,$ $q_3 = l_1x_2 + a_7x_3x_5 + a_{11}x_3x_8 + a_{12}x_7^2,$ $q_4 = l_1x_3 + a_7x_4x_5 + a_{11}x_4x_8 + a_{12}x_7x_8$ $q_5 = l_1x_4 + a_7x_5^2 + a_{11}x_5x_8 + a_{12}x_8^2$

deg(C)	e	Generating matrix for I_S	Generators of I_C modulo I_S
11	0	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_5 & x_6 & x_7 & x_8 \\ x_1 & x_2 & x_3 & x_4 & x_6 & x_7 & x_8 & x_9 \end{pmatrix}$	$q_1 = x_0 \sum_{i=0}^{3} a_{i+1}x_i$ $+ x_0 \sum_{i=5}^{8} a_i x_i$ $+ x_5 \sum_{i=5}^{8} a_{i+4}x_i,$ $q_2 = x_1 \sum_{i=0}^{3} a_{i+1}x_i$ $+ x_1 \sum_{i=5}^{8} a_i x_i$ $+ x_6 \sum_{i=5}^{8} a_{i+4}x_i,$ $q_3 = x_2 \sum_{i=0}^{3} a_{i+1}x_i$ $+ x_2 \sum_{i=5}^{8} a_i x_i$ $+ x_7 \sum_{i=5}^{8} a_{i+4}x_i,$ $q_4 = x_3 \sum_{i=0}^{3} a_{i+1}x_i$ $+ x_3 \sum_{i=5}^{8} a_i x_i$ $+ x_8 \sum_{i=5}^{8} a_{i+4}x_i,$ $q_5 = x_4 \sum_{i=0}^{3} a_{i+1}x_i$ $+ x_4 \sum_{i=5}^{8} a_i x_i$ $+ x_9 \sum_{i=5}^{8} a_{i+4}x_i,$ $q_6 = x_4 \sum_{i=1}^{4} a_i x_i$ $+ x_9 \sum_{i=1}^{4} a_{i+4}x_i$ $+ x_9 \sum_{i=6}^{9} a_{i+3}x_i$
11	2	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_6 & x_7 & x_8 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_7 & x_8 & x_9 \end{pmatrix}$	$q_1 = l_1x_0 + a_{11}x_6^2 + a_{12}x_6x_7,$ $q_2 = l_1x_1 + a_{11}x_6x_7 + a_{12}x_7^2,$ $q_3 = l_1x_2 + a_{11}x_6x_8 + a_{12}x_7x_8,$ $q_4 = l_1x_3 + a_{11}x_6x_9 + a_{12}x_7x_9,$ $q_5 = l_1x_4 + a_{11}x_7x_9 + a_{12}x_8x_9,$ $q_6 = l_1x_5 + a_{11}x_8x_9 + a_{12}x_9^2$
12	1	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_6 & x_7 & x_8 & x_9 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_7 & x_8 & x_9 & x_{10} \end{pmatrix}$	$q_1 = x_0 \sum_{i=0}^{4} a_{i+1}x_i$ $+ x_0 \sum_{i=6}^{9} a_i x_i$ $+ x_6 \sum_{i=6}^{8} a_{i+4}x_i,$ $q_2 = x_1 \sum_{i=0}^{4} a_{i+1}x_i$ $+ x_1 \sum_{i=6}^{9} a_i x_i$ $+ x_7 \sum_{i=6}^{8} a_{i+4}x_i,$ $q_3 = x_2 \sum_{i=0}^{4} a_{i+1}x_i$ $+ x_2 \sum_{i=6}^{9} a_i x_i$ $+ x_8 \sum_{i=6}^{8} a_{i+4}x_i,$ $q_4 = x_3 \sum_{i=0}^{4} a_{i+1}x_i$ $+ x_3 \sum_{i=6}^{9} a_i x_i$ $+ x_9 \sum_{i=6}^{8} a_{i+4}x_i,$ $q_5 = x_4 \sum_{i=0}^{4} a_{i+1}x_i$ $+ x_4 \sum_{i=6}^{9} a_i x_i$ $+ x_{10} \sum_{i=6}^{8} a_{i+4}x_i,$ $q_6 = x_4 \sum_{i=1}^{5} a_i x_i$ $+ x_9 \sum_{i=2}^{5} a_{i+4}x_i$ $+ x_9 \sum_{i=8}^{10} a_{i+2}x_i,$ $q_7 = x_5 \sum_{i=1}^{5} a_i x_i$ $+ x_{10} \sum_{i=2}^{5} a_{i+4}x_i$ $+ x_{10} \sum_{i=8}^{10} a_{i+2}x_i,$

deg(C)	e	Generating matrix for I_S	Generators of I_C modulo I_S
12	3	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_7 & x_8 & x_9 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_8 & x_9 & x_{10} \end{pmatrix}$	$q_1 = l_1x_0 + a_{12}x_7^2$, $q_2 = l_1x_1 + a_{12}x_7x_8$, $q_3 = l_1x_2 + a_{12}x_7x_9$, $q_4 = l_1x_3 + a_{12}x_7x_{10}$, $q_5 = l_1x_4 + a_{12}x_8x_{10}$, $q_6 = l_1x_5 + a_{12}x_9x_{10}$, $q_7 = l_1x_6 + a_{12}x_{10}^2$

A third presentation of generators, namely as matrix product

Using the generators of I_C modulo I_S as listed in the above tables we give, in the cases $d = 7$, $d = 8$ and $d = 9$, a matrix A with as few columns as possible such that the entries in the matrix product $M \cdot A$ give us all generators of I_C where M is the generating matrix of I_S as given in Table 3.1.

$d = 7$

In the case when S is maximally balanced, i.e. when $e = 0$, the matrix A is given as follows:

$$A = \begin{pmatrix} l_1 & x_2 & x_4 & x_5 & 0 \\ l_2 & -x_1 & -x_3 & -x_4 & 0 \\ l_3 & 0 & 0 & 0 & x_5 \\ l_4 & 0 & 0 & 0 & -x_4 \end{pmatrix}.$$

In the case $e = 2$ the matrix A has the following form:

$$A = \begin{pmatrix} l_1 & x_2 & x_3 & x_5 & 0 \\ l_2 & -x_1 & -x_2 & -x_4 & 0 \\ l_3 & 0 & 0 & 0 & x_5 \\ l_4 & 0 & 0 & 0 & -x_3 \end{pmatrix}.$$

$d = 8$

In the case $e = 1$ the matrix A is given as follows:

$$A = \begin{pmatrix} l_1 & 0 & x_2 & x_3 & x_5 & x_6 & 0 & 0 \\ l_2 & l_1 & -x_1 & -x_2 & -x_4 & -x_5 & 0 & 0 \\ 0 & l_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_3 & 0 & 0 & 0 & 0 & 0 & -x_3 & x_6 \\ 0 & l_3 & 0 & 0 & 0 & 0 & x_2 & -x_5 \end{pmatrix}.$$

In the case $e = 3$ the matrix A is given as follows:

$$A = \begin{pmatrix} l_1 & 0 & x_2 & x_3 & x_4 & x_6 & 0 & 0 \\ l_2 & l_1 & -x_1 & -x_2 & -x_3 & -x_5 & 0 & 0 \\ l_3 & l_2 & 0 & 0 & 0 & 0 & -x_3 & x_6 \\ 0 & l_3 & 0 & 0 & 0 & 0 & x_2 & -x_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$d = 9$

In the case $e = 0$ the matrix A is given as follows:

$$A = \begin{pmatrix} l_1 & 0 & x_2 & x_3 & x_5 & x_6 & x_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_1 & -x_2 & -x_4 & -x_5 & -x_6 & 0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_2 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_3 & -x_5 & -x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_4 & x_5 \end{pmatrix}.$$

In the case $e = 2$ the matrix A is given as follows:

$$A = \begin{pmatrix} l_1 & 0 & x_2 & x_3 & x_4 & x_6 & x_7 & 0 & 0 & 0 & 0 \\ l_2 & 0 & -x_1 & -x_2 & -x_3 & -x_5 & -x_6 & 0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 & 0 & 0 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & -x_4 & -x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_3 & x_5 \end{pmatrix}.$$

Chapter 4

The minimal free resolution of $\mathcal{O}_C$

Let C be a smooth curve of genus 2 and degree $d \geq 5$ embedded in $\mathbf{P}^{d-2}$, and let S be the $g_2^1(C)$-scroll. In this chapter we will find the minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module, using the minimal free resolution of $\mathcal{O}_S$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module and a mapping cone technique.

4.1 Free resolutions and Betti diagrams

First we recall the definition of a minimal free resolution of a module M over a ring R:

Definition 4.1. *Let R be a commutative ring and M an R-module. A free resolution of M as an R-module is an exact complex of R-modules:*

$$0 \to \oplus_{j \geq j_n} R(-j)^{\beta_{nj}} \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} \oplus_{j \geq j_1} R(-j)^{\beta_{1j}} \xrightarrow{\phi_1} \oplus_{j \geq j_0} R(-j)^{\beta_{0j}} \to M \to 0.$$

The resolution is minimal if the image of ϕ_i is contained in the maximal ideal of R for all i.
The non-negative integers $\beta_{i,j}$, i.e. the ranks of the R-modules, are called the graded Betti numbers of M.

For our varieties X, several Betti numbers of $\mathcal{O}_X$ are equal to 0. If X is a rational normal scroll, then $\mathcal{O}_X$ is resolved by the Eagon-Northcott complex, which means that the only non-zero Betti numbers are equal to $\beta_{00} = 1$ and $\beta_{i,i+1}$ for $i = 1, \ldots, \operatorname{codim}(X)$. Let now $V = H^0(C, \mathcal{O}_C(H))$, $R = \mathcal{S}\mathrm{ym}(V)$ and $R_C = R/I_C = \oplus_{q \in \mathbf{Z}} H^0(C, \mathcal{O}_C(qH))$. The following proposition states which Betti numbers of R_C definitely are equal to 0:

Proposition 4.2. *The only possible non-zero Betti numbers of R_C are β_{00}, $\beta_{i,i+1}$, $i = 1, \ldots, d-4$, $\beta_{d-4,d-2}$ and $\beta_{d-3,d-1}$.*

Proof. There is the following Koszul complex:

$$\cdots \longrightarrow \bigwedge^{p+1} V \otimes (R_C)_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^{p} V \otimes (R_C)_q \xrightarrow{d_{p,q}} \bigwedge^{p-1} V \otimes (R_C)_{q+1} \longrightarrow \cdots.$$

Let $\mathcal{K}_{p,q}(R_C, V)$ be the *Koszul cohomology group*

$$\mathcal{K}_{p,q}(R_C, V) := \frac{\ker d_{p,q}}{\operatorname{im} d_{p+1,q-1}}.$$

Moreover, let

$$\cdots \to \oplus_{q \geq q_1} M_{1,q} \otimes R(-q) \to \oplus_{q \geq q_0} M_{0,q} \otimes R(-q) \to R_C \to 0$$

be the minimal free resolution of R_C.
Then by the Syzygy Theorem ([MG1], Thm. (1.b.4)) there is the following isomorphism:

$$\mathcal{K}_{p,q}(R_C, V) \cong M_{p,p+q}.$$

Theorem (4.a.1) in [Gre84] then gives us that $\mathcal{K}_{p,q}(R_C, V) = 0$ for $q \geq 3$, since $h^1(\mathcal{O}_C(H)) = 0$, and $\mathcal{K}_{p,2}(R_C, V) = 0$ if $d \geq 5+p$. That is, we have two candidates for $\mathcal{K}_{p,2}$ not to be zero, namely for $p = d-3$ and $p = d-4$·
In order to show that $\mathcal{K}_{d-3,1}(R_C, V) = 0$ we need the following theorem:

Theorem 4.3. *(The Duality Theorem, Thm. (2.c.6) in [Gre84])*
Let X be a compact complex manifold of dimension n, let $\mathcal{L} \to X$ be a line bundle and let $\mathcal{F} \to X$ be a vector bundle.
Set $B = \oplus_{q \in \mathbf{Z}} H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes q})$. If $V \subseteq H^0(X, \mathcal{L})$ is basepoint-free and of dimension $r+1$ and

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{q-i}) = 0, \qquad i = 1, \ldots, n-1,$$
$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{q-i-1}) = 0, \qquad i = 1, \ldots, n-1,$$

then

$$\mathcal{K}_{p,q}(B, V)^* \cong \mathcal{K}_{r-n-p,n+1-q}(B', V),$$

where $B' := \oplus_{q \in \mathbf{Z}} H^0(X, \omega_X \otimes \mathcal{F}^* \otimes \mathcal{L}^{\otimes q})$.

Here we use the Duality Theorem with $X = C$, $\mathcal{L} = \mathcal{O}_C(H)$, $\mathcal{F} = \mathcal{O}_C$ and $V = H^0(C, \mathcal{O}_C(H))$ and obtain that

$$\mathcal{K}_{d-3,1}(R_C, V)^* = \mathcal{K}_{0,1}(\oplus_{q \in \mathbf{Z}} H^0(C, \mathcal{O}_C(qH + K_C), V)).$$

By definition we have that

$$\mathcal{K}_{0,1}(C, K_C, \mathcal{O}_C(H)) = \frac{\ker d_{0,1}}{\operatorname{im} d_{1,0}}$$

where

$$H^0(C, \mathcal{O}_C(H)) \otimes H^0(C, \mathcal{O}_C(K_C)) \xrightarrow{d_{1,0}} H^0(C, \mathcal{O}_C(K_C + H)) \xrightarrow{d_{0,1}} 0.$$

Consider the map

$$\psi : H^0(C, \mathcal{O}_C(H)) \otimes H^0(C, \mathcal{O}_C(K_C)) \to H^0(C, \mathcal{O}_C(K_C + H)),$$

$$D_1 \otimes D_2 \mapsto D_1 + D_2.$$

The image of ψ is spanned by those divisors in $H^0(C, \mathcal{O}_C(H + K_C))$ that contain a divisor in $H^0(\mathcal{O}_C(K_C))$. Each divisor in $H^0(\mathcal{O}_C(K_C))$ spans a fiber of S, i.e. the image of ψ is spanned by those hyperplanes in $\mathbf{P}(H^0(\mathcal{O}_C(H + K_C))) = \mathbf{P}^d$ that contain a fiber of $S \subseteq \mathbf{P}^d$. Consequently, ψ is surjective if and only if there is no common point for all fibres, i.e. if and only if $S \subseteq \mathbf{P}^d$ is smooth. By Proposition 2.13 in Chapter 2 $S \subseteq \mathbf{P}^d$ is smooth for all $d \geq 5$. This implies that $\mathcal{K}_{0,1}(C, K_C, \mathcal{O}_C(H)) = 0$ and thus also $\mathcal{K}_{d-3,1}(C, \mathcal{O}_C(H)) = 0$ for $d \geq 5$. □

We write the Betti numbers in a Betti diagram, and our notation will be the following, where a dash indicates that the corresponding Betti number is equal to 0:

–	–	–	...	–	$\beta_{d-4,d-2}$	$\beta_{d-3,d-1}$
–	β_{12}	β_{23}	...	$\beta_{d-5,d-4}$	$\beta_{d-4,d-3}$	–
β_{00}	–	–	...	–	–	–

For an $\mathcal{O}_{\mathbf{P}^{d-2}}$-module V we denote by S_iV the symmetric algebra of V, by D_iV the divided power algebra of V and by $\bigwedge^j V$ the jth wedge product of V. Notice that there is an isomorphism $D_iV \cong S_iV^*$. We will also use the following natural isomorphisms:

$$D_i\mathcal{O}^2_{\mathbf{P}^{d-2}} \cong S_i\mathcal{O}^2_{\mathbf{P}^{d-2}} \cong \mathcal{O}^{i+1}_{\mathbf{P}^{d-2}}$$

and

$$\bigwedge^j \mathcal{O}^{d-3}_{\mathbf{P}^{d-2}}(-1) \cong \mathcal{O}^{\binom{d-3}{j}}_{\mathbf{P}^{d-2}}(-j).$$

In this chapter we will use the following coordinates:
After possibly a coordinate change we can assume that I_S is generated by the (2×2)-minors of the following matrix:

$$M = \begin{pmatrix} x_0 & \dots & x_{e_1-1} & x_{e_1+1} & \dots & x_{d-3} \\ x_1 & \dots & x_{e_1} & x_{e_1+2} & \dots & x_{d-2} \end{pmatrix}.$$

Let $\Phi : \mathcal{O}^{d-3}_{\mathbf{P}^{d-2}}(-1) \to \mathcal{O}^2_{\mathbf{P}^{d-2}}$ be the map given by multiplication with the matrix M. Define a complex $\mathcal{C}^b$ by defining the jth term as

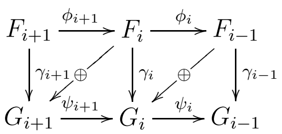
$$C^b_j = \begin{cases} \bigwedge^j \mathcal{O}^{d-3}_{\mathbf{P}^{d-2}}(-1) \otimes S_{b-j}\mathcal{O}^2_{\mathbf{P}^{d-2}}, & 0 \le j \le b \\ \bigwedge^{j+1} \mathcal{O}^{d-3}_{\mathbf{P}^{d-2}}(-1) \otimes S_{j-b-1}\mathcal{O}^2_{\mathbf{P}^{d-2}}, & j \ge b+1 \end{cases}$$

and with differentials $\mathcal{C}^b_j \to \mathcal{C}^b_{j-1}$ given by the multiplication with $\Phi \in H^0(\mathbf{P}^{d-2}, \mathcal{O}^{2(d-3)}_{\mathbf{P}^{d-2}}(1))$ for $j \ne b+1$ and $\bigwedge^2 \Phi \in H^0(\mathbf{P}^{d-2}, \mathcal{O}^{\binom{d-3}{2}}_{\mathbf{P}^{d-2}}(2))$ for $j = b+1$.

Theorem 4.4. *([Sch86], §1)*
For $a, b \in \mathbf{Z}$, $b \ge -1$, $\mathcal{C}^b(a)$ is a minimal resolution of $\mathcal{O}_S(aH+bF)$ as an $\mathcal{O}_{\mathbf{P}^{d-2}}$-module.

In particular, $\mathcal{C}^0$ is a minimal resolution of $\mathcal{O}_S$ as an $\mathcal{O}_{\mathbf{P}^{d-2}}$-module. Since $[C] = 2H + (6-d)F$ in $\mathrm{Pic}(S)$, we have $\mathcal{I}_{C,S} = \mathcal{O}_S(-C) = \mathcal{O}_S(-2H + (d-6)F)$, and thus $\mathcal{C}^{d-6}(-2)$ is a minimal resolution of $I_{C,S}$ as an $\mathcal{O}_{\mathbf{P}^{d-2}}$-module.

Definition 4.5. *(Mapping cone)(cf. [Eis95], Appendix A3.12): If $F_\bullet$ and $G_\bullet$ are complexes with differentials $\phi_i : F_i \to F_{i-1}$ and $\psi_i : G_i \to G_{i-1}$ and if $\gamma : F \to G$ is a map, then the mapping cone complex $H_\bullet$ is defined as follows:*



$H_i := F_i \oplus G_{i+1}$ with differential

$$\delta_i = \begin{pmatrix} -\phi_i & 0 \\ \gamma_i & \psi_{i+1} \end{pmatrix} : H_i \to H_{i-1}.$$

In the next section we will describe how we obtain a resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module from the mapping cone $\mathcal{C}^{d-6}(-2) \to \mathcal{C}^0$.
First we will describe in one example how we obtain the minimal free resolution of I_C with the mapping cone technique:

Example 4.6. $d = 6$*:*
A resolution of a curve of genus 2 *and degree* 6 *is given by the mapping cone* $\mathcal{C}^0(-2) \to \mathcal{C}^0$*.*
The complex $\mathcal{C}^0$ *is given by*

$$0 \to \bigwedge^3 \mathcal{O}^3_{\mathbf{P}^4}(-1) \otimes S_1\mathcal{O}^2_{\mathbf{P}^4} \to \bigwedge^2 \mathcal{O}^3_{\mathbf{P}^4}(-1) \otimes S_0\mathcal{O}^2_{\mathbf{P}^4} \to \bigwedge^0 \mathcal{O}^3_{\mathbf{P}^4}(-1) \otimes S_0\mathcal{O}^2_{\mathbf{P}^4} \to 0,$$

which is equal to

$$0 \to \mathcal{O}^2_{\mathbf{P}^4}(-3) \to \mathcal{O}^3_{\mathbf{P}^4}(-2) \to \mathcal{O}_{\mathbf{P}^4} \to 0.$$

Consequently $\mathcal{C}^0(-2)$ *is equal to*

$$0 \to \mathcal{O}^2_{\mathbf{P}^4}(-5) \to \mathcal{O}^3_{\mathbf{P}^4}(-4) \to \mathcal{O}_{\mathbf{P}^4}(-2) \to 0.$$

Now we form the mapping cone $\mathcal{C}^0(-2) \to \mathcal{C}^0$*:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^2_{\mathbf{P}^4}(-5) & \longrightarrow & \mathcal{O}^3_{\mathbf{P}^4}(-4) & \longrightarrow & \mathcal{O}_{\mathbf{P}^4}(-2) & \longrightarrow & 0 \\
\downarrow & \swarrow\oplus & \downarrow & \swarrow\oplus & \downarrow & \swarrow\oplus & \downarrow & \swarrow\oplus & \downarrow \\
0 & \longrightarrow & \mathcal{O}^2_{\mathbf{P}^4}(-3) & \longrightarrow & \mathcal{O}^3_{\mathbf{P}^4}(-2) & \longrightarrow & \mathcal{O}_{\mathbf{P}^4} & \longrightarrow & 0
\end{array}$$

and finally obtain the following resolution of $\mathcal{O}_C$ *as an* $\mathcal{O}_{\mathbf{P}^4}$*-module:*

$$0 \to \mathcal{O}^2_{\mathbf{P}^4}(-5) \to \mathcal{O}^3_{\mathbf{P}^4}(-4) \oplus \mathcal{O}^2_{\mathbf{P}^4}(-3) \to \mathcal{O}^4_{\mathbf{P}^4}(-2) \to \mathcal{O}_{\mathbf{P}^4} \to \mathcal{O}_C \to 0.$$

Since all entries in the matrices that give the differentials in the above complex have degree at least 1*, we observe that this resolution is in fact minimal.*

4.2 The resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module for $d \geq 5$

Let C be a smooth curve of genus 2 and degree $d \geq 5$ embedded in $\mathbf{P}^{d-2}$.
Set $\mathcal{O} := \mathcal{O}_{\mathbf{P}^{d-2}}$.
We will now describe a way to obtain a resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbf{P}^{d-2}}$-module. Having obtained this resolution we will see that it is in fact minimal.
We refer to [Sch86] for further details.

(1) Construct $\mathcal{C}^0$:

$$\begin{aligned} 0 \;\rightarrow\; & \bigwedge^{d-3} \mathcal{O}^{d-3}(-1) \otimes S_{d-5}\mathcal{O}^2 \rightarrow \bigwedge^{d-4} \mathcal{O}^{d-3}(-1) \otimes S_{d-6}\mathcal{O}^2 \\ \rightarrow\; & \bigwedge^{d-5} \mathcal{O}^{d-3}(-1) \otimes S_{d-7}\mathcal{O}^2 \rightarrow \cdots \rightarrow \bigwedge^{4} \mathcal{O}^{d-3}(-1) \otimes S_{2}\mathcal{O}^2 \\ \rightarrow\; & \bigwedge^{3} \mathcal{O}^{d-3}(-1) \otimes S_{1}\mathcal{O}^2 \rightarrow \bigwedge^{2} \mathcal{O}^{d-3}(-1) \otimes S_{0}\mathcal{O}^2 \\ \rightarrow\; & \bigwedge^{0} \mathcal{O}^{d-3}(-1) \otimes S_{0}\mathcal{O}^2 \rightarrow 0 \end{aligned}$$

which is equal to

$$\begin{aligned} 0 \;\rightarrow\; & \mathcal{O}(-(d-3))^{d-4} \rightarrow \mathcal{O}(-(d-4))^{(d-5)(d-3)} \rightarrow \mathcal{O}(-(d-5))^{(d-6)\binom{d-3}{2}} \\ \rightarrow\; & \cdots \rightarrow \mathcal{O}(-4)^{3\binom{d-3}{4}} \rightarrow \mathcal{O}(-3)^{2\binom{d-3}{3}} \rightarrow \mathcal{O}(-2)^{\binom{d-3}{2}} \rightarrow \mathcal{O} \rightarrow 0. \end{aligned}$$

(2) Find $\mathcal{C}^{d-6}(-2)$:

(a) $\mathcal{C}^{d-6}$ is given by

$$\begin{aligned} 0 \;\rightarrow\; & \bigwedge^{d-3} \mathcal{O}^{d-3}(-1) \otimes S_{1}\mathcal{O}^2 \rightarrow \bigwedge^{d-4} \mathcal{O}^{d-3}(-1) \otimes S_{0}\mathcal{O}^2 \\ \rightarrow\; & \bigwedge^{d-6} \mathcal{O}^{d-3}(-1) \otimes S_{0}\mathcal{O}^2 \rightarrow \cdots \rightarrow \bigwedge^{d-7} \mathcal{O}^{d-3}(-1) \otimes S_{1}\mathcal{O}^2 \\ \rightarrow\; & \bigwedge^{d-8} \mathcal{O}^{d-3}(-1) \otimes S_{2}\mathcal{O}^2 \rightarrow \cdots \rightarrow \bigwedge^{4} \mathcal{O}^{d-3}(-1) \otimes S_{d-10}\mathcal{O}^2 \\ \rightarrow\; & \bigwedge^{3} \mathcal{O}^{d-3}(-1) \otimes S_{d-9}\mathcal{O}^2 \rightarrow \bigwedge^{2} \mathcal{O}^{d-3}(-1) \otimes S_{d-8}\mathcal{O}^2 \\ \rightarrow\; & \bigwedge^{1} \mathcal{O}^{d-3}(-1) \otimes S_{d-7}\mathcal{O}^2 \rightarrow \bigwedge^{0} \mathcal{O}^{d-3}(-1) \otimes S_{d-6}\mathcal{O}^2 \rightarrow 0 \end{aligned}$$

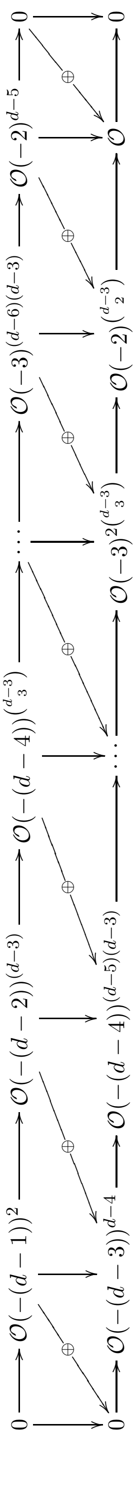
which is equal to

$$\begin{aligned} 0 \;\rightarrow\; & \mathcal{O}(-(d-3))^{2} \rightarrow \mathcal{O}(-(d-4))^{(d-3)} \rightarrow \mathcal{O}(-(d-6))^{\binom{d-3}{3}} \\ \rightarrow\; & \mathcal{O}(-(d-7))^{2\binom{d-3}{4}} \rightarrow \mathcal{O}(-(d-8))^{3\binom{d-3}{5}} \rightarrow \cdots \\ \rightarrow\; & \mathcal{O}(-4)^{(d-9)\binom{d-3}{4}} \rightarrow \mathcal{O}(-3)^{(d-8)\binom{d-3}{3}} \rightarrow \mathcal{O}(-2)^{(d-7)\binom{d-3}{2}} \\ \rightarrow\; & \mathcal{O}(-1)^{(d-6)(d-3)} \rightarrow \mathcal{O}^{d-5} \rightarrow 0. \end{aligned}$$

(b) Therefore $\mathcal{C}^{d-6}(-2)$ is the following complex:

$$\begin{aligned} 0 &\to \mathcal{O}(-(d-1))^2 \to \mathcal{O}(-(d-2))^{(d-3)} \to \mathcal{O}(-(d-4))^{\binom{d-3}{3}} \\ &\to \mathcal{O}(-(d-5))^{2\binom{d-3}{4}} \to \mathcal{O}(-(d-6))^{3\binom{d-3}{5}} \to \cdots \\ &\to \mathcal{O}(-6)^{(d-9)\binom{d-3}{4}} \to \mathcal{O}(-5)^{(d-8)\binom{d-3}{3}} \to \mathcal{O}(-4)^{(d-7)\binom{d-3}{2}} \\ &\to \mathcal{O}(-3)^{(d-6)(d-3)} \to \mathcal{O}(-2)^{d-5} \to 0. \end{aligned}$$

(3) Now form the mapping cone $\mathcal{C}^{d-6}(-2) \to \mathcal{C}^0$:


0
$\mathcal{O}(-(d-1))^2$
$\mathcal{O}(-(d-2))^{(d-3)}$
$\mathcal{O}(-(d-4))^{\binom{d-3}{3}}$
$\cdots$
$\mathcal{O}(-3)^{(d-6)(d-3)}$
$\mathcal{O}(-2)^{d-5}$
0
0
$\mathcal{O}(-(d-3))^{d-4}$
$\mathcal{O}(-(d-4))^{(d-5)(d-3)}$
$\cdots$
$\mathcal{O}(-3)^{2\binom{d-3}{3}}$
$\mathcal{O}(-2)^{\binom{d-3}{2}}$
$\mathcal{O}$
0
$\oplus$

From the above mapping cone we obtain the following resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module:

$$\begin{aligned}
0 &\rightarrow \mathcal{O}(-(d-1))^2 \rightarrow \mathcal{O}(-(d-2))^{(d-3)} \oplus \mathcal{O}(-(d-3))^{(d-4)} \\
&\rightarrow \mathcal{O}(-(d-4))^{\binom{d-3}{3}+(d-5)(d-3)} \rightarrow \mathcal{O}(-(d-5))^{2\binom{d-3}{4}+(d-6)\binom{d-3}{2}} \\
&\rightarrow \cdots \rightarrow \mathcal{O}(-4)^{(d-7)\binom{d-3}{2}+3\binom{d-3}{4}} \rightarrow \mathcal{O}(-3)^{(d-6)(d-3)+2\binom{d-3}{3}} \\
&\rightarrow \mathcal{O}(-2)^{\binom{d-3}{2}+d-5} \rightarrow \mathcal{O} \rightarrow \mathcal{O}_C \rightarrow 0.
\end{aligned}$$

Since all maps in the complexes are given by multiplication with matrices with polynomial entries of degree at least 1, this resolution is in fact minimal. Hence we obtain the following Betti numbers:

$$\begin{aligned}
\beta_{0,0} &= 1, \\
\beta_{i,i+1} &= i\binom{d-3}{i+1} + (d-4-i)\binom{d-3}{i-1}, i = 1, \ldots, d-4, \\
\beta_{d-4,d-2} &= d-3, \\
\beta_{d-3,d-1} &= 2.
\end{aligned}$$

Consequently, the Betti diagram looks like this:

–	–	–	$\cdots$	–	–	2
–	–	–	$\cdots$	–	$d-3$	–
–	$\binom{d-3}{2}$ $+(d-5)$	$2\binom{d-3}{3}$ $+(d-6)(d-3)$	$\cdots$	$(d-5)(d-3)$ $+\binom{d-3}{3}$	$d-4$	–
1	–	–	$\cdots$	–	–	–

Remark 4.7. *We can also find the Betti numbers β_{ij} via the Hilbert polynomial of $\mathcal{O}_C$, $H_C(t) = dt - 1$:*
By Proposition 4.2 we know that the only possible non-zero Betti numbers are β_{00}, β_{12}, β_{23}, ..., $\beta_{d-4,d-3}$, $\beta_{d-4,d-2}$, $\beta_{d-3,d-1}$. In particular, for each j there is at most one i such that $\beta_{ij} \neq 0$. Now, by [Eis05], Chapter 1B, Corollary 1.10., we have the following recursive formula:

$$B_j = H_C(j) - \sum_{k<j} B_k \binom{d-2+j-k}{j-k},$$

where $B_j = \sum_{i \geq 0} (-1)^i \beta_{ij}$.
We can start with $\beta_{00} = 1$ and obtain the other β_{ij} recursively.

4.3 The differentials in the mapping cone complex

$d = 5$

Let us consider a curve C of degree 5 on a smooth $g^1_2(C)$-scroll S, i.e. S is a smooth quadric $Q \cong \mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$. After possibly a coordinate change the ideal I_S is generated by the determinant of the matrix

$$M = \begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \end{pmatrix}.$$

Recall that if C is a curve of genus 2 and degree 5 in $\mathbf{P}^3$, then $I_{C,S} = \mathcal{O}_S(-2H - F)$. Hence by Theorem 4.4, a minimal free resolution of $I_{C,S}$ as an $\mathcal{O}_{\mathbf{P}^3}$-module is given by the complex $\mathcal{C}^{-1}(-2)$. Moreover, again by Theorem 4.4, $\mathcal{C}^0$ gives a minimal resolution of $\mathcal{O}_S$ as an $\mathcal{O}_{\mathbf{P}^3}$-module. The mapping cone $\mathcal{C}^{-1}(-2) \to \mathcal{C}^0$, corresponding to $I_{C,S} \to \mathcal{O}_S$, yields a minimal resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbf{P}^3}$-module.
Hence we look at the following double complex:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^2(-4) & \xrightarrow{M} & \mathcal{O}^2(-3) & \longrightarrow & I_{C,S} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C_1} & & \downarrow{\scriptstyle C_0} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-2) & \xrightarrow{\det(M)} & \mathcal{O} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0,
\end{array}$$

where

$$\begin{aligned} C_0 &= (-x_1q_2 + x_3q_1, -x_0q_2 + q_1x_2), \\ C_1 &= (q_1, q_2) \end{aligned}$$

and $q_1, q_2 \in k[x_0, x_1, x_2, x_3]$ are quadratic forms.
Taking the mapping cone complex we now obtain

$$0 \to \mathcal{O}(-4)^2 \xrightarrow{\phi} \mathcal{O}(-3)^2 \oplus \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O} \to 0,$$

where ψ is given by multiplication with the matrix

$$\begin{pmatrix} -x_1q_2 + x_3q_1, -x_0q_2 + q_1x_2, x_0x_3 - x_1x_2 \end{pmatrix}$$

and ϕ is given by multiplication with the matrix

$$\begin{pmatrix} -x_0 & -x_2 \\ -x_1 & -x_3 \\ q_1 & q_2 \end{pmatrix}.$$

$d = 6$

Let S be a smooth scroll in $\mathbf{P}^4$ which ideal is generated by the (2×2)-minors of the following matrix:

$$M = \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix}.$$

If we take a general quadric Q, then as we have seen before, $Q \cap S$ is a smooth curve C of degree 6 and genus 2. The resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbf{P}^4}$-module is given by the mapping cone $\mathcal{C}^0(-2) \to \mathcal{C}^0$.
That is, we consider the following complex:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{O}^2(-5) & \xrightarrow{A_1} & \mathcal{O}^3(-4) & \xrightarrow{A_0} & \mathcal{O}(-2) & \longrightarrow & I_{C,S} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C_1} & & \downarrow{\scriptstyle C_0} & & \downarrow{\scriptstyle Q} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^2(-3) & \xrightarrow{B_1} & \mathcal{O}^3(-2) & \xrightarrow{B_0} & \mathcal{O} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0,
\end{array}$$

where

$$\begin{aligned}
A_0 = B_0 &= \begin{pmatrix} x_0x_2 - x_1^2, & x_0x_4 - x_1x_3, & x_1x_4 - x_2x_3 \end{pmatrix}, \\
A_1 = B_1 &= \begin{pmatrix} x_3 & x_4 \\ -x_1 & -x_2 \\ x_0 & x_1 \end{pmatrix}, \\
C_0 &= \begin{pmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & Q \end{pmatrix}, \\
C_1 &= \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.
\end{aligned}$$

The mapping cone complex is then given by

$$0 \to \mathcal{O}(-5)^2 \xrightarrow{\phi} \mathcal{O}(-4)^3 \oplus \mathcal{O}(-3)^2 \xrightarrow{\psi} \mathcal{O}(-2)^4 \xrightarrow{\rho} I_C \to 0,$$

where ϕ is given by multiplication with the matrix

$$\begin{pmatrix} -x_3 & -x_4 \\ x_1 & x_2 \\ -x_0 & -x_1 \\ Q & 0 \\ 0 & Q \end{pmatrix},$$

ψ is given by multiplication with the matrix

$$\begin{pmatrix} -x_0x_2 + x_1^2 & -x_0x_4 + x_1x_3 & -x_1x_4 + x_2x_3 & 0 & 0 \\ Q & 0 & 0 & x_3 & x_4 \\ 0 & Q & 0 & -x_1 & -x_2 \\ 0 & 0 & Q & x_0 & x_1 \end{pmatrix}$$

and ρ is given by multiplication with the matrix

$$\begin{pmatrix} Q & x_0x_2 - x_1^2 & x_0x_4 - x_1x_3 & x_1x_4 - x_2x_3 \end{pmatrix}.$$

$d = 7$

Let

$$M = \begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix},$$

and let S be the two-dimensional rational normal scroll defined by the (2×2)-minors of M.

Let $q_1, \ldots, q_6$ denote the (2×2)-minors of M, let l_1, l_2, l_3, l_4 in $k[x_0, x_1, x_2, x_3, x_4, x_5]$ be general linear forms, and set

$$\begin{aligned}
Q_1 &= l_1x_0 + l_2x_1 + l_3x_3 + l_4x_4, \\
Q_2 &= l_1x_1 + l_2x_2 + l_3x_4 + l_4x_5.
\end{aligned}$$

In Section 3.5 we have seen that the ideal $(q_1, q_2, q_3, q_4, q_5, q_6, Q_1, Q_2) =: I_C$ defines a smooth curve C of genus 2 and degree 7 with associated $g_2^1(C)$-scroll S.
The mapping cone $\mathcal{C}^1(-2) \to \mathcal{C}^0$ is a minimal resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^5}$-module.
That is, we consider the following complex:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathcal{O}^2(-6) & \xrightarrow{A_2} & \mathcal{O}^4(-5) & \xrightarrow{A_1} & \mathcal{O}^4(-3) & \xrightarrow{A_0} & \mathcal{O}^2(-2) & \longrightarrow & I_{C,S} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C_2} & & \downarrow{\scriptstyle C_1} & & \downarrow{\scriptstyle C_0} & & \downarrow{\scriptstyle (-Q_2,Q_1)} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^3(-4) & \xrightarrow{B_2} & \mathcal{O}^8(-3) & \xrightarrow{B_1} & \mathcal{O}^6(-2) & \xrightarrow{B_0} & \mathcal{O} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}$$

where the maps are given by multiplication with the matrices A_0, A_1, A_2, B_0, B_1, B_2, C_0, C_1 and C_2 which can be found in Appendix A.1.

$d = 8$

Let S be a maximally balanced scroll, defined by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_5 & x_6 \end{pmatrix}.$$

In Section 3.5 we found the following description of the ideal I_C of a curve C of genus 2 and degree 8:
$I_C = I_S + (Q_1, Q_2, Q_3)$, where Q_1, Q_2 and Q_3 are the following quadrics:

$$\begin{aligned}
Q_1 &= l_1x_0 + l_2x_1 + l_3x_4, \\
Q_2 &= l_1x_1 + l_2x_2 + l_3x_5, \\
Q_3 &= l_1x_2 + l_2x_3 + l_3x_6,
\end{aligned}$$

with general linear forms $l_1, l_2, l_3 \in k[x_0, \dots, x_6]$.
The mapping cone $\mathcal{C}^{d-6}(-2) \to \mathcal{C}^0$ gives a resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^6}$-module.

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \mathcal{O}^2(-7) & \xrightarrow{A_3} & \mathcal{O}^5(-6) & \xrightarrow{A_2} & \mathcal{O}^{10}(-4) & \xrightarrow{A_1} & \mathcal{O}^{10}(-3) & \xrightarrow{A_0} & \mathcal{O}^3(-2) & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C_4} & & \downarrow{\scriptstyle C_3} & & \downarrow{\scriptstyle C_2} & & \downarrow{\scriptstyle C_1} & & \downarrow{\scriptstyle C_0} & & & & \\
0 & \longrightarrow & \mathcal{O}^4(-5) & \xrightarrow{B_3} & \mathcal{O}^{15}(-4) & \xrightarrow{B_2} & \mathcal{O}^{20}(-3) & \xrightarrow{B_1} & \mathcal{O}^{10}(-2) & \xrightarrow{B_0} & \mathcal{O} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}$$

The maps in the above complex are given by multiplication with the matrices that can be found in Appendix A.2.

Chapter 5

The ideal of C as a sum of scrollar ideals

In this chapter we will show that the ideal I_C of a linearly normal embedded curve $C \subseteq \mathbf{P}^{d-2}$ of genus 2 and degree $d \geq 6$ is generated by the ideals I_S and I_V where S is the $g^1_2(C)$-scroll and $V = V_{|D|}$ is a $g^1_3(C)$-scroll not containing S. In other words, we will prove the following main theorem in this section:

Theorem 5.1. *Let C be a non-singular and irreducible curve of genus 2, linearly normal embedded in $\mathbf{P}^{d-2}$ by a complete linear system $|H|$ of degree $d \geq 6$. Then*

$$I_S + I_V = I_C$$

for a $g^1_3(C)$-scroll V that does not contain the $g^1_2(C)$-scroll S.

We will first give an inductive proof of this theorem. We will then give a proof of a slightly weaker result, this proof goes via the quadric $\mathbf{P}^1 \times \mathbf{P}^1 \subseteq \mathbf{P}^3$.
Before we will take a look at the ideals we show that $S \cap V = C$:

Proposition 5.2. *Let $C \subseteq \mathbf{P}^{d-2}$ be a smooth and irreducible curve of genus 2 and degree $d \geq 6$. For a $g^1_3(C)$-scroll $V = V_{|D|}$ that does not contain the $g^1_2(C)$-scroll S the following holds:*

$$S \cap V = C.$$

Proof. Obviously, $C \subseteq S \cap V$. In the case $d = 6$ the claim follows by Bézout's Theorem, since then $S \cap V$ is of degree 6 and dimension 1. Let now $d \geq 7$. If $S \cap V$ is more than C, then it must at least contain one line: If $S \cap V \supseteq C \cup P$ for a point P that does not lie on C, then P lies on one fiber F_0 of the scroll S. But since P does not lie on the curve, each quadric that contains V intersects F_0 in at least three points, consequently the whole fiber F_0 must be contained in each quadric that contains V, and since the ideal I_V is generated by quadrics F_0 is contained in $S \cap V$. Now there are a priori two possibilities for a fiber F of S to be contained in V:

(1) F is contained in one of the fibers of $V = V_{|D|}$; this implies that the $g^1_3(C)$ $|D|$ has a basepoint, $|D| = |K_C + P|$, and consequently $S \subseteq V$.

(2) F is intersecting each fiber of V in one point. Since F is a fiber of S, the point of intersection lies on C for exactly two fibers of V. Projecting from F yields a curve C' of genus 2 and degree $d-2$, linearly normal embedded in $\mathbf{P}^{d-4}$ with

the linear system $|H - K_C|$. The curve C' lies on the surface scroll S' which is the image of V under the projection from F. A general fiber of V is projected to a fiber in S', and the three points in the intersection of C with a general fiber in V will be projected to three points on a fiber in S' which is impossible by Corollary 3.2 unless C was a curve of degree 7 in $\mathbf{P}^5$. If C is a curve of degree 7 that projects to a curve C' of degree 5 on $\mathbf{P}^1 \times \mathbf{P}^1$, then $|H| = |D + 2K_C|$ and the $g^1_3(C)$-scroll $V = V_{|D|}$ contains the $g^1_2(C)$-scroll S by Proposition 2.17.

This proves that the intersection $S \cap V$ cannot contain any line, i.e. in total we obtain $S \cap V = C$. □

In our proof of Theorem 5.1 we will give an inductive argument. We will divide the proof into the cases when d is an even number and when d is an odd number. When d is an even number the induction starts with the case $d = 6$, and when d is an odd number the induction start is the case $d = 7$.
Our strategy will be the following:
As usual, let S be the $g^1_2(C)$-scroll, and let $V = V_{|D|}$ be a $g^1_3(C)$-scroll that does not contain S. There is the following short exact sequence of ideal sheaves:

$$0 \to \mathcal{I}_{S\cup V} \to \mathcal{I}_V \to \mathcal{I}_{S\cap V}|_S \to 0.$$

By Propositon 5.2 we have $S \cap V = C$, and moreover we know that $\mathcal{I}_C|_S = \mathcal{O}_S(-C)$. We thus obtain the following short exact sequence:

$$0 \to \mathcal{I}_{S\cup V} \to \mathcal{I}_V \to \mathcal{O}_S(-C) \to 0.$$

Tensoring with $\mathcal{O}_{\mathbf{P}^{d-2}}(2H)$ yields the following exact sequence:

$$0 \to \mathcal{I}_{S\cup V}(2H) \to \mathcal{I}_V(2H) \to \mathcal{O}_S(2H - C) \to 0.$$

Taking the long exact sequence in cohomology yields

$$0 \to H^0(\mathcal{I}_{S\cup V}(2)) \to H^0(\mathcal{I}_V(2)) \to H^0(\mathcal{O}_S(2H - C)) \to H^1(\mathcal{I}_{S\cup V}(2)) \to 0.$$

Note that $h^1(\mathcal{I}_V(2)) = 0$ since V is projectively normal.
Since $[C] = 2H - (d - 6)F$ on S, we can write the above sequence in the following form:

$$0 \to H^0(\mathcal{I}_{S\cup V}(2)) \to H^0(\mathcal{I}_V(2)) \xrightarrow{\psi} H^0(\mathcal{O}_S((d - 6)F)) \to H^1(\mathcal{I}_{S\cup V}(2)) \to 0.$$

Our aim is now to show the following claim:

Claim 5.3. *For each $|D| \in G^1_3(C)$ such that $V_{|D|}$ does not contain S, the map ψ : $H^0(\mathcal{I}_V(2)) \to H^0(\mathcal{O}_S((d - 6)F))$ defined via*

$$\psi(Q) := \begin{cases} 0 & \text{if} \quad S \subseteq Q, \\ Q \cap S - C \in |(d-6)F| & \text{if} \quad S \not\subseteq Q \end{cases}$$

is surjective.

If the claim is true, then we have $h^1(\mathcal{I}_{S\cup V}(2)) = 0$, and thus the short exact sequence

$$0 \to \mathcal{I}_{S\cup V}(2) \to \mathcal{I}_S(2) \oplus \mathcal{I}_V(2) \to \mathcal{I}_{\underbrace{S\cap V}_{=C}}(2) \to 0$$

gives the following long exact sequence in cohomology:

$$0 \to H^0(\mathcal{I}_{S\cup V}(2)) \to H^0(\mathcal{I}_S(2)) \oplus H^0(\mathcal{I}_V(2)) \to H^0(\mathcal{I}_C(2)) \to 0.$$

This implies that

$$\begin{aligned} h^0(\mathcal{I}_C(2)) &= \dim(H^0(\mathcal{I}_S(2)) \oplus H^0(\mathcal{I}_V(2))) - h^0(\mathcal{I}_{S\cup V}(2)) \\ &= \dim(H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2))). \end{aligned}$$

This argument implies that, since $H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2)) \subseteq H^0(\mathcal{I}_C(2))$,

$$H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2)) = H^0(\mathcal{I}_C(2)),$$

but since all I_S, I_V and I_C are generated by quadrics, we obtain $I_S + I_V = I_C$.
Proof of Claim 5.3:
Now we will prove by induction that the map

$$\psi : H^0(\mathcal{I}_V(2)) \to H^0(\mathcal{O}_S((d-6)F))$$

as defined above is surjective. We divide our argument into the cases when d is even and when d is odd.
The case when d is even, $d = 2m$ for $m \geq 3$:
The induction start: $d = 6$:
For $d = 6$ the surjectivity of ψ is obvious. More precisely, if $|D|$ is a basepoint-free $g^1_3(C)$ such that $|H - D|$ also is basepoint-free, then by Proposition 2.17 $V_{|D|} =: Q_6$ is a quadric that does not contain S.
The induction step: $d = 2m \geq 8$:
In the induction step we have to consider the case $d = 8$ separately, since by Proposition 2.17 the condition that $|D|$ is basepoint-free is not enough to ensure that $V_{|D|}$ does not contain the $g^1_2(C)$-scroll S in the case when the degree of C is equal to 6. Let now $C \subseteq \mathbf{P}^6$ be a curve embedded by a linear system $|H|$ of degree 8. The problematic case is when $|H| = |D + 2K_C + P|$, since then $|H - K_C| = |D + D'|$ where $|D'|$ has one basepoint P. In this case let $R_1 \neq P$ and $R_2 \neq P$ be two points on C such that $R_1 + R_2$ is not a divisor in $|K_C|$, and let R'_1 and R'_2 be two points on C such that $R_1 + R'_1$ and $R_2 + R'_2$ are divisors in $|K_C|$. Projecting from the line L_R spanned by R_1 and R_2 yields a curve C' of degree 6, embedded with the system $|H'| := |H - R_1 - R_2| = |D + R'_1 + R'_2 + P| = |D + D'|$ with $|D'|$ basepoint-free. Under this projection the $g^1_2(C)$-scroll S maps to the $g^1_2(C')$-scroll S', and the $g^1_3(C)$-scroll $V_{|D|}$ maps to a $g^1_3(C')$-scroll $V'_{|D|}$ which does not contain S'. By the induction hypothesis there exists a quadric Q_6 that contains the $g^1_3(C')$-scroll $V'_{|D|}$ and that does not contain the $g^1_2(C')$-scroll S'. The cone over this quadric Q_6 with the line L_R as vertex is a quadric Q_8 that contains $V_{|D|}$ but not S and that in addition contains two fibers of S: The fiber F_{R_1} spanned by R_1 and R'_1 intersects the quadric Q_8 in three points, counted with multiplicity: The quadric Q_8 intersects this line in at least the two points R_1 and R'_1, and since the quadric is singular along the line L_R the quadric Q_8 intersects F_{R_1} in

the point R_1 with at least multiplicity 2. Consequently Q_8 must contain the line F_{R_1}, and the same argument applies to the fiber F_{R_2} which is spanned by the points R_2 and R_2'. By degree reasons $Q_8 \cap S = C \cup F_{R_1} \cup F_{R_2}$, and since the divisors $F_{R_1} + F_{R_2}$, where R_1 and R_2 run through the set of all points on C, span the linear system $|2F|$ and ψ is linear, ψ is surjective.
Let now C be a curve of degree $d = 2m \geq 8$ in $\mathbf{P}^{d-2}$ such that $|H| \neq |D + 2K_C + P|$ if $d = 8$. Pick $2m - 7$ fibers $F = F_0, F_1, \ldots, F_{2m-8}$ on S and project from F into $\mathbf{P}^{2m-4}$. Under this projection the curve C maps to a curve C' of degree $2m - 2$, the $g^1_2(C)$-scroll S maps to the $g^1_2(C')$-scroll S' and the $g^1_3(C)$-scroll $V_{|D|}$ maps to a $g^1_3(C')$-scroll $V'_{|D|}$ which does not contain S'. By the induction hypothesis we find a quadric $Q_{2m-2} \subseteq \mathbf{P}^{2m-4}$ which contains $V'_{|D|}$ but not S', and which contains the fibers $F_1', \ldots F_{2m-8}'$ where F_i' for $i = 1, \ldots, 2m - 8$ denotes the image of F_i under the projection.
The cone over Q_{2m-2} with F as vertex is then a quadric Q_{2m} which contains $V_{|D|}$ and not S. Moreover, Q_{2m} contains the fibers $F_1, \ldots, F_{2m-8}$ and the fiber F. Since F lies in the singular locus of Q_{2m}, F is contained in Q_{2m} with multiplicity at least 2. By degree reasons $Q_{2m} \cap S$ cannot contain more than C, F of multiplicity 2 and $F_1, \ldots, F_{2m-8}$. Consequently, $\psi(Q_{2m}) = 2F \cup F_1 \cup \ldots \cup F_{2m-8}$. Since the divisors $2F + F_1 + \cdots + F_{2m-8}$ where $F, F_1, \ldots, F_{2m-8}$ run through all fibers of S, span the linear system $|(2m-6)F|$, varying the fibers F and $F_1, \ldots, F_{2m-8}$ yields the surjectivity of ψ.
Alternatively, we can argue in the following way: Polynomials of degree $2m - 6$ with one double root generate all polynomials of degree $2m - 6$, using the isomorphism $H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}((d-6)F)) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d-6))$ gives the desired result.

The case when d is odd, $d = 2m + 1$ for $m \geq 3$:
The induction start: $d = 7$:
For a curve C of degree 7 let $V_{|D|}$ be a $g^1_3(C)$-scroll that does not contain the $g^1_2(C)$-scroll S. For any two quadrics $Q_1 \neq Q_2$ their intersection $Q_1 \cap Q_2$ is a complete intersection of dimension 3 and degree 4, hence if Q_1 and Q_2 both contained S and V, then we have $Q_1 \cap Q_2 = V \cup \mathbf{P}^3$. Since $S \subseteq Q_1 \cap Q_2$ and S is irreducible, we must have $S \subseteq V$ or $S \subseteq \mathbf{P}^3$, but since S spans all of $\mathbf{P}^5$ and by hypothesis S is not contained in V, both cases are impossible.
This shows that $h^0(\mathcal{I}_{S \cup V}(2)) \leq 1$, and consequently we obtain

$$\dim(H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2))) \geq h^0(\mathcal{I}_S(2)) + h^0(\mathcal{I}_V(2)) - 1 = 8 = h^0(\mathcal{I}_C(2)),$$

and thus ψ is surjective.

The induction step: $d = 2m + 1 \geq 9$:
The induction step goes analogously to the induction step in the case when d is even: In the case when C is a curve of degree $d = 9$ and $|H| = |D + 3K_C|$ we cannot use the projection from a fiber on S, spanned by a divisor in $|K_C|$, since if C' is embedded in $\mathbf{P}^5$ by the linear system $|D + 2K_C|$, the $g^1_3(C)$-scroll $V_{|D|}$ contains the $g^1_2(C)$-scroll S by Proposition 2.17. Consequently, analogous to the above case $d = 8$, $|H| = |D + 2K_C + P|$, we will deal with this case first using a different projection and then go over to the induction step in the remaining cases.
We pick two points R_1 and R_2 on C such that $R_1 + R_2$ is not a divisor in $|K_C|$, and moreover we pick two points R_1' and R_2' on C such that $R_1 + R_1'$ and $R_2 + R_2'$ are divisors

in $|K_C|$. We project from the line L_R spanned by the two points R_1 and R_2 and obtain a curve C' embedded with the system $|H'| := |H - R_1 - R_2| = |D + K_C + R_1' + R_2'|$. Under this projection the $g_2^1(C)$-scroll S maps to the $g_2^1(C')$-scroll S', and the $g_3^1(C)$-scroll $V_{|D|}$ maps to a $g_3^1(C')$-scroll $V'_{|D|}$.
By the induction hypothesis there exists a quadric Q_7 that contains $V'_{|D|}$ but not S' and that in addition contains one fiber F_0 of S'. The cone over this quadric Q_7 with the line L_R as vertex is a quadric Q_9 that contains $V_{|D|}$ but not S and that in addition contains a fiber F_0 of S. In addition, Q_9 contains two more fibers of S, namely the fiber F_{R_1} spanned by R_1 and R_1' and the fiber F_{R_2} spanned by R_2 and R_2', by the same reasons as in the case $d = 8$, $|H| = |D + 2K_C + P|$.
Let now $C \subseteq \mathbf{P}^{d-2}$ be a curve of degree $d = 2m + 1 \geq 9$ such that $|H| \neq |3K_C + P|$ if $d = 9$.
Let $F = F_0, F_1, \ldots, F_{2m-7}$ be fibers on S and project from F into $\mathbf{P}^{2m-3}$. Under this projection the curve C maps to a curve C' of degree $2m - 1$, the $g_2^1(C)$-scroll S maps to the $g_2^1(C')$-scroll S', and the $g_3^1(C)$-scroll $V_{|D|}$ maps to a $g_3^1(C')$-scroll $V'_{|D|}$ that does not contain S'. By the induction hypothesis we find a quadric $Q_{2m-1} \subseteq \mathbf{P}^{2m-3}$ which contains $V'_{|D|}$ but not S' and which moreover contains the fibers $F_1', \ldots, F_{2m-7}'$ of S'.
The cone over Q_{2m-1} with F as vertex is then a quadric Q_{2m+1} in $\mathbf{P}^{2m-1}$ which contains $V_{|D|}$ and not S and that in addition contains the fibers $F_1, \ldots, F_{2m-7}$. Moreover, Q_{2m+1} contains the fiber F with at least multiplicity 2 since F lies in the singular locus of Q_{2m+1}. Again, by degree reasons we have $Q_{2m+1} \cap S = C \cup 2F \cup F_1 \ldots \cup F_{2m-7}$, i.e. $\psi(Q_{2m+1}) = 2F + F_1 + \cdots + F_{2m-7}$.
Since divisors of the form $2F + F_1 + \cdots + F_{2m-7}$ where $F, F_1, \ldots, F_{2m-7}$ vary, span the linear system $|(2m - 5)F|$ and ψ is linear we conclude that ψ is surjective.

□

Remark 5.4. *As a biproduct we have shown in the proof for the case $d = 7$ that for every fiber F_0 on S and every $|D| \in G_3^1(C)$ such that $V_{|D|}$ does not contain S there is a quadric Q_7 which contains F_0 and $V_{|D|}$ but not S.*
We want to mention that we can find a quadric of this kind explicitly, we will pick such a quadric of rank 4:
Since the $g_3^1(C)$-scroll $V_{|D|}$ does not contain the $g_2^1(C)$-scroll S, $|D|$ is basepoint-free and $|H| = |D + K_C + P + Q|$ where $P + Q$ is not a divisor in $|K_C|$ (cf. Proposition 2.17). Let us pick a point R on $C \cap F_0$ such that $R \neq P$, $R \neq Q$. There is one more point in $C \cap F_0$ which we denote by R' (note that R' might be equal to P or Q). We project from the point R into $\mathbf{P}^4$. Under this projection the curve C maps to a curve C' of genus 2 and degree 6, the $g_2^1(C)$-scroll S maps to the $g_2^1(C')$-scroll S', and the $g_3^1(C)$-scroll $V_{|D|}$ maps to a $g_3^1(C')$-scroll $V'_{|D|}$. Since the curve C' is embedded into $\mathbf{P}^4$ with the linear system $|H'| := |H - R|$ and $|H - D - R| = |R' + P + Q|$ is basepoint-free, there exists, as we have seen above, a quadric Q_6 of rank 4 which is equal to $V'_{|D|}$ and which does not contain S'. Taking the cone over Q_6 with R as vertex gives a quadric Q_7 of rank 4 in $\mathbf{P}^5$ that contains $V_{|D|}$ but not S.
Now it remains to show that Q_7 contains the fiber F_0: Any line L which is not contained in Q_7 intersects Q_7 in 2 points, counted with multiplicity. Obviously, both R and R' lie on $Q_7 \cap F_0$, and since R is a singular point of Q_7, F_0 and Q_7 intersect in at least 3 points counted with multiplicity. Thus F_0 is contained in Q_7.

Now we will present a proof of a slightly weaker result. Here we need to decompose $d = 3a + 2b$, with a and b in $\mathbf{Z}$, $a \geq 2$, $b \geq 3$, i.e. the proof only works for $d = 12$ and

$d \geq 14$.

Theorem 5.5. *Let C be a non-singular and irreducible curve of genus 2, linearly normal embedded in $\mathbf{P}^{d-2}$ by a complete linear system $|H|$ of degree $d = 12$ or $d \geq 14$. Suppose that we can find a basepoint-free $|D| \in G^1_3(C)$ such that $|H| = |aD + bK_C|$ for integers $a \geq 2$, $b \geq 3$.*
Then

$$I_S + I_V = I_C$$

where $V = V_{|D|}$ is the scroll associated to $|D|$.

Remark 5.6. *Since the map*

$$\mathrm{Pic}^3(C) \to \mathrm{Pic}^{3a}(C),$$

$$\mathcal{L} \mapsto a\mathcal{L},$$

is surjective, there exists a $|D| \in G^1_3(C)$ such that $|H - bK_C| = a|D|$, i.e. $|H| = |aD + bK_C|$. In Chapter 6.1 we will see that if $a = 2$ there actually exists a basepoint-free $|D|$ such that $|H| = |2D + bK_C|$. For $a \geq 3$ we have to make the assumption in Theorem 5.5 that we can find a basepoint-free $|D|$.

In order to prove Theorem 5.5 our strategy will be to proceed in the following steps:

(1) Let C be a non-singular curve of genus 2, and let $|H|$ be a complete linear system of degree d on C, where $d = 12$ or $d \geq 14$. Moreover, let $a \geq 2$ and $b \geq 3$ be integers such that $3a + 2b = d$.
Embed now the curve C into $\mathbf{P}^3$ with the linear system $|D + K_C|$. The image curve in $\mathbf{P}^3$, which we also will denote by C, lies on exactly one smooth quadric $Q' \cong \mathbf{P}^1 \times \mathbf{P}^1$. On $\mathbf{P}^1 \times \mathbf{P}^1$ C is of type $(2, 3)$.

(2) The hyperplane class $|H'| = (a, b)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ restricts to $|H|$ on C. Use $|H'|$ to embed into $\mathbf{P}^{h^0(H')-1}$. Denote the embedding by $\Phi_{H'}$ and set $Q = \Phi_{H'}(Q')$. The embedding of a divisor in the system $E = (0, 1)$ is a rational normal curve of degree a which spans a $\mathbf{P}^a$, i.e. the linear system E gives rise to an $(a+1)$-dimensional scroll X_E over $\mathbf{P}^1$. In a similar way the linear system $D' = (1, 0)$ gives rise to a $(b+1)$-dimensional scroll $X_{D'}$ over $\mathbf{P}^1$. Both scrolls are linearly normal since they contain the linearly normal Q. Hence by Proposition 2.5 X_E and $X_{D'}$ are rational normal scrolls.
The degree of X_E is equal to $(a+1)(b+1) - (a+1) = (a+1)b$, and the degree of $X_{D'}$ is equal to $(a+1)(b+1) - (b+1) = (b+1)a$.

Note that the image of the curve C under the embedding $\Phi_{H'}$ is a curve on Q of degree $3a + 2b = d$. We will denote this image curve also by C.

Let $k[s,t] \otimes k[u,v]$ denote the homogeneous coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^1$. We choose coordinates x_{ij}, $i = 0, \ldots, a$, $j = 0, \ldots, b$, in $\mathbf{P}^{h^0(H')-1}$ in such a way that the restriction of x_{ij} to the quadric Q is given by

$$x_{i,j}|_Q = s^{a-i}t^i u^{b-j} v^j.$$

The ideal of a rational normal scroll is generated by the (2×2)-minors of a $(2 \times m)$-matrix, where m is the degree of the scroll.

For $i = 0, \dots, a$ and $j = 0, \dots, b$ define the following blocks:

$$\alpha_i := \begin{bmatrix} s^{a-i}t^iu^b & s^{a-i}t^iu^{b-1}v & \cdots & s^{a-i}t^iuv^{b-1} \\ s^{a-i}t^iu^{b-1}v & s^{a-i}t^iu^{b-2}v^2 & \cdots & s^{a-i}t^iv^b \end{bmatrix}$$

and

$$\beta_j := \begin{bmatrix} s^au^{b-j}v^j & s^{a-1}tu^{b-j}v^j & \cdots & st^{a-1}u^{b-j}v^j \\ s^{a-1}tu^{b-j}v^j & s^{a-2}t^2u^{b-j}v^j & \cdots & t^au^{b-j}v^j \end{bmatrix}.$$

The (2×2)-minors of the following matrices M_E and $M_{D'}$ vanish:

$$M_E = \left(\ \alpha_0 \ \middle| \ \alpha_1 \ \middle| \cdots \middle| \ \alpha_a \ \right),$$

$$M_{D'} = \left(\ \beta_0 \ \middle| \ \beta_1 \ \middle| \cdots \middle| \ \beta_b \ \right).$$

In the coordinates x_{ij} of $\mathbf{P}^{h^0(H')-1}$ the blocks take the following form:

$$\mu_i := \begin{bmatrix} x_{i0} & x_{i1} & \cdots & x_{i,b-1} \\ x_{i1} & x_{i2} & \cdots & x_{ib} \end{bmatrix}$$

and

$$\nu_j := \begin{bmatrix} x_{0j} & x_{1j} & \cdots & x_{a-1,j} \\ x_{1j} & x_{2j} & \cdots & x_{aj} \end{bmatrix}.$$

In the coordinates x_{ij} of $\mathbf{P}^{h^0(H')-1}$, I_{X_E} and $I_{X_{D'}}$ are in fact generated by the (2×2)-minors of M_E and $M_{D'}$ respectively:

$$M_E = \left(\ \mu_0 \ \middle| \ \mu_1 \ \middle| \cdots \middle| \ \mu_a \ \right),$$

$$M_{D'} = \left(\ \nu_0 \ \middle| \ \nu_1 \ \middle| \cdots \middle| \ \nu_b \ \right).$$

We notice that M_E consists of $a+1$ blocks of length b each. The (2×2)-minors in each block μ_i generate the ideal of a rational normal curve of degree b. So we obtain $a+1$ rational normal curves of degree b which are directrix curves of the scroll X_E, i.e. if $\phi_i : \mathbf{P}^1 \to C_i$, $i = 0, \dots, a$, parametrize these rational normal curves, then for each point $P \in \mathbf{P}^1$, $\text{span}\{\phi_i(P) | i = 0, \dots, a\}$ is a fiber in the scroll X_E, and moreover, all fibers in the scroll can be described in this way, i.e.

$$X_E = \overline{\bigcup_{P \in \mathbf{P}^1} \text{span}\{\phi_i(P) | i = 0, \dots, a\}}.$$

The number of blocks in the matrix M_E is equal to the dimension of X_E which is equal to $a+1$.
In the same way we can take a closer look at the scroll $X_{D'}$: The matrix $M_{D'}$ consists of $b+1$ blocks of length a each. The (2×2)-minors in each block ν_j

generate the ideal of a rational normal curve of degree a. These $b+1$ rational normal curves of degree a are directrix curves of the scroll $X_{D'}$. In this way we can describe the scroll as

$$X_{D'} = \overline{\bigcup_{P \in \mathbf{P}^1} \operatorname{span}\{\psi_i(P) | i = 0, \ldots, b\}},$$

where $\psi_i : \mathbf{P}^1 \to C_i$, $i = 0, \ldots, b$, parametrize the directrix curves.
Also, the number of blocks in $M_{D'}$ is equal to the dimension of $X_{D'}$, which is equal to $b+1$.
Notice that these observations imply that the scrolls X_E and $X_{D'}$ are maximally balanced, in particular these scrolls are smooth, and as bundles over $\mathbf{P}^1$ we have

$$X_E \cong \mathbf{P}(\underbrace{\mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \cdots \mathcal{O}_{\mathbf{P}^1}(b)}_{a+1})$$

and

$$X_{D'} \cong \mathbf{P}(\underbrace{\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(a) \oplus \cdots \mathcal{O}_{\mathbf{P}^1}(a)}_{b+1}).$$

(3) We show that

$$\dim(H^0(\mathbf{P}^{h^0(H')-1}, \mathcal{I}_{X_E}(2)) + H^0(\mathbf{P}^{h^0(H')-1}, \mathcal{I}_{X_{D'}}(2))) = h^0(\mathbf{P}^{h^0(H')-1}, \mathcal{I}_Q(2))$$

, and thus we obtain that $I_{X_E} + I_{X_{D'}} = I_Q$.

(4) Finally, we intersect with a linear subspace $\mathbf{P} \cong \mathbf{P}^{h^0(H')-h^0(H'-C)-1} = \mathbf{P}^{d-2}$. We show that $\mathbf{P} \cap Q = C$, $\mathbf{P} \cap X_E = S$ and $\mathbf{P} \cap X_{D'} = V_{|D|}$ with $|D| \cong D'$ and obtain in this way that $I_S + I_{V_{|D|}} = I_C$.

The situation is the following:

$$\begin{array}{rl}
(2,3) = C, (a,b) = H' \subseteq & \mathbf{P}^1 \times \mathbf{P}^1 \\
& \big\downarrow \Phi_{H'} \\
Q \subseteq X_E, X_{D'} \subseteq & \mathbf{P}^{h^0(H')-1} \\
& \big\uparrow \\
C \subseteq S, V \subseteq & \mathbf{P}
\end{array}$$

Proposition 5.7. *With the notations as above we have*

$$I_{X_E} + I_{X_{D'}} = I_Q.$$

Before we will prove this proposition, we will need some preliminary lemmata and definitions:

Lemma 5.8. *We have the following:*

$$h^0(\mathcal{I}_Q(2)) \quad = \quad \binom{(a+1)(b+1)+1}{2} - (2a+1)(2b+1).$$

Proof. From the exact sequence

$$0 \to \mathcal{I}_Q(2) \to \mathcal{O}_{\mathbf{P}^{h^0(H')-1}}(2) \to \mathcal{O}_Q(2) \to 0$$

we obtain the long exact sequence in cohomology

$$0 \to H^0(\mathcal{I}_Q(2)) \to H^0(\mathcal{O}_{\mathbf{P}^{h^0(H')-1}}(2)) \to H^0(\mathcal{O}_Q(2)) \to H^1(\mathcal{I}_Q(2)) \to \cdots .$$

Since Q is projectively normal we have $h^1(\mathcal{I}_Q(2)) = 0$, so we obtain the short exact sequence

$$0 \to H^0(\mathcal{I}_Q(2)) \to H^0(\mathcal{O}_{\mathbf{P}^{h^0(H')-1}}(2)) \to H^0(\mathcal{O}_Q(2)) \to 0$$

and thus

$$\begin{aligned} h^0(\mathcal{I}_Q(2)) &= h^0(\mathcal{O}_{\mathbf{P}^{h^0(H')-1}}(2)) - h^0(\mathcal{O}_Q(2)) \\ &= \binom{(a+1)(b+1)+1}{2} - (2a+1)(2b+1). \end{aligned}$$

□

Definition 5.9. *We define an order $\leq$ on the set*

$$\{x_{\alpha\beta}x_{\gamma\delta} - x_{\alpha'\beta'}x_{\gamma'\delta'} | \alpha, \alpha'\gamma, \gamma' \in \{0, \ldots, a\}, \beta, \beta', \delta, \delta' \in \{0, \ldots, b\}\}$$

in the following way:

(1) First we will consider the set of monomials $\{x_{\alpha\beta}x_{\gamma\delta}\}$ ordered in such a way that

$$(\alpha < \gamma) \text{ or } (\alpha = \gamma \text{ and } \beta \leq \delta).$$

(2) Now assume that the set of monomials $\{x_{\alpha\beta}x_{\gamma\delta}\}$ is ordered according to (1). Then for two elements in this set we can define

$$\begin{aligned} x_{\alpha\beta}x_{\gamma\delta} \leq x_{\alpha'\beta'}x_{\gamma'\delta'} \quad :\Leftrightarrow \quad & (\alpha < \alpha') \text{ or } (\alpha = \alpha' \text{ and } \beta < \beta') \\ & \text{or } (\alpha = \alpha' \text{ and } \beta = \beta' \text{ and } \gamma < \gamma') \\ & \text{or } (\alpha = \alpha' \text{ and } \beta = \beta' \text{ and } \gamma = \gamma' \text{ and } \delta \leq \delta'). \end{aligned}$$

We write $x_{\alpha\beta}x_{\gamma\delta} < x_{\alpha'\beta'}x_{\gamma'\delta'}$ if at least one of the strict inequalities $\alpha < \alpha'$, $\beta < \beta'$, $\gamma < \gamma'$, $\delta < \delta'$ holds.

(3) Now we can define for two binomials $x_{\alpha\beta}x_{\gamma\delta} - x_{\alpha'\beta'}x_{\gamma'\delta'}$ and $x_{\kappa\lambda}x_{\mu\nu} - x_{\kappa'\lambda'}x_{\mu'\nu'}$:

$$\begin{aligned} & x_{\alpha\beta}x_{\gamma\delta} - x_{\alpha'\beta'}x_{\gamma'\delta'} \leq x_{\kappa\lambda}x_{\mu\nu} - x_{\kappa'\lambda'}x_{\mu'\nu'} \\ :\Leftrightarrow \quad & (x_{\alpha\beta}x_{\gamma\delta} < x_{\kappa\lambda}x_{\mu\nu}) \\ & \text{or } (x_{\alpha\beta}x_{\gamma\delta} = x_{\kappa\lambda}x_{\mu\nu} \text{ and } x_{\alpha'\beta'}x_{\gamma'\delta'} \leq x_{\kappa'\lambda'}x_{\mu'\nu'}). \end{aligned}$$

Definition 5.10. *(1) We define an equivalence relation* $\sim_E$ *on the set of monomials* $\{x_{\alpha\beta}x_{\gamma\delta} | x_{\alpha\beta}x_{\gamma\delta}$ *is a monomial in a minor in* $M_E\}$ *in the following way:*

$$\begin{aligned} & x_{\alpha\beta}x_{\gamma\delta} \sim_E x_{\alpha'\beta'}x_{\gamma'\delta'} \\ :\Leftrightarrow \quad & x_{\alpha\beta}x_{\gamma\delta} - x_{\alpha'\beta'}x_{\gamma'\delta'} \in I_{X_E}. \end{aligned}$$

(2) Similarly we define an equivalence relation $\sim_{D'}$ *on the set of monomials*

$$\{x_{\alpha\beta}x_{\gamma\delta} | x_{\alpha\beta}x_{\gamma\delta} \text{ is a monomial in a minor in } M_{D'}\}:$$

$$\begin{aligned} & x_{\alpha\beta}x_{\gamma\delta} \sim_{D'} x_{\alpha'\beta'}x_{\gamma'\delta'} \\ :\Leftrightarrow \quad & x_{\alpha\beta}x_{\gamma\delta} - x_{\alpha'\beta'}x_{\gamma'\delta'} \in I_{X_{D'}}. \end{aligned}$$

Lemma 5.11. *Let* $\sim$ *be one of the equivalence relations* $\sim_E$*,* $\sim_{D'}$ *as given in Definition 5.10. Then we have the following:*
If two monomials $x_{kl}x_{mn}$ *and* $x_{pq}x_{rs}$ *are in the same equivalence class with respect to the equivalence relation* $\sim$*, then* $k+m = p+r$ *and* $l+n = q+s$*.*

Proof. (1) If $x_{kl}x_{mn}$ and $x_{pq}x_{rs}$ are in the same equivalence class with respect to the equivalence relation $\sim_E$, then

$$\begin{bmatrix} x_{kl} \\ x_{pq} \end{bmatrix}$$

is a column in M_E, and

$$\begin{bmatrix} x_{rs} \\ x_{mn} \end{bmatrix}$$

is another column in M_E.
By the structure of M_E we obtain

$$\begin{aligned} k &= p, \\ l &= q-1, \\ m &= r, \\ n &= s+1. \end{aligned}$$

Consequently, $k+m = p+r$ and $l+n = q+s$.

(2) Similarly, if $x_{kl}x_{mn}$ and $x_{pq}x_{rs}$ are in the same equivalence class with respect to the equivalence relation $\sim_{D'}$, then

$$\begin{bmatrix} x_{kl} \\ x_{pq} \end{bmatrix}$$

is a column in $M_{D'}$, and

$$\begin{bmatrix} x_{rs} \\ x_{mn} \end{bmatrix}$$

is another column in $M_{D'}$.
By the structure of $M_{D'}$ we obtain

$$\begin{aligned} k &= p-1, \\ l &= q, \\ m &= r+1, \\ n &= s. \end{aligned}$$

Consequently, $k+m = p+r$ and $l+n = q+s$.

□

Remark 5.12. *(1) We will denote the equivalence class of $x_{kl}x_{mn}$ with respect to the equivalence relation $\sim_E$ by $\alpha_{k+m,l+n}$.*
For some indices (i,j) there exist several (disjoint) equivalence classes α_{ij}, we will distinguish between those in the following way:
All elements in the equivalence class of $x_{kl}x_{mn}$ are of the form $x_{k,\gamma}x_{m,\delta}$, $\gamma+\delta = l+n$, so denote the equivalence class containing $x_{kl}x_{mn}$ by $\alpha^{(k,m)}_{k+m,l+n}$.

(2) Similarly, we will denote the equivalence class of $x_{kl}x_{mn}$ with respect to the equivalence relation $\sim_{D'}$ by $\beta_{k+m,l+n}$. All elements in this equivalence class are of the form $x_{\gamma,l}x_{\delta,n}$, $\gamma+\delta = k+m$, so if there exist more than one equivalence class with a fixed index (i,j), we will distinguish between these by denoting the one containing $x_{kl}x_{mn}$ by $\beta^{(l,n)}_{k+m,l+n}$.

Now we are able to formulate the proof of Proposition 5.7:

Proof of Proposition 5.7:
Obviously we have $I_{X_E} + I_{X_{D'}} \subseteq I_Q$. Our strategy is now to determine the dimension of $H^0(\mathcal{I}_{X_E}(2)) + H^0(\mathcal{I}_{X_{D'}}(2))$ and consequently show that $H^0(\mathcal{I}_{X_E}(2)) + H^0(\mathcal{I}_{X_{D'}}(2)) = H^0(\mathcal{I}_Q(2))$. More precisely, we will study the (2×2)-minors of $M_{D'}$ and find the ones independent over the ideal generated by the (2×2)-minors of M_E.
In order to pick enough minors in $M_{D'}$ that are independent over $H^0(\mathcal{I}_{X_E}(2))$, we will study the sets $\mathcal{Q}_{kl}$, $k = 0,\dots,a-2$, $l = 0,\dots,b$, of quadrics, where $\mathcal{Q}_{kl}$ contains minors in $M_{D'}$ that start with x_{kl}, i.e. that are of the form $x_{kl}x_{rs} - x_{k+1,l}x_{r-1,s}$, ordered according to the order in Definition 5.9, and that are independent over $H^0(\mathcal{I}_{X_E}(2))$.
We choose the elements in the sets $\mathcal{Q}_{kl}$ in the following way:
First we will study those minors of the form

$$x_{k0}x_{rs} - x_{k+1,0}x_{r-1,s},$$

$$k = 0,\dots,a-2, r = k+2,\dots,a, s = 0,\dots,b,$$

i.e. those minors where the first variable in the first monomial appears in the first row of the first block in $M_{D'}$.
We start with the coordinate x_{00} and consider all minors of the form

$$x_{00}x_{rs} - x_{10}x_{r-1,s},$$

where now $r = 2, \dots, a$, i.e. we skip the minors which start with the monomial $x_{00}x_{1s}$, $s = 0, \dots, b$ where x_{1s} apparently is the first entry in the second row of each block in $M_{D'}$.
In general we will consider the minors of the form

$$x_{k0}x_{rs} - x_{k+1,0}x_{r-1,s},$$

$$k = 0, \dots, a-2, r = k+2, \dots, a, s = 0, \dots, b,$$

i.e. we will skip the minors where the second variable in the first monomial is among the first $k+1$ entries in the second row of each block of $M_{D'}$.
Afterwards we will study the minors that start with the coordinates x_{kl}, $k = 1, \dots, a-2$, $l = 0, \dots, b$. Here we will only study the minors where the second coordinate in the first monomial appears in the second row of the last block in $M_{D'}$, i.e. we will consider the minors of the form

$$x_{kl}x_{rb} - x_{k+1,l}x_{r-1,b},$$

$$r = k+2, \dots, a.$$

Summarizing, we are considering the following quadrics which are sorted in ascending order in the sense of the order $\leq$ from Definition 5.9:

$\mathcal{Q}_{00}$	$\mathcal{Q}_{10}$	$\dots$	$\mathcal{Q}_{a-2,0}$
$x_{00}x_{20} - x_{10}^2$		$\dots$	
$x_{00}x_{30} - x_{10}x_{20}$	$x_{10}x_{30} - x_{20}^2$	$\dots$	
$\vdots$	$x_{10}x_{40} - x_{20}x_{30}$	$\dots$	
$\vdots$	$\vdots$	$\dots$	
$x_{00}x_{a0} - x_{10}x_{a-1,0}$	$x_{10}x_{a0} - x_{20}x_{a-1,0}$	$\dots$	$x_{a-2,0}x_{a0} - x_{a-1,0}x_{a-1,0}$
$x_{00}x_{21} - x_{10}x_{11}$		$\dots$	
$x_{00}x_{31} - x_{10}x_{21}$	$x_{10}x_{31} - x_{20}x_{21}$	$\dots$	
$\vdots$	$\vdots$	$\dots$	
$x_{00}x_{a1} - x_{10}x_{a-1,1}$	$x_{10}x_{a1} - x_{20}x_{a-1,1}$	$\dots$	$x_{a-2,0}x_{a1} - x_{a-1,0}x_{a-1,1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_{00}x_{2b} - x_{10}x_{1b}$		$\dots$	
$x_{00}x_{3b} - x_{10}x_{2b}$	$x_{10}x_{3b} - x_{20}x_{2b}$	$\dots$	
$\vdots$	$\vdots$	$\dots$	
$x_{00}x_{ab} - x_{10}x_{a-1,b}$	$x_{10}x_{ab} - x_{20}x_{a-1,b}$	$\dots$	$x_{a-2,0}x_{ab} - x_{a-1,0}x_{a-1,b}$

For $l = 1, \dots, b$:

$\mathcal{Q}_{0l}$	$\mathcal{Q}_{1l}$	$\dots$	$\mathcal{Q}_{a-2,l}$
$x_{0l}x_{2b} - x_{1l}x_{1b}$		$\dots$	
$x_{0l}x_{3b} - x_{1l}x_{2b}$	$x_{1l}x_{3b} - x_{2l}x_{2b}$	$\dots$	
$\vdots$	$\vdots$	$\dots$	
$x_{0l}x_{ab} - x_{1l}x_{a-1,b}$	$x_{1l}x_{ab} - x_{2l}x_{a-1,b}$	$\dots$	$x_{a-2,l}x_{ab} - x_{a-1,l}x_{a-1,b}$

(1) First we list the equivalence classes α_{ij} with respect to the equivalence relation $\sim_E$ as defined in Definition 5.10, part (1). We will order the monomials within each

equivalence class in ascending order with respect to the order given in Definition 5.9:

$$\begin{aligned}
\alpha^{(k,k)}_{2k,s} &= \{x_{k0}x_{ks}, x_{k1}x_{k,s-1}, \ldots, x_{k,\lfloor\frac{s}{2}\rfloor}x_{k,\lfloor\frac{s+1}{2}\rfloor}\},\\
&\quad k = 0,\ldots,a, s = 2,\ldots,b,\\
\alpha^{(k,r)}_{k+r,s} &= \{x_{k0}x_{rs}, x_{k1}x_{r,s-1}, \ldots, x_{ks}x_{r0}\},\\
&\quad k = 0,\ldots,a-1, r = k+1,\ldots,a, s = 1,\ldots,b,\\
\alpha^{(k,k)}_{2k,l+b} &= \{x_{kl}x_{kb}, x_{k,l+1}x_{k,b-1}, \ldots, x_{k,\lfloor\frac{b+l}{2}\rfloor}x_{k,\lfloor\frac{b+l+1}{2}\rfloor}\},\\
&\quad k = 0,\ldots,a, l = 1,\ldots,b-2,\\
\alpha^{(k,r)}_{k+r,l+b} &= \{x_{kl}x_{rb}, x_{k,l+1}x_{r,b-1}, \ldots, x_{kb}x_{rl}\},\\
&\quad k = 0,\ldots,a-1, r = k+1,\ldots,a, l = 1,\ldots,b-1.
\end{aligned}$$

(2) For the sets $\mathcal{Q}_{kl}$ we have:

$$\begin{aligned}
\mathcal{Q}_{k0} &= \{x_{k0}x_{rs} - x_{k+1,0}x_{r-1,s} | r = k+2,\ldots,a, s = 0,\ldots,b\},\\
&\quad k = 0,\ldots,a-2,\\
\mathcal{Q}_{kl} &= \{x_{kl}x_{rb} - x_{k+1,l}x_{r-1,b} | r = k+2,\ldots,a\},\\
&\quad k = 0,\ldots,a-2, l = 1,\ldots,b.
\end{aligned}$$

Now we form equivalence classes from the sets $\mathcal{Q}_{kl}$ with respect to the equivalence relations $\sim_{D'}$. Order the monomials in each of these equivalence classes in ascending order with respect to the order as defined in Definition 5.9. Note that each equivalence class $\beta_{k+r,j}$ consists of exactly two elements and the difference of these two monomials gives exactly one of the minors in $\mathcal{Q}_{kl}$. This minor we will refer to as *the minor defined by* $\beta_{k+r,j}$.

$$\begin{aligned}
\beta_{k+r,s} &= \{x_{k0}x_{rs}, x_{k+1,0}x_{r-1,s}\},\\
&\quad k = 0,\ldots,a-2, r = k+2,\ldots,a, s = 0,\ldots,b,\\
\beta_{k+r,l+b} &= \{x_{kl}x_{rb}, x_{k+1,l}x_{r-1,b}\},\\
&\quad k = 0,\ldots,a-2, r = k+2,\ldots,a, l = 1,\ldots,b.
\end{aligned}$$

(3) Now we will prove that the minor defined by the class $\beta_{k+r,j}$ is independent over I_{X_E} of the minors defined by the other classes $\beta_{m,n}$.
Pick an equivalence class $\beta_{k+r,j}$. One of the following three things will happen:

(I) $j \in \{0, 2b\}$: There exists no class $\alpha_{k+r,j}$, $k+r = 2,\ldots,2a-2$, which implies, by Lemma 5.11, that the minor defined by $\beta_{k+r,j}$ does not lie in I_{X_E} and is moreover independent of all the other minors in $M_{D'}$ over I_{X_E}.

(II) $j \in \{1,\ldots,2b-1\}$:
For each $k+r = 0,\ldots,2a-2$ there exist either only one class $\alpha_{k+r,j}$ or at least two classes $\alpha^{(k,r)}_{k+r,j}$ and $\alpha^{(k+1,r-1)}_{k+r,j}$.

(II.1) If there exists only one class $\alpha_{k+r,j}$, then the class $\beta_{k+r,j}$, $k+r = 0,\dots,2a-2$, has exactly one monomial in common with this class $\alpha_{k+r,j}$. Since the second monomial in $\beta_{k+r,j}$ does not appear in $\alpha_{k+r,j}$, we know by Lemma 5.11 that the minor defined by $\beta_{k+r,j}$ does not lie in I_{X_E}. Since the second monomial in $\beta_{k+r,j}$ does not appear in any of the other classes α_{mn}, the minor defined by $\beta_{k+r,j}$ is also independent of all other minors over I_{X_E}.

(II.2) If there exist at least two classes $\alpha_{k+r,j}^{(k,r)}$ and $\alpha_{k+r,j}^{(k+1,r-1)}$, then the class $\beta_{k+r,j}$ has one monomial in common with the class $\alpha_{k+r,j}^{(k,r)}$ and one with the class $\alpha_{k+r,j}^{(k+1,r-1)}$.
Since $\alpha_{k+r,j}^{(k,r)}$ and $\alpha_{k+r,j}^{(k+1,r-1)}$ are disjoint, the minor defined by $\beta_{k+r,j}$ does not lie in the ideal I_{X_E}. Moreover, by Lemma 5.11 this minor is independent of all other minors in $M_{D'}$ over I_{X_E}.

In total we obtain at least

$$\begin{aligned}\sum_{k,l}\#\mathcal{Q}_{kl} &= \sum_{k=0}^{a-2}\#\mathcal{Q}_{k0}+\sum_{k=0}^{a-2}\sum_{l=1}^{b}\#\mathcal{Q}_{kl}\\ &= \sum_{k=0}^{a-2}(a-(k+1))(b+1)+b\sum_{k=0}^{a-2}(a-(k+1))\\ &= (2b+1)\left((a-1)^2-\frac{(a-2)(a-1)}{2}\right)\\ &= a^2b-ab+\frac{1}{2}a^2-\frac{1}{2}a\end{aligned}$$

quadrics in $H^0(\mathcal{I}_{X_{D'}}(2))$ that are independent over $H^0(\mathcal{I}_{X_E}(2))$.
Notice that we have only given a lower bound for the number of elements of all sets $\mathcal{Q}_{kl}$, since we actually did not show that the minors in $M_{D'}$ which we skipped are dependent of the ones we chose over $H^0(\mathcal{I}_{X_E}(2))$. But since we have by construction $H^0(\mathcal{I}_{X_E}(2))+H^0(\mathcal{I}_{X_D}(2))\subseteq H^0(\mathcal{I}_Q(2))$, the inequality $\dim(H^0(\mathcal{I}_{X_E}(2))+H^0(\mathcal{I}_{X_{D'}}(2)))\leq h^0(\mathcal{I}_Q(2))$ always holds. Consequently we are done as soon as we can show that $h^0(\mathcal{I}_Q(2))-h^0(\mathcal{I}_{X_E}(2))$ is equal to the above number $a^2b-ab+\frac{1}{2}a^2-\frac{1}{2}a$.
By Lemma 5.8 we have that

$$\begin{aligned}h^0(\mathcal{I}_Q(2)) &= \binom{(a+1)(b+1)+1}{2}-(2a+1)(2b+1)\\ &= \frac{1}{2}a^2b^2+a^2b+ab^2+\frac{1}{2}a^2+\frac{1}{2}b^2-\frac{3}{2}ab-\frac{1}{2}a-\frac{1}{2}b\end{aligned}$$

and thus

$$\begin{aligned}h^0(\mathcal{I}_Q(2))-h^0(\mathcal{I}_{X_{E(2)}}) &= \frac{1}{2}a^2b^2+a^2b+ab^2+\frac{1}{2}a^2+\frac{1}{2}b^2-\frac{3}{2}ab\\ &- \frac{1}{2}a-\frac{1}{2}b-\binom{(a+1)b}{2}\\ &= a^2b+\frac{1}{2}a^2-ab-\frac{1}{2}a.\end{aligned}$$

This proves the claim. □

Let us illustrate the proof of Proposition 5.7 with an example:

Example 5.13. *Let $H = (2,3)$, then we consider the following matrices M_E and $M_{D'}$:*

$$M_E = \begin{pmatrix} x_{00} & x_{01} & x_{02} & x_{10} & x_{11} & x_{12} & x_{20} & x_{21} & x_{22} \\ x_{01} & x_{02} & x_{03} & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \end{pmatrix}$$

and

$$M_{D'} = \begin{pmatrix} x_{00} & x_{10} & x_{01} & x_{11} & x_{02} & x_{12} & x_{03} & x_{13} \\ x_{10} & x_{20} & x_{11} & x_{21} & x_{12} & x_{22} & x_{13} & x_{23} \end{pmatrix}.$$

(1) Let us first list the equivalence classes with respect to the equivalence relation $\sim_E$ from Definition 5.10, (1):

$$\begin{aligned}
\alpha_{02} &= \{x_{00}x_{02}, x_{01}^2\}, \\
\alpha_{03} &= \{x_{00}x_{03}, x_{01}x_{02}\}, \\
\alpha_{04} &= \{x_{01}x_{03}, x_{02}^2\}, \\
\alpha_{11} &= \{x_{00}x_{11}, x_{01}x_{10}\}, \\
\alpha_{12} &= \{x_{00}x_{12}, x_{01}x_{11}, x_{02}x_{10}\}, \\
\alpha_{13} &= \{x_{00}x_{13}, x_{01}x_{12}, x_{02}x_{11}, x_{03}x_{10}\}, \\
\alpha_{14} &= \{x_{01}x_{13}, x_{02}x_{12}, x_{03}x_{11}\}, \\
\alpha_{15} &= \{x_{02}x_{13}, x_{03}x_{12}\}, \\
\alpha_{21} &= \{x_{00}x_{21}, x_{01}x_{20}\}, \\
\alpha_{22}^{(0,2)} &= \{x_{00}x_{22}, x_{01}x_{21}, x_{02}x_{20}\}, \\
\alpha_{22}^{(1,1)} &= \{x_{10}x_{12}, x_{11}^2\}, \\
\alpha_{23}^{(0,2)} &= \{x_{00}x_{23}, x_{01}x_{22}, x_{02}x_{21}, x_{03}x_{20}\}, \\
\alpha_{23}^{(1,1)} &= \{x_{10}x_{13}, x_{11}x_{12}\}, \\
\alpha_{24}^{(0,2)} &= \{x_{01}x_{23}, x_{02}x_{22}, x_{03}x_{21}\}, \\
\alpha_{24}^{(1,1)} &= \{x_{11}x_{13}, x_{12}^2\}, \\
\alpha_{25} &= \{x_{02}x_{23}, x_{03}x_{22}\}, \\
\alpha_{31} &= \{x_{10}x_{21}, x_{11}x_{20}\}, \\
\alpha_{32} &= \{x_{10}x_{22}, x_{11}x_{21}, x_{12}x_{20}\}, \\
\alpha_{33} &= \{x_{10}x_{23}, x_{11}x_{22}, x_{12}x_{21}, x_{13}x_{20}\}, \\
\alpha_{34} &= \{x_{11}x_{23}, x_{12}x_{22}, x_{13}x_{21}\}, \\
\alpha_{35} &= \{x_{12}x_{23}, x_{13}x_{22}\}, \\
\alpha_{42} &= \{x_{20}x_{22}, x_{21}^2\}, \\
\alpha_{43} &= \{x_{20}x_{23}, x_{21}x_{22}\}, \\
\alpha_{44} &= \{x_{21}x_{23}, x_{22}^2\}.
\end{aligned}$$

(2) Here the sets Q_{kl} to be considered are the following:

$$\begin{aligned}
Q_{00} &= \{x_{00}x_{20} - x_{10}^2, x_{00}x_{21} - x_{10}x_{11}, x_{00}x_{22} - x_{10}x_{12}, x_{00}x_{23} - x_{10}x_{13}\}, \\
Q_{01} &= \{x_{01}x_{23} - x_{11}x_{13}\}, \\
Q_{02} &= \{x_{02}x_{23} - x_{12}x_{13}\}, \\
Q_{03} &= \{x_{03}x_{23} - x_{13}^2\}.
\end{aligned}$$

From the minors in $M_{D'}$ we listed above we form equivalence classes with respect to the equivalence relation $\sim_{D'}$ as defined in Definition 5.10, (2):

$$\begin{aligned}
\beta_{20} &= \{x_{00}x_{20}, x_{10}^2\}, \\
\beta_{21} &= \{x_{00}x_{21}, x_{10}x_{11}\}, \\
\beta_{22} &= \{x_{00}x_{22}, x_{10}x_{12}\}, \\
\beta_{23} &= \{x_{00}x_{23}, x_{10}x_{13}\}, \\
\beta_{24} &= \{x_{01}x_{23}, x_{11}x_{13}\}, \\
\beta_{25} &= \{x_{02}x_{23}, x_{12}x_{13}\}, \\
\beta_{26} &= \{x_{03}x_{23}, x_{13}^2\}.
\end{aligned}$$

(3) *Now we go through the process as described in the proof of Proposition 5.7:*

- *We consider the class β_{20}. There is no equivalence class α_{20}, consequently the minor defined by β_{20}, namely $x_{00}x_{20} - x_{10}^2$, is not an element in the ideal I_{X_E}.*

- *The next class we consider is equal to β_{21}. the first monomial in β_{21}, $x_{00}x_{21}$, is also an element of α_{21}, while the second one, $x_{10}x_{11}$, is not. This implies that the minor $x_{00}x_{21} - x_{10}x_{11}$ defined by β_{21} does not lie in I_{X_E}.*

- *Next we consider the class β_{22}: The first monomial in β_{22}, $x_{00}x_{22}$ is also an element of $\alpha_{22}^{(0,2)}$, while the second monomial in β_{22}, $x_{10}x_{12}$, also lies in the class $\alpha_{22}^{(1,1)}$. Since $\alpha_{22}^{(0,2)}$ and $\alpha_{22}^{(1,1)}$ are disjoint, the minor $x_{00}x_{22} - x_{10}x_{12}$ defined by β_{22} is not an element of I_{X_E}.*

- *The next class we consider is β_{23}. The first monomial in β_{23} is also an element of $\alpha_{23}^{(0,2)}$, and the second one lies also in $\alpha_{23}^{(1,1)}$. Since $\alpha_{23}^{(0,2)}$ and $\alpha_{23}^{(1,1)}$ are disjoint, the minor $x_{00}x_{23} - x_{10}x_{13}$ defined by β_{23} does not lie in I_{X_E}.*

- *Next we study β_{24}. The first monomial in β_{24} is also an element of $\alpha_{24}^{(0,2)}$ and the second one lies also in $\alpha_{24}^{(1,1)}$. Since $\alpha_{24}^{(0,2)}$ and $\alpha_{24}^{(1,1)}$ are disjoint, the minor $x_{01}x_{23} - x_{11}x_{13}$ defined by β_{24} does not lie in I_{X_E}.*

- *The first monomial in the next class, β_{25}, is also an element of α_{25}, while the other one is not. This implies that the minor $x_{02}x_{23} - x_{12}x_{13}$ defined by β_{25} does not lie in the ideal I_{X_E}.*

- *The last class we consider is β_{26}. Since β_{26} is disjoint with all the classes α_{ij}, the minor defined by β_{26}, $x_{03}x_{23} - x_{13}^2$, is not an element of the ideal I_{X_E} and is independent of all the preceding minors we studied over I_{X_E}.*

Since the indices (i,j) of the equivalence classes β_{ij} are increasing, the independence of some minor of all the preceding minors over I_{X_E} follows automatically.

In total we obtain 7 quadrics in $H^0(\mathcal{I}_{X_D}(2))$ that are independent over $H^0(\mathcal{I}_{X_E}(2))$, *and by Lemma 5.8 this number is exactly equal to* $h^0(\mathcal{I}_Q(2)) - h^0(\mathcal{I}_{X_E}(2)) = \binom{13}{2} - 35 - \binom{9}{2}$.

Now we are able to prove Theorem 5.5:
Proof of Theorem 5.5:

With X_E, $X_{D'}$ and Q as above we have shown that $I_{X_E} + I_{X_{D'}} = I_Q$. We will now show the following:

(a) We can choose a linear space $\mathbf{P} \cong \mathbf{P}^{d-2}$ in such a way that $Q \cap \mathbf{P} = C$.

With exactly the same $\mathbf{P}$ as in (a) we obtain:

(b) $X_E \cap \mathbf{P} = S$, S being the $g^1_2(C)$-scroll in $\mathbf{P}$.

(c) $X_{D'} \cap \mathbf{P} = V$, where $V = V_{|D|}$ is a $g^1_3(C)$-scroll in $\mathbf{P}$ with $|D| \cong D'$.

(d) $H^0(\mathcal{I}_{X_E \cap \mathbf{P}}(2)) + H^0(\mathcal{I}_{X_{D'} \cap \mathbf{P}}(2)) \to H^0(\mathcal{I}_{Q \cap \mathbf{P}}(2))$ is surjective.

From this it then follows that

$$\begin{aligned} h^0(\mathcal{I}_C(2)) &= h^0(\mathcal{I}_{Q\cap\mathbf{P}}(2)) \leq \dim(H^0(\mathcal{I}_{X_E\cap\mathbf{P}}(2)) + H^0(\mathcal{I}_{X_{D'}\cap\mathbf{P}}(2))) \\ &= \dim(H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2))), \end{aligned}$$

but since the other inequality always holds, we obtain equality.
Ad(a): To prove (a) we use the coordinates $(s,t;u,v)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ and the monomials $s^{a-i}t^iu^{b-j}v^j$, $i = 0,\ldots,a$, $j = 0,\ldots,b$ which are the restrictions of the coordinates x_{ij} in $\mathbf{P}^{h^0(H')-1}$ to Q.
The equation of the curve C in the system $(2,3)$ on $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$ is given by

$$\begin{aligned} f_C &= a_1s^2u^3 + a_2s^2u^2v + a_3s^2uv^2 + a_4s^2v^3 + a_5stu^3 + a_6stu^2v \\ &+ a_7stuv^2 + a_8stv^3 + a_9t^2u^3 + a_{10}t^2u^2v + a_{11}t^2uv^2 + a_{12}t^2v^3, \end{aligned}$$

with $a_1,\ldots,a_{12} \in k$.
Set $\mathbf{P} := \mathbf{P}(H^0(H)) = V(\overline{f_C}H^0(H'-C))$.
The sections in $H^0(H'-C)$ can be represented as monomials $s^{a-2-i}t^iu^{b-3-j}v^j$, $i = 0,\ldots,a-2$, $j = 0,\ldots,b-3$, and the polynomials

$$f_{ij} := fs^{a-2-i}t^iu^{b-3-j}v^j, \quad i = 0,\ldots,a-2, j = 0,\ldots,b-3$$

define all hyperplane sections of Q that contain C.
Observe that $h^0(H') - h^0(H'-C) - 1 = 3a + 2b - 2 = d - 2$.

Ad (b): The embedding of a divisor E_0 in the system $E = (0,1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^{h^0(H')-1}$ is a rational normal curve of degree a, which spans a $\mathbf{P}^a$. The equation of this rational normal curve on Q is given by

$$g = b_1u + b_2v$$

with $b_1, b_2 \in k$. The sections in $H^0(H'-E)$ can be represented as monomials $s^{a-i}t^iu^{b-1-j}v^j$, $i = 0,\ldots,a$, $j = 0,\ldots,b-1$. Then

$$g_{ij} := gs^{a-i}t^iu^{b-1-j}v^j, \qquad i = 0,\ldots,a, j = 0,\ldots,b-1$$

are all hyperplane sections of Q that contain E_0.
Now the vector space spanned by $\{f_{ij}|i = 0,\ldots,a-2, j = 0,\ldots,b-3\}$ has dimension $h^0(H'-C) = (a-1)(b-2)$, and the polynomials $g_{i,j}, i = 0,\ldots,a,\ j = 0,\ldots,b-1$, span a vector space of dimension $h^0(H'-E) = (a+1)b$.
For a smooth curve C these two vector spaces intersect in a vector space of dimension $h^0(H-C-E) = (a-1)(b-3)$.
Now we can consider the intersection of a fiber in X_E with the space $\mathbf{P}^{d-2}$: The dimension of this intersection as vector space is equal to

$$h^0(H') - h^0(H'-C) - h^0(H'-E) + h^0(H'-C-E) = 2.$$

This implies that the linear space $\mathbf{P}$ intersects each fiber of the scroll X_E in a projective line. Since $C = Q \cap \mathbf{P} \subseteq X_E \cap \mathbf{P}$ and C intersects each rational normal curve that spans a fibre in two points, $X_E \cap \mathbf{P}$ is equal to the $g^1_2(C)$-scroll S.
Notice that the degree of S is equal to $d-1-2 = d-3$, and this number is equal to $h^0(H'-E) - h^0(H'-C-E) = \deg X_E - h^0(H'-C-E)$.

Ad (c): Observations similar to the ones in (b) yield the following for $D' = (1,0)$:
Each rational normal curve of degree b which is the embedding of a divisor in the system $D' = (1,0)$ is given by an equation $h = c_1 s + c_2 t,\ c_1, c_2 \in k$ on Q.
The sections in $H^0(H'-D')$ can be represented as monomials $s^{a-1-i}t^i u^{b-j} v^j$, $i = 0,\ldots,a-1,\ j = 0,\ldots,b$, and the polynomials

$$h_{ij} := hs^{a-1-i}t^i u^{b-j} v^j, \qquad i = 0,\ldots,a-1, j = 0,\ldots,b$$

define all hyperplane sections of Q that contain D'.
The sets $\{f_{ij}|i = 0,\ldots,a-2, j = 0,\ldots,b-3\}$ and $\{h_{ij}|i = 0,\ldots,a-1, j = 0,\ldots,b\}$ span vector spaces of dimensions $h^0(H'-C)$ and $h^0(H'-D')$ respectively. These two vector spaces intersect in a vector space of dimension $h^0(H'-C-D') = (a-2)(b-2)$.
Consequently, each fiber in the scroll $X_{D'}$ intersects the linear space $\mathbf{P}$ in a vector space of dimension

$$h^0(H') - h^0(H'-C) - h^0(H'-D') + h^0(H'-C-D') = 3,$$

i.e. the projective dimension of each such intersection is equal to 2. This implies that, since $C \subseteq X_{D'} \cap \mathbf{P}$ and C intersects each fiber of the scroll $X_{D'}$ in 3 points, $X_{D'} \cap \mathbf{P}$ is equal to the $g^1_3(C)$-scroll $V_{|D|}$ where $|D|$ is isomorphic to D'.
Observe also that $\deg(X_{D'}) - h^0(H'-C-D') = h^0(H'-D') - h^0(H'-C-D') = a(b+1) - (a-2)(b-2) = 3a + 2b - 4 = d - 4 = \deg V_{|D'|}$.

Ad (d):
In order to show that

$$H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_{V_{|D|}}(2)) \to H^0(\mathcal{I}_C(2))$$

is surjective, we need to prove that $h^1(\mathcal{I}_{S\cup V_{|D|}}(2)) = 0$. This was already shown in the proof of Theorem 5.1, but we will sketch an alternative proof below:
Since $S = X_E \cap \mathbf{P}$ and $V_{|D|} = X_{D'} \cap \mathbf{P}$ where $\mathbf{P}$ is an intersection of $ab + a + b - (3a + 2b - 2) = (a-1)(b-2)$ hyperplanes, we are done as soon as we have shown the following proposition:

Proposition 5.14. *With $X = X_E \cup X_{D'}$ and $r = (a-1)(b-2)$ we have*

$$h^1(\mathcal{I}_{X\cap H_1\cap\ldots\cap H_r}(2)) = 0.$$

In order to be able to prove this proposition we need some preliminary lemmata:

Lemma 5.15. *Let Z be a scheme and H a hyperplane that does not contain any component of Z. Moreover, assume that $h^i(\mathcal{I}_Z(k)) = 0$ for all $1 \le i \le m$, $n \le k \le 2$ such that $i + k = 3$.*
Then $h^i(\mathcal{I}_{Z\cap H}(k)) = 0$ for all $1 \le i \le m-1$, $n+1 \le k \le 2$ such that $i+k = 3$.

Proof. We have the following short exact sequence:

$$0 \to \mathcal{I}_Z(k-1) \xrightarrow{\cdot f} \mathcal{I}_Z(k) \longrightarrow \mathcal{I}_{Z\cap H}|_H(k) \to 0, \tag{5.1}$$

where f is a linear form defining the hyperplane H.
The associated long exact sequence in cohomology is as follows:

$$\cdots \to H^i(\mathcal{I}_Z(k)) \to H^i(\mathcal{I}_{Z\cap H}|_H(k)) \to H^{i+1}(\mathcal{I}_Z(k-1)) \to \cdots .$$

From this the result follows immediately. □

Lemma 5.16. *Let $Z = Z_1 \cup F$ where Z_1 is a scheme and $F \cong \mathbf{P}^N$, and let H be a hyperplane that contains Z_1 but not F.*
If $h^1(\mathcal{I}_Z(2)) = 0$, then $h^1(\mathcal{I}_{Z\cap H}(2)) = 0$.

Proof. We have the following short exact sequence

$$0 \to \mathcal{I}_F(1) \xrightarrow{\cdot f} \mathcal{I}_Z(2) \longrightarrow \mathcal{I}_{Z\cap H}|_H(2) \to 0, \tag{5.2}$$

where f is a linear form defining H, and the associated long exact sequence in cohomology:

$$\cdots \to \underbrace{H^1(\mathcal{I}_F(1))}_{=0} \to H^1(\mathcal{I}_Z(2)) \to H^1(\mathcal{I}_{Z\cap H}(2)) \to \underbrace{H^2(\mathcal{I}_F(1))}_{=0} \cdots .$$

Consequently, $h^1(\mathcal{I}_Z(2)) = h^1(\mathcal{I}_{Z\cap H}|_H(2))$. □

Now in order to prove Proposition 5.14 we want to apply Lemma 5.15 and Lemma 5.16. We have to intersect with $r := (a-1)(b-2)$ hyperplanes,

$$S \cup V_{|D|} = X \cap \mathbf{P} = X \cap H_1 \cap \ldots \cap H_r,$$

so we want to proceed by induction and start with $Z = X$, i.e. we need to check that $h^i(\mathcal{I}_X(k)) = 0$ for all $1 \le i \le r+1$, $2-r \le k \le 2$, $i+k = 3$. We will do so in the following lemma.
We will then use the two methods of intersecting with a hyperplane as described in Lemma 5.15 and Lemma 5.16. The method in Lemma 5.15 reduces the dimension but not the degree, and the method in Lemma 5.16, taking F to be a fiber in X_E or in $X_{D'}$, reduces the degree but not the dimension.
In total we obtain that $h^i(\mathcal{I}_{X\cap H_1\cap\ldots\cap H_r}(k)) = 0$ for all $1 \le i \le r+1-r = 1$ and $2-r+r \le k \le 2$, i.e. $0 = h^1(\mathcal{I}_{X\cap H_1\cap\ldots\cap H_r}(2)) = h^1(\mathcal{I}_{S\cup V_{|D|}}(2))$.

Lemma 5.17. *(i)* $h^1(\mathcal{I}_X(2)) = 0$.

(ii) $h^i(\mathcal{I}_X(k)) = 0$
for $2 \leq i \leq (a-1)(b-2)+1$, $-ab+2a+b \leq k \leq 1$, $i+k=3$.

For the proof of Lemma 5.17 we will need another lemma:

Lemma 5.18. *(i)* $h^i(\mathcal{I}_{X_E}(k)) = 0$
for $2 \leq i \leq (a-1)(b-2)+1$, $-ab+2a+b \leq k \leq 1$, $i+k=3$.

(ii) $h^i(\mathcal{I}_{X_D}(k)) = 0$
for $2 \leq i \leq (a-1)(b-2)+1$, $-ab+2a+b \leq k \leq 1$, $i+k=3$.

(iii) $h^i(\mathcal{I}_Q(k)) = 0$ *for* $1 \leq i \leq (a-1)(b-2)+1$, $-ab+2a+b \leq k \leq 1$, $i+k=2$.

Proof. (i) We consider the short exact sequence

$$0 \to \mathcal{I}_{X_E}(k) \to \mathcal{O}_{\mathbf{P}^{ab+a+b}}(k) \to \mathcal{O}_{X_E}(k) \to 0$$

and the associated long exact sequence in cohomology

$$\cdots \to H^{i-1}(\mathcal{O}_{X_E}(k)) \to H^i(\mathcal{I}_{X_E}(k)) \to H^i(\mathcal{O}_{\mathbf{P}^{ab+a+b}}(k)) \to \cdots.$$

Since $2 \leq i \leq (a-1)(b-2)+1 < ab+a+b$, we have $h^i(\mathcal{O}_{\mathbf{P}^{ab+a+b}}(k)) = 0$.

Thus it remains to check that $h^i(\mathcal{O}_{X_E}(k)) = 0$ for $1 \leq i \leq (a-1)(b-2)$, $-ab+2a+b \leq k \leq 1$ and $i+k=2$.

To prove this we use the fact that X_E is totally maximally balanced of dimension $a+1$,

$$X_E \cong \mathbf{P}(\underbrace{\mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(b)}_{a+1}),$$

thus X_E is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^a$.

By Künneth's formula we have

$$h^i(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^a}(k)) = \sum_{j+l=i} h^j(\mathcal{O}_{\mathbf{P}^1}(k)) h^l(\mathcal{O}_{\mathbf{P}^a}(k)).$$

Consequently, $h^i(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^a}(k)) = 0$ if $i = j+l$ with $j \geq 2$ or $l \notin \{0, a\}$. Thus the cases we have to check are the following:

$$\begin{aligned} j = 0 \quad &, \quad l = a, \\ j = 1 \quad &, \quad l = 0, \\ j = 1 \quad &, \quad l = a. \end{aligned}$$

Let us check each of them:

- $j = 0$, $l = a$:
  Then $i = a$, $k = 2 - a \leq 0$, so here we have anyway only a problem if $a = 2$. We thus obtain:

$$\begin{aligned} h^a(\mathcal{O}_{\mathbf{P}^a}(k)) &= h^0(\mathcal{O}_{\mathbf{P}^a}(-a-1-k)) \\ &= h^0(\mathcal{O}_{\mathbf{P}^a}(-a-1-2+a)) = h^0(\mathcal{O}_{\mathbf{P}^a}(-3)) = 0. \end{aligned}$$

- $j = 1$, $l = 0$:
  Then $i = 1$, $k = 1$. We obtain the following:

$$h^1(\mathcal{O}_{\mathbf{P}^1}(k)) = h^0(\mathcal{O}_{\mathbf{P}^1}(-2-k)) = h^0(\mathcal{O}_{\mathbf{P}^1}(-3)) = 0.$$

- $j = 1$, $l = a$:
  Then $i = a + 1$, $k = 1 - a$. Hence we obtain the following:

$$\begin{aligned} h^a(\mathcal{O}_{\mathbf{P}^a}(k)) &= h^0(\mathcal{O}_{\mathbf{P}^a}(-a-1-k)) \\ &= h^0(\mathcal{O}_{\mathbf{P}^a}(-a-1-1+a)) \\ &= h^0(\mathcal{O}_{\mathbf{P}^a}(-2)) = 0. \end{aligned}$$

(ii) The proof of (ii) is completely analogous to the proof of (i). Notice that the cases we have to check here are the following:

$$\begin{aligned} j = 0 \quad &, \quad l = b, \\ j = 1 \quad &, \quad l = 0, \\ j = 1 \quad &, \quad l = a. \end{aligned}$$

If $j = 0$ and $l = b$, then $i = b$, $k = 2 - b < 0$, so $h^0(\mathcal{O}_{\mathbf{P}^1}(k)) = 0$.

(iii) To prove (iii) we proceed in a similar way as in (i) and use the fact that $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$.
We have the following short exact sequence

$$0 \to \mathcal{I}_Q(k) \to \mathcal{O}_{\mathbf{P}^{ab+a+b}}(k) \to \mathcal{O}_Q(k) \to 0$$

and the associated long exact sequence in cohomology

$$\cdots \to H^{i-1}(\mathcal{O}_Q(k)) \to H^i(\mathcal{I}_Q(k)) \to H^i(\mathcal{O}_{\mathbf{P}^{ab+a+b}}(k)) \to \cdots.$$

Again, $2 \leq i < ab + a + b$, consequently $h^i(\mathcal{O}_{\mathbf{P}^{ab+a+b}}(k)) = 0$.

It remains to check that $h^i(\mathcal{O}_Q(k)) = 0$ for $1 \leq i \leq (a-1)(b-2)$, $-ab+2a+b \leq k \leq 1$ and $i + k = 2$.

By Künneth's formula we have

$$h^i(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(k)) = \sum_{j+l=i} h^j(\mathcal{O}_{\mathbf{P}^1}(k)) h^l(\mathcal{O}_{\mathbf{P}^1}(k)).$$

This implies that $h^i(\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(k)) = 0$ if $i = j + l$ with $j \geq 2$ or $l \geq 2$. Hence the cases we have to check are the following:

$$\begin{aligned} j = 0 \quad &, \quad l = 1, \\ j = 1 \quad &, \quad l = 0, \\ j = 1 \quad &, \quad l = 1. \end{aligned}$$

- If one of j or l is equal to 1 and the other one is equal to 0, then $i = 1$, $k = 1$. Consequently we obtain:

$$h^1(\mathcal{O}_{\mathbf{P}^1}(k)) = h^0(\mathcal{O}_{\mathbf{P}^1}(-2-k)) = h^0(\mathcal{O}_{\mathbf{P}^1}(-3)) = 0.$$

- If both j and l are equal to 1, then $i = 2$, $k = 0$. We thus obtain the following:

$$h^1(\mathcal{O}_{\mathbf{P}^1}(k)) = h^0(\mathcal{O}_{\mathbf{P}^1}(-2-k)) = h^0(\mathcal{O}_{\mathbf{P}^1}(-2)) = 0.$$

□

Proof of Lemma 5.17:
We have the following short exact sequence:

$$0 \to \mathcal{I}_X(k) \to \mathcal{I}_{X_E}(k) \oplus \mathcal{I}_{X_D}(k) \to \mathcal{I}_Q(k) \to 0 \tag{5.3}$$

In the proof of Proposition 5.7 we have shown that the map

$$H^0(\mathcal{I}_{X_E}(2)) \oplus H^0(\mathcal{I}_{X_D}(2)) \to H^0(\mathcal{I}_Q(2))$$

is surjective, this implies that $h^1(\mathcal{I}_X(2)) = 0$, i.e. (i) is proven.
For (ii), i.e. for the cases when $2 \leq i \leq (a-1)(b-2)+1$, we can use the long exact sequence in cohomology

$$\cdots \to H^{i-1}(\mathcal{I}_Q(k)) \to H^i(\mathcal{I}_X(k)) \to H^i(\mathcal{I}_{X_E}(k)) + H^i(\mathcal{I}_{X_D}(k)) \to \cdots$$

and apply Lemma 5.18. □

Explicit construction of $H^0(\mathcal{I}_{S\cup V}(2))$ in the case $d = 7$ and when S is not contained in V

Let now $C \subseteq \mathbf{P}^5$ be a curve of genus 2 and degree 7 and S be the $g^1_2(C)$-scroll. In the proof of Theorem 5.1 we saw that $h^0(\mathcal{I}_{S\cup V}(2)) \leq 1$ for a $g^1_3(C)$-scroll V that does not contain S. In this section we will describe how we actually can find a quadric of rank 4 that contains both S and V.
Let $|D|$ be a $g^1_3(C)$ such that $V_{|D|}$ does not contain S. In this case $|H| = |D+K+P+Q|$ where P and Q are two points on C such that $P+Q$ is not a divisor in $|K_C|$ (cf. Proposition 2.17). We project from the line that P and Q span to $\mathbf{P}^3$. The curve projects to a curve of degree 5, embedded in $\mathbf{P}^3$ by the linear system $|D + K_C|$. The image curve lies on one smooth quadric which is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. The cone over this quadric with the line spanned by P and Q as vertex is a quadric of rank 4 that contains both S and V.

Chapter 6

Quadrics of low rank containing a curve of degree 6

In this chapter we will study curves of genus 2 and degree 6 in $\mathbf{P}^4$ more closely. Our aim is to study the quadrics in the ideal I_C which by Proposition 3.1 is generated by 4 quadrics.
To a smooth curve C we can associate the $g_2^1(C)$-scroll S, by the equation (2.1) the degree of S is equal to 3. Moreover, C is the complete intersection of S and a quadric that contains C but not S.
If C is a singular curve, then there is no obvious way to define a $g_2^1(C)$. Still C is the complete intersection of a quadric with a scroll of degree 3 and dimension 2 that contains the curve. In this way the singular curves become natural degenerations of smooth curves. We will in this case also denote this scroll by S.
Since a general quadric in $\mathbf{P}^4$ has rank 5 and a quadric of rank 2 or rank 1 is reducible and does not contain any smooth curve of degree 6, we are in the case of smooth curves interested in the quadrics of rank 3 and rank 4 that contain the curve, respectively the scroll S.
For a smooth curve C we will introduce the Kummer surface associated to C and discover a connection to quadrics of rank 3 and 4 in I_C.
For a singular curve C we cannot define the Kummer surface, but we still have an analogous description of the locus of quadrics of rank ≤ 4 or ≤ 3 in I_C.
After having discussed some general theory for smooth curves we will give examples of singular and reducible curves.

6.1 Smooth curves of degree 6

In this section we will study smooth curves C. From a smooth curve C we can construct a Kummer surface as described below. We will distinguish between the cases when the $g_2^1(C)$-scroll S is smooth and when the $g_2^1(C)$-scroll S has one singular point.
For any integer $k \geq 0$ we denote by $\mathrm{Pic}^k(C)$ the set of all line bundles of degree k on C modulo isomorphism.
Here we are interested in the cases $k = 0$ and $k = 3$.

Definition 6.1. *The Jacobian variety of C, $\mathrm{Jac}(C)$, is defined as $\mathrm{Pic}^0(C)$.*

Note that by fixing a divisor D_0 of degree 3 we obtain an isomorphism

$$\begin{array}{rcl}\mu : \mathrm{Pic}^0(C) & \to & \mathrm{Pic}^3(C), \\ \mathcal{O}_C(D) & \mapsto & \mathcal{O}_C(D + D_0).\end{array}$$

Hence $\mathrm{Pic}^3(C) \cong \mathrm{Jac}(C)$, and since $h^0(\mathcal{O}_C(D)) = 2$ for each divisor D on C of degree 3, each $g^1_3(C)$ is a complete linear system, i.e. in total we obtain

$$G^1_3(C) \cong \mathrm{Pic}^3(C) \cong \mathrm{Jac}(C).$$

Definition 6.2. *Let C be a smooth curve of genus 2. The Kummer variety of the Jacobian variety* $\mathrm{Jac}(C)$ *is the quotient of* $\mathrm{Jac}(C)$ *by the Kummer involution $x \mapsto -x$.*

Remark 6.3. *For a general curve C the Kummer involution is the only automorphism with fixed points on* $\mathrm{Jac}(C)$*, an abelian surface.*

Definition 6.4. *A quartic Kummer surface is a hypersurface of degree 4 in $\mathbf{P}^3$ with 16 singularities.*

It is known that any such surface is the Kummer variety of the Jacobian variety of a smooth curve of genus 2. The singular points of the quartic Kummer surface are exactly the 16 fixed points of the Kummer involution.
Let now $C \subseteq \mathbf{P}^4$ be a smooth curve of degree 6 and genus 2. By Proposition 3.1 we know that I_C is generated by 4 quadrics.
Moreover, for any quadric Q in $H^0(\mathcal{I}_C(2)) - H^0(\mathcal{I}_S(2))$ we have $I_C = I_S + (Q)$.
Each $g^1_3(C)$-scroll $V_{|D|}$ is a quadric of rank 4 or less.
Consider the map

$$\begin{array}{c}\iota : G^1_3(C) \cong \mathrm{Jac}(C) \to \mathrm{Jac}(C) \cong G^1_3(C), \\ |D| \mapsto |H - D|.\end{array}$$

Since $\iota^2 = \iota$, ι is an involution on the Jacobian variety. A fixed point $|D|$ of ι has the property that $|H| = |2D|$. The associated $g^1_3(C)$-scroll $V_{|D|}$ to a fixed point $|D|$ of ι is a quadric of rank 3 in $\mathbf{P}^4$.
We will now prove that the locus $\mathcal{X}_3 := \{\text{quadrics of rank } \leq 3 \text{ in } H^0(\mathcal{I}_C(2))\}$ is either of dimension 0 and of degree 20 (as scheme) or infinite:

Lemma 6.5. *Let $\mathcal{X}_3 = \{$quadrics of rank ≤ 3 in $H^0(\mathcal{I}_C(2))\}$. If $\mathcal{X}_3$ is 0-dimensional, then the degree of $\mathcal{X}_3$ is equal to 20.*

Proof. Following [Ful98], Example 14.4.11, let $\sigma : \mathcal{O}_{\mathbf{P}^3}(-\frac{1}{2})^5 \to \mathcal{O}_{\mathbf{P}^3}(\frac{1}{2})^5$ be the map given by multiplication with the symmetric coefficient matrix (that defines a quadric in $\mathbf{P}^4$). Set $c_i := c_i(\mathcal{O}(\frac{1}{2})^5)$. The locus $\mathcal{X}_3$ is equal to the the locus where σ has at most rank 3.
By Example 14.4.11 in [Ful98] $\mathcal{X}_3$ is 0-dimensional, and the degree of $\mathcal{X}_3$ is equal to:

$$\begin{array}{rcl}2^2 \cdot \Delta_{2,1}(c(\mathcal{O}(\frac{1}{2}))) & = & 4 \cdot \det \begin{pmatrix} c_2 & c_3 \\ 1 & c_1 \end{pmatrix} \\ & = & 4(c_1c_2 - c_3) = 4\left(\frac{5}{2} \cdot \frac{5}{2} - \frac{5}{4}\right) \\ & = & 20.\end{array}$$

(Cf. also [HT81]). ☐

We have now shown that $\iota : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ is an involution with at least one fixed point, hence it is by Remark 6.3 isomorphic to the Kummer involution, and $K := \mathrm{Jac}(C)/(\iota(|D|) \sim |D|)$ maps birationally to a Kummer variety in $\mathbf{P}^3_C = \mathbf{P}(H^0(\mathcal{I}_C(2)))$, and the 16 fixed points of ι are exactly the singular points of K, which parametrize quadrics of rank 3 in $H^0(\mathcal{I}_C(2))$.
All quadrics that contain S have rank at most 4, so we can describe the locus $\{q \in H^0(\mathcal{I}_C(2)) | \operatorname{rk}(q) \leq 4\}$ as the union of two components, $\mathbf{P}(H^0(\mathcal{I}_S(2))) \cup K = \mathbf{P}^2_S \cup K$. The component K is isomorphic to the Kummer variety, and moreover $\mathbf{P}^2_S \cap K$ is birationally equivalent to C.

We will now give a more explicit construction of the Kummer surface K in $\mathbf{P}(H^0(\mathcal{I}_C(2)))$: We pick three quadrics q_1, q_2 and q_3 such that $\langle q_1, q_2, q_3 \rangle = H^0(\mathcal{I}_S(2))$ and a general quadric Q in $\mathbf{P}^4$ which is not contained in I_S and then form a general quadric in I_C, i.e. we take $q = a_1q_1 + a_2q_2 + a_3q_3 + a_4Q$ with parameters a_1, a_2, a_3 and $a_4 \in k$. The projective space $\mathbf{P}(k[a_1, a_2, a_3, a_4]) = \mathbf{P}H^0(\mathcal{I}_C(2))$ we denote by $\mathbf{P}^3_{\underline{a}}$.
In terms of the a_i we can form a parametrization of the locus of quadrics of rank less or equal to 3 containing C (respectively S) and the quadrics of rank less or equal to 4 containing C (respectively S): We form the symmetric (5×5)-coefficient matrix of q, the determinant of this matrix describes the locus of quadrics of rank 4 or less, the (4×4)-minors describe the locus of quadrics of rank 3 or less. The determinant will have the factor a_4 and when dividing the determinant by a_4, then we obtain the equation of the quartic Kummer surface K in $\mathbf{P}^3_{\underline{a}}$. The variety $V(a_4)$ describes $\mathbf{P}^2_S = \mathbf{P}(H^0(\mathcal{I}_S(2)))$.
Let now X_3 denote the variety of quadrics of rank ≤ 3 in $H^0(\mathcal{I}_C(2))$ and Z_3 denote the variety of quadrics of rank ≤ 3 in $H^0(\mathcal{I}_C(2)) - H^0(\mathcal{I}_S(2))$. By Lemma 6.5 X_3 has degree 20.
Since a $|D|$ with $|H| = |2D|$ gives a quadric $V_{|D|}$ of rank 3, we obtain $\operatorname{sing}(K) \subseteq X_3$, and moreover $Z_3 \subseteq \operatorname{sing}(K)$.

6.1.1 Smooth curves C having a smooth $g^1_2(C)$-scroll

In this section we will study smooth curves C where the associated $g^1_2(C)$-scroll S is smooth.
After possibly a coordinate change the ideal I_S is generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix}.$$

Set $q_1 = x_0x_2 - x_1^2$, $q_2 = x_0x_4 - x_1x_3$, $q_3 = x_1x_4 - x_2x_3$ and a general quadric in I_S to be $q = a_1q_1 + a_2q_2 + a_3q_3$ with parameters a_1, a_2 and a_3 in k.
The locus of quadrics of rank 3 in I_S is one point in $\mathbf{P}^3_{\underline{a}}$, namely given by $V(a_2, a_3, a_4)$.
The singular locus of K consists of 16 points. As discussed below, the point in X_3 that corresponds to the point $V(a_2, a_3, a_4)$ can have multiplicity 4 or 5.
In order to illustrate the 16 singular points of the Kummer surface K, let us pick an example of a smooth curve on a smooth scroll: Let C be given by the ideal $I_C = I_S + (x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2)$. We found the following singular points of the Kummer surface K (found with the computer algebra system [GS], in characteristic 32749):

$$
\begin{aligned}
&(0,0,\pm 2,1),\\
&(0,\pm 2,0,1),\\
&(4,\pm 15331,0,1),\\
&(4,0,\pm 15331,1),\\
&(8710,\pm 639,\pm 639,1),\\
&(-8708,\pm 820,\pm 820,1).
\end{aligned}
$$

It can easily be checked that all these 16 points correspond to quadrics of rank 3 that do not contain S.
The curve intersects the directrix line $V(x_0,x_1,x_2)$ of the scroll S in two different points, i.e. $|H| = |D_1 + D_2|$ where $|D_1| \neq |D_2|$, both $|D_1|$ and $|D_2|$ have basepoints and the point $(1,0,0,0)$ which corresponds to the quadric $q_1 = x_0x_2 - x_1^2$ does not correspond to a fixed point of the Kummer involution.
In general, if $I_C = I_S + (f(\underline{x}) + g(\underline{x}))$, where $f(\underline{x}) \in (x_0,x_1,x_2) \subseteq k[x_0,\ldots,x_4]$ is a quadratic polynomial and $g(\underline{x}) \in (x_3,x_4) \subseteq k[x_0,\ldots,x_4]$ is a quadratic polynomial with two different roots such that the corresponding curve C is smooth, then the curve C intersects the directrix line of the scroll $V(x_0,x_1,x_2)$ in two different points and $|H| = |D_1 + D_2|$, $|D_1| \neq |D_2|$. The point $(1,0,0,0)$ is thus not a singular point of the Kummer surface, and the multiplicity of the point in X_3 corresponding to $(1,0,0,0)$ is equal to 4.
On the other hand, if $I_C = I_S + (f(\underline{x}) + g(\underline{x})^2)$, where $f(\underline{x}) \in (x_0,x_1,x_2) \subseteq k[x_0,\ldots,x_4]$ is a quadratic polynomial and $g(\underline{x}) \in (x_3,x_4) \subseteq k[x_0,\ldots,x_4]$ is a linear polynomial such that the corresponding curve C is smooth, then the curve C intersects the directrix line of the scroll $V(x_0,x_1,x_2)$ in one point, $V(x_0,x_1,x_2,g(x))$, of multiplicity 2 and $|H| = |2D|$. The point $(1,0,0,0)$ is thus among the 16 singular points of the Kummer surface, and the multiplicity of the point in X_3 corresponding to $(1,0,0,0)$ is equal to 5.

6.1.2 Smooth curves C with a singular $g_2^1(C)$-scroll

In this section we will study smooth curves C where the $g_2^1(C)$-scroll S is singular.
After possibly a coordinate change the ideal I_S is generated by the (2×2)-minors of the matrix

$$
\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.
$$

Set $q_1 = x_0x_2 - x_1^2$, $q_2 = x_0x_3 - x_1x_2$, $q_3 = x_1x_3 - x_2^2$ and a general quadric in I_S to be $q = a_1q_1 + a_2q_2 + a_3q_3$ with parameters a_1, a_2 and a_3 in k.
Notice that q_1 and q_3 are two independent quadrics of rank 3 in $H^0(\mathcal{I}_S(2))$. The variety that parametrizes the quadrics of rank 3 or less in I_S is equal to a conic in $\mathbf{P}^2_{a_1,a_2,a_3}$, given by $E := V(a_1a_3 - a_2^2)$.
The singular locus of K, $\operatorname{sing}(K)$, is again of dimension 0 and degree 16, $X_3 = \operatorname{sing}(K) \cup E$, and $\operatorname{sing}(K) \cap E$ consists of 6 points which are equal to the ramification points of the map $\Phi_{|K_C|} : C \to \mathbf{P}^1$.

6.2 Examples of singular curves of degree 6

In this section we will study examples of singular curves C which are complete intersections of a two-dimensional scroll S of degree 3 and a quadric. In the case of a singular curve C there is no obvious way to construct a $g_2^1(C)$ or a $g_3^1(C)$ and we cannot construct a Kummer surface as in Section 6.1, but still we can consider the loci of quadrics of rank 4 or less which lie in $H^0(\mathcal{I}_C(2))$ or $H^0(\mathcal{I}_S(2))$. We can construct an analogue of the Kummer surface in the following way:
Choose three quadrics q_1, q_2 and q_3 in such a way that $H^0(\mathcal{I}_S(2)) = \langle q_1, q_2, q_3 \rangle$, and let Q be a quadric such that $H^0(\mathcal{I}_C(2)) = \langle q_1, q_2, q_3, Q \rangle$. Form a general quadric $q = a_1 q_1 + a_2 q_2 + a_3 q_3 + a_4 Q$, $a_1, \ldots, a_4 \in k$, in $H^0(\mathcal{I}_C(2))$. The projective space $\mathbf{P}(k[a_1, a_2, a_3, a_4])$ we denote by $\mathbf{P}^3_{\underline{a}}$. The determinant of the symmetric coefficient matrix of the general quadric q is a polynomial of degree 5 which contains a_4 as factor. We divide now by a_4 and obtain a polynomial of degree 4 which defines a hypersurface in $\mathbf{P}^3_{\underline{a}}$, unless the determinant is equal to 0.
In the following two sections we will use the following notation:

- X_3 denotes the variety of quadrics of rank ≤ 3 in $H^0(\mathcal{I}_C(2))$.
- Y denotes the locus of quadrics of rank ≤ 4 in I_C obtained by taking the determinant of the coefficient matrix of a general quadric $q = a_1 q_1 + a_2 q_2 + a_3 q_3 + a_4 Q$ in I_C and then dividing by a_4. Unless Y is all of $\mathbf{P}^3_{\underline{a}}$, Y is of degree 4 and dimension 2.
- Since the locus of quadrics of rank ≤ 3 is contained in the singular locus of the quadrics of rank 4, $X_3 \subseteq \operatorname{sing}(Y)$.

6.2.1 Examples of singular curves C as the complete intersection of a smooth scroll and a quadric

In this section we list singular curves on a smooth scroll S which ideal is given by the (2×2)-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix}.$$

If C is the complete intersection of S with a quadric, then the arithmetic genus of C is equal to 2 by the adjunction formula.
We will list curves with 0-dimensional singular locus and curves with 1-dimensional singular locus, the latter being necessarily non-reduced, in our cases also reducible.
We have the following:

- There is only one quadric of rank 3 in $\mathbf{P}H^0(\mathcal{I}_S(2))$, namely $x_0x_2 - x_1^2$, corresponding to $V(a_2, a_3)$.

(1) $Q = 3x_0x_2 - 2x_0x_3 + 2x_0x_4 - x_1x_3 + 3x_1x_4 + 4x_4^2$:
The curve C is irreducible, it has one singular point $V(x_0, x_1, x_3, x_4)$ which lies on the fiber $V(x_0, x_1, x_3)$ of S.

The locus Y is irreducible, the singular locus of Y is equal to the line given by $V(a_1+3a_4, a_3)$ and 16 points (found with the computer algebra system Macaulay 2 ([GS]) in characteristic 32749):

$$
\begin{aligned}
&(1,0,0,0),\\
&(-6141,-1,0,1),\\
&(16130,-4390,627,1),\\
&(10864,4386,-1567,1),\\
&(-10059,-11257,3284,1),\\
&(-3,-2,359,1),\\
&(12239,11253,14031,1),\\
&(-3,-2,-365,1),\\
&(-3,-7823,0,1)(\text{multiplicity } 2),\\
&(-3,-3435,0,1)(\text{multiplicity } 2),\\
&(-3,3432,0,1)(\text{multiplicity } 2),\\
&(-3,7820,0,1)(\text{multiplicity } 2).
\end{aligned}
$$

Notice that the last 4 points already lie on the line $V(a_1+3a_4, a_3)$.

The locus of X_3 is a 0-dimensional scheme of degree 20, $(X_3)_{\text{red}}$ is equal to the above 16 points, the point $(1,0,0,0)$ counts with multiplicity 5.

(2) $Q = 3x_0^2 + x_0x_1 + 2x_1^2 + 2x_1x_3 + 3x_1x_4 - 3x_3x_4$:
The curve C is the union of the fibers $V(x_0, x_1, x_3)$ and $V(x_1, x_2, x_4)$ of S and a rational curve of degree 4, it has four singular points $V(x_0, x_1, x_3, x_4)$, $V(x_0, x_1, x_2 - x_4, x_3)$, $V(x_0 - x_3, x_1, x_2, x_4)$ and $V(x_0 + 3x_3, x_1, x_2, x_4)$ which are the intersection points between the fibers and the rational curve.

Y is irreducible, its singular locus is equal to the two lines given by $V(a_1+a_2, a_3+3a_4)$ and $V(a_1, a_3)$ and 16 points.

The locus X_3 is a 0-dimensional scheme of degree 20.

(3) $Q = x_1x_2 + x_3x_4$:
The curve C is the union of two fibers of S, $V(x_0, x_1, x_3)$ and $V(x_1, x_2, x_4)$ and two conics K_1 and K_2, it has the following singular points:
$V(x_1, x_2, x_3, x_4)$ with multiplicity 4, lying on the fiber $V(x_1, x_2, x_4)$ of S, and the intersection points $V(x_0, x_1, x_3) \cap K_1$ and $V(x_0, x_1, x_3) \cap K_2$.

The locus Y decomposes into two planes $V(a_1+15645a_2)$ and $V(a_1-15645a_2)$ and a quadric $V(a_2a_3+a_1a_4)$. The singular locus of Y consists of $V(a_3-15645a_4, a_1+15645a_2)$, $V(a_3+15645a_4, a_1-15645a_2)$ and $V(a_1, a_2)$.

The locus X_3 consists of 20 points, counted with multiplicity.

(4) $Q = x_0^2 + x_1^2 + x_2^2$:
The curve C is the union of six lines: The directrix line of S, $V(x_0, x_1, x_2)$, with multiplicity 2 and four other lines.

The locus Y decomposes into four planes: $V(a_2-4354a_3)$, $V(a_2+4354a_3)$, $V(a_2-4355a_3)$ and $V(a_2+4355a_3)$.

The singular locus of Y is equal to the locus X_3, and both are equal to the line $V(a_2, a_3)$, which is the intersection of all four planes.

(5) $Q = x_0x_2 - x_1^2 + x_3x_4$:
The curve C is the union of two fibers of the scroll S, $V(x_0, x_1, x_3)$ and $V(x_1, x_2, x_4)$, and the conic $K := V(x_0x_2 - x_1^2, x_3, x_4)$ with multiplicity 2, the singular locus of C is thus equal to this conic and the points of intersection $K \cap V(x_0, x_1, x_3) = V(x_0, x_1, x_3, x_4)$ and $K \cap V(x_1, x_2, x_4) = V(x_1, x_2, x_3, x_4)$.

The locus Y decomposes into the plane $V(a_1+a_4)$ (of multiplicity 2) and a quadric $V(a_2a_3 + a_1a_4 + a_4^2)$, the singular locus of Y is given by the plane $V(a_1 + a_4)$.

The locus X_3 is 0-dimensional and has degree 20, it consists of the points $V(a_1 + a_4, a_2, a_3)$ (of multiplicity 16) and $V(a_2, a_3, a_4)$ (of multiplicity 4).

6.2.2 Examples of singular curves as the complete intersection of a singular scroll and a quadric

Let now S be the singular scroll which ideal is generated by the (2×2)-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

This scroll has one singular point, given by $V(x_0, x_1, x_2, x_3)$.
Let C be a curve which is the complete intersection of the scroll S and a quadric Q. By the adjunction formula the arithmetic genus of C is equal to 2.
We have the following:

- The quadrics of rank ≤ 3 in $H^0(\mathcal{I}_S(2))$ are parametrized by a conic $V(a_2^2 - a_1a_3, a_4)$.
- Y is of dimension 2, degree 4 or all of $\mathbf{P}^3_{\underline{a}}$.
- $X_3 \subseteq \operatorname{sing}(Y)$.

(1) $Q = x_0x_4 + x_1x_2 + x_4^2$:
The curve is irreducible, it has one singular point $V(x_0, x_1, x_3, x_4)$ and does not pass through the singular point of S.

Y is irreducible, its singular locus is given by the line $V(a_2, a_3)$ and 18 points (with multiplicity).

The locus X_3 decomposes into the conic $V(a_1a_3 - a_2^2, a_4)$ plus 10 points.

(2) $Q = x_0^2 + x_0x_1 + x_0x_2 + x_1x_4 + x_2x_4 + x_3^2$:
The curve is irreducible, it has one singular point $V(x_0, x_1, x_2, x_3)$ which is equal to the singular point of S.

The locus Y decomposes into the plane $V(a_4)$ and a cubic hypersurface, its singular locus is given by the intersection of these two varieties, $V(a_2, a_4)$, $V(a_1-a_2, a_4)$ and $V(a_2 - a_3, a_4)$ and 19 points.

The locus X_3 decomposes into the conic $V(a_1a_3 - a_2^2, a_4)$ and 20 points.

(3) $Q = x_0^2 + x_1^2 + x_2^2$:
The curve is the union of six lines: the fiber $V(x_0, x_1, x_2)$ with multiplicity 2 and four more lines.

Since the quadric $x_0^2 + x_1^2 + x_2^2$ itself is a quadric of rank 3 that passes through the singular locus of S, the locus Y is all of $\mathbf{P}^3_{\underline{a}}$.

X_3 is an irreducible hypersurface of degree 4, its singular locus consists of the line $V(a_2, a_3)$ and 20 points not on this line.

(4) $Q = x_2^2 + x_3^2 + x_4^2$:
The curve is irreducible, it has one singular point, $V(x_1, x_2, x_3, x_4)$.

Y is irreducible, singular along the line $V(a_1, a_2)$ and singular in 17 points not on this line.

As set X_3 is equal to the conic $V(a_1a_3 - a_2^2, a_4)$ and the points $V(a_1, a_2, a_3)$, $V(a_1, a_2, a_3 - a_4)$.

(5) $Q = x_0x_2 - x_1^2 + x_3x_4$:
The curve is the union of one fiber of S, $V(x_1, x_2, x_3)$ (multiplicity 3), and the rational curve of degree 3 which is the intersection of S with the hyperplane $V(x_4)$.

The locus Y decomposes into the plane $V(a_1 + a_4)$ (with multiplicity 3) and the plane $V(a_4)$, the singular locus of Y is thus equal to $V(a_1 + a_4)$.

Chapter 7

The first syzygies of I_C where C is a curve of degree 7

In Chapter 5 we showed that the ideal of C is generated by the ideal of the $g^1_2(C)$-scroll S and the ideal of a $g^1_3(C)$-scroll $V_{|D|}$ that does not contain S. Defining the ideal to be the 0th syzygy-module we have shown that the 0th syzyzgies of I_C are generated by the 0th syzygies of rational normal scrolls.
Aiming at a pursuit of generalization a natural question to pose is the following:

Are the ith syzygies of I_C generated by the ith syzygies of ideals of rational normal scrolls?

We will here restrict ourselves to the first syzygies and study the following question:

Are the first syzygies of I_C generated by the first syzygies of I_S and the first syzygies of I_V, where V runs through the family of all $g^1_3(C)$-scrolls?

7.1 Preliminary definitions and motivation

Definition 7.1. *For any variety $X \subseteq \mathbf{P}^N$ let the minimal free resolution of its ideal I_X be given as follows:*

$$0 \to \oplus_{j \geq j_n} \mathcal{O}_{\mathbf{P}^N}(-j)^{\beta_{nj}} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} \oplus_{j \geq j_1} \mathcal{O}_{\mathbf{P}^N}(-j)^{\beta_{1j}} \xrightarrow{\phi_0} I_X \to 0.$$

The ith syzygy-module of I_X, $\mathrm{Syz}_i(I_X)$, is defined to be the image of ϕ_i.

An alternative definition is the following:

Definition 7.2. *Let $X \subseteq \mathbf{P}^N$ be a variety and $I_X = (f_1, \dots, f_l)$ be its ideal. If $l \geq 2$, then there exists at least one relation between the generators f_i, $i = 1, \dots, l$, i.e. there exists polynomials $g_1, \dots, g_l \in k[x_0, \dots, x_N]$ such that*

$$\sum_{i=1}^{l} g_i f_i = 0.$$

The coefficients $(g_1, \dots, g_l)$ form a first syzygy of I_X. There might be relations between these g_i which form a second syzygy of I_X and so on.
In the case when all coefficients g_i have degree 1, we say that the corresponding syzygy is a linear syzygy.

Our motivating question is now the following:

Are the first syzygies of I_C generated by the first syzygies of I_S and the first syzygies of I_V, where V runs through the family of all $g^1_3(C)$-scrolls?

Let us first consider the cases of low degree:
If $d = 5$, then I_C is generated by a quadric, which gives the $g^1_2(C)$-scroll S, and two cubics. There are two first syzygies of I_C and since S is a hypersurface there are no first syzygies of I_S.
If $d = 6$, then I_C is generated by I_S and one additional quadric. A $g^1_3(C)$-scroll is a hypersurface, so there are no first syzygies of I_V.
The first interesting case is thus given when the degree of C is equal to 7. In this chapter we will focus on this case.
If C is a curve of genus 2 and degree 7 in $\mathbf{P}^5$, then we can see from the resolution of I_C in Chapter 4 that the first syzygies of I_C are generated by linear syzygies. Moreover, since the resolution of I_X, where X is a rational normal scroll, is given by the Eagon-Northcott complex described as $\mathcal{C}_0$ in Chapter 4, the first syzygies of I_X are linear.
Since the family $G^1_3(C) = \{g^1_3(C)'s\}$ is isomorphic to $\mathrm{Jac}(C)$, this family is two-dimensional, as we also already have seen in Chapter 2.
Our motivating question becomes thus the following:

Is the vector space of first syzygies of I_C spanned by the first syzygies of I_S and the first syzygies of I_V, where V runs through the two-dimensional family of all $g^1_3(C)$-scrolls?

In Chapter 4 we saw that the vector space of linear first syzygies of I_C is 12-dimensional. Also, again since the resolution of I_X, where X is a rational normal scroll, is given by the Eagon-Northcott complex, we can verify that the space of linear first syzygies of I_S, where S is the $g^1_2(C)$-scroll, is 8-dimensional and that the space of linear first syzygies of I_V, where V is a $g^1_3(C)$-scroll, is of dimension 2. Thus we immediately see that, contrary to the case of the ideal, we cannot find just one $g^1_3(C)$-scroll such that the first syzygies of I_C are generated by the first syzygies of I_S and the first syzygies of the ideal of this $g^1_3(C)$-scroll. However, in most of our cases, two $g^1_3(C)$-scrolls are sufficient to give all first syzygies of I_C.

We introduce the rank of a linear syzygy:

Definition 7.3. *Let $s \in \mathrm{Syz}_i(I_X)$ be a linear syzygy. The rank of s is defined to be the dimension of the vector space that the linear forms in s span.*

A basis for the vector space of first syzygies of I_S can be chosen to be syzygies of rank 3. The same applies to the vector space of first syzygies of I_V where V is a $g^1_3(C)$-scroll, since its ideal is generated by the (2×2)-minors of a (2×3)-matrix.
The main aim in this section is to prove the following theorem:

Theorem 7.4. *Consider the following condition for a given curve $C \in \mathcal{M}_2$ and a linear system $|H|$ on C of degree 7 which embeds C into $\mathbf{P}^5$:*

($\diamondsuit$) There exist two $g^1_3(C)$-scrolls V_1 and V_2 such that the space of first syzygies of I_C is generated by the first syzygies of I_S, the first syzygies of I_{V_1} and the first syzygies of I_{V_2}.

The condition ($\diamondsuit$) is satisfied for a general curve C in $\mathcal{M}_2$, the moduli space of non-singular curves of genus 2, and a general $\mathcal{O}_C(H) \in \mathrm{Pic}^7(C)$ such that the complete linear system $|H|$ embeds C into $\mathbf{P}^5$ as a smooth curve.

More precisely: The condition $(\diamondsuit)$ *is satisfied for a general curve* $C \overset{|H|}{\hookrightarrow} \mathbf{P}^5$ *such that the* $g^1_2(C)$*-scroll is of scroll type* $(2,2)$*, and the condition* $(\diamondsuit)$ *is satisfied for a general curve* $C \overset{|H|}{\hookrightarrow} \mathbf{P}^5$ *such that the* $g^1_2(C)$*-scroll is of scroll type* $(3,1)$*.*

We will prove Theorem 7.4 by giving one example of a smooth curve that satisfies the claim for each scroll type of S and using the fact that condition $(\diamondsuit)$ is an open condition.
In this way Theorem 7.4 leads us to the following conjecture:

Conjecture 7.5. *For every curve* $C \in \mathcal{M}_2$ *and every* $\mathcal{O}_C(H) \in \mathrm{Pic}^7(C)$ *there exist two* $g^1_3(C)$*-scrolls* V_1 *and* V_2 *such that the first syzygies of* I_{V_1} *and the first syzygies of* I_{V_2} *together with the first syzygies of* I_S*, where* S *is the* $g^1_2(C)$*-scroll, generate the space of first syzygies of* I_C*, where* $C \subseteq \mathbf{P}^5$ *is embedded as a smooth curve with the complete linear system* $|H|$*.*

In addition we will give families of reducible curves C, a two-dimensional rational normal scroll S and three-dimensional scrolls, which all contain C, such that the syzygies of the ideals of these scrolls span the space of first syzygies of I_C.
In three of our four examples we are able to find two three-dimensional-scrolls V_1 and V_2 such that the first syzygies of I_{V_1} and the first syzygies of I_{V_2} together with the first syzygies of I_S generate the vector space of first syzygies of I_C. In the last example three three-dimensional scrolls are enough.
In some natural sense our examples give rise to the following conjecture:

Conjecture 7.6. *Let* $C \overset{|H|}{\hookrightarrow} \mathbf{P}^5$ *be a variety of pure dimension* 1*, arithmetic genus* 2 *and degree* 7*.*
Then C *lies on a non-degenerate surface* S *of degree* 4 *and a family* $\mathcal{V}$ *of non-degenerate* 3*-dimensional varieties of degree* 3 *such that the first syzygies of* I_C *are generated by the first syzygies of* I_S *together with the first syzygies of all* I_Y *where* Y *runs through the family* $\mathcal{V}$*.*

7.2 Curves C on a two-dimensional scroll of type $(2,2)$

In this section we will consider a curve C of genus 2 and degree 7 lying on a two-dimensional scroll $S \cong \mathbf{P}^1 \times \mathbf{P}^1$. After a coordinate change we might assume that the ideal of S is generated by the (2×2)-minors of the following matrix:

$$M = \begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix}.$$

In Section 3.5 we have seen that the following ideal is the ideal of a curve C of genus 2 and degree 7 in $\mathbf{P}^5$:

$$I_C = I_S + (l_1x_0 + l_2x_1 + l_3x_3 + l_4x_4, l_1x_1 + l_2x_2 + l_3x_4 + l_4x_5),$$

where the $l_i \in k[x_0, x_1, x_2, x_3, x_4, x_5]$ are linear forms. For general l_i the curve C is smooth.
We put the generators of I_C in the following order:

$$
\begin{aligned}
Q_1 &= l_1x_0 + l_2x_1 + l_3x_3 + l_4x_4, \\
Q_2 &= l_1x_1 + l_2x_2 + l_3x_4 + l_4x_5, \\
q_1 &= x_0x_2 - x_1^2, \\
q_2 &= x_0x_4 - x_1x_3, \\
q_3 &= x_0x_5 - x_1x_4, \\
q_4 &= x_1x_4 - x_2x_3, \\
q_5 &= x_1x_5 - x_2x_4, \\
q_6 &= x_3x_5 - x_4^2.
\end{aligned}
$$

The vector space of the linear first syzygies of I_C is 12-dimensional. We write down a basis for this vector space as a matrix where each column represents a syzygy. The first 8 syzygies form a basis for the space of first syzygies of I_S.

$$
(\heartsuit)\quad \left(\begin{array}{c|cccccccccccc}
Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_4 & -x_5 \\
Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_3 & x_4 \\
q_1 & x_3 & x_4 & x_4 & x_5 & 0 & 0 & 0 & 0 & -l_2 & l_1 & 0 & 0 \\
q_2 & -x_1 & -x_2 & 0 & 0 & x_4 & x_5 & 0 & 0 & -l_3 & 0 & l_1 & 0 \\
q_3 & 0 & 0 & -x_1 & -x_2 & -x_3 & -x_4 & 0 & 0 & -l_4 & 0 & 0 & l_1 \\
q_4 & x_0 & x_1 & 0 & 0 & 0 & 0 & x_4 & x_5 & 0 & -l_3 & l_2 & 0 \\
q_5 & 0 & 0 & x_0 & x_1 & 0 & 0 & -x_3 & -x_4 & 0 & -l_4 & 0 & l_2 \\
q_6 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_1 & x_2 & 0 & 0 & -l_4 & l_3
\end{array}\right).
$$

Our aim is now to find, for particular choices of the linear forms l_i, three-dimensional rational normal scrolls such that the first syzygies of their ideals together with the first syzygies of I_S generate the vector space of first syzygies of I_C, i.e. we want to find 4 first syzygies of rank 3 that together with the first syzygies of I_S generate the above 12-dimensional vector space and that actually are syzygies of three-dimensional rational normal scrolls.

7.2.1 A smooth example

Let

$$
\begin{aligned}
l_1 &= x_0, \\
l_2 &= x_5, \\
l_3 &= -x_3, \\
l_4 &= x_4,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
Q_1 &= x_0^2 + x_1x_5 - x_3^2 + x_4^2, \\
Q_2 &= x_0x_1 + x_2x_5 - x_3x_4 + x_4x_5.
\end{aligned}
$$

The resulting curve is smooth, and the first syzygies of I_C are generated by the first syzygies of I_S and the first syzygies of I_{V_1} and I_{V_2} where V_1 and V_2 are $g^1_3(C)$-scrolls and the ideals are given as follows:

Ideal of the $g^1_3(C)$-scroll	Generating matrix
$I_{V_1} = (Q_1 - q_5, Q_2 - q_2, q_3 - q_6)$	$M_{V_1} = \begin{pmatrix} x_0+x_3 & -x_4 & -x_5 \\ x_2+x_4 & x_0-x_3 & x_1-x_4 \end{pmatrix}$
$I_{V_2} = (Q_1 + q_6, -q_1 - q_2 + q_4, Q_2 + q_2)$	$M_{V_2} = \begin{pmatrix} x_0+x_3 & -x_5 & -x_1-x_4 \\ x_1+x_3 & x_0-x_3 & -x_2-x_4 \end{pmatrix}$

7.2.2 A family of curves where $l_1 = l_2$ and $l_3 = l_4$

Let

$$\begin{aligned} l_1 &= l_2 = a_0x_0 + (a_0+1)x_1 + a_3x_3 + (a_3+a_4)x_4, \\ l_3 &= l_4 = b_0x_0 + (b_0+b_1)x_1 + b_3x_3 + (b_3+1)x_4, \end{aligned}$$

where $a_0, a_3, a_4, b_0, b_1, b_3$ are elements in k.
Set $l'_1 := a_0x_1 + (a_0+1)x_2 + a_3x_4 + (a_3+a_4)x_5$
and $l'_3 := b_0x_1 + (b_0+b_1)x_2 + b_3x_4 + (b_3+1)x_5$.
For a general choice of $a_0, a_3, a_4, b_0, b_1, b_3$ the resulting curve C is reducible, it is the union of two lines L_1 and L_2 which are fibers of the scroll, and a rational normal curve C_5. The ideals are given as follows:

$$\begin{aligned} I_{L_1} &= (x_0, x_1, x_3, x_4), \\ I_{L_2} &= (x_0+x_1, x_1+x_2, x_3+x_4, x_4+x_5), \\ I_{C_5} &= I_S + (l_1x_0 + l_3x_3, l_1x_1 + l_3x_4, l_1x_2 + l_3x_5, l'_1x_2 + l'_3x_5). \end{aligned}$$

The curve has 4 singular points, two of those are the intersection points between the fibre L_1 and the curve C_5, the other two are the intersection points between L_2 and C_5.
The aim is now to find 4 linearly independent syzygies of I_C among those which are not syzygies of I_S that are syzygies of scrolls of three-dimensional rational normal scrolls that contain C.
The first two syzygies, which are syzygies of the scroll V_1 listed below, come naturally from the reduction of the syzygy matrix (♡).
We give the conclusion: The space of first syzygies of I_C is spanned by the first syzygies of I_S and the first syzygies of I_{V_1} and I_{V_2}, where V_1 and V_2 are three-dimensional scrolls which ideals are given as follows:

Ideal of the $g^1_3(C)$-scroll	Generating matrix
$I_{V_1} = (Q_1, Q_2, q_2 + q_3 + q_4 + q_5)$	$M_{V_1} = \begin{pmatrix} l_3 & x_0+x_1 & x_1+x_2 \\ -l_1 & x_3+x_4 & x_4+x_5 \end{pmatrix}$
$I_{V_2} = (Q_1 + (a_0+1)q_1 + a_3q_2 + (a_3+a_4)q_3,$ $Q_2 - a_0q_1 + a_3q_4 + (a_3+a_4)q_5,$ $-q_2 - q_3)$	$M_{V_2} = \begin{pmatrix} l_1+l'_1 & x_3+x_4 & x_4+x_5 \\ -l_3 & x_0 & x_1 \end{pmatrix}$

Note that for general choices of the a_i and b_i, more precisely for $l_1 \neq \pm l_3$, $l_1 + l'_1 \neq \pm l_3$ and $l_1, l_3 \neq \pm(x_3 + x_4)$, both scrolls V_1 and V_2 are smooth and irreducible.

In Chapter 5 we had already seen that $h^0(\mathcal{I}_{S\cup V}(2)) = 1$ for any $g^1_3(C)$-scroll V that does not contain S. We also found a quadric of rank 4 which vertex line intersects the curve in two points and that contains both S and a given V. We will now give the quadrics which each generate $H^0(\mathcal{I}_S(2)) \cap H^0(\mathcal{I}_{V_i}(2))$, $i = 1, 2$, explicitly, both are of rank 4:

- $H^0(\mathcal{I}_{S\cup V_1}(2)) = \langle q_2 + q_3 + q_4 + q_5 \rangle$,
- $H^0(\mathcal{I}_{S\cup V_2}(2)) = \langle q_2 + q_3 \rangle$.

7.2.3 A family of curves in the case $l_1 = l_3$ and $l_2 = l_4$

For comparison we give a family of reducible curves analogous to the family of curves in Section 7.2.2, where $l_1 = l_3$ and $l_2 = l_4$.
Let

$$
\begin{aligned}
l_1 &= l_3 = a_0x_0 + a_1x_1 + (a_0+1)x_3 + (a_1+a_4)x_4,\\
l_2 &= l_4 = b_0x_0 + b_1x_1 + (b_0+b_3)x_3 + (b_1+1)x_4,
\end{aligned}
$$

where $a_0, a_1, a_4, b_0, b_1, b_3 \in k$.
Set $l_1' := a_0x_1 + a_1x_2 + (a_0+1)x_4 + (a_1+a_4)x_5$
and $l_2' := b_0x_1 + b_1x_2 + (b_0+b_3)x_4 + (b_1+1)x_5$.
For general $a_0, a_1, a_4, b_0, b_1, b_3$ the resulting curve C is reducible, it is the union of a line L, which is a fiber of the scroll S, a conic Y and a rational curve C_4 of degree 4 which is a hyperplane section of S. The ideals are as follows:

$$
\begin{aligned}
I_L &= (x_0, x_1, x_3, x_4),\\
I_Y &= (x_0+x_3, x_1+x_4, x_2+x_5, x_3x_5 - x_4^2),\\
I_{C_4} &= I_S + (l_1 + l_2').
\end{aligned}
$$

The curve C has 4 singular points:
The intersection point of the line L with the conic Y, the intersection point of the line L with the curve C_4 and the two intersection points of the conic with the curve C_4.
Analogous to the examples in the previous section we want to find two three-dimensional scrolls V_1 and V_2 containing C such that the first syzygies of the ideals I_{V_1} and I_{V_2} together with the first syzygies of I_S generate the vector space of the first syzygies of I_C. The first two syzygies, which are syzygies of the scroll V_1, come naturally from the reduction of the syzygy matrix (♡).
The result is the following:

Ideal of the $g^1_3(C)$-scroll	Generating matrix
$I_{V_1} = (Q_1, Q_2, q_1 + q_3 - q_4 + q_6)$	$M_{V_1} = \begin{pmatrix} l_2 & x_0+x_3 & x_1+x_4 \\ -l_1 & x_1+x_4 & x_2+x_5 \end{pmatrix}$
$I_{V_2} = (Q_1 + b_1q_1 + b_1q_3$ $-(b_1+1)q_4 + (b_1+1)q_6,$ $Q_2 - b_0q_1 - b_0q_3$ $+(b_0+b_3-1)q_4 + q_5$ $-(b_0+b_3)q_6,$ $q_2)$	$M_{V_2} = \begin{pmatrix} l_1 + l_2' - b_3x_4 - x_5 & -x_3 & -x_4 \\ l_1 + l_2' + b_3x_1 + x_2 & x_0 & x_1 \end{pmatrix}$

For general $a_0, a_1, a_4, b_0, b_1, b_3$, more precisely for $l_1 \neq \pm l_2$, $l_1 \neq \pm(x_1 + x_4)$ and $l_2 \neq \pm(x_0 + x_3)$, the scroll V_1 has one singular point $V(x_0, x_1, x_3, x_4, x_2 + x_5)$, but is irreducible, and the scroll V_2 is smooth.

7.3 Curves C on a two-dimensional scroll of type $(3,1)$

Let now C be a curve of genus 2 and degree 7 in $\mathbf{P}^5$ on a two-dimensional scroll of type $(3,1)$.
After possibly a coordinate change the ideal I_S is generated by the (2×2)-minors of the following matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 \\ x_1 & x_2 & x_3 & x_5 \end{pmatrix}.$$

By the results in Section 3.5 we can write the ideal of C in the following way:

$$I_C = I_S + (l_1x_0 + l_2x_1 + l_3x_2 + l_4x_4, l_1x_1 + l_2x_2 + l_3x_3 + l_4x_5),$$

where $l_1, \ldots, l_4$ are linear forms in $k[x_0, x_1, x_2, x_3, x_4, x_5]$. For general l_i the curve C is smooth.
We put the generators of I_C in the following order:

$$\begin{aligned}
Q_1 &= l_1x_0 + l_2x_1 + l_3x_2 + l_4x_4, \\
Q_2 &= l_1x_1 + l_2x_2 + l_3x_3 + l_4x_5, \\
q_1 &= x_0x_2 - x_1^2, \\
q_2 &= x_0x_3 - x_1x_2, \\
q_3 &= x_0x_5 - x_1x_4, \\
q_4 &= x_1x_3 - x_2^2, \\
q_5 &= x_1x_5 - x_2x_4, \\
q_6 &= x_2x_5 - x_3x_4.
\end{aligned}$$

As we have seen before, the vector space of first syzygies of I_C is 12-dimensional. We will write a basis for this vector space as a matrix, where each column represents a first syzygy. The first 8 syzygies form a basis for the space of first syzygies of I_S:

$$(\clubsuit) \quad \left(\begin{array}{c|cccccccccccc}
Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_3 & -x_5 \\
Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & x_4 \\
q_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 & 0 & 0 & -l_2 & l_1 & 0 & 0 \\
q_2 & -x_1 & -x_2 & 0 & 0 & x_4 & x_5 & 0 & 0 & -l_3 & 0 & l_1 & 0 \\
q_3 & 0 & 0 & -x_1 & -x_2 & -x_2 & -x_3 & 0 & 0 & -l_4 & 0 & 0 & l_1 \\
q_4 & x_0 & x_1 & 0 & 0 & 0 & 0 & x_4 & x_5 & 0 & -l_3 & l_2 & 0 \\
q_5 & 0 & 0 & x_0 & x_1 & 0 & 0 & -x_2 & -x_3 & 0 & -l_4 & 0 & l_2 \\
q_6 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_1 & x_2 & 0 & 0 & -l_4 & l_3
\end{array}\right).$$

7.3.1 A smooth example

Let

$$\begin{aligned} l_1 &= x_0, \\ l_2 &= x_2, \\ l_3 &= -x_3, \\ l_4 &= x_5. \end{aligned}$$

The resulting curve C is smooth, and the first syzygies of I_C are generated by the first syzygies of I_S and the first syzygies of I_{V_1} and I_{V_2} where V_1 and V_2 are $g^1_3(C)$-scrolls which ideals are given as follows:

Ideal of the $g^1_3(C)$-scroll	Generating matrix
$I_{V_1} = (Q_1 + q_1 - q_6, Q_2 + q_1, -q_2 + q_3 - q_4 + q_5)$	$M_{V_1} = \begin{pmatrix} x_0 - x_1 + x_2 & x_2 - x_4 & x_3 - x_5 \\ x_3 + x_5 & x_0 + x_1 & x_1 + x_2 \end{pmatrix}$
$I_{V_2} = (Q_1 + q_2 + q_6, Q_2 + q_4, -q_2 - q_3)$	$M_{V_2} = \begin{pmatrix} x_0 + x_3 & x_2 + x_4 & x_3 + x_5 \\ x_3 - x_5 & x_0 & x_1 \end{pmatrix}$

Analogous to the families of curves in Sections 7.2.2 and 7.2.3 we will give families of reducible curves in the following, one family for the case $l_1 = l_2$ and $l_3 = l_4$ and one family for the case $l_1 = l_3$ and $l_2 = l_4$.

7.3.2 A family of curves in the case $l_1 = l_2$ and $l_3 = l_4$

Let

$$\begin{aligned} l_1 &= l_2 = a_0x_0 + (a_0 + 1)x_1 + a_2x_2 + (a_2 + a_4)x_4, \\ l_3 &= l_4 = b_0x_0 + (b_0 + b_1)x_1 + b_2x_2 + (b_2 + 1)x_4. \end{aligned}$$

The resulting curve is reducible and singular, it is the union of the line $V(x_0, x_1, x_2, x_4)$ and an elliptic normal curve C_6, the singular points being the intersection points of C_6 and the line.
Set $l'_1 := a_0x_1 + (a_0 + 1)x_2 + a_2x_3 + (a_2 + a_4)x_5$ and
$l'_3 := b_0x_1 + (b_0 + b_1)x_2 + b_2x_3 + (b_2 + 1)x_5$.
The first two syzygies, which are syzygies of I_{V_1} as described below, come naturally from the reduction of the syzygy matrix (♣).
The first syzygies of I_C are generated by the first syzygies of I_S and the first syzygies of I_{V_1} and I_{V_2} where V_1 and V_2 are three-dimensional scrolls containing C which ideals are given as follows:

Ideal of the $g^1_3(C)$-scroll	Generating matrix
$I_{V_1} = (Q_1, Q_2, q_2 + q_3 + q_4 + q_5)$	$M_{V_1} = \begin{pmatrix} l_3 & x_0 + x_1 & x_1 + x_2 \\ -l_1 & x_2 + x_4 & x_3 + x_5 \end{pmatrix}$
$I_{V_2} = (Q_1 + (a_0 + 1)q_1 + a_2q_2 + (a_2 + a_4)q_3, Q_2 - a_0q_1 + a_2q_4 + (a_2 + a_4)q_5, -q_2 - q_3)$	$M_{V_2} = \begin{pmatrix} l_1 + l'_1 & x_2 + x_4 & x_3 + x_5 \\ -l_3 & x_0 & x_1 \end{pmatrix}$

Notice that for general $a_0, a_1, a_4, b_0, b_1, b_3$, more precisely for $l_1 \neq \pm l_3$, $l_1 + l'_1 \neq \pm l_3$, $l_1 \notin \{\pm(x_1 + x_2), \pm(x_2 + x_4)\}$ and $l_3 \notin \{\pm(x_0 + x_1), \pm(x_1 + x_2)\}$, both scrolls V_1 and V_2 are smooth and irreducible.

7.3.3 A family of curves where $l_1 = l_3$ and $l_2 = l_4$

Let

$$\begin{aligned} l_1 &= l_3 = a_0x_0 + a_1x_1 + (a_0+1)x_2 + (a_1+a_4)x_4, \\ l_2 &= l_4 = b_0x_0 + b_1x_1 + (b_0+b_2)x_2 + (b_1+1)x_4. \end{aligned}$$

The resulting curve is reducible and singular, it is the union of the line $V(x_0, x_1, x_2, x_4)$ and an elliptic normal curve C_6, the singular points being the intersection points of C_6 and the line.
Set $l_1' := a_0x_1 + a_1x_2 + (a_0+1)x_3 + (a_1+a_4)x_5$ and
$l_2' := b_0x_1 + b_1x_2 + (b_0+b_2)x_3 + (b_1+1)x_5$.
In this example we could not find two three-dimensional scrolls such that the first syzygies of their ideals together with the first syzygies of I_S generate the space of first syzygies of I_C. By performing reduction operations on the above syzygy matrix (♣) we found three three-dimensional scrolls V_1, V_2 and V_3 such that the first syzygies of I_C are generated by the first syzygies of I_S and the first syzygies of I_{V_1}, I_{V_2} and I_{V_3}. The ideals and the generating matrices are given as follows:

Ideal of the $g^1_3(C)$-scroll	Generating matrix
$I_{V_1} = (Q_1, Q_2, q_1 + q_3 - q_4 + q_6)$	$M_{V_1} = \begin{pmatrix} l_2 & x_0+x_2 & x_1+x_3 \\ -l_1 & x_1+x_4 & x_2+x_5 \end{pmatrix}$
$I_{V_2} = (Q_1,$ $Q_2 + a_1q_1 + q_2 + (a_1+a_4)q_3$ $-a_1q_4 + (a_1+a_4)q_6,$ $-a_0q_1 - a_0q_3 + (a_0+1)q_4$ $+a_4q_5 - (a_0+1)q_6)$	$M_{V_2} = \begin{pmatrix} l_2 & -l_1 & -l_1' \\ x_0+x_2 & x_1+x_4 & x_2+x_5 \end{pmatrix}$
$I_{V_3} = (Q_1,$ $Q_2 - b_0q_1 - b_0q_3 + (b_0+b_2)q_4$ $+q_5 - (b_0+b_2)q_6,$ $b_1q_1 + b_2q_2 + (b_1+1)q_3$ $-b_1q_4 + (b_1+1)q_6)$	$M_{V_3} = \begin{pmatrix} l_1 & -l_2 & -l_2' \\ x_1+x_4 & x_0+x_2 & x_1+x_3 \end{pmatrix}$

Notice again that for general $a_0, a_1, a_4, b_0, b_1, b_2$, more precisely for $l_1 \neq \pm l_2$, $l_1' \neq -l_2$ and $l_1 \neq -l_2'$, all three scrolls V_1, V_2 and V_3 are smooth and irreducible.

Conclusion: Proof of Theorem 7.4:

For each pair $(e_1, e_2) \in \{(2,2),(3,1)\}$ we have the following:
The condition (♢) in Theorem 7.4 is an open condition in the sense that the negated condition is closed by the Semicontinuity Theorem (cf. [Har77], Chapter III, Theorem 12.8). This together with the facts that the moduli space $\mathcal{M}_2$ of non-singular curves of genus 2 is irreducible and that we found an example of a curve C and a system $|H|$ that embeds C into $\mathbf{P}^5$ as a smooth curve and such that the scrolltype of the $g^1_2(C)$-scroll S is equal to (e_1, e_2) and that satisfies (♢) implies that the condition (♢) is satisfied for a general curve and a general complete linear system $|H|$ of degree 7 on C that embeds the curve as a smooth curve into $\mathbf{P}^5$.

Chapter 8

The degree of $Sec_3(C)$

The aim of this chapter is to find the degree of the third secant variety $\mathrm{Sec}_3(C)$ of a smooth curve C of genus 2 and degree $d \geq 8$ embedded in $\mathbf{P}^{d-2}$. We will present three methods to obtain this degree: The first one, which is presented in Section 8.2.1, uses the presentation of $\mathrm{Sec}_3(C)$ as the union of all $g_3^1(C)$-scrolls.
After that we present in Section 8.2.2 a method that counts the number of all divisors on C of degree 3 that impose at most two conditions on a given linear system of degree d and dimension 4.
Finally, in Section 8.2.3, we introduce Berzolari's formula which computes the number of trisecant lines to a curve of genus g and degree d in $\mathbf{P}^4$ and show for $g = 2$ and $d \geq 8$ the equality of the number this formula yields and the number we obtain from our first two methods.

8.1 Preliminaries

Let C be a non-singular and irreducible curve of genus 2.
For any integer $k \geq 0$ we denote by $\mathrm{Pic}^k(C)$ the set of all line bundles on C of degree k modulo isomorphism. Here we will consider $k = 0$ and $k = 3$.
We define C_3 to be the set of all effective divisors on C of degree 3. Notice that there is a natural isomorphism $C_3 \cong C^3/S_3 = (C \times C \times C)/S_3$, where S_3 denotes the symmetric group on three letters. Thus the dimension of C_3 is equal to 3.
Let $u : C_3 \to \mathrm{Pic}^3(C)$ be the map given by $u(D) := \mathcal{O}_C(D)$ for each $D \in C_3$. All fibers of u are isomorphic to $\mathbf{P}^1$, in this way C_3 is a projective bundle over $\mathrm{Pic}^3(C)$, a fact we will return to later.

Definition 8.1. *The Jacobian variety of C, $\mathrm{Jac}(C)$, is defined as $\mathrm{Pic}^0(C)$.*

Note that by fixing a divisor D_0 of degree 3 we obtain an isomorphism

$$\begin{aligned} \mu : \mathrm{Pic}^0(C) &\to \mathrm{Pic}^3(C), \\ \mathcal{O}_C(D) &\mapsto \mathcal{O}_C(D + D_0). \end{aligned}$$

Hence $\mathrm{Pic}^3(C) \cong \mathrm{Jac}(C)$.
Fixing a point P_0 on C gives an embedding

$$\begin{array}{rcl} \nu : C & \to & \mathrm{Jac}(C), \\ R & \mapsto & [\mathcal{O}_C(R - P_0)]. \end{array}$$

The dimension of $\mathrm{Jac}(C)$ is equal to the genus of C, which is equal to 2. Hence $\mathrm{Jac}(C)$ is an abelian surface with theta divisor Θ which is the image of C under the above map ν. This theta divisor on $\mathrm{Jac}(C)$ is thus isomorphic to C.
For fixed points P and Q on C we define

$$\Theta_{P,Q} := \{[\mathcal{O}_C(P + Q + R)] | R \in C\}.$$

$\Theta_{P,Q}$ is a divisor on $\mathrm{Pic}^3(C)$ and using the above isomorphism μ with $D_0 = P+Q+P_0$, we see that the divisor $\Theta_{P,Q}$ is isomorphic to Θ.
It is this $\Theta_{P,Q}$ we will use later when we consider Θ on $\mathrm{Pic}^3(C)$.

Proposition 8.2. *The divisor Θ has self-intersection $\Theta^2 = 2$.*

Proof. Choose points P, P', Q_1 and Q_2, $Q_1 \neq Q_2$, on C such that $P + P'$ is a divisor in $|K_C|$ and such that $Q_1 + Q_2$ is not a divisor in $|K_C|$. There exists points Q_1' and Q_2' on C such that $Q_1 + Q_1' \in |K_C|$ and $Q_2 + Q_2' \in |K_C|$, and we obtain the following:

$$\begin{array}{rcl} \Theta^2 & = & \Theta_{P,Q_1}.\Theta_{P',Q_2} \\ & = & \#\{[\mathcal{O}_C(Q_1 + Q_2 + R)] | R \in \{Q_1', Q_2'\}\} \\ & = & 2. \end{array}$$

□

Let now C be embedded in $\mathbf{P}^{d-2}$ with a complete linear system $|H|$ of degree $d \geq 8$. The main aim of this section is to compute the degree of the third secant variety of C, which is defined as

$$\mathrm{Sec}_3(C) = \overline{\bigcup_{D \in C_3} \mathrm{span}(D)},$$

where by $\mathrm{span}(D)$ we denote the plane spanned by the three points in the effective divisor D on C.

We will need Chern classes and Todd classes of vector bundles. Recall the definitions from [Ful98], Chapter 3:

Definition 8.3. *For a nonsingular variety X and each $r \in \mathbf{N}$ let $A^r(X)$ denote the group of cycles of codimension r on X modulo rational equivalence and set*

$$A^*(X) := \oplus_{r=0}^{\dim X} A^r(X).$$

For each r and s in $\mathbf{N}$ the intersection product gives a map:

$$A^r(X) \times A^s(X) \to A^{r+s}(X).$$

In this way $A^*(X)$ turns into a group which is called the Chow group of X.

Definition 8.4. *Let E be a vector bundle of rank r on a variety X. The Chern polynomial of E is defined as a formal power series*

$$c_t(E) := \sum_{i=0}^{\infty} c_i(E)t^i.$$

Notice that $c_i(E) = 0$ for $i > \min\{r, \dim(X)\}$.
The Chern roots $\alpha_1, \ldots, \alpha_r$ of E are defined via the formal factorization

$$c_t(E) = \Pi_{i=1}^r (1 + \alpha_i t).$$

Definition 8.5. *Let E be a vector bundle of rank r on a variety X.*
The Chern character of E is defined as

$$\mathrm{ch}(E) = \Pi_{i=1}^r e^{\alpha_i},$$

where $\alpha_1, \ldots, \alpha_r$ are the Chern roots of E. Expanding this product yields the first terms

$$\mathrm{ch}(E) = r + c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)) + \frac{1}{6}(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)) + \cdots .$$

Definition 8.6. *The Todd class of a vector bundle E of rank r on a variety X is defined as*

$$\mathrm{td}(E) = \Pi_{i=1}^r \frac{\alpha_i}{1 - e^{-\alpha_i}},$$

where $\alpha_1, \ldots, \alpha_r$ are the Chern roots of E. Expanding this product yields the first terms:

$$\mathrm{td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)) + \frac{1}{24}c_1(E)c_2(E) + \cdots .$$

If Y is a variety, then by $\mathrm{td}(Y)$ we denote $\mathrm{td}(T_Y)$, the Todd class of the tangent bundle.

We will need Todd classes only in the cases when the dimension of X is equal to 1 or 2, i.e. in these cases we have

$$\mathrm{td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)).$$

Definition 8.7. *([Ful98], §1.4)*
Let $f : X \to Y$ be a proper morphism of varieties. For any subvariety $V \subseteq X$, the image $W := f(V)$ is a closed subvariety of Y. If W has the same dimension as V, then the induced embedding $K(W) \hookrightarrow K(V)$ is a finite field extension. Now set

$$\deg(V/W) := \begin{cases} [K(V) : K(W)] & \textit{if } \dim(W) = \dim(V) \\ 0 & \textit{if } \dim(W) < \dim(V) \end{cases}$$

Then the pushforward of the class of V is defined to be

$$f_*([V]) := \deg(V/W)[W].$$

Proposition 8.8. *(The projection formula, cf. [Ful98], Prop. 2.5(c)) Let $f : X \to Y$ be a proper morphism of non-singular varieties, $\alpha \in A^k(X)$ and $\beta \in A^l(Y)$. Then*

$$f_*(\alpha . f^*\beta) = (f_*\alpha).\beta$$

in $A^*(Y)$.

In this chapter we need some Todd classes which we will find in Lemma 8.9.
Consider now the projections

$$\begin{array}{ccc} & C \times \operatorname{Pic}^3(C) & \\ {}^{p}\swarrow & & \searrow^{q} \\ C & & \operatorname{Pic}^3(C) \end{array}$$

and

$$\begin{array}{ccc} & C \times C_3 & \\ {}^{\rho}\swarrow & & \searrow^{\pi} \\ C & & C_3 \end{array} ,$$

let P be a point on C such that $2P \in |K_C|$ and set $F := \rho^*(P)$, $f := p^*(P) = (1_C \times u)_*(F)$.
In the rest of this chapter we will use the notation P, f and F both as varieties and as classes.

Lemma 8.9. *We have the following Todd classes:*

(1) $\operatorname{td}(\operatorname{Pic}^3(C)) = 1$.

(2) $\operatorname{td}(C) = 1 - P$.

(3) $\operatorname{td}(C \times \operatorname{Pic}^3(C)) = 1 - f$.

Proof. (1) Since $\operatorname{Pic}^3(C) \cong \operatorname{Jac}(C)$ is an abelian variety, we have $K_{\operatorname{Pic}^3(C)} = 0$ and thus also $c_1(T_{\operatorname{Pic}^3(C)}) = 0$.

(2) $\operatorname{td}(C) = 1 + \frac{1}{2}c_1(T_C) = 1 - \frac{1}{2}K_C = 1 - P$.

(3) $\operatorname{td}(C \times \operatorname{Pic}^3(C)) = \operatorname{td}(p^*(C)).\operatorname{td}(q^* \operatorname{Pic}^3(C)) = 1 - f$.

□

Let $\Delta \subseteq C \times C_3$ be the universal divisor, i.e. $\Delta|_{C\times\{D\}} \cong D$ for all $D \in C_3$, or writing this in term of incidences,

$$\Delta = \{(R, D) \in C \times C_3 | R \in D\}.$$

For each point $Q \in C$ there is a divisor X_Q on C_3, namely given by

$$X_Q = \{Q + D' | D' \in C_2\}.$$

Now we are able to define a line bundle $\mathcal{L}$ on $C \times \operatorname{Pic}^3(C)$ which turns out to be a Poincaré line bundle. Our first method for computing the degree of $\operatorname{Sec}_3(C)$ uses the identification of $\operatorname{Sec}_3(C)$ as a degeneracy locus of a map of vector bundles involving a Poincaré line bundle of degree 3.

8.1.1 The Poincaré line bundle $\mathcal{L}$

We will first give the definition of a Poincaré line bundle:

Definition 8.10. *A Poincaré line bundle of degree k is a line bundle $\mathcal{L}$ on $C \times \mathrm{Pic}^k(C)$ such that $\mathcal{L}|_{C\times[\mathcal{O}_C(D)]} \cong \mathcal{O}_C(D)$ for all points $[\mathcal{O}_C(D)]$ in $\mathrm{Pic}^k(C)$.*

As mentioned above, in this section we will only consider Poincaré line bundles of degree 3.
Now fix a point Q on C and set

$$\mathcal{L} := (1\times u)_*(\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q))).$$

Notice that $\mathcal{L}$ is a line bundle, since $[\Delta - \pi^*(X_Q)]$ is trivial on each fiber $(1\times u)^*(P \times [\mathcal{O}_C(D_0)]) = P \times [|D_0|]$ of $(1\times u)$ such that neither P_0 nor Q is a basepoint of $|D_0|$. Since all fibers of $(1 \times u)$ are algebraically equivalent, $[\Delta - \pi^*(X_Q)]$ is trivial on all fibers of $(1\times u)$.

Claim 8.11. *The line bundle $\mathcal{L}$ is a Poincaré line bundle which is trivial on $Q \times \mathrm{Pic}^3(C)$, i.e. the following holds:*

(1) $\mathcal{L}|_{C\times[\mathcal{O}_C(D)]} \cong \mathcal{O}_C(D)$ for every point $[\mathcal{O}_C(D)]$ in $\mathrm{Pic}^3(C)$.

(2) $\mathcal{L}|_{Q\times \mathrm{Pic}^3(C)} \cong \mathcal{O}_{C\times\mathrm{Pic}^3(C)}|_{Q\times\mathrm{Pic}^3(C)} \cong \mathcal{O}_{Q\times\mathrm{Pic}^3(C)} \cong \mathcal{O}_{\mathrm{Pic}^3(C)}$.

Proof. In order to verify (1) our strategy is as follows:
We will first show that $\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q))|_{C\times|D|} \cong \mathcal{O}_{C\times|D|}(C\times|D|)$ for all $|D| \in G^1_3(C)$. It is enough to check this isomorphism for basepoint-free $|D| \in G^1_3(C)$, since the set of all basepoint-free systems $|D| \in G^1_3(C)$ is an open and dense set in the set of all systems $|D| \in G^1_3(C)$ and the condition that $\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q))|_{C\times|D|} \cong \mathcal{O}_{C\times|D|}(C \times |D|)$ is a closed condition. Let now $|D| \in G^1_3(C)$ be basepoint-free.
Set $\mathcal{X} := \Delta|_{C\times|D|}$ and $\mathcal{Y} := \pi^*(X_Q)|_{C\times|D|}$. Then $\mathcal{X}$ and $\mathcal{Y}$ are divisors on the ruled surface $C \times |D| \cong C \times \mathbf{P}^1$. Let

$$\begin{array}{ccc} & C\times|D| & \\ {\scriptstyle\kappa}\swarrow & & \searrow{\scriptstyle\lambda} \\ C & & |D| \end{array}$$

be the two projections. The group $\mathrm{Num}(C\times|D|)$ is generated by a a fiber $F_0 = \kappa^*(P_0)$ with $P_0 \in C$ and a section $C_0 := \lambda^*(D_0)$ with $D_0 \in |D|$, satisfying $F_0^2 = 0$, $C_0^2 = 0$ and $C_0.F_0 = 1$.
Now we want to find the linear equivalence classes of $\mathcal{X}$ and $\mathcal{Y}$: We set $[\mathcal{X}] = aC_0 + bF_0$ and $[\mathcal{Y}] = cC_0 + dF_0$.
As sets we can write:

$$\begin{aligned} \mathcal{X} &= \{(R,E) \in C\times|D| \,|\, R \in E\}, \\ \mathcal{Y} &= \{(R,E) \in C\times|D| \,|\, Q \in E\}. \end{aligned}$$

We conclude:

$$
\begin{aligned}
a &= [\mathcal{X}].F_0 = \text{number of elements in } |D - P_0| = 1, \\
c &= [\mathcal{Y}].F_0 = \text{number of elements in } |D - Q| = 1.
\end{aligned}
$$

Moreover, $\mathcal{X}|_{C_0} = \kappa^*(D_0)$ and $\mathcal{Y}|_{C_0} = 0$.
Since on a fiber $F_0 \cong \mathbf{P}^1$ linear equivalence is the same as numerical equivalence, we obtain the linear equivalence classes $[\mathcal{X}] = C_0 + \kappa^*(D)$ and $[\mathcal{Y}] = C_0$, which implies that $\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q))|_{C\times|D|} \cong \mathcal{O}_{C\times|D|}(\kappa^*(D))$.
Consider now the following commutative diagram

$$
\begin{array}{ccc}
C \times |D| & \xrightarrow{\kappa} & C \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \nu} \\
C \times C_3 & \xrightarrow{1\times u} & C \times \mathrm{Pic}^3(C),
\end{array}
$$

where ι is the inclusion and ν is the map $1 \times [\mathcal{O}_C(D)]$.
Finally, we show that $\mathcal{L}|_{C\times[\mathcal{O}_C(D)]} \cong \mathcal{O}_C(D)$ for all $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$.
By the projection formula for line bundles we obtain

$$\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q)) \cong (1 \times u)^*(\mathcal{L})$$

and consequently

$$
\begin{aligned}
\kappa_*\iota^*(\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q))) &= \kappa_*\iota^*(1\times u)^*(\mathcal{L}) \\
&= \kappa_*\kappa^*\nu^*(\mathcal{L}) = \nu^*(\mathcal{L}).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\mathcal{L}|_{C\times[\mathcal{O}_C(D)]} &\cong \nu^*\mathcal{L} \\
&= \kappa_*\iota^*\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q)) \\
&= \kappa_*\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q))|_{C\times|D|} \\
&\cong \kappa_*\mathcal{O}_{C\times|D|}(\kappa^*(D)) \\
&\cong \kappa_*\kappa^*\mathcal{O}_C(D) = \mathcal{O}_C(D).
\end{aligned}
$$

In order to prove (2) it suffices to show that $\mathcal{O}_{C\times C_3}(\Delta - \pi^*(X_Q))|_{Q\times C_3} \cong \mathcal{O}_{C\times C_3}|_{Q\times C_3}$.
This follows from the fact that $\Delta|_{Q\times C_3} = X_Q$. □

Remark 8.12. *We have shown that $\mathcal{L}$ as defined above is a Poincaré line bundle of degree 3 that is trivial on $Q \times \mathrm{Pic}^3(C)$. Moreover, $\mathcal{L}$ is also unique with the property that $\mathcal{L}|_{Q\times\mathrm{Pic}^3(C)} \cong \mathcal{O}_{\mathrm{Pic}^3(C)}$. The uniqueness here comes from the fact that if $\mathcal{L}$ and $\mathcal{L}'$ are two Poincaré line bundles, i.e. $\mathcal{L}|_{C\times[\mathcal{O}_C(D)]} \cong \mathcal{O}_C(D)$ and $\mathcal{L}'|_{C\times[\mathcal{O}_C(D)]} \cong \mathcal{O}_C(D)$, then $\mathcal{L} \otimes (\mathcal{L}')^{-1} = q^*(\mathcal{R})$ for some bundle $\mathcal{R}$ on $\mathrm{Pic}^3(C)$ and $\mathcal{R} \cong q^*(\mathcal{R})|_{Q\times\mathrm{Pic}^3(C)} = \mathcal{L} \otimes (\mathcal{L}')^{-1}|_{Q\times\mathrm{Pic}^3(C)} = \mathcal{O}_{\mathrm{Pic}^3(C)}$.*

We define the following vector bundles on $\mathrm{Pic}^3(C) \cong \mathrm{Jac}(C)$:

$$\begin{aligned}\mathcal{H} &= q_*(\mathcal{L}),\\ \mathcal{G} &= q_*(p^*\mathcal{O}_C(H)\otimes\mathcal{L}^{-1}).\end{aligned}$$

Since the fibre of $\mathcal{H}$ over a point $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ is equal to $H^0(\mathcal{O}_C(D))$, the rank of $\mathcal{H}$ is equal to $h^0(\mathcal{O}_C(D)) = 2$, and since the fibre of $\mathcal{G}$ over a point $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ is equal to $H^0(\mathcal{O}_C(H-D))$, the rank of $\mathcal{G}$ is equal to $h^0(\mathcal{O}_C(H-D)) = d-4$.
We will use these vector bundles $\mathcal{H}$ and $\mathcal{G}$ in Section 8.2 to define a map of vector bundles which degeneracy locus is equal to the third secant variety of C, $\mathrm{Sec}_3(C)$. We will need the Chern classes of $\mathcal{H}$ and $\mathcal{G}$, and for this purpose we need the Chern classes of $\mathcal{L}$, which is our next aim to find.

8.1.2 The Chern classes of the Poincaré line bundle $\mathcal{L}$

Following [ACGH85], Chapter VIII, §2 (pp. 333-336) we will now find the Chern classes of the Poincaré line bundle $\mathcal{L}$ as defined in Section 8.10.
We set

$$c_1(\mathcal{L}) = c^{2,0} + c^{1,1} + c^{0,2},$$

where $c^{i,j}$ is the component of $c_1(\mathcal{L})$ in the (i,j)th term of the Künneth decomposition

$$\begin{aligned}H^2(C\times \mathrm{Pic}^3(C)) &= (H^2(C)\otimes H^0(\mathrm{Pic}^3(C)))\\ &\quad\oplus(H^1(C)\otimes H^1(\mathrm{Pic}^3(C)))\\ &\quad\oplus(H^0(C)\otimes H^2(\mathrm{Pic}^3(C))).\end{aligned}$$

Since $\mathcal{L}|_{Q\times\mathrm{Pic}^3(C)} \cong \mathcal{O}_{\mathrm{Pic}^3(C)}$ we find that $c^{0,2} = 0$, and the fact that $\mathcal{L}|_{C\times[\mathcal{O}_C(D)]} \cong \mathcal{O}_C(D)$ for all $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ implies that $c^{2,0} = 3f$. From [ACGH85], p.335, we find that $c^{1,1} =: \gamma$, with

$$\gamma^2 = -2f.q^*(\Theta), \gamma^3 = f.\gamma = 0.$$

We obtain

$$c_1(\mathcal{L}) = 3f + \gamma$$

and consequently

$$\mathrm{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + 3f + \gamma - f.q^*(\Theta).$$

Soon we are able to compute the Chern classes of the bundles $\mathcal{H}$ and $\mathcal{G}$, before we will do so we need a lemma:

Lemma 8.13. *We have the following pushforwards:*

(1) $q_*(1) = 0.$

(2) $q_*(f) = 1.$

(3) $q_*(\gamma) = 0.$

Proof.

(1) $q_*(1) = q_*([C \times \mathrm{Pic}^3(C)]) = 0$ since $\dim(q(C \times \mathrm{Pic}^3(C))) = \dim(\mathrm{Pic}^3(C)) < \dim(C \times \mathrm{Pic}^3(C))$.

(2) Since $q(f) = \mathrm{Pic}^3(C)$ has the same dimension as f, we have $q_*(f) = a[\mathrm{Pic}^3(C)]$ for some positive integer a. By the projection formula we obtain for every point $[\mathcal{O}_C(D_0)] \in \mathrm{Pic}^3(C)$: $a = q_*(f).[\mathcal{O}_C(D_0)] = q_*(f.q^*[\mathcal{O}_C(D_0)]) = f.q^*[\mathcal{O}_C(D_0)] = [P \times \mathrm{Pic}^3(C)].[C \times \mathcal{O}_C(D_0)] = 1$, where we could use the equality $q_*(f.q^*[\mathcal{O}_C(D_0)]) = f.q^*[\mathcal{O}_C(D_0)]$ since $f.q^*[\mathcal{O}_C(D_0)]$ is 0-dimensional.

(3) Since γ is a divisor on $C \times \mathrm{Pic}^3(C)$, we must have $q_*(\gamma) = a[\mathrm{Pic}^3(C)]$ for some non-negative integer a. By the projection formula we have $a = q_*(\gamma).[\mathcal{O}_C(D_0)] = q_*(\gamma.q^*[\mathcal{O}_C(D_0)]) = \gamma.q^*[\mathcal{O}_C(D_0)] = c_1(\mathcal{L}).q^*[\mathcal{O}_C(D_0)] - q_*(3f) = 3 - 3 = 0$, where again we could use the equality $q_*(\gamma.q^*[\mathcal{O}_C(D_0)]) = \gamma.q^*[\mathcal{O}_C(D_0)]$ since $\gamma.q^*[\mathcal{O}_C(D_0)]$ is 0-dimensional. □

8.1.3 The Chern classes of $\mathcal{H}$

The Chern classes of $\mathcal{H}$ we obtain by the Grothendieck-Riemann-Roch Theorem:

$$\mathrm{ch}(q_*(\mathcal{L})).\,\mathrm{td}(\mathrm{Pic}^3(C)) = q_*(\mathrm{ch}(\mathcal{L}).\,\mathrm{td}(C \times \mathrm{Pic}^3(C))),$$

i.e. by Lemma 8.9 we obtain

$$\begin{aligned} \mathrm{ch}(\mathcal{H}) &= \mathrm{ch}(q_*\mathcal{L}) \\ &= q_*(\mathrm{ch}(\mathcal{L}).(1-f)) = q_*((1+3f+\gamma - f.q^*(\Theta)).(1-f)) \\ &= q_*(1+2f+\gamma - f.q^*(\Theta)). \end{aligned}$$

By Lemma 8.13 and the projection formula we can conclude:

$$\mathrm{ch}(\mathcal{H}) = 2 - q_*(f).\Theta = 2 - \Theta.$$

Consequently we obtain for the Chern polynomial

$$c_t(\mathcal{H}) = e^{-\Theta t}.$$

8.1.4 The Chern classes of $\mathcal{G}$

In order to find the Chern classes of $\mathcal{G}$ we again use the Grothendieck-Riemann-Roch formula:

$$\begin{aligned} &\mathrm{ch}(q_*(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1}))).\,\mathrm{td}(\mathrm{Pic}^3(C)) \\ = \;& q_*(\mathrm{ch}(p^*(\mathcal{O}_C(H) \otimes \mathcal{L}^{-1})).\,\mathrm{td}(C \times \mathrm{Pic}^3(C))). \end{aligned}$$

We obtain by Lemma 8.9, Lemma 8.13 and the projection formula

$$\begin{aligned}
\mathrm{ch}(\mathcal{G}) &= \mathrm{ch}(q_*(p^*\mathcal{O}_C(H)\otimes\mathcal{L}^{-1}))\\
&= q_*(\mathrm{ch}(p^*\mathcal{O}_C(H)\otimes\mathcal{L}^{-1}).(1-f))\\
&= q_*(p^*(\mathrm{ch}(\mathcal{O}_C(H))).\,\mathrm{ch}(\mathcal{L}^{-1}).(1-f))\\
&= q_*((1+p^*(H)).(1-3f-\gamma-f.q^*(\Theta)).(1-f))\\
&= q_*((1+df).(1-4f-\gamma-f.q^*(\Theta)))\\
&= q_*(1+(d-4)f-\gamma-f.q^*(\Theta))\\
&= d-4-\Theta.
\end{aligned}$$

This yields for the Chern polynomial:

$$c_t(\mathcal{G}) = e^{-\Theta t}.$$

8.1.5 C_3 as a $\mathbf{P}^1$-bundle over $\mathrm{Jac}(C)$

A fiber of u over a point $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ is equal to the linear system $|D| \cong \mathbf{P}^1$. In this way C_3 is a projective line bundle over $\mathrm{Pic}^3(C)$. That is, $C_3 = \mathbf{P}(E)$, where E is a bundle of rank 2 over $\mathrm{Pic}^3(C)$. If we tensorize a vector bundle E which satisfies $\mathbf{P}(E) \cong C_3$ with a line bundle L, then we still have $\mathbf{P}(E\otimes L) \cong C_3$, but, on the other hand, E is also unique up to isomorphism and tensorizing with a line bundle.
By Proposition (2.1)(i) in Chapter VII, §2 in [ACGH85] we may take $E = q_*(\mathcal{L})$. With $E = q_*\mathcal{L}$ we have in addition by Proposition (2.1)(ii), Loc. cit., the identification $\mathcal{O}_{\mathbf{P}(E)}(1) = \mathcal{O}_{C_3}(X_Q)$.
Consequently we can also take $E = u_*(\mathcal{O}_{C_3}(X_Q))$.
Set $x := c_1(\mathcal{O}_{C_3}(X_Q)) = c_1(\mathcal{O}_E(1))$, let Θ be the Theta divisor on $\mathrm{Pic}^3(C)$, as described as $\Theta_{P,Q}$ in Chapter 8.1, and set $\theta := u^*(\Theta) \subseteq C_3$.
By [Fulton], Ex. 4.3.3, or using the fact that E can be chosen as $q_*(\mathcal{L})$ and the computations of $\mathrm{ch}(q_*\mathcal{L})$ in Section 8.1.3, we have:

$$\begin{aligned}
c_1(u^*(E)) &= -\theta,\\
c_2(u^*(E)) &= \frac{1}{2}\theta^2.
\end{aligned}$$

Remark 8.14. *From Remark 3.2.4 in [Ful98] (cf. also [Har77], Appendix A) we obtain the relation*

$$x^2 + c_1(u^*E) + c_2(u^*E) = 0.$$

Using our above remark this turns into

$$x^2 - u^*(\Theta) + \frac{1}{2}u^*(\Theta^2) = 0.$$

Now we come to our main aim of this chapter, namely the computation of the degree of the third secant variety of C, $\mathrm{Sec}_3(C)$.

8.2 Computations of the degree of $\mathrm{Sec}_3(C)$

Our aim in the following sections is to compute the degree of the third secant variety of a curve C of genus 2 and degree $d \geq 8$, embedded in $\mathbf{P}^{d-2}$,

$$\mathrm{Sec}_3(C) = \overline{\bigcup_{D \in C_3} \mathrm{span}(D)},$$

using three different methods.
The expected dimension of $\mathrm{Sec}_3(C)$ where C is a curve in $\mathbf{P}^{d-2}$ is equal to $\min\{5, d-2\}$, i.e. for $d = 5$, $d = 6$ and $d = 7$ the third secant variety is equal to the ambient space $\mathbf{P}^{d-2}$, consequently the degree of $\mathrm{Sec}_3(C)$ is equal to 1 in these cases. Hence we will now restrict ourselves to the cases when the degree of C is bigger or equal to 8.
The three different methods will be:

(I) Set $E := \mathcal{G} \boxtimes \mathcal{O}_{\mathbf{P}^{d-2}}(-1)$ and $F := \mathcal{H}^* \boxtimes \mathcal{O}_{\mathbf{P}^{d-2}}$. These are two vector bundles on $\mathrm{Pic}^3(C) \times \mathbf{P}^{d-2}$ of rank $d-4$ and 2 respectively.
There is a map $\Phi : E \to F$ which is induced by the multiplication of fibers:

$$H^0(\mathcal{O}_C(H - D)) \otimes H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_C(H)).$$

Set

$$X_1 := X_1(\Phi) := \{x \in \mathrm{Pic}^3(C) \times \mathbf{P}^{d-2} | \, \mathrm{rk}(\Phi_x) \leq 1\}$$

Consider the two projections

$$\begin{array}{ccccc} & & \mathrm{Pic}^3(C) \times \mathbf{P}^{d-2} & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathrm{Pic}^3(C) & & & & \mathbf{P}^{d-2} \end{array}$$

and their restrictions to X_1:

$$\begin{array}{ccccc} & & X_1 & & \\ & \swarrow^{p_1|_{X_1}} & & \searrow^{p_2|_{X_2}} & \\ \mathrm{Pic}^3(C) & & & & \mathbf{P}^{d-2}. \end{array}$$

We have the following:

(i) Over every point $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ the fibre of $p_1|X_1$ is the 3-dimensional rational normal scroll $V_{|D|} \subseteq [\mathcal{O}_C(D)] \times \mathbf{P}^{d-2} \cong \mathbf{P}^{d-2}$ associated to $|D|$.

(ii) The image of such a fibre under the projection p_2 is thus the rational normal scroll $V_{|D|}$ in $\mathbf{P}^{d-2}$.

(iii) Thus $p_2(X_1)$ is the union of all $g^1_3(C)$-scrolls $V_{|D|}$ in $\mathbf{P}^{d-2}$ which again is equal to $\mathrm{Sec}_3(C)$.

Set x_1 to be the class of X_1. From the above we have $(p_2)_*(x_1) = [\mathrm{Sec}_3(C)]$. Let $h' \subseteq \mathbf{P}^{d-2}$ be a hyperplane class, and set $h := (p_2)^*(h') \subseteq \mathrm{Pic}^3(C) \times \mathbf{P}^{d-2}$.
Since $\mathrm{Sec}_3(C) \subseteq \mathbf{P}^{d-2}$ has dimension 5, we obtain the degree of $\mathrm{Sec}_3(C)$ by

intersecting with $(h')^5$.
Now we have

$$\begin{aligned} \deg(\mathrm{Sec}_3(C)) &= [Sec^3(C)].(h')^5 = (p_2)_*(x_1).(h')^5 \\ &= (p_2)_*(x_1.p_2^*(h')^5) = (p_2)_*(x_1.h^5) = x_1.h^5. \end{aligned}$$

That is, with this approach we have to find the class x_1.

(II) A second method is given by counting the number of all divisors of degree 3 on C that impose at most two conditions on a linear system on C of degree d and dimension 4. A very ample linear system of degree d and dimension 4 on C gives an embedding of C into $\mathbf{P}^4$ as a curve of degree d. The divisors of degree 3 that impose dependent conditions on such a system give a trisecant line to the curve in $\mathbf{P}^4$. Thus we see that for very ample linear systems of degree d and dimension 4 this method is equivalent to the third method.

(III) The number of trisecant lines to a smooth curve of genus g and degree d in $\mathbf{P}^4$ is well-known and given by Berzolari's formula. We show that for $g = 2$ and $d \geq 8$ this number is equal to the degree of $\mathrm{Sec}_3(C)$ we found with the first method and thus justify that all the three methods yield the same number.

8.2.1 First method

Here the aim is to find the class x_1 of $X_1(\Phi)$.
Since $X_1(\Phi)$ has expected dimension $5 = \dim(\mathrm{Pic}^3(C) \times \mathbf{P}^{d-2}) - (d-4-1)(2-1)$, by Porteous' formula (cf. [ACGH], Chapter II, (4.2)) we obtain the following:

$$\begin{aligned} x_1 &= \Delta_{1,d-5}(c_t(F-E)) \\ &= \det \underbrace{\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{d-6} & c_{d-5} \\ 1 & c_1 & c_2 & \cdots & c_{d-7} & c_{d-6} \\ 0 & 1 & c_1 & \cdots & c_{d-8} & c_{d-7} \\ & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 1 & c_1 \end{pmatrix}}_{=:\mathcal{A}_{d-5}}, \end{aligned}$$

where $c_i := c_i(F-E)$ and $c_i(F-E)$ is defined via $c_t(F-E) := \frac{c_t(F)}{c_t(E)}$.
In Sections 8.1.3 and 8.1.4 we found

$$c_t(\mathcal{H}) = c_t(\mathcal{G}) = e^{-\Theta t}.$$

We thus obtain

$$c_t(F) = c_t(p_1^*\mathcal{H}^*) = c_{-t}(p_1^*\mathcal{H}) = e^{p_1^*\Theta t}.$$

We compute $c_t(E)$:
Let α_i be the Chern roots of $\mathcal{G}$ and set $\beta_i := p_1^*(\alpha_i)$. Then

$$c_t(\mathcal{G}) = \Pi_{i=1}^{d-4}(1 + \alpha_i t),$$

and we obtain:

$$\begin{aligned} c_t(E) &= \Pi_{i=1}^{d-4}(1+(\beta_i - h)t) \\ &= \Pi_{i=1}^{d-4}(1-ht)(1+\beta_i\frac{t}{1-ht}) \\ &= (1-ht)^{d-4}\Pi_{i=1}^{d-4}(1+\beta_i\frac{t}{1-ht}) \\ &= (1-ht)^{d-4}c_{\frac{t}{1-ht}}(p_1^*\mathcal{G}) \\ &= (1-ht)^{d-4}e^{\frac{-p_1^*\Theta t}{1-ht}}. \end{aligned}$$

In the following we will identify Θ with $p_1^*(\Theta)$, it will be clear from the context if we mean Θ on $\mathrm{Pic}^3(C)$ or Θ on $\mathrm{Pic}^3(C)\times\mathbf{P}^{d-2}$.
We conclude now:

$$\begin{aligned} c_t(F-E) &= e^{\Theta t}(1-ht)^{4-d}e^{\frac{\Theta t}{1-ht}} \\ &= (1-ht)^{4-d}e^{\frac{2\Theta t-\Theta.ht^2}{1-ht}} \\ &= (1-ht)^{4-d}\sum_{j=0}^{\infty}\frac{1}{j!}(2\Theta t-\Theta.ht^2)^j(1-ht)^{-j} \\ &= \sum_{j=0}^{\infty}(1-ht)^{4-d-j}\frac{1}{j!}(2\Theta t-\Theta.ht^2)^j. \end{aligned}$$

Since $\Theta^3 = 0$, we only get some contribution from $j = 0, 1, 2$, and thus obtain the following:

$$\begin{aligned} c_t(F-E) &= (1-ht)^{4-d}+(1-ht)^{3-d}(2\Theta t-\Theta.ht^2) \\ &+ \frac{1}{2}(1-ht)^{2-d}(4\Theta^2t^2-4\Theta^2.ht^3+\Theta^2.h^2t^4) \\ &= (1-ht)^{2-d}((1-ht)^2+(1-ht)(2\Theta t-\Theta.ht^2)+2\Theta^2t^2-2\Theta^2.ht^3) \\ &+ \frac{1}{2}(1-ht)^{2-d}\Theta^2.h^2t^4 \\ &= \left(\frac{1}{1-ht}\right)^{d-2}(1+(2\Theta-2h)t+(2\Theta^2-3\Theta.h+h^2)t^2) \\ &+ \left(\frac{1}{1-ht}\right)^{d-2}((\Theta.h^2-2\Theta^2.h)t^3+\frac{1}{2}\Theta^2.h^2t^4) \\ &= \sum_{k=0}^{\infty}\binom{d-2+k-1}{k}h^kt^k(1+(2\Theta-2h)t+(2\Theta^2-3\Theta.h+h^2)t^2) \\ &+ \sum_{k=0}^{\infty}\binom{d-2+k-1}{k}h^kt^k((\Theta.h^2-2\Theta^2.h)t^3+\frac{1}{2}\Theta^2.h^2t^4) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\binom{d-2+k-1}{k}h^k t^k \\
&+ \sum_{k=0}^{\infty}\binom{d-2+k-1}{k}(2\Theta.h^k - 2h^{k+1})t^{k+1} \\
&+ \sum_{k=0}^{\infty}\binom{d-2+k-1}{k}(2\Theta^2.h^k - 3\Theta.h^{k+1} + h^{k+2})t^{k+2} \\
&+ \sum_{k=0}^{\infty}\binom{d-2+k-1}{k}(\Theta.h^{k+2} - 2\Theta^2.h^{k+1})t^{k+3} \\
&+ \sum_{k=0}^{\infty}\frac{1}{2}\binom{d-2+k-1}{k}\Theta^2.h^{k+2}t^{k+4}.
\end{aligned}$$

This implies that

$$\begin{aligned}
c_i(F-E) &= \left(\binom{d-3+i}{i} - 2\binom{d-4+i}{i-1} + \binom{d-5+i}{i-2}\right)h^i \\
&+ \left(2\binom{d-4+i}{i-1} - 3\binom{d-5+i}{i-2} + \binom{d-6+i}{i-3}\right)\Theta.h^{i-1} \\
&+ \left(2\binom{d-5+i}{i-2} - 2\binom{d-6+i}{i-3} + \frac{1}{2}\binom{d-7+i}{i-4}\right)\Theta^2.h^{i-2} \\
&= \binom{d-5+i}{i}h^i + \left(\binom{d-5+i}{i-1} + \binom{d-6+i}{i-1}\right)\Theta.h^{i-1} \\
&+ \left(2\binom{d-6+i}{i-2} + \frac{1}{2}\binom{d-7+i}{i-4}\right)\Theta^2.h^{i-2}.
\end{aligned}$$

Now the last step in the computation of x_1 is to find the determinant of the matrix $\mathcal{A}_{d-5}$:

Proposition 8.15. *For $d \geq 8$ the determinant of the matrix*

$$\mathcal{A}_{d-5} = \begin{pmatrix}
c_1 & c_2 & c_3 & \cdots & c_{d-6} & c_{d-5} \\
1 & c_1 & c_2 & \cdots & c_{d-7} & c_{d-6} \\
0 & 1 & c_1 & \cdots & c_{d-8} & c_{d-7} \\
 & & & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_1 & c_2 \\
0 & 0 & 0 & \cdots & 1 & c_1
\end{pmatrix}$$

is equal to

$$\begin{aligned}
\mathcal{D}_{d-5} &= \left(\frac{1}{2}\binom{d-2}{3} - (d-4)\right)\Theta^2.h^{d-7} \\
&+ \left(\binom{d-3}{2} - 1\right)\Theta.h^{d-6} + (d-4)h^{d-5}.
\end{aligned}$$

Proof. For fixed d and $n = 1, \dots, d-5$, $k = 2, \dots, d-6$, let

$$d_n := \det \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ 1 & c_1 & c_2 & \cdots & c_{n-1} \\ & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & \cdots & 1 & c_1 \end{pmatrix}$$

and

$$b_{n,k} := \det \begin{pmatrix} c_k & c_{k+1} & c_{k+2} & \cdots & c_{n-1} & c_n \\ 1 & c_1 & c_2 & \cdots & c_{n-(k+1)} & c_{n-k} \\ 0 & 1 & c_1 & \cdots & c_{n-(k+2)} & c_{n-(k+1)} \\ & & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 1 & c_1 \end{pmatrix}.$$

By expansion with respect to the first column we have for each n, k:

$$d_n = c_1 d_{n-1} - b_{n,2}$$

and

$$b_{n,k} = c_k d_{n-k} - b_{n,k+1}.$$

This gives us by induction:

$$d_n = \sum_{i=1}^{n} (-1)^{i-1} c_i d_{n-i}.$$

Let d be fixed. We had computed that

$$\begin{aligned} c_i &= \binom{d-5+i}{i} h^i \\ &+ \left(\binom{d-5+i}{i-1} + \binom{d-6+i}{i-1} \right) \Theta . h^{i-1} \\ &+ \left(2\binom{d-6+i}{i-2} + \frac{1}{2}\binom{d-7+i}{i-4} \right) \Theta^2 . h^{i-2}. \end{aligned}$$

So we can compute d_i for low i:

$$
\begin{aligned}
d_0 &:= 1,\\
d_1 &= c_1 = (d-4)h + 2\Theta,\\
d_2 &= c_1^2 - c_2 = \binom{d-4}{2}h^2 + (2d-9)\Theta.h + 2\Theta^2,\\
d_3 &= c_1^3 - 2c_1c_2 + c_3\\
&= \binom{d-4}{3}h^3 + (d-5)^2\Theta.h^2 + (2d-10)\Theta^2.h.
\end{aligned}
$$

This leads us to the following proposition:

Proposition 8.16. *For $n \geq 3$ we have*

$$
\begin{aligned}
d_n &= \binom{d-4}{n}h^n + \left(\binom{d-3}{n} - \binom{d-5}{n}\right)\Theta.h^{n-1}\\
&+ \left(\frac{1}{2}\binom{d-2}{n} - \binom{d-4}{n} + \frac{1}{2}\binom{d-6}{n}\right)\Theta^2.h^{n-2}.
\end{aligned}
$$

Proof. By induction over n:

$$
\begin{aligned}
d_n &= \sum_{i=1}^{n}(-1)^{i-1}c_i d_{n-i}\\
&= \sum_{i=1}^{n}(-1)^{i-1}\left(\binom{d-5+i}{i}h^i + \left(\binom{d-6+i}{i-1} + \binom{d-5+i}{i-1}\right)\Theta.h^{i-1}\right.\\
&\qquad \left. + \left(2\binom{d-6+i}{i-2} + \frac{1}{2}\binom{d-7+i}{i-4}\right)\Theta^2.h^{i-2}\right)\\
&\qquad .\left(\binom{d-4}{n-i}h^{n-i} + \left(\binom{d-3}{n-i} - \binom{d-5}{n-i}\right)\Theta.h^{n-i-1}\right.\\
&\qquad \left. + \left(\frac{1}{2}\binom{d-2}{n-i} - \binom{d-4}{n-i} + \frac{1}{2}\binom{d-6}{n-i}\right)\Theta^2.h^{n-i-2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i}\binom{d-4}{n-i}h^n \\
+ & \sum_{i=1}^{n}(-1)^{i-1}\left(\binom{d-5+i}{i}\binom{d-3}{n-i}-\binom{d-5+i}{i}\binom{d-5}{n-i}\right. \\
& \left. +\binom{d-4}{n-i}\binom{d-6+i}{i-1}+\binom{d-4}{n-i}\binom{d-5+i}{i-1}\right)\Theta.h^{n-1} \\
+ & \sum_{i=1}^{n}(-1)^{i-1}\left(\frac{1}{2}\binom{d-5+i}{i}\binom{d-2}{n-i}-\binom{d-5+i}{i}\binom{d-4}{n-i}\right. \\
& +\frac{1}{2}\binom{d-5+i}{i}\binom{d-6}{n-i}+\binom{d-6+i}{i-1}\binom{d-3}{n-i} \\
& -\binom{d-6+i}{i-1}\binom{d-5}{n-i}+\binom{d-5+i}{i-1}\binom{d-3}{n-i} \\
& -\binom{d-5+i}{i-1}\binom{d-5}{n-i}+2\binom{d-6+i}{i-2}\binom{d-4}{n-i} \\
& \left.+\frac{1}{2}\binom{d-7+i}{i-4}\binom{d-4}{n-i}\right)\Theta^2.h^{n-2}.
\end{aligned}
$$

Using the binomial identities

(a) Upper negation: $\binom{-r}{m} = (-1)^m\binom{r+m-1}{m}$ for $r, m \in \mathbf{N}$,

(b) Vandermonde's identity: $\sum_{k=0}^{r}\binom{m}{k}\binom{s}{r-k} = \binom{m+s}{r}$ for $m, r, s \in \mathbf{N}$

we obtain the following:

(1) The coefficient in front of h^n is equal to

$$
\begin{aligned}
& \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i}\binom{d-4}{n-i} = \sum_{i=1}^{n}(-1)^{2i-1}\binom{4-d}{i}\binom{d-4}{n-i} \\
= & -\binom{0}{n}+\binom{d-4}{n} = \binom{d-4}{n}
\end{aligned}
$$

for $n \geq 1$.

(2) The coefficient in front of $\Theta.h^{n-1}$ is equal to

$$
\begin{aligned}
& \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i}\binom{d-3}{n-i} - \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i}\binom{d-5}{n-i} \\
+ & \sum_{i=1}^{n}(-1)^{i-1}\binom{d-6+i}{i-1}\binom{d-4}{n-i} + \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i-1}\binom{d-4}{n-i} \\
= & \sum_{i=1}^{n}(-1)^{2i-1}\binom{4-d}{i}\binom{d-3}{n-i} - \sum_{i=1}^{n}(-1)^{2i-1}\binom{4-d}{i}\binom{d-5}{n-i} \\
+ & \sum_{i=1}^{n}(-1)^{2i-2}\binom{4-d}{i-1}\binom{d-4}{n-i} + \sum_{i=1}^{n}(-1)^{2i-2}\binom{3-d}{i-1}\binom{d-4}{n-i} \\
= & -\binom{1}{n} + \binom{d-3}{n} + \binom{-1}{n} - \binom{d-5}{n} + \sum_{i=0}^{n-1}\binom{4-d}{i}\binom{d-4}{n-i-1} \\
+ & \sum_{i=0}^{n-1}\binom{3-d}{i}\binom{d-4}{n-i-1} \\
= & \binom{d-3}{n} + \binom{-1}{n} - \binom{d-5}{n} + \binom{0}{n-1} + \binom{-1}{n-1} \\
= & \binom{d-3}{n} - \binom{d-5}{n} + (-1)^n + (-1)^{n-1} \\
= & \binom{d-3}{n} - \binom{d-5}{n}
\end{aligned}
$$

for $n \geq 2$.

(3) The coefficient in front of $\Theta^2.h^{n-2}$ is equal to

$$
\begin{aligned}
& \frac{1}{2}\sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i}\binom{d-2}{n-i} - \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i}\binom{d-4}{n-i} \\
+ & \frac{1}{2}\sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i}\binom{d-6}{n-i} + \sum_{i=1}^{n}(-1)^{i-1}\binom{d-6+i}{i-1}\binom{d-3}{n-i} \\
- & \sum_{i=1}^{n}(-1)^{i-1}\binom{d-6+i}{i-1}\binom{d-5}{n-i} + \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i-1}\binom{d-3}{n-i} \\
- & \sum_{i=1}^{n}(-1)^{i-1}\binom{d-5+i}{i-1}\binom{d-5}{n-i} + 2\sum_{i=1}^{n}(-1)^{i-1}\binom{d-6+i}{i-2}\binom{d-4}{n-i} \\
+ & \frac{1}{2}\sum_{i=1}^{n}(-1)^{i-1}\binom{d-7+i}{i-4}\binom{d-4}{n-i}
\end{aligned}
$$

$$\begin{aligned}
= \;& \frac{1}{2}\sum_{i=1}^{n}(-1)^{2i-1}\binom{4-d}{i}\binom{d-2}{n-i} - \sum_{i=1}^{n}(-1)^{2i-1}\binom{4-d}{i}\binom{d-4}{n-i} \\
+ \;& \frac{1}{2}\sum_{i=1}^{n}(-1)^{2i-1}\binom{4-d}{i}\binom{d-6}{n-i} + \sum_{i=1}^{n}(-1)^{2i-2}\binom{4-d}{i-1}\binom{d-3}{n-i} \\
- \;& \sum_{i=1}^{n}(-1)^{2i-2}\binom{4-d}{i-1}\binom{d-5}{n-i} + \sum_{i=1}^{n}(-1)^{2i-2}\binom{3-d}{i-1}\binom{d-3}{n-i} \\
- \;& \sum_{i=1}^{n}(-1)^{2i-2}\binom{3-d}{i-1}\binom{d-5}{n-i} + 2\sum_{i=1}^{n}(-1)^{2i-3}\binom{3-d}{i-2}\binom{d-4}{n-i} \\
+ \;& \frac{1}{2}\sum_{i=1}^{n}(-1)^{2i-5}\binom{2-d}{i-4}\binom{d-4}{n-i} \\
= \;& -\frac{1}{2}\binom{2}{n} + \frac{1}{2}\binom{d-2}{n} + \binom{0}{n} - \binom{d-4}{n} - \frac{1}{2}\binom{-2}{n} + \frac{1}{2}\binom{d-6}{n} \\
+ \;& \sum_{i=0}^{n-1}\binom{4-d}{i}\binom{d-3}{n-i-1} - \sum_{i=0}^{n-1}\binom{4-d}{i}\binom{d-5}{n-i-1} \\
+ \;& \sum_{i=0}^{n-1}\binom{3-d}{i}\binom{d-3}{n-i-1} - \sum_{i=0}^{n-1}\binom{3-d}{i}\binom{d-5}{n-i-1} \\
- \;& 2\sum_{i=0}^{n-2}\binom{3-d}{i}\binom{d-4}{n-i-2} - \frac{1}{2}\sum_{i=0}^{n-4}\binom{2-d}{i}\binom{d-4}{n-i-4} \\
= \;& \frac{1}{2}\binom{d-2}{n} - \binom{d-4}{n} - \frac{1}{2}\binom{-2}{n} + \frac{1}{2}\binom{d-6}{n} + \binom{1}{n-1} \\
- \;& \binom{-1}{n-1} + \binom{0}{n-1} - \binom{-2}{n-1} - 2\binom{-1}{n-2} - \frac{1}{2}\binom{-2}{n-4} \\
= \;& \frac{1}{2}\binom{d-2}{n} - \binom{d-4}{n} + \frac{1}{2}\binom{d-6}{n} + \frac{1}{2}(-1)^{n-1}(n+1) + (-1)^{n+2} \\
+ \;& (-1)^n n + 2(-1)^{n-1} + \frac{1}{2}(-1)^{n-3}(n-3) \\
= \;& \frac{1}{2}\binom{d-2}{n} - \binom{d-4}{n} + \frac{1}{2}\binom{d-6}{n}
\end{aligned}$$

for $n \geq 3$.

$\square$

To finish the proof of Proposition 8.15 we put $n = d - 5 \geq 3$ in the above expressions for the coefficients:

$$\begin{aligned}
\mathcal{D}_{d-5} &= d_{d-5} = \binom{d-4}{d-5}h^{d-5} + \left(\binom{d-3}{d-5} - \binom{d-5}{d-5}\right)\Theta.h^{d-6} \\
&+ \left(\frac{1}{2}\binom{d-2}{d-5} - \binom{d-4}{d-5} + \frac{1}{2}\binom{d-6}{d-5}\right)\Theta^2.h^{d-7} \\
&= (d-4)h^{d-5} + \left(\binom{d-3}{2} - 1\right)\Theta.h^{d-6} \\
&+ \left(\frac{1}{2}\binom{d-2}{3} - (d-4)\right)\Theta^2.h^{d-7}.
\end{aligned}$$

□

Now we are able to deduce the formula for the degree of $\mathrm{Sec}_3(C)$ where C is a curve of genus 2 and degree $d \geq 8$ in $\mathbf{P}^{d-2}$:

Proposition 8.17. *The degree of the third secant variety* $\mathrm{Sec}_3(C)$ *of a curve of genus 2 and degree* $d \geq 8$ *in* $\mathbf{P}^{d-2}$ *is equal to*

$$\binom{d-2}{3} - 2(d-4).$$

Proof. Since $\mathrm{Sec}_3(C)$ has dimension 5, we have to intersect with $(h')^5$ where h' is a hyperplane class in $\mathbf{P}^{d-2}$ in order to obtain the degree of $\mathrm{Sec}_3(C)$. From the above remarks we now have to find $\deg x_1.h^5$.
We have

$$\begin{aligned}
\deg x_1.h^5 &= \deg \mathcal{D}_{d-5}.h^5 \\
&= \left(\frac{1}{2}\binom{d-2}{3} - (d-4)\right)\deg \Theta^2.h^{d-2}.
\end{aligned}$$

Since $\deg \Theta^2.h^{d-2} = 2$ (cf. Proposition 8.2) we finally obtain

$$\deg \mathcal{D}_{d-5}.h^5 = 2\left(\frac{1}{2}\binom{d-2}{3} - (d-4)\right) = \binom{d-2}{3} - 2(d-4).$$

□

8.2.2 Second method

Now we take a general linear system $|D|$ of degree d and dimension 4, and we count the number of all divisors of degree 3 on C that impose at most 2 conditions on $|D|$. In the first section we introduced the map

$$u : C_3 \to \mathrm{Pic}^3(C),$$

$$D \mapsto \mathcal{O}_C(D).$$

Let Θ be the theta divisor on $\mathrm{Jac}(C) \cong \mathrm{Pic}^3(C)$ and set $\theta := u^*(\Theta)$.
Moreover, fix a point $Q \in C$, and let x be the class of the divisor $X_Q := \{D \in C_3 | Q \in D\}$ on C_3, and let

$$Y := \{\text{divisors in } C_3 \text{ that impose at most two conditions on } |D|\}.$$

The expected dimension of Y is equal to 0, and by Lemma 4.1 in [ACGH85], Chapter VIII, §4, we know that the class of Y is given by

$$[Y] \;=\; \Delta_{3,1}((1+xt)^{d-4}e^{\frac{\theta t}{1+xt}}).$$

Before we compute the above class we observe the following:

$$\begin{aligned}
(1+xt)^{d-4}e^{\frac{\theta t}{1+xt}} &= (1+xt)^{d-4}\sum_{j=0}^{\infty}\frac{1}{j!}(1+xt)^{-j}\theta^j t^j \\
&= \sum_{j=0}^{\infty}\frac{1}{j!}(1+xt)^{d-4-j}\theta^j t^j \\
&= \sum_{i,j=0}^{\infty}\frac{1}{j!}\binom{d-4-j}{i}\theta^j.x^i t^{i+j} \\
&= \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\frac{1}{j!}\binom{d-4-j}{m-j}\theta^j.x^{m-j}\right)t^m.
\end{aligned}$$

Now $\Delta_{3,1}((1+xt)^{d-4}e^{\frac{\theta t}{1+xt}})$ is just the coefficient in front of t^3, i.e.

$$\begin{aligned}
\Delta_{3,1}((1+xt)^{d-4}e^{\frac{\theta t}{1+xt}}) &= \sum_{j=0}^{3}\frac{1}{j!}\binom{d-4-j}{3-j}\theta^j.x^{3-j} \\
&= \binom{d-4}{3}x^3 + \binom{d-5}{2}\theta.x^2 + \frac{1}{2}(d-6)\theta^2.x.
\end{aligned}$$

Our aim now is to find the degree of $[Y]$, i.e. we have to find $\deg(x^3)$, $\deg(\theta.x^2)$ and $\deg(\theta^2.x)$. In order to do so we need a lemma:

Lemma 8.18. *We have the following pushforwards:*

(1) $u_*(1) = 0$.

(2) $u_*(x) = 1$.

(3) $u_*(x^2) = \Theta$.

(4) $u_*(x^3) = \frac{1}{2}\Theta^2$.

Proof. (1) Since $u(C_3) = \mathrm{Pic}^3(C)$ and $\dim(\mathrm{Pic}^3(C)) = 2 < 3 = \dim(C_3)$, we obtain $u_*(1) = u_*([C_3]) = 0$.

(2) Since the dimension of X_Q is equal to 2, $u_*(x)$ will either be zero or some positive multiple of $[\mathrm{Pic}^3(C)]$, i.e. $u_*(x) = a[\mathrm{Pic}^3(C)]$ for some non-negative integer a. Using the projection formula we obtain for every point $[\mathcal{O}_C(D)] \in \mathrm{Pic}^3(C)$ such that Q is not a basepoint of $|D|$:
$a = u_*(x).[\mathcal{O}_C(D)] = u_*(x.u^*[\mathcal{O}_C(D)]) = x.u^*[\mathcal{O}_C(D)] = x.|D| = 1.$

(3) This equality we obtain by Remark 8.14: Taking the pushforward of both sides of the equation $x^2 = u^*(\Theta).x - \frac{1}{2}u^*(\Theta^2)$ and using the projection formula we obtain

$$\begin{aligned} u_*(x^2) &= u_*(u^*(\Theta).x) - \frac{1}{2}u_*u^*(\Theta^2) \\ &= \Theta.u_*(x) - \frac{1}{2}\Theta^2.u_*(1) \\ &= \Theta. \end{aligned}$$

(4) Also here we use Remark 8.14: From $x^2 = u^*(\Theta).x - \frac{1}{2}u^*(\Theta^2)$ we obtain

$$\begin{aligned} x^3 &= u^*(\Theta).x^2 - \frac{1}{2}u^*(\Theta^2).x \\ &= \frac{1}{2}u^*(\Theta^2).x \end{aligned}$$

and thus

$$\begin{aligned} u_*(x^3) &= \frac{1}{2}u_*(u^*(\Theta^2).x) \\ &= \frac{1}{2}\Theta^2.u_*(x) \\ &= \frac{1}{2}\Theta^2. \end{aligned}$$

□

Now we can conclude the following by Lemma 8.18 and Proposition 8.2:

$$\begin{aligned} \deg(\theta^2.x) &= \deg(u_*(\theta^2.x)) = \deg(u_*(u^*(\Theta^2).x)) \\ &= \deg(\Theta^2.u_*(x)) = \deg(\Theta^2) = 2, \end{aligned}$$

$$\begin{aligned} \deg(\theta.x^2) &= \deg(u_*(\theta.x^2)) = \deg(u_*(u^*(\Theta).x^2)) \\ &= \deg(\Theta.u_*(x^2)) = \deg(\Theta^2) = 2 \end{aligned}$$

and

$$\deg(x^3) = \deg(u_*(x^3)) = \deg(\frac{1}{2}\Theta^2) = 1.$$

Consequently, we obtain the following:

$$\begin{aligned}
\deg[Y] &= \deg(x^3)\binom{d-4}{3} + \deg(\theta.x^2)\binom{d-5}{2} + \deg(\theta^2.x)\frac{1}{2}(d-6) \\
&= \binom{d-4}{3} + 2\binom{d-5}{2} + (d-6) \\
&= \binom{d-2}{3} - \binom{d-3}{2} + \left[\binom{d-5}{2} - \binom{d-4}{2}\right] + \binom{d-5}{2} + (d-6) \\
&= \binom{d-2}{3} - \binom{d-3}{2} + \binom{d-5}{2} - (d-5) + (d-6) \\
&= \binom{d-2}{3} - \left[\binom{d-4}{2} + (d-4)\right] + \left[\binom{d-4}{2} - (d-5)\right] - 1 \\
&= \binom{d-2}{3} - 2(d-4).
\end{aligned}$$

8.2.3 Third method: Berzolari's formula for the number of trisecant lines to a smooth curve of genus g and degree d in $\mathbf{P}^4$

The number of trisecant lines to a smooth curve of genus g and degree d in $\mathbf{P}^4$ is well-known and given by Berzolari's formula (cf. e.g. [BC99], §4).
This number is equal to $\binom{d-2}{3} - g(d-4)$. In our situation g is equal to 2, and for $d \geq 8$ the number Berzolari's formula yields is exactly equal to the degree of $\mathrm{Sec}_3(C)$, where C is a curve of degree d and genus 2 in $\mathbf{P}^{d-2}$, which we found in Sections 8.2.1 and 8.2.2.
Why are these two numbers equal?
Let C be a curve of degree $d \geq 8$ and genus 2 embedded in $\mathbf{P}^{d-2}$. Since the dimension of $\mathrm{Sec}_3(C)$ is equal to 5, in order to find the degree of $\mathrm{Sec}_3(C)$ we have to intersect $\mathrm{Sec}_3(C)$ with 5 general hyperplanes. Let now V denote the intersection of 5 general hyperplanes in $\mathbf{P}^{d-2}$, i.e. V is a general space of codimension 5 in $\mathbf{P}^{d-2}$.
Since $\dim(Sec_2(C)) = 3$ and $\mathrm{codim}(V) = 5$, V and $\mathrm{Sec}_2(C)$ do not intersect. This implies that V cannot intersect any plane in $\mathrm{Sec}_3(C)$ in a line, since every plane in $\mathrm{Sec}_3(C)$ contains three lines in $\mathrm{Sec}_2(C)$, and so if V intersects a plane in a line L, then L intersects at least one of those lines in $\mathrm{Sec}_2(C)$ in a point which obviously lies in $\mathrm{Sec}_2(C)$.
The last step is now to project from V down to $\mathbf{P}^4$. Since V was chosen to be a general space of codimension 5, V does not intersect the curve C, and thus the curve in $\mathbf{P}^4$ which is the image of C under the projection from V is also a curve of degree d and genus 2.
Moreover, the fact that V does not intersect $\mathrm{Sec}_2(C)$ implies that the image curve is smooth.
A trisecant plane to $C \subseteq \mathbf{P}^{d-2}$ which intersects V in one point projects down to a trisecant line to the image curve in $\mathbf{P}^4$.
Summarizing, the number of trisecant planes to $C \subseteq \mathbf{P}^{d-2}$ that intersect V in one point is exactly equal to the number of trisecant lines to the image curve in $\mathbf{P}^4$, and thus

it follows that the degree of $\mathrm{Sec}_3(C)$ is equal to the number of trisecant lines to the image curve in $\mathbf{P}^4$.

Remark 8.19. *For $d = 6$ and $d = 7$ the formula for the degree of $\mathrm{Sec}_3(C)$ does not make sense, since for these values of d the third secant variety $\mathrm{Sec}_3(C)$ is equal to the ambient space $\mathbf{P}^{d-2}$. That is the reason we restrict to the cases $d \geq 8$ in our three methods. However, Berzolari's formula is still valid for the cases $d = 6$ and $d = 7$ since this formula counts the number of trisecant lines to curves of degree d and genus 2 in $\mathbf{P}^4$. This number is equal to 0 for $d = 6$, since in this case C is linearly normal embedded in $\mathbf{P}^4$, and thus $C \subseteq \mathbf{P}^4$ has no trisecant lines by Corollary 3.2. In the case $d = 7$ this number is equal to 4, which reflects the fact that through each general point on the curve there are 4 trisecant planes.*
For a curve $C \subseteq \mathbf{P}^3$ of genus 2 and degree 5 there are infinitely many trisecant lines, since C is of type $(2, 3)$ on a smooth quadric which is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.

Appendix A

Appendix to Chapter 4

In this appendix we list the matrices that give the maps in the resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^{d-2}}$-module as presented in Chapter 4, for $d = 7$ and $d = 8$.

A.1 $d = 7$

Let

$$M = \begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix}$$

and let S be the two-dimensional rational normal scroll defined by the (2×2)-minors of M.
Let

$$\begin{aligned} q_1 &= x_0x_2 - x_1^2, \\ q_2 &= x_0x_4 - x_1x_3, \\ q_3 &= x_0x_5 - x_1x_4, \\ q_4 &= x_1x_4 - x_2x_3, \\ q_5 &= x_1x_5 - x_2x_4, \\ q_6 &= x_3x_5 - x_4^2 \end{aligned}$$

denote the (2×2)-minors of M, let $l_1, l_2, l_3, l_4 \in k[x_0, x_1, x_2, x_3, x_4, x_5]$ be general linear forms and set

$$\begin{aligned} Q_1 &= l_1x_0 + l_2x_1 + l_3x_3 + l_4x_4, \\ Q_2 &= l_1x_1 + l_2x_2 + l_3x_4 + l_4x_5. \end{aligned}$$

In Section 3.5 we have seen that the ideal $(q_1, q_2, q_3, q_4, q_5, q_6, Q_1, Q_2) =: I_C$ defines a smooth curve C of genus 2 and degree 7 with associated $g^1_2(C)$-scroll S.
The mapping cone $\mathcal{C}^1(-2) \to \mathcal{C}^0$ is a minimal resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^5}$-module.
That is, we consider the following complex:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathcal{O}^2(-6) & \xrightarrow{A_2} & \mathcal{O}^4(-5) & \xrightarrow{A_1} & \mathcal{O}^4(-3) & \xrightarrow{A_0} & \mathcal{O}^2(-2) & \longrightarrow & I_{C,S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle C_2} & & \downarrow{\scriptstyle C_1} & & \downarrow{\scriptstyle C_0} & & \downarrow{\scriptstyle (-Q_2,Q_1)} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}^3(-4) & \xrightarrow{B_2} & \mathcal{O}^8(-3) & \xrightarrow{B_1} & \mathcal{O}^6(-2) & \xrightarrow{B_0} & \mathcal{O} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0, \end{array}$$

where the maps are given by multiplication with the following matrices:

$$A_0 = M = \begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} q_4 & q_5 & q_6 & 0 \\ -q_2 & -q_3 & 0 & q_6 \\ q_1 & 0 & -q_3 & -q_5 \\ 0 & q_1 & q_2 & q_4 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} x_4 & x_5 \\ -x_3 & -x_4 \\ x_1 & x_2 \\ -x_0 & -x_1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & x_4 & x_3 & 0 & 0 & x_5 & x_4 \\ 0 & x_4 & 0 & -x_1 & 0 & x_5 & 0 & -x_2 \\ 0 & -x_3 & -x_1 & 0 & 0 & -x_4 & -x_2 & 0 \\ x_4 & 0 & 0 & x_0 & x_5 & 0 & 0 & x_1 \\ -x_3 & 0 & x_0 & 0 & -x_4 & 0 & x_1 & 0 \\ x_1 & x_0 & 0 & 0 & x_2 & x_1 & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} x_0 & 0 & x_1 \\ -x_1 & 0 & -x_2 \\ x_3 & 0 & x_4 \\ -x_4 & 0 & -x_5 \\ 0 & x_1 & x_0 \\ 0 & -x_2 & -x_1 \\ 0 & x_4 & x_3 \\ 0 & -x_5 & -x_4 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -l_2 & l_1 & 0 & 0 \\ -l_3 & 0 & l_1 & 0 \\ -l_4 & 0 & 0 & l_1 \\ 0 & -l_3 & l_2 & 0 \\ 0 & -l_4 & 0 & l_2 \\ 0 & 0 & -l_4 & l_3 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} l_4x_1 & -l_3x_1 & l_2x_1 & Q_2 - l_1x_1 \\ -l_4x_2 & l_3x_2 & Q_2 - l_2x_2 & l_1x_2 \\ l_4x_4 & Q_2 - l_3x_4 & l_2x_4 & -l_1x_4 \\ Q_2 - l_4x_5 & l_3x_5 & -l_2x_5 & l_1x_5 \\ 0 & 0 & 0 & -Q_1 \\ 0 & 0 & -Q_1 & 0 \\ 0 & -Q_1 & 0 & 0 \\ -Q_1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -Q_2 & 0 \\ 0 & Q_1 \\ Q_1 & 0 \end{pmatrix}.$$

A.2 $d = 8$

Let

$$M = \begin{pmatrix} x_0 & x_1 & x_2 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_5 & x_6 \end{pmatrix},$$

and let S denote the rational normal scroll which ideal is generated by the (2×2)-minors of M. Moreover, let

$$
\begin{aligned}
q_1 &= x_0x_2 - x_1^2, \\
q_2 &= x_0x_3 - x_1x_2, \\
q_3 &= x_0x_5 - x_1x_4, \\
q_4 &= x_0x_6 - x_1x_5, \\
q_5 &= x_1x_3 - x_2^2, \\
q_6 &= x_1x_5 - x_2x_4, \\
q_7 &= x_1x_6 - x_2x_5, \\
q_8 &= x_2x_5 - x_3x_4, \\
q_9 &= x_2x_6 - x_3x_5, \\
q_{10} &= x_4x_6 - x_5^2
\end{aligned}
$$

denote the (2×2)-minors of M, let $l_1, l_2, l_3 \in k[x_0, x_1, x_2, x_3, x_4, x_5, x_6]$ be general linear forms and set

$$
\begin{aligned}
Q_1 &= l_1x_0 + l_2x_1 + l_3x_4, \\
Q_2 &= l_1x_1 + l_2x_2 + l_3x_5, \\
Q_3 &= l_1x_2 + l_2x_3 + l_3x_6.
\end{aligned}
$$

In Section 3.5 we had seen that the ideal $I_C := I_S + (Q_1, Q_2, Q_3)$ is the ideal of a smooth curve C of genus 2 and degree 8 in $\mathbf{P}^6$.

The mapping cone $\mathcal{C}^2(-2) \to \mathcal{C}^0$ gives a resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbf{P}^6}$-module, i.e. we consider the following complex:

$$
\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \mathcal{O}^2(-7) & \xrightarrow{A_3} & \mathcal{O}^5(-6) & \xrightarrow{A_2} & \mathcal{O}^{10}(-4) & \xrightarrow{A_1} & \mathcal{O}^{10}(-3) & \xrightarrow{A_0} & \mathcal{O}^3(-2) & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C_4} & & \downarrow{\scriptstyle C_3} & & \downarrow{\scriptstyle C_2} & & \downarrow{\scriptstyle C_1} & & \downarrow{\scriptstyle C_0} & & & & \\
0 & \longrightarrow & \mathcal{O}^4(-5) & \xrightarrow{B_3} & \mathcal{O}^{15}(-4) & \xrightarrow{B_2} & \mathcal{O}^{20}(-3) & \xrightarrow{B_1} & \mathcal{O}^{10}(-2) & \xrightarrow{B_0} & \mathcal{O} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0
\end{array}
$$

The maps in the above complex are given by multiplication with the matrices we list below. We will use the same ordering of the columns of the matrices as the computer algebra system Macaulay 2 ([GS]).

$$
A_0 = \begin{pmatrix}
x_0 & x_1 & x_2 & x_4 & x_5 & 0 & 0 & 0 & 0 & 0 \\
x_1 & x_2 & x_3 & x_5 & x_6 & x_0 & x_1 & x_2 & x_4 & x_5 \\
0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_5 & x_6
\end{pmatrix},
$$

$$A_1 = \begin{pmatrix} -x_1 & 0 & -x_2 & 0 & 0 & -x_4 & 0 & 0 & 0 & -x_5 \\ x_0 & -x_2 & 0 & 0 & -x_4 & 0 & 0 & 0 & -x_5 & 0 \\ 0 & x_1 & x_0 & -x_4 & 0 & 0 & 0 & -x_5 & 0 & 0 \\ 0 & 0 & 0 & x_2 & x_1 & x_0 & -x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_4 & x_2 & x_1 & x_0 \\ -x_2 & 0 & -x_3 & 0 & 0 & -x_5 & 0 & 0 & 0 & -x_6 \\ x_1 & -x_3 & 0 & 0 & -x_5 & 0 & 0 & 0 & -x_6 & 0 \\ 0 & x_2 & x_1 & -x_5 & 0 & 0 & 0 & -x_6 & 0 & 0 \\ 0 & 0 & 0 & x_3 & x_2 & x_1 & -x_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_5 & x_3 & x_2 & x_1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -q_8 & 0 & 0 & -q_{10} & -q_9 \\ -q_3 & -q_{10} & 0 & 0 & -q_4 \\ q_6 & 0 & -q_{10} & 0 & q_7 \\ -q_1 & -q_7 & -q_4 & 0 & 0 \\ q_2 & q_9 & 0 & -q_4 & 0 \\ -q_5 & 0 & q_9 & q_7 & 0 \\ 0 & -q_5 & -q_2 & -q_1 & 0 \\ 0 & q_6 & q_3 & 0 & -q_1 \\ 0 & -q_8 & 0 & q_3 & q_2 \\ 0 & 0 & -q_8 & -q_6 & -q_5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} x_6 & x_5 \\ x_1 & x_0 \\ -x_2 & -x_1 \\ x_3 & x_2 \\ -x_5 & -x_4 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7 & q_8 & q_9 & q_{10} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} x_3 & -x_2 & -x_4 & 0 & x_6 & -x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_6 & x_6 & 0 & 0 & 0 & 0 \\ -x_2 & x_1 & 0 & 0 & 0 & 0 & 0 & x_6 & -x_4 & 0 & 0 & x_6 & 0 & 0 & -x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_6 & 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & 0 & -x_3 & 0 & 0 & x_6 & -x_5 \\ 0 & 0 & 0 & -x_5 & -x_2 & x_1 & x_1 & -x_3 & x_1 & 0 & 0 & -x_3 & 0 & x_2 & 0 & 0 & 0 & 0 & -x_5 & x_4 \\ x_1 & -x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_6 & 0 & -x_5 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_0 & -x_5 & 0 & 0 & x_1 & -x_3 & x_1 & x_6 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & x_4 & x_1 & -x_0 & -x_0 & x_2 & -x_0 & -x_5 & -x_3 & 0 & x_2 & 0 & x_1 & x_1 & 0 & -x_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_0 & -x_5 & 0 & 0 & 0 & 0 & 0 & x_1 & x_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 & x_2 & x_1 & -x_1 & -x_0 & -x_0 & -x_0 & -x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & x_1 & -x_0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix}
-x_6 & x_4 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x_5 & 0 & x_6 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_3 & 0 & x_6 & 0 & 0 & 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & x_6 & x_5 \\
0 & 0 & -x_3 & -x_1 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & -x_1 & -x_0 \\
x_3 & 0 & -x_6 & 0 & x_1 & 0 & 0 & 0 & -x_0 & 0 & 0 & x_5 & 0 & x_4 & 0 \\
0 & 0 & 0 & 0 & x_2 & x_6 & 0 & x_3 & 0 & 0 & 0 & x_6 & 0 & 0 & -x_4 \\
0 & 0 & 0 & 0 & 0 & -x_6 & -x_3 & -x_3 & -x_1 & 0 & 0 & -x_6 & -x_6 & -x_5 & 0 \\
0 & -x_0 & x_5 & 0 & 0 & 0 & -x_1 & -x_1 & 0 & -x_6 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_2 & 0 & -x_5 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & x_6 & 0 & 0 \\
0 & 0 & x_2 & x_0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1 & 0 & -x_4 & 0 & 0 & 0 & x_0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\
-x_2 & x_0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & x_6 & x_5 & 0 & x_4 & 0 & 0 \\
0 & 0 & 0 & 0 & x_0 & x_4 & 0 & x_1 & 0 & x_6 & 0 & 0 & x_4 & 0 & 0 \\
0 & x_1 & -x_6 & 0 & 0 & -x_5 & 0 & 0 & -x_0 & 0 & 0 & 0 & -x_5 & 0 & 0 \\
0 & 0 & x_6 & x_4 & -x_1 & 0 & 0 & -x_2 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_5 & x_2 & x_2 & x_0 & 0 & -x_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_1 & 0 & x_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & 0 & x_1 & 0 & x_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & -x_2 & 0 & 0 & x_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & x_1
\end{pmatrix},$$

$$B_3 = \begin{pmatrix} -x_5 & -x_4 & 0 & 0 \\ 0 & -x_6 & -x_5 & 0 \\ 0 & -x_1 & -x_0 & 0 \\ 0 & 0 & x_2 & x_1 \\ 0 & -x_6 & 0 & x_4 \\ 0 & x_2 & 0 & -x_0 \\ 0 & -x_5 & -x_4 & 0 \\ -x_6 & 0 & x_4 & 0 \\ 0 & 0 & x_6 & x_5 \\ x_1 & x_0 & 0 & 0 \\ -x_2 & 0 & x_0 & 0 \\ x_3 & 0 & 0 & x_0 \\ 0 & -x_2 & -x_1 & 0 \\ 0 & x_3 & 0 & -x_1 \\ 0 & 0 & x_3 & x_2 \end{pmatrix}, \quad C_0 = \begin{pmatrix} Q_3 & -Q_2 & Q_1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} l_1 & 0 & 0 & 0 & 0 & -l_2 & l_1 & 0 & 0 & 0 \\ l_2 & 0 & 0 & 0 & 0 & 0 & 0 & l_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -l_3 & 0 & 0 & l_1 & 0 \\ l_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_1 \\ 0 & l_2 & -l_1 & 0 & 0 & 0 & 0 & l_2 & 0 & 0 \\ 0 & 0 & 0 & -l_1 & 0 & 0 & -l_3 & 0 & l_2 & 0 \\ 0 & l_3 & 0 & 0 & -l_1 & 0 & 0 & 0 & 0 & l_2 \\ 0 & 0 & 0 & -l_2 & 0 & 0 & 0 & -l_3 & 0 & 0 \\ 0 & 0 & l_3 & 0 & -l_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_3 & 0 & 0 & 0 & 0 & 0 & l_3 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & -l_1 & l_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -l_2 & 0 & l_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & l_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -l_3 & 0 & -l_1 & 0 & 0 & 0 \\ 0 & 0 & l_3 & -l_1 & l_2 & 0 & 0 & 0 & -l_1 & 0 \\ 0 & 0 & 0 & 0 & l_1 & -l_2 & 0 & 0 & 0 & l_1 \\ -l_3 & 0 & 0 & 0 & -l_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -l_3 & 0 & -l_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & l_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l_3 & 0 & 0 & -l_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_2 & 0 & 0 \\ 0 & l_3 & 0 & l_2 & 0 & 0 & 0 & -l_1 & 0 & 0 \\ 0 & 0 & 0 & l_2 & 0 & 0 & 0 & -l_1 & l_2 & 0 \\ 0 & 0 & 0 & -l_1 & l_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_1 & 0 & 0 & 0 & 0 & 0 & l_2 \\ 0 & 0 & -l_3 & 0 & -l_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_3 & 0 & 0 \\ 0 & 0 & 0 & l_3 & 0 & 0 & 0 & 0 & l_3 & 0 \\ 0 & 0 & 0 & 0 & l_3 & 0 & 0 & 0 & 0 & l_3 \\ 0 & 0 & 0 & 0 & 0 & -l_3 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & -l_1x_4 & 0 & 0 & -Q_1+l_3x_4 \\ -Q_2 & 0 & -l_2x_6 & 0 & 0 \\ 0 & -Q_2 & -l_2x_1 & 0 & 0 \\ l_3x_2 & 0 & Q_3-l_1x_2 & l_2x_2 & 0 \\ -Q_2+l_3x_5 & 0 & -l_1x_5-l_2x_6 & l_2x_5 & Q_3 \\ -l_3x_1 & Q_3 & -l_3x_5 & -l_2x_1 & 0 \\ 0 & 0 & -l_2x_5 & 0 & Q_2 \\ Q_1 & -l_1x_5 & l_2x_5 & 0 & -Q_2+l_3x_5 \\ -Q_3+l_3x_6 & 0 & -l_1x_6 & l_2x_6 & 0 \\ 0 & -Q_1+l_1x_0 & 0 & 0 & -l_3x_0 \\ 0 & Q_2-l_1x_1 & -Q_1+l_2x_1 & 0 & l_3x_1 \\ l_3x_1 & -Q_3+l_1x_2 & l_3x_5 & -Q_1+l_2x_1 & -l_3x_2 \\ 0 & 0 & Q_2-l_2x_2 & 0 & 0 \\ -l_3x_2 & 0 & -l_3x_6 & Q_2-l_2x_2 & 0 \\ l_3x_3 & 0 & -l_1x_3 & -Q_3+l_2x_3 & 0 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -Q_3 & 0 \\ Q_2-l_2x_2 & -l_2x_1 \\ 0 & Q_2 \\ 0 & -Q_3 \end{pmatrix}.$$

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