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LOCAL RISK-MINIMIZATION UNDER A PARTIALLY OBSERVED MARKOV-MODULATED EXPONENTIAL LÉVY MODEL

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Abstract. In this paper, the option hedging problem for a Markov-modulated exponential Lévy model is examined. We employ the local risk-minimization approach to study optimal hedging strategies for Europeans derivatives under both full information and then partial information.

1. Introduction

Unpredictable structural changes in the trend of asset prices or stock indexes on financial markets is a current reality nowadays. They are not usually caused by internal events of the market itself but are more related to the global socioeconomic and political environment. To account for these features, Markov-modulated (or regime-switching) models have since been widely used in econometrics and financial mathematics. See for instance, Hamilton [24] for exhibiting the non-stationarity of macroeconomic times series, Elliott and Van der Hoek [14] for asset allocation, Pliska [29] and Elliott et *al.* [10] for short rate models, Naik [26], Guo [23] and Buffington and Elliott [2] for option valuation.

The Markov-modulated exponential Lévy model is very attractive as alternative to the classical Black-Scholes model because they couple the benefit of an exponential Lévy model (notably the presence of jumps) with the possibility, thanks to the Markov chain, to having long-term variability of some characteristics of the return distribution. However, in the context of derivative pricing these models lead to incomplete markets. Therefore, the question of hedging becomes a crucial one.

In this paper, we consider the problem of optimal quadratic hedging of an European derivative contract in a market driven by a Markov-modulated Lévy model. Typically, in this model the full information on the modulating factor X is not available in the market and the agent has only access to the information contained in past asset prices. Consequently, we will deal with an optimal quadratic hedging problem for a partially observed model (or partial information scenario).

This kind of problem has been extensively studied in the literature. Di Masi, Platen and Runggaldier [7] were the first to discuss the problem of risk-minimizing (mean-variance) hedging under restricted information when the stock price is a martingale and the prices are observed only at discrete time instants. In [33], Schweizer explicited for general filtrations

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$\mathbf{G} := \{\mathcal{G}_t\}_{t\in[0,T]} \subseteq \{\mathcal{F}_t\}_{t\in[0,T]} := \mathbf{F}$ a risk-miminizing strategy based on $\mathbf{G}$-predictable projections. Pham [28] solved the problem of mean-variance hedging for partially observed drift processes. Frey and Runggaldier [17] determined a locally risk-minimizing hedging strategy when the asset price process follows a stochastic model and is observed only at discrete random times. Frey [18] considered risk-minimization with incomplete information in a model for high-frequency data. In the same framework but for more general model, Ceci [3] computed the optimal hedge strategy under the criterion of risk-minimization. In all these papers, the methodology consists first, to determine the optimal strategy under the full information and second, determine the final solution by projecting on the filtration available in to the investor. Then a natural question arises that given a Markov-modulated Lévy model, can we applied the above methodology to study the problem of local risk-minimization under partial information?

The aim of this paper is to give an answer to the previous question. In fact, we show that under some restrictive conditions on our Lévy model, we can apply the same technics used by the precedent authors to obtain an optimal hedging strategy for local risk-minimization under partial information. In fact, we first derive a martingale representation for the wealth process under full information. Then we proceed as in the classical setting by solving a local risk minimization under full information. Let us mention that the optimal strategy obtained under full information is quit explicit. Finally, using the fact that our processes do not jumps simultaneously, we can deduce an orthogonal projection of the claim with respect to smaller filtration and therefore the optimal strategy.

The paper is organized as follows. Section 2 describe in details our model setup and build two different filtrations that characterized the situation where investor have full or partial information. In Section 3, we recall some basic notions and results on risk-minimization. Section 4 contains the main results, namely the martingale representation property for the value process and the existence of optimal strategies in our market model under full and partial information.

2. The model

2.1. **Framework.**

We consider a financial market with two primary securities, namely a money market account B and a stock S which are traded continuously over the time horizon $\mathcal{T} := [0,T]$, where $T \in (0,\infty)$, is fixed and represents the maturity time for all economic activities. To formalize this market, we fix a (complete) filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ satisfying the usual conditions. We suppose also that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ contains only the null sets of $\mathcal{F}$ and their complements. All processes are defined on the stochastic basis above. Further, we will add to this setup a filtration which specifies the flow of informations available for the investors.

Let $X := \{X_t : t \in \mathcal{T}\}$ an irreducible homogeneous continuous-time Markov chain with a finite state space $\mathbb{S} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_M\} \subset \mathbb{R}^M$ characterized by a rate (or intensity) matrix $A := \{a_{ij} : 1 \le i,j \le M\}$. Following Dufour and Elliott [8], we can identify $\mathbb{S}$ with the basis set of the linear space $\mathbb{R}^M$. From now, we set $\mathbf{e}_i = (0,0,\dots,\underbrace{1}_{i-th},\dots,0)$. It follows from

Elliott [11] that X admits the following semimartingale representation

$$X_t = X_0 + \int_0^t AX_s + \Gamma_t, \tag{2.1}$$

where $\Gamma := \{(\Gamma_t^i)_{i=1}^M : t \in [0,T]\}$ is a vector-martingale in $\mathbb{R}^M$ with respect to the filtration generated by X.

Let r_t denote the instantaneous interest rate of the money market account B at time t. If we suppose that $r_t := r(t, X_t) = \langle \underline{r} | X_t \rangle$, where $\langle \cdot | \cdot \rangle$ is the usual scalar product in $\mathbb{R}^M$ and $\underline{r} = (r_1, r_2, \dots, r_M) \in {\mathbb{R}_+}^M$, then the price dynamics of B is given by:

$$dB_t = r_t B_t \, dt, \; B(0) = 1 \quad \text{for} \quad t \in \mathcal{T}. \tag{2.2}$$

The appreciation rate μ_t and the volatility σ_t of the stock S at time time $t$are defined by

$$\begin{aligned} \mu_t &:= \mu(t, X_t) = \langle \underline{\mu} | X_t \rangle, \\ \sigma_t &:= \sigma(t, X_t) = \langle \underline{\sigma} | X_t \rangle, \quad t \in \mathcal{T} \end{aligned} \tag{2.3}$$

where $\underline{\mu} = (\mu_1, \mu_2, \dots, \mu_M) \in \mathbb{R}^M$ and $\underline{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_M) \in {\mathbb{R}_+}^M$.

The stock price process S is described by this following Markov modulated Lévy process:

$$dS_t = S_{t^-}\Big(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}\backslash\{0\}} (e^z - 1)\widetilde{N}^X(dt; dz)\Big), \quad S(0) = S_0 > 0 \tag{2.4}$$

Here $W := (W_t)_{t\in\mathcal{T}}$ is a one-dimensional standard Brownian motion or Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, independent of X and N^X,

$$\widetilde{N}^X(dt, dz) := \begin{cases} N^X(dt, dz) - \rho^X(dz)dt & \text{if } |z| < 1 \\ N^X(dt, dz) & \text{if } |z| \geq 1, \end{cases} \tag{2.5}$$

with $N^X(dt, dz)$ is the differential form of a Markov-modulated random measure on $\mathcal{T} \times \mathbb{R}\backslash\{0\}$. We recall from Elliott and Osakwe [12] and Elliott and Royal [13] that a Markov-modulated random measure on $\mathcal{T} \times \mathbb{R}\backslash\{0\}$ is a family $\{N^X(dt, dz; \omega) : \omega \in \Omega\}$ of non-negative measures on the measurable space $(\mathcal{T} \times \mathbb{R}\backslash\{0\}, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}(\mathbb{R}\backslash\{0\}))$, which satisfies$N^X(\{0\}, \mathbb{R}\backslash\{0\}; \omega) = 0$ and has the following compensator, or dual predictable projection

$$\rho^X(dz)dt := \sum_{i=1}^M \langle X_{t^-} | \mathbf{e}_i \rangle \rho_i(dz)dt, \tag{2.6}$$

where $\rho_i(dz)$ is the density for the jump size when the Markov chain X is in state $\mathbf{e}_i$ and satisfying

$$\int_{|z|\geq 1} (e^z - 1)^2 \rho_i(dz) < \infty. \tag{2.7}$$

The general setting considered here can be seen as an extension of the exponential-Lévy model described in Cont and Tankov [6] where a factor of modulation is introduced. Hence, we can retrieve in a simple way most of some current models which exist in the literature as for example the classical Black-Scholes model and the family of exponential-Lévy models.

The subsequent assumption will be fundamental for obtaining our results, particularly in Section 4.1 to obtain a martingale representation for the value process.

Assumption 2.1. *We assume that a transition of Markov chain X from state $\mathbf{e}_j$ to state $\mathbf{e}_k$ and a jump of S do not happen simultaneously almost surely.*

Let $\xi := \{\xi_t\}_{t\in\mathcal{T}}$ denoting the discounted stock price. Then,

$$\xi_t := \frac{S_t}{B_t} = e^{-\int_0^t r_u du} S_t.$$

If $R_t = e^{\int_0^t r_u du}$, for each $t \in \mathcal{T}$. Then, the discounted stock price process is given by :

$$\begin{aligned} d\xi_t &= F_\mu(t,\xi_{t^-},X_t)dt + F_\sigma(t,\xi_{t^-},X_t)dW_t + \int_{\mathbb{R}\setminus\{0\}} F_\gamma(t,\xi_{t^-},X_t)\widetilde{N}^X(dt;dz), \\ \xi(0) &= S_0 > 0 \ \mathbb{P}\ a.s, \end{aligned} \tag{2.8}$$

or the following integral decomposition

$$\xi_t = S_0 + \underbrace{\int_0^t F_\mu(s,\xi_{s^-},X_s)ds}_{\text{finite variation part}} + \underbrace{\int_0^t F_\sigma(s,\xi_{s^-},X_s)dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} F_\gamma(s,\xi_{s^-},X_s)\widetilde{N}^X(ds;dz)}_{\text{local-martingale part}}, \tag{2.9}$$

where

$$\left\{ \begin{array}{ll} F_\mu(t,\xi_t,X_t) := & \Big(\mu(t,R_t\xi_t,X_t) - r(t,R_t\xi_t,X_t)\Big)\xi_t \\ F_\sigma(t,\xi_t,X_t) := & \sigma(t,R_t\xi_t,X_t)\xi_t \\ F_\gamma(t,\xi_t,X_t) := & \xi_t(e^z-1), \end{array} \right. \tag{2.10}$$

The theory of stochastic flows will also be used to identify the integrands in the stochastic integrals involved in the martingale representation property in Section 4.1. Let now consider a general form of stochastic differential equation (SDE) (2.8):

$$\left\{ \begin{array}{ll} d\xi_t & = F_\mu(t,\xi_{t^-},X_t)dt + F_\sigma(t,\xi_{t^-},X_t)dW_t + \int_{\mathbb{R}\setminus\{0\}} F_\gamma(t,\xi_{t^-},X_t)\widetilde{N}^X(dt;dz), \\ \xi_s & = x > 0 \quad \mathbb{P}\ a.s.\ for\ 0 \le s < t \le T. \end{array} \right. \tag{2.11}$$

We assume that the coefficients $F_\mu, F_\sigma, F_\gamma$ are smooth enough to guaranty the existence and uniqueness of a strong adapted *càdlàg* solution $\xi_{s,\ t}(x)$ (see Fujiwara and Kunita [21]). Furthermore, this solution forms a stochastic flow of diffeomorphisms $\Phi_{s,\ t} : (0,+\infty)\times\Omega \to (0,+\infty)$ given by

$$\Phi_{s,\ t}(x,\omega) = \xi_{s,\ t}(x)(\omega), \tag{2.12}$$

for each (s,t) such that $0 \le s < t \le T$, $x \in (0,+\infty)$ and $\omega \in \Omega$. $(\Phi_{s,\ t})_{s<t}$ verifies the following properties:

- $\Phi_{s,\ t} = \Phi_{0,\ t} \circ \Phi_{0,\ s}^{-1}$ for all $s < t$;
- Cocycle property : $\Phi_{s,\ u} = \Phi_{t,\ u} \circ \Phi_{s,\ t}$ for all $s < t < u$;
- Conditional independent increments: for $t_0 \le t_1 \le \dots \le t_n$, $\Phi_{t_0,\ t_1}, \Phi_{t_1,\ t_2}, \dots, \Phi_{t_{n-1},\ t_n}$ are conditionally independent given $\mathcal{F}_T^X$.

Let $x = \xi_{0,\ t}(x_0)$, for each $t \in [0,T]$. By the uniqueness of solutions of SDE and the semi-group property, we get

$$\xi_{0,\ T}(x_0) = \xi_{t,\ T}(\xi_{0,\ t}(x_0)) = \xi_{t,\ T}(x). \tag{2.13}$$

Differentiating (2.13) with respect to x_0, we obtain:

$$\frac{\partial \xi_{0,T}(x_0)}{\partial x_0} = \frac{\partial \xi_{t,\ T}(x)}{\partial x}\frac{\partial \xi_{0,\ t}(x_0)}{\partial x_0}. \tag{2.14}$$

2.2. Market information.

In general, the Markov-modulated Lévy model as described by Equation (2.4) is based on the mathematical framework of the Markov additive processes (MAP). This last object is an old and widely studied subject in stochastic analysis (see, e.g, [4, 5, 16, 22] for a few.) In particular, the couple (X, S) is a Markov additive process and yields to two important filtrations as we will see below.

Let $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t\in\mathcal{T}}$ and $\mathcal{F}^S := \{\mathcal{F}_t^S\}_{t\in\mathcal{T}}$ denote the right-continuous, $\mathbb{P}-$complete filtrations generated by X et S respectively. We define for $t \in \mathcal{T}$,

$$\mathcal{G}_t := \mathcal{F}_t^S \tag{2.15}$$

and

$$\overline{\mathcal{G}}_t := \mathcal{F}_T^X \vee \mathcal{F}_t^S. \tag{2.16}$$

The filtration $\mathbf{G} := \{\mathcal{G}_t\}_{t\in\mathcal{T}}$ represents all the information up to time t gained from the observations of the price fluctuations S. The strict larger filtration $\overline{\mathbf{G}} := \{\overline{\mathcal{G}}_t\}_{t\in\mathcal{T}}$ denotes the information about the stock price history up to time t and the information about the entire path $\mathcal{F}_T^X$ of the modulation factor process X.

We will assume in the last section of is paper that the investors in the market only have access to the first filtration which is thus the one used practically whereas the last serves mainly theoretical purposes.

2.3. Esscher transform change of measure.

One of the main features of the Markov-modulated Lévy model is that it leads to an incomplete market. We shall therefore employ the regime-switching Esscher transform as in Elliott *et al.* [11] to determine an equivalent martingale measure.

For doing so, we define the process Y by

$$Y_t = \int_0^t \Big(\mu_r - \frac{1}{2}\sigma_r^2\Big)dr + \int_0^t \sigma_r dW_r + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\widetilde{N}^X(dr;dz) - \int_0^t \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\rho^X(dz)dr \tag{2.17}$$

As in [35], let consider the following set

$$\Theta := \left\{ (\theta_t)_{t\in\mathcal{T}} \mid \theta_t := \sum_{i=1}^N \theta_i \langle X_{t-}|\mathbf{e}_i\rangle \text{ with } (\theta_1, \theta_2, ..., \theta_N) \in \mathbb{R}^N \text{such that } \mathbb{E}^{\mathbb{P}}\Big[e^{-\int_0^t \theta_r dY_r}\Big|\mathcal{F}_T^X\Big] < \infty \right\}.$$

For $\theta := (\theta_t)_{t\in\mathcal{T}} \in \Theta$, the generalized Laplace transform of a $\overline{\mathbb{G}}$-adapted process Y is defined as

$$\mathcal{M}_Y(\theta)_t := E^{\mathbb{P}}\Big[e^{-\int_0^t \theta_r dY_r}\Big|\mathcal{F}_T^X\Big]. \tag{2.18}$$

Notice that contrary to the usual Esscher transform, the expectation involved here is taken conditionally on the information of all the future of the Markov chain X. With this extended definition of a Laplace transform, we can now define the generalized Esscher transform (with respect to the parameter θ called *Esscher parameter*).

Let $\Lambda^\theta = \{\Lambda^\theta_t\}_{t\in\mathcal{T}}$ denote a $\overline{\mathbf{G}}$-adapted stochastic process defined as

$$\Lambda^\theta_t := \frac{e^{-\int_0^t \theta_r dY_r}}{\mathcal{M}_Y(\theta)_t}, \quad t \in \mathcal{T};\ \theta \in \Theta. \tag{2.19}$$

It can be shown that (see for example, [11])

$$\Lambda^\theta_t = \exp\Bigg[-\int_0^t \theta_r\sigma_r dW_r - \frac{1}{2}\int_0^t \theta_r^2\sigma_r^2 dr - \int_0^t\int_{\mathbb{R}\setminus\{0\}} \theta_{r^-} z\widetilde{N}^X(dr;dz) \\ - \int_0^t\int_{\mathbb{R}\setminus\{0\}} \Big(e^{-z\theta_r} - 1 + \theta_r z\Big)\rho^X(dz)dr\Bigg]. \tag{2.20}$$

Moreover, as proven in [35], the stochastic process $\Lambda^\theta = \{\Lambda^\theta_t\}_{t\in\mathcal{T}}$ defined by (2.19) is a positive $(\overline{\mathbf{G}}, \mathbb{P})$-martingale and

$$E^{\mathbb{P}}[\Lambda^\theta_t] = 1, \quad \forall t \in \mathcal{T}. \tag{2.21}$$

From Equation 2.21, we deduce that the process $\Lambda^\theta = \{\Lambda^\theta_t\}_{t\in\mathcal{T}}$ given by Equation (2.20) is a density process inducing a change of measure in the probability space $(\Omega, \overline{\mathcal{G}}_T)$. Indeed, by setting

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\Big|_{\overline{\mathcal{G}}_t} = \Lambda^\theta_t \quad t \in \mathcal{T}, \tag{2.22}$$

we define for each process θ in Θ a new probability measure $\mathbb{Q}^\theta$ equivalent to $\mathbb{P}$. Actually, $\mathbb{Q}^\theta$ is just an equivalent probability measure, to transform it into a martingale equivalent measure we need to impose some conditions generally known as *martingale condition*. It stipulates that the discounted stock price $\{\xi_t\}_{t\in\mathcal{T}}$ would be a $\overline{\mathbf{G}}$-martingale under $\mathbb{Q}^\theta$. Then,

$$E^{\mathbb{Q}^\theta}\Big[\xi_t|\overline{\mathcal{G}}_0\Big] = \xi(0),\ \forall t \in \mathcal{T}. \tag{2.23}$$

Hence, we have

Proposition 2.2. *An equivalent probability measure $\mathbb{Q}^\theta$ defined through (2.22) is an equivalent martingale measure on $(\Omega, \overline{\mathcal{G}}_T)$, i.e. it satisfies condition (2.23), if and only if the process θ satisfies the following equation*

$$\mu_t - r_t - \theta_t\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)(e^{-z\theta_t} - 1)\rho^X(dz) = 0,\ \forall t \in \mathcal{T}. \tag{2.24}$$

Proof. The proof is a straightforward adaptation of that of Proposition 2.2 in Elliott *et al.* [11]. The main ingredient is an explicit computation of the generalized Laplace transform defined by (2.18). □

However, the process θ is completely determined by the vector $(\theta_1, \theta_2, \dots, \theta_M)$ solution of the system of equations

$$\mu_i - r_i - \theta_i\sigma_i^2 + \int_{\mathbb{R}} (e^z - 1)(e^{-z\theta_i} - 1)\rho_i(z)dz = 0, \tag{2.25}$$

for $i = 1, 2, \dots, N$.

For pricing purposes, we need to know the dynamics of the discounted stock price under the martingale probability measure $\mathbb{Q}^\theta$. The following proposition states a result in this direction.

Proposition 2.3. *Under risk-neutral probability measure $\mathbb{Q}^\theta$, the discounted stock price process ξ is solution to the following stochastic differential equation*

$$\begin{cases} d\xi_t & = F_\sigma(t,\xi_{t^-},X_t)dW_t^\theta + \int_{\mathbb{R}\setminus\{0\}} F_\gamma(t,\xi_{t^-},X_t)\underline{\widetilde{N}}^\theta(dt;dz) \\ \xi(0) & = S_0 > 0 \quad \mathbb{P}\text{-a.s. for } 0 \le t \le T, \end{cases} \tag{2.26}$$

where

- W^θ *defined by*

$$W_t^\theta := W_t + \int_0^t \theta_r \sigma_r dr, \tag{2.27}$$

  is the standard Brownian motion under $\mathbb{Q}^\theta$*;*

- $\underline{\widetilde{N}}^\theta$ *defined by*

$$\underline{\widetilde{N}}^\theta(dr;dz) = N^X(dr;dz) - \underline{\rho}^{\theta^X}(dz)dr, \tag{2.28}$$

  is the compensated measure of N^X under $\mathbb{Q}^\theta$ with $\underline{\rho}^{\theta^X}(dz) := e^{-\theta z}\rho^X(dz)$.

Proof. This follows easily from Equation (2.8) by the application of Girsanov-Meyer Theorem (See Øksendal and Sulem [27], Protter [30]). □

3. The locally risk-minimizing hedging problem

In this section, we recall some terminology on local risk minimization. We shall simply give necessary results; for further informations, the reader is referred to the survey of Schweizer [34] from which our presentation owes much.

3.1. Review of some notions on the risk-minimization approach.

This concept has been introduced by Föllmer and Sondermann [20] for nonredundant (or non-attainable) contingent claim written on a one-dimensional, square-integrable discounted risky asset ξ which is a martingale under the original measure $\mathbb{P}$. Concretely, given a stochastic basis as above the goal consist to minimize the conditional remaining risk : $\mathcal{R}_t := \mathbb{E}^\mathbb{P}[(C_T - C_t)^2|\mathcal{F}_t]$ for all $t \in \mathcal{T}$. Here C_t stands for the cost process and is defined as the difference between the value of the (portfolio) strategy detained by the investor at time t and the gains made from trading in the financial market up to time t. Let $\mathcal{L}^2(\xi)$ the space of all $\mathbb{R}$-valued predictable process ϕ such that

$$||\phi||_{\mathcal{L}^2(\xi)} := \Big(\mathbb{E}^\mathbb{P}\Big[\int_0^T \phi_u^2 d[\xi,\xi]_u\Big]\Big)^{\frac{1}{2}} < \infty,$$

A trading strategy is a pair of processes $\varphi = (\phi,\psi)$ where ψ is an adapted process and $\phi \in \mathcal{L}^2(\xi)$ is a $\mathbf{F}$-predictable process, such that the value process $V := \phi\xi + \psi$ has right continuous sample paths and $\mathbb{E}^\mathbb{P}[V_t^2] < \infty$ for every $t \in \mathcal{T}$(i.e $V_t \in \mathcal{L}^2(\Omega,\mathbb{P})$ for every $t \in \mathcal{T}$).

For a trading strategy $\varphi = (\phi,\psi)$, where $\phi = (\phi_t)_{t\in\mathcal{T}}$ denotes at time t, the number of stocks held and $\psi = (\psi_t)_{t\in\mathcal{T}}$ the amount invested in the money market account.

Let H be a claim which is $\mathcal{F}_T$-measurable and square-integrable. Consider a strategies that replicate the contingent claim H at time T; that is the strategies with the assumption

$$V_T = H \;\; \mathbb{P}\text{-a.s.}$$

Such strategies are called *H-admissible*.

A trading strategy φ such that $C_t(\varphi) = C_0(\varphi)$ for all $t \in \mathcal{T}$ is called *self-financing*. Furthermore, if the cost process $C_t(\varphi)$ is a $\mathbb{P}$-martingale then φ is said to be *mean self-financing*.

Definition 3.1. *Let (ϕ, ψ) and $(\widetilde{\phi}, \widetilde{\psi})$ be H-admissible strategies. Then $(\widetilde{\phi}, \widetilde{\psi})$ is called a H-admissible strategy continuation of (ϕ, ψ) at time $t \in [0, T)$ if $\widetilde{\phi}_s = \phi_s$ for $s \in [0, t]$ and $\widetilde{\psi}_s = \psi_s$ for $s \in [0, t)$.*

The following result obtained by Föllmer and Sondermann [20] is based on the Galtchouk-Kunita-Watanabe (GKW) decomposition (see Kunita-Watanabe [25]) of H and gives a risk-minimizing hedging strategy under full information.

Theorem 3.2. *Assume the GKW decomposition of the claim $H \in \mathcal{L}^2(\Omega, \mathbb{P})$ given by*

$$H = H_0 + \int_0^T \phi_s^H d\xi_s + L_T^H,$$

with $\phi^H \in \mathcal{L}^2(\xi)$, L^H a square-integrable $\mathbb{P}$-martingale orthogonal to ξ with $H_0 = \mathbb{E}^{\mathbb{P}}[H]$ $\mathbb{P}$-a.s. Then, the trading strategy $\varphi^{\otimes} = (\phi^{\otimes}, \psi^{\otimes})$ defined by

$$(\phi_t^{\otimes}, \psi_t^{\otimes}) := (\phi_t^H, H_0 + \int_0^t \phi_s^H d\xi_s - \phi_t^H \xi_t + L_t^H), \quad \forall t \in [0, T] \tag{3.1}$$

is H-admissible and risk-minimizing. Its associated risk process $\mathcal{R}^{\otimes}$ is given by

$$\mathcal{R}_t^{\otimes} = \mathbb{E}^{\mathbb{P}}[(L_T^H - L_t^H)^2 | \mathcal{F}_t], \ \mathbb{P} - a.s. \quad \forall t \in [0, T]. \tag{3.2}$$

Furthermore, this strategy is unique.

From now on, we assume that the one-dimensional discounted asset ξ is no longer a martingale under the measure $\mathbb{P}$ but only a semimartingale with the following decomposition

$$\xi = \xi_0 + Z + A \tag{3.3}$$

where Z a square-integrable martingale for which $Z_0 = 0$, and A a predictable process of finite variation $|A|$(i.e $\sup_\tau \sum_{i=1}^{N_\tau} |A_{t_i} - A_{t_{i-1}}| < \infty$) for every partition τ of $\mathcal{T}$. In this situation, we cannot longer apply the preceding result of Föllmer and Sondermann [20]. To deal with such a case, Schweizer [33, 34] introduced the concept of locally risk-minimizing strategy where the conditional variances are kept as small as possible but now in a local manner. Now, to adapt the definition of a trading strategy in this case we need that $\phi \in \mathcal{L}^2(Z)$ and that $\int_0^T |\phi_u dA_u| \in \mathcal{L}^2(\Omega, \mathbb{P})$.

Definition 3.3. *(small perturbation). A trading strategy $\Delta = (\delta, \epsilon)$ is called a* small perturbation *if it satisfies the following conditions:*

- *δ is bounded;*
- *$\int_0^T |\delta_u||dA_u|$ is bounded;*
- *$\delta_T = \epsilon_T = 0$.*

For any subinterval $(s, T] \subset [0, T]$, we define the small perturbation $\Delta\big|_{(s,T]} := (\delta 1_{(s,T]}, \epsilon 1_{(s,T]})$.

Now we can define

Definition 3.4. *(locally risk-minimizing strategy). For a trading strategy φ, a small perturbation Δ and a partition τ of $[0,T]$ the* risk-quotient (R-quotient) $r^\tau[\varphi,\Delta]$ *which is a sort of relative local risk is defined as*

$$r^\tau[\varphi,\Delta] := \sum_{t_i,t_{i+1}\in\tau} \frac{\mathcal{R}_{t_i}(\varphi+\Delta\big|_{(t_i,t_{i+1}]}) - \mathcal{R}_{t_i}(\varphi)}{\mathbb{E}^{\mathbb{P}}[\langle Z\rangle_{t_{i+1}} - \langle Z\rangle_{t_i}|\mathcal{F}_{t_i}]} 1_{(t_i,t_{i+1}]}. \tag{3.4}$$

A trading strategy φ is called locally risk-minimizing if

$$\liminf_{n\to\infty} r^{\tau_n}[\varphi,\Delta] \geq 0, \quad \mathbb{P}\times\langle Z\rangle - a.s.$$

for every small perturbation Δ and every increasing sequence (τ_n) of partitions of $\mathcal{T}$ such that $||\tau_n|| \to 0$.

To present the main results, we need the following technical assumptions:

Assumption 3.5.

- **(A1)** *For $\mathbb{P}$-almost all ω the measure on $[0,T]$ induced by $\langle Z\rangle(\omega)$ has the whole interval $[0,T]$ as its support, i.e $\langle Z\rangle$ should be $\mathbb{P}$-almost surely strictly increasing on the whole interval $[0,T]$.*
- **(A2)** *A is continuous.*
- **(A3)** *A is absolutely continuous with respect to $\langle Z\rangle$ with a density α satisfying*

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\alpha_u| \max(\log|\alpha_u|,0) d\langle Z\rangle_u\Big] < \infty.$$

A sufficient condition for **(A3)** is that $\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\alpha_u|^2 d\langle Z\rangle_u\Big] < \infty$ and one refers to that by saying: ξ satisfies *the Structure Condition (SC)*. We can remark that with assumption **(A2)**, ξ is a *special* semimartingale. We can now state the optimality result.

Theorem 3.6. *A contingent claim $H \in \mathcal{L}^2(\Omega,\mathbb{P})$ admits a (pseudo-optimal) locally risk-minimizing strategy $\varphi^\odot = (\phi^\odot,\psi^\odot)$ with $V_T(\varphi^\odot) = H$ $\mathbb{P}$ a.s. if and only if H can be written as*

$$H = H_0 + \int_0^T \phi_s^H d\xi_s + L_T^H \quad \mathbb{P} \ a.s. \tag{3.5}$$

with $H_0 \in \mathcal{L}^2(\Omega,\mathbb{P})$, $\phi^H \in \mathcal{L}^2(\xi)$, L^H a square-integrable $\mathbb{P}$-martingale null at the origin and $\mathbb{P}$-strongly orthogonal to M. The strategy φ is then given by

$$\phi_t^\odot = \phi_t^H, \quad t\in[0,T]$$

and

$$C_t(\varphi^\odot) = H_0 + L_t^H, \quad t\in[0,T];$$

its value process is

$$V_t(\varphi^\odot) = C_t(\varphi) + \int_0^t \phi_s^\odot d\xi_s = H_0 + \int_0^t \phi_s^H d\xi_s + L_t^H, \quad t\in[0,T]. \tag{3.6}$$

Proof. See Proposition 3.4 of Schweizer [34]. □

Equation (3.5) is called *Föllmer-Schweizer decomposition (FS)* for the contingent claim H. In practice, to obtain this decomposition is very difficult so the more natural approach introduced by Föllmer and Schweizer [19] consist to use a Girsanov transformaton to shift the problem back to a martingale measure where standard techniques as Galchouk-Kunita-Watanabe projection is available.

4. Main results

4.1. **A martingale representation property.**

In this section, we give an explicit representation of a martingale which is useful for the problem of hedging in the context of a Markov-modulated Lévy model. The proof of the result is similar to the one given by Elliott *et al.* [15]. We give an explicit martingale representation of the wealth process which will be useful later on in the finding of an optimal strategy the proof of our main result.

First, it is easy to see that the Esscher transform change of measure Λ^θ introduced in Section 2.3 is solution to this following SDE

$$\begin{cases} \Lambda_{t,\ u}(x) & = 1 + \int_t^u \Lambda_{t,\ r^-}(x)(-\theta_r\sigma_r)(r, \xi_{t,\ r^-}(x), X_r)dW_r \\ & \quad + \int_t^u \int_{\mathbb{R}\setminus\{0\}} \Lambda_{t,\ r^-}(x)(e^{-z\theta_r(r,\xi_{t,\ r^-}(x),X_r)} - 1)\widetilde{N}^X(dr; dz) \\ \Lambda_{t,\ t}(x) & = 1 \quad \mathbb{P}\ a.s.\ for\ 0 \le t < u \le T. \end{cases} \tag{4.1}$$

Indeed, for all $t \in [0,T]$, $\Lambda_t^\theta = \Lambda_{0,\ t}(x)$.

Now, consider a function $c(\cdot) : (0, +\infty) \to \mathbb{R}$ such that $c(\cdot)$ is twice differentiable and $c(\cdot)$ and $\frac{\partial c(\cdot)}{\partial x}$ are at most linear growth in x. We shall determine the current price at time t of a contingent claim of the form $c(S_T)$, which is the payoff of the claim at maturity $T > t$. In the sequel, we have to work with the discounted claim as function of the discounted stock price, that is:

$$\hat{c}(\xi_{0,T}) := R_T^{-1} c(R_T \xi_{0,T}(x_0)) = R_T^{-1} c(S_T). \tag{4.2}$$

So, we assume that the process θ is chosen such that $\mathbb{E}^{\mathbb{Q}^\theta}[\hat{c}^2(\xi_{0,T}(x_0))] < \infty$ and then we define the square-integrable $(\overline{\mathbf{G}}, \mathbb{Q}^\theta)$-martingale $\{V_t\}_{t\in[0,T]}$ as:

$$V_t := \mathbb{E}^{\mathbb{Q}^\theta}[\hat{c}(\xi_{0,T}(x_0))|\overline{\mathcal{G}}_t], \quad t \in [0,T]. \tag{4.3}$$

As (X, ξ) and (X, Λ) are Markov additive processes (See Çinlar [4]) we have that they verify the Markov property with respect to the large filtration $\overline{\mathbf{G}}$. Hence, we obtain by using

Bayes'rule

$$\begin{aligned}
V_t &:= \mathbb{E}^{\mathbb{Q}^\theta}[\hat{c}(\xi_{0,\ T}(x_0))|\overline{\mathcal{G}}_t] \\
&= \frac{\mathbb{E}^{\mathbb{P}}[\Lambda_{0,\ T}(x_0)\hat{c}(\xi_{0,\ T}(x_0))|\mathcal{G}_t]}{\mathbb{E}^{\mathbb{P}}[\Lambda_{0,\ T}(x_0)|\overline{\mathcal{G}}_t]} \\
&= \mathbb{E}^{\mathbb{P}}\Big[\frac{\Lambda_{0,\ t}(x_0)\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))}{\Lambda_{0,\ t}(x_0)}\Big|\overline{\mathcal{G}}_t\Big], \quad \text{because } \mathbb{E}^{\mathbb{P}}[\Lambda_{t,\ T}(x)|\overline{\mathcal{G}}_t] = 1; \\
&= \mathbb{E}^{\mathbb{P}}[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))|\overline{\mathcal{G}}_t] \\
&= \mathbb{E}^{\mathbb{P}}[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))|X_t = \mathbf{e}, \xi_{0,\ t}(x_0) = x]. \qquad (4.4)
\end{aligned}$$

Thus, we define for each $x \in (0, +\infty)$ and $\mathbf{e} \in \mathbb{S}$,

$$\begin{aligned}
V(t, x, \mathbf{e}) &:= \mathbb{E}^{\mathbb{P}}[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))|X_t = \mathbf{e}, \xi_{0,\ t}(x_0) = x] \qquad (4.5)\\
&(= \mathbb{E}^{\mathbb{Q}^\theta}[\hat{c}(\xi_{t,\ T}(x))|X_t = \mathbf{e}, \xi_{0,\ t}(x_0) = x]).
\end{aligned}$$

For each (t, u) such that $0 \le t < u \le T$, let introduce the following processes:

(1) L defined by

$$\begin{aligned}
L_{t,\ u} &:= \int_t^u \frac{\partial(-\theta_r\sigma_r)}{\partial \xi}(r,\ \xi_{t,\ r}(x), X_r) \times \frac{\partial \xi_{t,\ r}}{\partial x} dW_r^\theta \\
&+ \int_t^u \int_{\mathbb{R}\setminus\{0\}} \Big[e^{z\theta_r(r,\ \xi_{t,\ r^-}(x),\ X_r)} \frac{\partial e^{-z\theta_r(r,\ \xi_{t,\ r^-}(x),\ X_r)}}{\partial \xi} \times \frac{\partial \xi_{t,\ r^-}}{\partial x}(x)\Big] \underline{\widetilde{N}}^\theta(dr, dz),
\end{aligned}$$

(2) K defined by

$$\begin{aligned}
K_{t,\ u} &:= \int_t^u \frac{\Lambda_{t,\ r}(x + \zeta_t(y))}{\Lambda_{t,\ r}(x)} \Big[(-\theta_r\sigma_r)(r,\ \xi_{t,\ r}(x + \zeta_t(y)), X_r) + (\theta_r\sigma_r)(r,\ \xi_{t,\ r}(x), X_r)\Big] dW_r^\theta \\
&+ \int_t^u \int_{\mathbb{R}\setminus\{0\}} \frac{\Lambda_{t,\ r^-}(x + \zeta_t(y))}{\Lambda_{t,\ r^-}(x)} \Big[\frac{e^{-z\theta_r(r,\ \xi_{t,\ r^-}(x+\zeta_t(y)),\ X_r)} - e^{-z\theta_r(r,\ \xi_{t,\ r^-}(x),\ X_r)}}{e^{-z\theta_r(r,\ \xi_{t,\ r^-}(x),\ X_r)}}\Big] \underline{\widetilde{N}}^\theta(dr, dz)
\end{aligned}$$

with $\xi_{t^-} = x,\ \zeta_t(y) := \zeta(t, x, y)$,

(3) $\mathbf{V}$ the vector process defined by

$$\mathbf{V}(t, \xi_{0,\ t}(x_0)) := \Big(V(t, \xi_{0,\ t}(x_0), \mathbf{e}_1), V(t, \xi_{0,\ t}(x_0), \mathbf{e}_2), \dots, V(t, \xi_{0,\ t}(x_0), \mathbf{e}_M)\Big).$$

Now, we are able to give an martingale representation for the $\{V_t\}_{t\in\mathcal{T}}$.

Proposition 4.1. *The $(\overline{\mathbf{G}}, \mathbb{Q}^\theta)$-martingale $\{V_t\}_{t\in\mathcal{T}}$ has the representation*

$$V_t = V_0 + \int_0^t \phi_r^c(\xi_r, X_r) dW_r^\theta + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \phi_r^d(z, \xi_{r^-}, X_{r^-}) \underline{\widetilde{N}}^\theta(dr, dz) + \int_0^t \langle \alpha_r, d\Gamma_r \rangle, \qquad (4.6)$$

where ϕ^c, ϕ^d and α are such that,
$\mathbb{E}^{\mathbb{Q}^\theta}\Big[\int_0^T (\Phi_r^c)^2 dr\Big] < \infty$, $\mathbb{E}^{\mathbb{Q}^\theta}\Big[\int_0^T ||\alpha_r||^2 dr\Big] < \infty$ *and* $\mathbb{E}^{\mathbb{Q}^\theta}\Big[\int_0^T \int_{\mathbb{R}\setminus\{0\}} (\phi_r^d(z))^2 \rho^X(dz) dr\Big] < \infty$,

with the following explicit expressions

$$\phi^c_r(\xi_r, X_r) = \mathbb{E}^{\mathbb{Q}^\theta}\left[L_{r,\ T}\hat{c}(\xi_{r,\ T}(x)) + \frac{\partial \hat{c}}{\partial \xi}(\xi_{r,\ T}(x))\frac{\partial \xi_{r,\ T}}{\partial x}(x)\Big| X_r = \mathbf{e}, \xi_{0,\ r}(x_0) = x\right]\sigma_r(r, \xi_r, X_r); \tag{4.7}$$

$$\phi^d_r(y, \xi_{r^-}, X_r) = \mathbb{E}^{\mathbb{Q}^\theta}\left[(K_{r,\ T}+1)\hat{c}(\xi_{r,\ T}(x_- + \zeta_r(z))) - \hat{c}(\xi_{r,\ T}(x))\Big| X_r = \mathbf{e}, \xi_{0,\ r}(x_0) = x\right]; \tag{4.8}$$

$$\alpha_t = \mathbf{V}(t, \xi_{0,\ t}(x_0)) \in \mathbb{R}^M. \tag{4.9}$$

with $x = \xi_{0,\ r}(x_0)$ and $x_- = \xi_{0,\ r^-}(x_0)$.

In order to prove Proposition 4.1, we need the subsequent result

Lemma 4.2. *The following identities hold*

$$\frac{\partial \Lambda_{t,\ T}}{\partial x}(x) = \Lambda_{t,\ T}(x) \times L_{t,\ T} \tag{4.10}$$

and

$$\Lambda_{t,\ T}(x + \zeta(z)) - \Lambda(x) = \Lambda_{t,\ T}(x) \times K_{t,\ T}. \tag{4.11}$$

Proof. See Appendix. □

Now, we give the proof of the Proposition 4.1.

Proof. (Proposition 4.1)

Noting that

$$V(t, \xi_t, X_t) = \langle \mathbf{V}(t, \xi_t)|X_t\rangle, \tag{4.12}$$

we obtain dy differentiation

$$dV(t, \xi_t, X_t) = \langle d\mathbf{V}(t, \xi_t)|X_t\rangle + \langle \mathbf{V}(t, \xi_t)|dX_t\rangle, \tag{4.13}$$

and from Itô differentiation rule

$$\begin{aligned} dV(t, \xi_t, X_t) = \left\langle \mathbf{V}(t, \xi_t)\Big| dX_t\right\rangle + \Big\langle \frac{\partial \mathbf{V}}{\partial t}dt + \frac{\partial \mathbf{V}}{\partial \xi}d\xi_t + \frac{1}{2}\frac{\partial^2 \mathbf{V}}{\partial \xi^2}d[\xi, \xi]^c_t \\ + \int_{\mathbb{R}\setminus\{0\}}\left[\mathbf{V}(t, \xi_{t^-}e^z) - \mathbf{V}(t, \xi_{t^-}) - \Delta\xi_t\frac{\partial \mathbf{V}}{\partial \xi}\right]N^X(dt, dz)\Big| X_t\Big\rangle \end{aligned} \tag{4.14}$$

From (3.3), we deduce that

$$dX_t = AX_{t^-}dt + d\Gamma_t. \tag{4.15}$$

By replacing this last expression in (4.14), we obtain

$$\begin{aligned}
&dV(t,\xi_t,X_t)\\
&=\left\langle \left[\frac{\partial \mathbf{V}}{\partial t}+\frac{1}{2}\sigma_t^2\xi_{t-}^2\frac{\partial^2\mathbf{V}}{\partial \xi^2}+\int_{\mathbb{R}\setminus\{0\}}\Big[\mathbf{V}(t,\xi_{t-}e^z)-\mathbf{V}(t,\xi_{t-})-\xi_{t-}(e^z-1)\frac{\partial \mathbf{V}}{\partial \xi}\Big]\rho^{\theta^X}(dz)\right]dt\,\middle|\, X_t\right\rangle\\
&+\left\langle \mathbf{V}(t,\xi_t)\,\middle|\, AX_{t-}\right\rangle dt+\left\langle \mathbf{V}(t,\xi_t)\,\middle|\, d\Gamma_t\right\rangle\\
&+\left\langle \sigma_t\xi_{t-}\frac{\partial \mathbf{V}}{\partial \xi}dW_t^\theta+\int_{\mathbb{R}\setminus\{0\}}\Big[\mathbf{V}(t,\xi_{t-}e^z)-\mathbf{V}(t,\xi_{t-})\Big]\underline{\tilde{N}}^\theta(dt,dz)\,\middle|\, X_t\right\rangle
\end{aligned}\tag{4.16}$$

As $\{V_t=V(t,\xi_t,X_t)\}_{t\in\mathcal{T}}$ is a $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$-martingale, his continuous finite variation part would be identically equal to zero $\mathbb{Q}^\theta$ a.s, thus

$$\begin{aligned}
&\left\langle \frac{\partial \mathbf{V}}{\partial t}+\frac{1}{2}\sigma_t^2\xi_t^2\frac{\partial^2\mathbf{V}}{\partial \xi^2}+\int_{\mathbb{R}\setminus\{0\}}\Big[\mathbf{V}(t,\xi_{t-}e^z)-\mathbf{V}(t,\xi_{t-})-\xi_{t-}(e^z-1)\frac{\partial \mathbf{V}}{\partial \xi}\Big]\rho^{\theta^X}(dz)\,\middle|\, X_t\right\rangle\\
&+\left\langle \mathbf{V}(t,\xi_t)\,\middle|\, AX_{t-}\right\rangle=0
\end{aligned}\tag{4.17}$$

which is equivalent with $X_t=\mathbf{e}$ to:

$$\begin{aligned}
&\frac{\partial V}{\partial t}(t,\xi_t,\mathbf{e})+\frac{1}{2}\sigma_t^2\xi_t^2\frac{\partial^2 V}{\partial \xi^2}(t,\xi_t,\mathbf{e})+\left\langle \mathbf{V}(t,\xi_t)\,\middle|\, AX_{t-}\right\rangle\\
&+\int_{\mathbb{R}\setminus\{0\}}\Big[V(t,\xi_{t-}e^z,\mathbf{e})-V(t,\xi_{t-},\mathbf{e})-\xi_{t-}(e^z-1)\frac{\partial V}{\partial \xi}(t,\xi_t,\mathbf{e})\Big]\rho^{\theta^X}(dz)=0.
\end{aligned}\tag{4.18}$$

Hence, back to Equation (4.16), we deduce that

$$\begin{aligned}
V(t,\xi_t,\mathbf{e})&=V(0,\xi_0,X_0)+\int_0^t\sigma_s\xi_s\frac{\partial V}{\partial \xi}(s,\xi_s,X_s)dW_s^\theta\\
&\quad+\int_0^t\int_{\mathbb{R}\setminus\{0\}}\Big[V(s,\xi_{s-}e^z,X_s)-V(s,\xi_{s-},X_s)\Big]\underline{\tilde{N}}^\theta(ds,dz)+\int_0^t\left\langle \mathbf{V}(s,\xi_s)\,\middle|\, d\Gamma_s\right\rangle.
\end{aligned}\tag{4.19}$$

We deduce from the uniqueness of the decomposition of the special semimartingale V that

- $\Phi_t^c(\xi_t)=\sigma_t\xi_t\frac{\partial V}{\partial \xi}(t,\xi_t,\mathbf{e})$;
- $\Phi_t^d(z,\xi_{t-})=V(t,\xi_{t-}e^z,\mathbf{e})-V(t,\xi_t,\mathbf{e})$;
- $\alpha_t=\mathbf{V}(t,\xi_t)$.

To obtain a more explicit expressions for these quantities, we write by noting that $\xi_{0,\,t}=x$ and $\xi_{0,\,t^-}=x_-$

$$\begin{aligned}
\Phi^c_t(\xi_t) &= x\sigma_t(t,x,\mathbf{e})\frac{\partial V}{\partial x}(t,x,\mathbf{e})\\
&= x\sigma_t(t,x,\mathbf{e})\frac{\partial}{\partial x}\mathbb{E}^{\mathbb{P}}[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x] \quad \text{by (4.5)}\\
&= x\sigma_t(t,x,\mathbf{e})\mathbb{E}^{\mathbb{P}}\Big[\frac{\partial \Lambda_{t,\ T}}{\partial x}(x)\hat{c}(\xi_{t,\ T}(x))+\Lambda_{t,\ T}(x)\frac{\partial \hat{c}}{\partial \xi}(\xi_{t,\ T}(x))\frac{\partial \xi_{t,\ T}}{\partial x}(x)\Big|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big]\\
&= x\sigma_t(t,x,\mathbf{e})\mathbb{E}^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x)L_{t,\ T}\hat{c}(\xi_{t,\ T}(x))\\
&+ \Lambda_{t,\ T}(x)\frac{\partial \hat{c}}{\partial \xi}(\xi_{t,\ T}(x))\frac{\partial \xi_{t,\ T}}{\partial x}(x)\Big|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big] \quad \text{by Lemma 4.2}\\
&= x\sigma_t(t,x,\mathbf{e})\mathbb{E}^{\mathbb{Q}^\theta}\Big[L_{t,\ T}\hat{c}(\xi_{t,\ T}(x))+\frac{\partial \hat{c}}{\partial \xi}(\xi_{t,\ T}(x))\frac{\partial \xi_{t,\ T}}{\partial x}(x)\Big|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big]. \qquad (4.20)
\end{aligned}$$

In the same way,

$$\begin{aligned}
\Phi^d_t(z,\xi_{t^-}) &= V(t,\xi_{t^-}e^z,\mathbf{e})-V(t,\xi_{t^-},\mathbf{e})\\
&= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x_-+\zeta_r(z))\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))\Big|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big]\\
&- \mathbb{E}^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x)\hat{c}(\xi_{t,\ T}(x))\Big|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big]\\
&= \mathbb{E}^{\mathbb{P}}\Big[\Big(\Lambda_{t,\ T}(x_-+\zeta_r(z))-\Lambda_{t,\ T}(x)\Big)\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))\Big|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big]\\
&+ \mathbb{E}^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x)\Big(\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))-\hat{c}(\xi_{t,\ T}(x))\Big)|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big]\\
&= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_{t,\ T}(x)K_{t,T}\Big(\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z))\Big)\\
&+ \Lambda_{t,\ T}(x)\Big(\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))-\hat{c}(\xi_{t,\ T}(x))\Big)\Big|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big] \quad \text{by Lemma 4.2}\\
&= \mathbb{E}^{\mathbb{Q}^\theta}\Big[(K_{t,T}+1)\hat{c}(\xi_{t,\ T}(x_-+\zeta_r(z)))-\hat{c}(\xi_{t,\ T}(x))\Big|X_t=\mathbf{e},\xi_{0,\ t}(x_0)=x\Big]. \qquad (4.21)
\end{aligned}$$

Finally, we have to show that the different component involved in (4.19) are mutually orthogonal $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$-local martingale, that is, the different product $W^\theta\cdot\underline{\tilde{N}}^\theta(\cdot,dz)$, $W^\theta\cdot\Gamma$ and $\Gamma\cdot\underline{\tilde{N}}^\theta(\cdot,dz)$ are $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$-local martingale. The claim is easy verified for the first ones by noting that W^θ is an continuous $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$ local-martingale such that $W^\theta_0=0$ whereas $\underline{\tilde{N}}^\theta(\cdot,dz)$ and Γ are pure jump $(\overline{\mathbf{G}},\mathbb{Q}^\theta)$ local-martingales. For the last, we have $\forall t\in\mathcal{T}$ and $\forall i\in\{1,2,...,M\}$

$$\begin{aligned}
[\Gamma^i,\underline{\tilde{N}}^\theta(\cdot,dz)]_t &= \sum_{0\le s\le t}\Delta\Gamma^i_s\Delta\underline{\tilde{N}}^\theta(s,dz)\\
&= 0. \qquad (4.22)
\end{aligned}$$

This result comes from Assumption 2.1 and the decomposition theorem of the (additive) component of the MAP (X,S) given in Çinlar [5], theorem 2.23. □

4.2. **The locally risk-minimizing hedging Problem under full information for the model (2.4)-(2.2).**

In this section, we consider the problem of hedging a contingent claim H in the Markov-modulated exponential Lévy model given by (2.2)-(2.4) given that the information set is $\overline{\mathbf{G}}$. In general, in such a market the claim H cannot be perfectly hedged. Therefore, we need to take into account the market participant's attitude toward risk in the search of the viable market transactions. One way of doing this in the literature consists to optimize a given criterion based or not on the preference of the market participant. In particular, the choice of quadratic criterion is quite natural and pertinent because it leads to a linear pricing rule which is very meaningful in financial economics.

Let B be a contingent claim with a discounted payoff $H = \hat{c}(\xi_{0,\,T}(x_0)) \in \mathcal{L}^2(\Omega, \mathbb{P})$. Following Schweizer [32], a locally risk-minimizing strategy $\varphi = (\phi, \psi)$ which generates $\hat{c}(\xi_{0,\,T}(x_0))$ must be such that

(1) $V_T = \hat{c}(\xi_{0,T}(x_0))$ $\mathbb{P}$-a.s.;
(2) $V_t(\varphi) = V_0(\varphi) + \int_0^t \phi_r d\xi_r + \Upsilon_t, \quad$ for all $t \in [0,T]$;
(3) Υ is a martingale under $\mathbb{P}$ and Υ is orthogonal to the martingale part Z of ξ under $\mathbb{P}$.

We shall require that $(V_t(\varphi))_{0\le t\le T}$ is a $(\overline{\mathbf{G}}, \mathbb{Q}^\theta)$-martingale. With this assumption and Equation (4.5), we have

$$\begin{aligned} V_t(\varphi) &= \mathbb{E}^{\mathbb{Q}^\theta}[V_T(\varphi)|\overline{\mathcal{G}}_t] \\ &= \mathbb{E}^{\mathbb{Q}^\theta}[\hat{c}(\xi_{0,\,T}(x_0))|X_t = \mathbf{e}, \xi_{0,\,t} = x] \\ &= V(t, x, \mathbf{e}). \end{aligned}$$

Now we can state the main proposition if this section.

Proposition 4.3. *Assume $\sigma_t > 0$ for all $t \in [0,T]$. If there exists a process θ^* satisfying (2.24) and such that*

$$\theta_t^* = \frac{\mu_t - r_t}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}} (e^x - 1)^2 \rho^X(dx)}, \tag{4.23}$$

$$e^{-z\theta_t^*} - 1 = -\frac{(\mu_t - r_t)(e^z - 1)}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}} (e^x - 1)^2 \rho^X(dx)}, \quad \forall z \in \mathbb{R} \tag{4.24}$$

then there exists a minimal martingale measure *defined by the Esscher transform Λ^{θ^*}. Furthermore, the locally risk-minimizing strategy for the contingent claim H is given by*

$$\phi_t^* = \frac{1}{\xi_{t^-}} \times \frac{\sigma_t \phi_t^c(\xi_t, X_t) + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)\phi_t^d(y, \xi_{t^-}, X_{t^-})\rho^X(dz)}{\sigma_t^2 + \int_{\mathbb{R}\setminus\{0\}} (e^x - 1)^2 \rho^X(dx)}, \tag{4.25}$$

and

$$\begin{aligned} \psi_t^* &:= V_t(\varphi) - \phi_t^* \xi_t \\ &= \mathbb{E}^{\mathbb{Q}^{\theta^*}}[\hat{c}(\xi_{0,\,T}(x_0))|X_t = \mathbf{e}, \xi_{0,\,t} = x] - \phi_t^* \xi_t. \end{aligned} \tag{4.26}$$

Proof.
1- We have to show that if there exists a process θ^* satisfies the Equations (2.24), (4.23) and (4.24) then the process Λ^{θ^*} defines a minimal martingale measure in the sense of Schweizer [31].

Indeed, under these assumptions we have from Equation (4.1)

$$\begin{aligned}\Lambda_t^{\theta^*} &= 1+\int_0^t \Lambda_{s^-}^{\theta^*}(-\theta_s^*\sigma_s)dW_s + \int_0^t\int_{\mathbb{R}\setminus\{0\}}(e^z-1)\widetilde{N}^X(ds,dz)\\ &= 1-\int_0^t \Lambda_{s^-}^{\theta^*}\Big[\frac{\mu_s-r_s}{\sigma_s^2+\int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)}\Big]\Big[\sigma_s dW_s + \int_{\mathbb{R}\setminus\{0\}}(e^z-1)\widetilde{N}^X(ds,dz)\Big]\\ &= 1-\int_0^t \Lambda_{s^-}^{\theta^*}\frac{1}{\xi_{s^-}}\times\Big[\frac{\mu_s-r_s}{\sigma_s^2+\int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)}\Big]dZ_s, \qquad (4.27)\end{aligned}$$

where Z denoted the martingale part of the (special) semimartingale ξ. Using Assumptions 4.23 and 4.24, it is easy to see that the process λ given for $t\in\mathcal{T}$ by:

$$\begin{aligned}\lambda_t &:= \frac{dA_t}{d\langle Z\rangle_t}\\ &= \frac{1}{\xi_{t^-}}\times\frac{\sigma_t^2\theta_t^*+\int_{\mathbb{R}\setminus\{0\}}(e^z-1)(e^{-\theta_t^* z}-1)\rho^X(dz)}{\sigma_s^2+\int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)}\\ &= \frac{1}{\xi_{t^-}}\times\frac{\mu_t-r_t}{\sigma_t^2+\int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)} \qquad (4.28)\end{aligned}$$

is $\overline{\mathbf{G}}$-predictable and verifies $\int_0^t\lambda_s^2 d\langle Z\rangle_s<\infty$ $\mathbb{P}$-a.s. Hence, we see that

$$\Lambda_t^{\theta^*} = 1-\int_0^t\Lambda_{s^-}^{\theta^*}\lambda_s dZ_s. \qquad (4.29)$$

This defines precisely the minimal martingale measure according Föllmer and Schweizer [19]. In the sequel we will denote it by $\mathbb{Q}^{\theta^*}$.
2- From Föllmer and Schweizer ([19]) we now that once a MMM is found, the locally risk-minimizing strategy of the contingent claim is uniquely determined from the $(\overline{\mathbf{G}},\mathbb{Q}^{\theta^*})$-projection of Galtchouk-Kunita-Watanabe decomposition of $\hat{c}(\xi_{0,\ T}(x_0))$. From Proposition 4.1, we have for all $t\in[0,T]$,

$$V_t = V_0+\int_0^t\phi_r^c(\xi_r,X_r)dW_r^{\theta^*}+\int_0^t\int_{\mathbb{R}\setminus\{0\}}\phi_r^d(z,\xi_{r^-},X_{r^-})\underline{\widetilde{N}}^{\theta^*}(dr,dz)+\int_0^t\langle\alpha_r|d\Gamma_r\rangle \qquad (4.30)$$

where ϕ^c and ϕ^d are given by Equations (4.7) and (4.8) respectively. Therefore, we have from (2)

$$\begin{aligned}\Upsilon_t &= V_t(\varphi)-\int_0^t\phi_r d\xi_r - V_0(\varphi)\\ &= \int_0^t\Big[\phi_r^c(\xi_r,X_r)-\sigma_r\xi_r\phi_r\Big]\Big[dW_r+\theta_t^*\sigma_r dr\Big]\\ &+ \int_0^t\int_{\mathbb{R}\setminus\{0\}}\Big[\phi_r^d(z,\xi_{r^-},X_{r^-})-\xi_{r^-}(e^z-1)\phi_r\Big]\Big[N^X(dr,dz)-\rho^X(dz)dr-(e^{-\theta_r^* z}-1)\rho^X(dz)dr\Big]\\ &+ \int_0^t\langle\alpha_r|d\Gamma_r\rangle. \qquad (4.31)\end{aligned}$$

From (3), Υ should be a $(\overline{\mathbf{G}},\mathbb{P})$-martingale thus the drift term in (4.31) should be zero or equivalently

$$\begin{aligned}\xi_{t-}\phi_t\Big[\int_{\mathbb{R}\backslash\{0\}}(e^z-1)(e^{-\theta_t^* z}-1)\rho^X(dz)-\theta_t^*\sigma_t^2\Big]\\ =\int_{\mathbb{R}\backslash\{0\}}\phi_t^d(z,\xi_{t-},X_{t-})(e^{-\theta_t^* z}-1)\rho^X(dz)-\phi_t^c(\xi_t,X_t)\theta_t^*\sigma_t. \end{aligned}\tag{4.32}$$

Hence

$$\begin{aligned}\Upsilon_t=&\int_0^t[\phi_r^c(\xi_r,X_r)-\sigma_r\xi_r\phi_r]dW_r+\int_0^t\langle\alpha_r|d\Gamma_r\rangle\\ &+\int_0^t\int_{\mathbb{R}\backslash\{0\}}[\phi_r^d(z,\xi_{r-},X_{r-})-\xi_{r-}(e^z-1)\phi_r]\widetilde{N}^X(dr,dz).\end{aligned}\tag{4.33}$$

The requirement (3) stipulates also that Υ is orthogonal to the martingale part Z of ξ under $\mathbb{P}$. This is verified if and only if ΥZ is a $(\overline{\mathbf{G}},\mathbb{P})$-martingale, therefore

$$\xi_{t-}\phi_t\Big[\int_{\mathbb{R}\backslash\{0\}}(e^x-1)^2\rho^X(dx)+\sigma_t^2\Big]=\phi_t^c(\xi_t,X_t)\sigma_t+\int_{\mathbb{R}\backslash\{0\}}\phi_t^d(z,\xi_{r-},X_{r-})(e^z-1)\rho^X(dz).\tag{4.34}$$

Recalling the martingale condition (2.24) and substituting it in Equation (4.32) we obtain

$$\xi_{t-}\phi_t(r_t-\mu_t)=\int_{\mathbb{R}\backslash\{0\}}\phi_t^d(z,\xi_{t-},X_{t-})(e^{-\theta_t^* z}-1)\rho^X(dz)-\phi_t^c(\xi_t,X_t)\theta_t^*\sigma_t,\tag{4.35}$$

and using Equation (4.34), we get that θ^* satisfies

$$\begin{aligned}&\left[\theta_t^*-\frac{\mu_t-r_t}{\sigma_t^2+\int_{\mathbb{R}\backslash\{0\}}(e^x-1)^2\rho^X(dx)}\right]\phi_t^c(\xi_t,X_t)\sigma_t\\ &-\int_{\mathbb{R}\backslash\{0\}}\left[(e^{-\theta_t^* z}-1)+\frac{(\mu_t-r_t)(e^z-1)}{\sigma_t^2+\int_{\mathbb{R}\backslash\{0\}}(e^x-1)^2\rho^X(dx)}\right]\phi_t^d(y,\xi_{t-},X_{t-})\rho^X(dz)=0.\end{aligned}\tag{4.36}$$

Thus, if there exists a process θ^* verifying (2.24) and such that $\forall t\in[0,T]$

$$\theta_t^*=\frac{\mu_t-r_t}{\sigma_t^2+\int_{\mathbb{R}\backslash\{0\}}(e^x-1)^2\rho^X(dx)}$$

and

$$e^{-\theta_t^* z}-1=-\frac{(\mu_t-r_t)(e^z-1)}{\sigma_t^2+\int_{\mathbb{R}\backslash\{0\}}(e^x-1)^2\rho^X(dx)},\quad \forall z\in\mathbb{R}\backslash\{0\}.$$

Then a locally risk-minimizing strategy exists (independently of the claim to be hedged) and is deduced from Equations (4.34) and (3.1)

$$\left\{\begin{array}{rl}\phi_t^*= & \frac{1}{\xi_{t-}}\times\frac{\sigma_t\phi_t^c+\int_{\mathbb{R}\backslash\{0\}}\phi_{t-}^d(z)(e^z-1)\rho^X(dz)}{\sigma_t^2+\int_{\mathbb{R}\backslash\{0\}}(e^x-1)^2\rho^X(dx)}\\ \psi_t^*= & V_0+\int_0^t\phi_s^* d\xi_s-\phi_t^*\xi_t+\Gamma_t.\end{array}\right.\tag{4.37}$$

The expression of ψ^* follows from the definition of the portfolio value process V and this ends the proof . ☐

We can derive easily the expression of the residual $\overline{\mathbf{G}}$-risk process Υ for all $t \in \mathcal{T}$ as

$$\Upsilon_t = \int_0^t \frac{1}{\sigma_r^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)} \times \Big[\phi_r^c \int_{\mathbb{R}\setminus\{0\}} (e^z-1)^2\rho^X(dz) - \sigma_r \int_{\mathbb{R}\setminus\{0\}} \phi_{r-}^d(z)(e^z-1)\rho^X(dz)\Big] dW_r$$
$$+ \int_0^t \int_{\mathbb{R}\setminus\{0\}} \frac{1}{\sigma_r^2 + \int_{\mathbb{R}\setminus\{0\}}(e^x-1)^2\rho^X(dx)} \times \Big[\sigma_r^2 \phi_{r-}^d(z) - (e^z-1)\sigma_r\phi_r^c\Big] \widetilde{N}^X(dr,dz) + \int_0^t \langle \alpha_r | d\Gamma_r \rangle. \tag{4.38}$$

Remark 4.4. *It is possible that Equation 4.36 has not a unique solution, for example if* $\phi^c \equiv 0 \equiv \phi^d$.

4.3. **The locally risk-minimizing hedging Problem under partial information.**

This section considers the problem of the local risk-minimization of the contingent claim H when the asset dynamics follows Equation (2.4) from the viewpoint of an investor/hedger which does not have at his disposal the full information as described by the filtration $\overline{\mathbf{G}} = \{\overline{\mathcal{G}}_t\}_{t\in\mathcal{T}}$ but only the information set $\mathbf{G} = \{\mathcal{G}_t\}_{t\in\mathcal{T}}$; with $\mathcal{G}_t \subset \overline{\mathcal{G}}_t$ for all $t \in \mathcal{T}$. We have that $\mathcal{G}_T = \overline{\mathcal{G}}_T$ thus the contingent claim $H = \hat{c}(\xi_{0,T}(x_0))$ which is $\overline{\mathcal{G}}_T$-measurable will also $\mathcal{G}_T$-measurable.

We aim at finding a $\mathbf{G}$-locally risk-minimizing strategy. From the previous section, we have the following representation

$$V_t(\varphi^*) = V_0(\varphi^*) + \int_0^t \phi_r^* d\xi_r + \Upsilon_t, \quad \text{for all } t \in [0,T] \tag{4.39}$$

where Υ is a $(\overline{\mathbf{G}}, \mathbb{P})$-martingale which is orthogonal to the martingale part Z of ξ under $\mathbb{P}$. Since we only admit strategies $\varphi = (\phi, \psi)$ such that the process $(V)_{t\in[0,T]}$ is square-integrable, has right continuous paths and satisfy $V_T = H$. So, we have that

$$H = \widetilde{H}_0 + \int_0^T \phi_r^* d\xi_r + \Upsilon_T, \tag{4.40}$$

where $\widetilde{H}_0 = V_0(\varphi^*)$ is $\overline{\mathcal{G}}_0$-measurable and $\phi^* = (\phi_t^*)_{t\in[0,T]}$ is $\overline{\mathbf{G}}$-predictable.

In the sequel, we make the following assumption

$$\mathbb{E}^{\mathbb{P}}\Big[\widetilde{H}_0^2 + \int_0^T (\phi_r^*)^2 d\langle\xi\rangle_r + \Big(\int_0^T |\phi_r^*| dA_r\Big)^2\Big] < \infty. \tag{4.41}$$

Let $\mathcal{P}$ (resp.$\widetilde{\mathcal{P}}$) the σ-field of predictable subsets on $\overline{\Omega} = \Omega \times [0,T]$ associated to the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ (resp.$(\overline{\mathcal{G}}_t)_{t\in[0,T]}$). We denote by $\overline{\mathbb{P}}$ the finite measure on $\mathcal{P}$ defined by

$$\overline{\mathbb{P}}(d\omega, dt) = \mathbb{P}(d\omega) \times d\langle\xi\rangle_t(\omega).$$

$\overline{\mathbb{Q}}^{\theta^*}$ is defined in the same way. We can now state a Föllmer-Schweizer type decomposition result. This result is adapted from Föllmer and Schweizer [19]

Theorem 4.5.
Giving the decomposition (4.40), H admits the following representation (Föllmer- Schweizer decomposition)

$$H = H_0 + \int_0^T \phi_r^H d\xi_r + L_T^H \tag{4.42}$$

with $H_0 := \mathbb{E}^{\mathbb{P}}[\widetilde{H}_0|\mathcal{G}_0]$, *where*

$$\phi^H = \mathbb{E}^{\overline{\mathbb{P}}}[\phi^*|\mathcal{P}] \tag{4.43}$$

is the conditional expectation of ϕ^* *with respect to* $\mathcal{P}$ *and* $\overline{\mathbb{P}}$, *and where* $L^H := (L_t^H)_{t\in[0,T]}$ *is the square-integrable* $\mathbf{G}$*-martingale orthogonal to* Z *associated to*

$$L_T^H = \widetilde{H}_0 - H_0 + \int_0^T (\phi_r^* - \phi_r^H) d\xi_r + \Upsilon_T \in \mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P}). \tag{4.44}$$

Proof.
1- We need to show in a similar way as in Fllmer and Schweizer [19] that all component in (4.42) are square-integrable. From Assumption 4.41 $\phi^* \in \mathcal{L}^2(\overline{\Omega}, \mathcal{P}, \overline{\mathbb{P}})$ and thus $\phi^H \in \mathcal{L}^2(\overline{\Omega}, \mathcal{P}, \overline{\mathbb{P}})$ by Jensen inequality. Since $\phi^H \in \mathcal{L}^2(\overline{\Omega}, \mathcal{P}, \overline{\mathbb{P}})$, by Doob's maximal inequality $\int_0^T \phi_r^H dM_r \in \mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$.
To show that $\int_0^T \phi_r^H dA_r \in \mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$, we have by predictable projection, Assumption 4.41 and Doob's maximal inequality that the application $\vartheta \longrightarrow \mathbb{E}^{\mathbb{P}}\Big[\vartheta \int_0^T \phi_r^H dA_r\Big]$ defined on $\mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$ is an element of the dual of this space but this dual is exactly (up to an isomorphism) $\mathcal{L}^2(\Omega, \mathcal{G}_T, \mathbb{P})$.
2- Now, Let us show that L_T^H is orthogonal to all square-integrable stochastic integrals of Z. It is sufficient to show that for any bounded $\mathcal{P}$-measurable process $\chi = (\chi)_{t\in[0,T]}$ the following holds:

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T (\phi_r^* - \phi_r^H) d\xi_r\Big).\Big(\int_0^T \chi_r dM_r\Big)\Big] = 0 \\ \Leftrightarrow\ &\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \phi_r^* d\xi_r\Big).\Big(\int_0^T \chi_r dM_r\Big)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \phi_r^H d\xi_r\Big).\Big(\int_0^T \chi_r dM_r\Big)\Big]. \end{aligned}$$

But the left hand side can be decomposed into two components. So, by Itô-type isometry

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \phi_r^* dM_r\Big).\Big(\int_0^T \chi_r dM_r\Big)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \phi_r^H \chi_r d\langle \xi \rangle_r\Big]$$

and by Predictable Projection

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T \phi_r^* dA_r\Big).\Big(\int_0^T \chi_r dM_r\Big)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \phi_r^*.\Big(\int_0^r \chi_s dM_s\Big) d\langle \xi \rangle_r\Big].$$

Now, we can replace in both parts ϕ^* by ϕ^H which finally gives the result.
3- It remains to show that Υ_T is orthogonal to all square-integrable stochastic integrals of Z. This follows from the fact that $(\Upsilon_t)_{t\in[0,T]}$ is orthogonal to Z. Therefore, L is orthogonal to Z. □

Remark 4.6. *The last result states that the contingent claim H has an orthogonal decomposition with respect to the smaller filtration. This result follows from the fact that a same decomposition is available with respect to the larger filtration. However, has pointed by Arai*

[1] *it is not always true in general that the contingent claim will have an orthogonal decomposition when dealing with discontinuous market model. Such orthogonal decomposition holds for instance when making the restrictive assumption that jumps of processes Z, L and Λ^θ do not happen simultaneously almost surely. Our model is one of those where the orthogonal decomposition (4.42) holds, this leads to the following proposition.*

Proposition 4.7. *Under the hypothesis of Proposition 4.3 and Theorem 4.5, there exists a unique* $\mathbf{G}$*-locally risk-minimizing hedging strategy* $({}^{\mathbf{G}}\phi^*, {}^{\mathbf{G}}\psi^*)$ *given by*

$$\begin{array}{ll} {}^{\mathbf{G}}\phi^* & = \overline{\mathbb{E}}^{\mathbb{Q}^{\theta^*}}[\phi^*|\mathcal{P}] \\ {}^{\mathbf{G}}\psi^* & = {}^{\mathbf{G}}V - {}^{\mathbf{G}}\phi^*.\xi \end{array} \tag{4.45}$$

with ${}^{\mathbf{G}}V_t := \mathbb{E}^{\mathbb{Q}^{\theta^*}}[H|\mathcal{G}_t]$ *for* $t \in [0,T]$.

Proof. The existence and the uniqueness of the $\mathbf{G}$-locally risk-minimizing hedging strategy follows from Theorem 4.5 and Proposition 3.6. For the explicit expression of this strategy we need to show that

$$\phi^H = \overline{\mathbb{E}}^{\mathbb{Q}^{\theta^*}}[\phi^*|\mathcal{P}]$$

where

$$\phi^H = \mathbb{E}^{\overline{\mathbb{P}}}[\phi^*|\mathcal{P}].$$

Without lost the generality, we can suppose that $\phi^* \geq 0$ otherwise we can decompose it into the difference of two non-negative terms. So, it is equivalent to showing that

$$\mathbb{E}^{\mathbb{Q}^{\theta^*}}\Big[\int_0^T \vartheta_r \phi_r^* d\langle X\rangle_s\Big] = \mathbb{E}^{\mathbb{Q}^{\theta^*}}\Big[\int_0^T \vartheta_r \phi_r^H d\langle X\rangle_s\Big]$$

for any non-negative $\mathcal{P}$-measurable process ϑ. By the definition of $\mathbb{Q}^{\theta^*}$, the left hand side equals

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{\theta^*}}\Big[\int_0^T \vartheta_r \phi_r^* d\langle X\rangle_s\Big] &= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\theta^*}\int_0^T \vartheta_r \phi_r^* d\langle X\rangle_s\Big] \\ &= \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \Lambda_r^{\theta^*}\vartheta_r \phi_r^* d\langle X\rangle_s\Big] \quad \text{by predictable projection} \\ &= \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \Lambda_r^{\theta^*}\vartheta_r \phi_r^H d\langle X\rangle_s\Big] \quad \text{by definition of } \phi^H \\ &= \mathbb{E}^{\mathbb{P}}\Big[\Lambda_T^{\theta^*}\int_0^T \vartheta_r \phi_r^H d\langle X\rangle_s\Big] \\ &= \mathbb{E}^{\mathbb{Q}^{\theta^*}}\Big[\int_0^T \vartheta_r \phi_r^H d\langle X\rangle_s\Big] \end{aligned} \tag{4.46}$$

□

Remark 4.8. *The problem of local risk-minimization under a Markov-modulated exponential Lévy model was studied. By noting that it consists to finding a locally risk-minimizing strategy for a partially observed model (or partial information scenario), we first solve the problem in the case of full information by providing an useful explicit martingale representation for the contingent claim. After that,we give a solution to the main problem by using the predictable projection.*
For practical purpose, it would be interesting to give a computational algorithm for the optimal

strategy. This proceed by using the techniques of filtering theory and will be our focus in the future.

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Appendix

Proof. (Lemma 4.2) a) First, from (4.1)

$$\Lambda_{t,\ T}(x) = 1 + \int_t^T \Lambda_{t,\ r}(x)(-\theta_r\sigma_r)(r, \xi_{t,\ r}(x), X_r)dW_r + \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}(x)(e^{-z\theta_r(r,\xi_{t,\ r^-}(x),X_r)} - 1)\widetilde{N}^X(dr, dz)$$

and by differentiation we obtain that

$$\begin{aligned}
&\frac{\partial \Lambda_{t,\ T}}{\partial x}(x) \\
&= \int_t^T (-\theta_r\sigma_r)\frac{\partial \Lambda_{t,\ r}}{\partial x}(x)dW_r + \int_t^T \int_{\mathbb{R}\backslash\{0\}} \frac{\partial \Lambda_{t,\ r^-}}{\partial x}(x)(e^{-z\theta_{r^-}} - 1)\widetilde{N}^X(dr, dz) \\
&+ \int_t^T \Lambda_{t,\ r}\frac{\partial(-\theta_r\sigma_r)}{\partial \xi} \times \frac{\partial \xi_{t,\ r}}{\partial x}(x)dW_r + \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}\frac{\partial(e^{-z\theta_r})}{\partial \xi} \times \frac{\partial \xi_{t,\ r^-}}{\partial x}(x)\widetilde{N}^X(dr, dz).
\end{aligned} \tag{4.47}$$

Also, by applying the Itô differentiation rule to the product $\Lambda_{t,\ T}(x)L_{t,\ T}$, we have

$$\begin{aligned}
&\Lambda_{t,\ T}(x)L_{t,\ T} \\
&= \int_t^T (-\theta_r\sigma_r)\Lambda_{t,\ r}(x)L_{t,\ r}dW_r + \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}(x)L_{t,\ r^-}(e^{-z\theta_{r^-}} - 1)\widetilde{N}^X(dr, dz) \\
&+ \int_t^T \Lambda_{t,\ r}\frac{\partial(-\theta_r\sigma_r)}{\partial \xi} \times \frac{\partial \xi_{t,\ r}}{\partial x}(x)dW_r + \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}\frac{\partial(e^{-z\theta_r})}{\partial \xi} \times \frac{\partial \xi_{t,\ r^-}}{\partial x}(x)\widetilde{N}^X(dr, dz).
\end{aligned} \tag{4.48}$$

Comparing Equations (4.47) and (4.48), we have by the unicity of solution of SDE that

$$\frac{\partial \Lambda_{t,\ T}}{\partial x}(x) = \Lambda_{t,\ T}(x) \times L_{t,\ T}. \tag{4.49}$$

For the second, we remark that

$$\begin{aligned}
&\Lambda_{t,\ T}(x_- + \zeta(z)) - \Lambda_{t,T}(x) \\
&= \int_t^T \Big[(-\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t + \zeta(z)), X_r) + (\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t), X_r)\Big]dW_r \\
&+ \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Lambda_{t,\ r^-}(\xi_{t^-} + \zeta(z)) \times \Big[e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_{t^-}+\zeta(z)),\ X_r)} - e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_t),\ X_r)}\Big]\widetilde{N}(dr, dz) \\
&+ \int_t^T \Big[\Lambda_{t,\ r}(x + \zeta(z)) - \Lambda_{t,r}(x)\Big]\Big[(-\theta_r\sigma_r)(r, \xi_{t,\ r}(\xi_t), X_r)\Big]dW_r \\
&+ \int_t^T \int_{\mathbb{R}\backslash\{0\}} \Big[\Lambda_{t,\ r^-}(x_- + \zeta(z)) - \Lambda_{t,r^-}(x)\Big] \times \Big[e^{-z\theta_{r^-}(r,\ \xi_{t,\ r^-}(\xi_t),\ X_r)} - 1\Big]\widetilde{N}(dr, dz).
\end{aligned} \tag{4.50}$$

On the other hand, by applying Itô differentiation rule

$$
\begin{aligned}
\Lambda_{t,\,T} K_{t,\,T} = & \int_t^T \Big[(-\theta_r\sigma_r)(r, \xi_{t,\,r}(\xi_t + \zeta(z)), X_r) + (\theta_r\sigma_r)(r, \xi_{t,\,r}(\xi_t), X_r)\Big] dW_r \\
& + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \Lambda_{t,\,r^-}(\xi_{t^-} + \zeta(z)) \\
& \times \Big[e^{-z\theta_{r^-}(r,\,\xi_{t,\,r^-}(\xi_{t^-}+\zeta(z)),\,X_r)} - e^{-z\theta_{r^-}(r,\,\xi_{t,\,r^-}(\xi_t),\,X_r)}\Big] \widetilde{N}(dr, dz) \\
& + \int_t^T \Lambda_{t,\,r^-}(x) K_{t,\,r} \Big[(-\theta_r\sigma_r)(r, \xi_{t,\,r}(\xi_t), X_r)\Big] dW_r \\
& + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \Lambda_{t,\,r^-}(x) K_{t,\,r^-} \Big[e^{-z\theta_{r^-}(r,\,\xi_{t,\,r^-}(\xi_t),\,X_r)} - 1\Big] \widetilde{N}(dr, dz).
\end{aligned}
$$

As above, we deduce the second identity from the uniqueness of solution of SDE. □

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