THE BUBBLE TRANSFORM AND THE DE RHAM COMPLEX

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Abstract. The purpose of this paper is to discuss a generalization of the bubble transform to differential forms. The bubble transform was discussed in [19] for scalar valued functions, or zero-forms, and represents a new tool for the understanding of finite element spaces of arbitrary polynomial degree. The present paper contains a similar study for differential forms. From a simplicial mesh $\mathcal{T}$ of the domain Ω, we build a map which decomposes piecewise smooth k forms into a sum of local bubbles supported on appropriate macroelements. The key properties of the decomposition are that it commutes with the exterior derivative and preserves the piecewise polynomial structure of the standard finite element spaces of k-forms. Furthermore, the transform is bounded in L^2 and also on the appropriate subspace consisting of k-forms with exterior derivatives in L^2.

1. Introduction

The bubble transform for scalar functions, or zero forms, was presented in [19]. In this paper, we will generalize this construction to differential forms. More precisely, our goal is to extend the construction of the bubble transform to the complete de Rham complex. Potentially, our results will have a number of applications for the analyses of finite element methods of high polynomial degree, such as for domain decomposition methods and the construction of uniformly bounded projection operators. In fact, our techniques can also be adopted to the setting of mesh refinements, and as a consequence, it may also be possible to obtain results for general hp–methods. However, to make the present paper as simple as possible, we will, throughout this paper, restrict the discussion to the basic properties of the transform, without considering possible applications.

Throughout this paper, Ω will be a bounded polyhedral domain in $\mathbb{R}^n$, and for $0 \le k \le n$, we will use $\Lambda^k(\Omega)$ to denote the space of smooth differential k–forms on Ω. If $\mathcal{T}$ is a simplicial triangulation of Ω, we will use $\Lambda^k(\mathcal{T})$ to denote the space of k–forms on Ω which are piecewise smooth with respect to $\mathcal{T}$. More precisely, the elements of $\Lambda^k(\mathcal{T})$ are smooth on the closed simplices T in the triangulation and have single-valued traces on each subsimplex of $\mathcal{T}$. We denote by $\Delta(\mathcal{T})$ the set of

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all subsimplices of $\mathcal{T}$, while $\Delta_m(\mathcal{T})$ is the set of simplices of dimension m. For each $f \in \Delta(\mathcal{T})$, the macroelement Ω_f consists of the union of all n–simplexes in $\Delta(\mathcal{T})$ containing f as a subsimplex. Furthermore, $\mathcal{T}_f$ is the restriction of the mesh $\mathcal{T}$ to the macroelement Ω_f, and $\mathring{\Lambda}^k(\mathcal{T}_f)$ is the subspace of $\Lambda^k(\mathcal{T}_f)$ consisting of forms with vanishing trace on the part of the boundary of Ω_f that is in the interior of Ω.

In the setting of finite element exterior calculus, there are two fundamental families of piecewise polynomial subspaces of $\Lambda^k(\mathcal{T})$. These are the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$, where $r \ge 1$. The spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ consist of all piecewise polynomial k-forms of degree r, while the spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ consist of piecewise polynomial k-forms which locally on each subsimplex contain $\mathcal{P}_{r-1}\Lambda^k$, but are contained in $\mathcal{P}_r\Lambda^k$. In the special case $r = 1$, the space $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ is exactly the Whitney forms associated to the mesh $\mathcal{T}$. For both these families of finite element spaces, there exist sets of degrees of freedom associated to elements of $\Delta(\mathcal{T})$ which uniquely determine the elements of the space. More precisely, an element u is uniquely determined by functionals of the form

$$u \mapsto \int_f \operatorname{tr}_f u \wedge \eta, \quad \eta \in \mathcal{P}'(f,k,r), \quad f \in \Delta(\mathcal{T}), \dim f \ge k, \tag{1.1}$$

where the test space $\mathcal{P}'(f,k,r) \subset \Lambda^{\dim f - k}(f)$. We refer to [1, Chapter 7], [2, Chapter 4], [4, Theorem 5.5], or [3] for more details. The degrees of freedom of the form (1.1) correspond to a decomposition of the dual space into local subspaces, and lead to a local basis, referred to as the dual basis for the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$. A further consequence is that the spaces themselves admit a decomposition of the form

$$V^k(\mathcal{T}) = \bigoplus_{\substack{f\in\Delta_m(\mathcal{T})\\ m\ge k}} V_f^k, \quad V_f^k \subset \mathring{\Lambda}^k(\mathcal{T}_f), \tag{1.2}$$

where $V^k(\mathcal{T})$ is a space of the form $\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$, and V_f^k is a corresponding local space associated to the simplex f. The space V_f^k consists of functions in $V^k(\mathcal{T})$ with all degrees of freedom taken to be zero except the ones associated to the simplex f. More precisely, a function $u \in V^k(\mathcal{T})$ admits a decomposition

$$u = \sum_{\substack{f\in\Delta_m(\mathcal{T})\\ m\ge k}} u_f, \quad u_f \in V_f^k,$$

and the map $u \mapsto \{u_f\}$ is implicitly given by the degrees of freedom (1.1). In particular,

$$\operatorname{tr}_f \sum_{j=k}^m u_j = \operatorname{tr}_f u, \quad f \in \Delta_m(\mathcal{T}),\ k \le m \le n, \tag{1.3}$$

where $u_j = \sum_{g\in\Delta_j(\mathcal{T})} u_g$ and where tr denotes the trace operator. The map $u \mapsto \{u_f\}$ depends heavily on the particular space $V^k(\mathcal{T})$, and in particular on the polynomial degree r. On the other hand, the geometry of the decomposition (1.2), represented by the macroelements Ω_f and the associated mesh $\mathcal{T}_f$, is independent of the choice of discrete spaces. This indicates that it might be possible to define the map $u \mapsto \{u_f\}$ independent of the choice of discrete spaces. With some modifications explained below, this is what we achieve by the construction given in this paper.

The bubble transform $\mathcal{B}^k$ that we will construct is made up of local operators $B^k_{m,f} : \Lambda^k(\mathcal{T}) \to \mathring{\Lambda}^k_m(\mathcal{T}, f)$, where $f \in \Delta_j(\mathcal{T})$, $m \le j \le m+k$, and $\mathring{\Lambda}^k_m(\mathcal{T}, f)$ is a space of rational k–forms with support in Ω_f. The functions in this space are piecewise smooth, but are allowed to be singular at the boundary of f. The precise definition of $\mathring{\Lambda}^k_m(\mathcal{T}, f)$ will be given in Section 2.4 below. The corresponding sum

$$B^k_m = \sum_{\substack{f \in \Delta_{m+j}(\mathcal{T}) \\ 0 \le j \le k}} B^k_{m,f}$$

will be a global operator which maps the space of piecewise smooth k forms, $\Lambda^k(\mathcal{T})$, to itself. In other words, the singular components that may be present in the local functions $B^k_{m,f}u$ will cancel when we sum over all f. The maps B^k_m will have a trace property similar to (1.3), i.e., for any $u \in \Lambda^k(\mathcal{T})$,

$$\operatorname{tr}_f \sum_{j=0}^{m} B^k_j u = \operatorname{tr}_f u, \quad f \in \Delta_m(\mathcal{T}),\ k \le m \le n.$$

The end result is that we can write

$$u = \sum_{m=0}^{n} B^k_m u = \sum_{m=0}^{n} \sum_{\substack{f \in \Delta_{m+j}(\mathcal{T}) \\ 0 \le j \le k}} B^k_{m,f} u. \tag{1.4}$$

Since there are no subsimplexes of $\mathcal{T}$ of dimension greater than n, the sum over j above should be restricted to $0 \le j \le n-m$. However, for simplicity we adopt the notation above throughout the paper, where $\Delta_{m+j}(\mathcal{T})$ is empty for $j > n-m$. To sum up, each operator $B^k_{m,f}$ will map u into a local bubble, and the complete collection, $\mathcal{B}^k = \{B^k_{m,f}\}$, produces a local decomposition of u. Although the operators $B^k_{m,f}$ will not commute with the exterior derivative, the operators B^k_m will have this key property. More precisely, the diagram

$$\begin{array}{ccc} \Lambda^k(\mathcal{T}) & \xrightarrow{d} & \Lambda^{k+1}(\mathcal{T}) \\ \Big\downarrow B^k_m & & \Big\downarrow B^{k+1}_m \\ \Lambda^k(\mathcal{T}) & \xrightarrow{d} & \Lambda^{k+1}(\mathcal{T}) \end{array} \tag{1.5}$$

commutes. The bubble transform also preserves the piecewise polynomial spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}^-_r\Lambda^k(\mathcal{T})$ in the sense that

$$B^k_m(V^k(\mathcal{T})) \subset V^k(\mathcal{T}), \tag{1.6}$$

where $V^k(\mathcal{T})$ can be either $\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}^-_r\Lambda^k(\mathcal{T})$. Finally, we will show that the bubble transform is bounded in L^2 in the sense that

$$\|B^k_m u\|_{L^2(\Omega)},\ \Big(\sum_{j=0}^{k} \sum_{f \in \Delta_{m+j}(\mathcal{T})} \|B^k_{m,f} u\|^2_{L^2(\Omega)}\Big)^{1/2} \le c\|u\|_{L^2(\Omega)}, \tag{1.7}$$

for $0 \le m \le n$, where the constant c depends on the *shape regularity* constant of the mesh $\mathcal{T}$. The operator $B^k_{m,f}$ will be defined by a recursive procedure. The key tool for the construction is a family of operators, $C^k_{m,f}$, which we will refer to as cut–off operators, since functions in the range have support in Ω_f. By using these operators, $B^k_{m,f}u$ is defined recursively by

$$B^k_{m,f}u = C^k_{m,f}(u - \sum_{j=0}^{m-1} B^k_j u), \quad m = 0, 1, \dots, n. \tag{1.8}$$

Hence, the properties of the operators $B^k_{m,f}$ will be derived from corresponding properties of the operators $C^k_{m,f}$.

The present study is partly motivated by the hp-finite element method, i.e., where both piecewise polynomials of arbitrary high degree and arbitrary small mesh cells are allowed. The analysis of finite element methods based on mesh refinements and a fixed polynomial degree, i.e., the h-method, is by now very well understood, with a number of finite element spaces developed for approximating all the spaces comprising the de Rham complex, A key step in this analysis has been the development of bounded projections that commute with the exterior derivative, e.g., see [2], [5], [10], [17], [18], and [22].

The corresponding analysis for the p-method, where the polynomial degree is unbounded, is so far less canonical. Pioneering results for the p-method applied to second order elliptic problems in two space dimensions were obtained by Babuška and Suri [6], while a corresponding analysis in three dimensions can be found in [21]. The study of the p-method for Maxwell equations was initiated in [11], and inspired the later work presented in [12, 13, 14, 15, 20] on discretization of the de Rham complex in three space dimensions. A crucial step in the analysis presented in these papers is the use of projection-based interpolation operators, as proposed in [7, Chapter 3], to construct projection operators which commute with the exterior derivative. The results of these papers can be used to derive a number of convergence results for the p-method, including for eigenvalue problems [8]. However, the approach using projection-based interpolation will usually not lead to projection operators that are bounded in appropriate Sobolev norms, and a common challenge is to show that desired bounds are independent of the polynomial degree. An alternative approach to the construction of commuting projections is discussed in [16]. These operators are L^2 bounded, but so far the construction is limited to the last part of the de Rham complex in two and three dimensions. A further discussion and additional references for interpolation operators and approximation in the hp-setting can also be found in this paper.

Preconditioners based on domain decomposition for the operators arising from finite element approximation of second order elliptic equations are considered in [23]. For the two-level Schwarz method, it is shown that the condition number is bounded uniformly in both the mesh size h and polynomial degree. However, so far the problem of establishing a similar bound with respect to the polynomial degree for Schwarz methods applied to more general Hodge-Laplace problems seems to be open.

The theory developed in this paper indicates an alternative path towards the understanding of finite element methods of high polynomial degree. In fact, the theory presented here is developed without reference to any specific piecewise polynomial space. The setting is simply a given domain, with a given simplicial mesh, and all the operators defining the basic decompositions depend only on the domain Ω and the mesh $\mathcal{T}$. In particular, the bounds we obtain only depend on these objects. However, the relation to more specific piecewise polynomial spaces appears as a consequence of the invariance property expressed by (1.6). The discussion in the present paper is restricted to basic properties of the bubble transform, without considering possible applications to more specific problems related to finite element methods. However, the use of the theory presented in this paper to analyze domain decomposition methods and to construct projections that commute with the exterior derivative appears to be a promising new approach, although not a straightforward one.

This paper is organized as follows. In Section 2 we introduce some basic notation and present some of the tools we will need for the construction. In particular, in Section 2.4, we will show how the main results will follow from corresponding properties of the cut–off operators $C^k_{m,f}$. As a consequence, the rest of the paper will be devoted almost entirely to analysis of these cut–off operators. A brief review of some results for scalar valued functions, or zero–forms, is given in Section 3, while Section 4 contains a preliminary discussion of corresponding results for k forms. In particular, this discussion motivates the need for a new family of order reduction operators which will be defined and analyzed in Section 5. These operators comprise a new tool developed in this paper, and their construction is based on the double complex idea introduced in [17, 18]; see also [5]. Using the order reduction operators, the general definition of the operators $C^k_{m,f}$ will then be given in Section 6. Section 7 is devoted to invariance properties, i.e., we derive the key results leading to the invariance property (1.6) and the commuting relation (1.5). At the end of that section, we briefly consider a possible approach for constructing projection operators, with desired properties, that are defined from local projections into pure polynomial spaces. Finally, in Section 8, we verify the basic bounds in appropriate operator norms.

2. Preliminaries and the main results

2.1. **Assumptions.** Throughout the paper we assume that Ω is a bounded polyhedral domain in $\mathbb{R}^n$ which is partitioned into a finite set of n simplexes. Furthermore, the simplicial triangulation $\mathcal{T}$, frequently referred to as a mesh, is assumed to be a simplicial decomposition of Ω, i.e., the union of these simplices is the closure of Ω, and the intersection of any two is either empty or a common subsimplex of each. As in [17] , cf. also [5], we will assume that the extended macroelement Ω^E_f, defined by

$$\Omega^E_f = \cup_{i\in I(f)}\Omega_{x_i},$$

is contractible for all $f \in \Delta(\mathcal{T})$. Finally, in the beginning of Section 8 we will make an additional topological assumption on the mesh $\mathcal{T}$ which will be used to obtain the bound (1.7).

2.2. **Notation.** We start by recalling some standard notation for differential forms. If $u \in \Lambda^k(\Omega)$, the space of smooth k forms on the domain Ω, the *exterior derivative*

$d = d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ is given by

$$du_x(v_1, \dots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial_{v_j} u_x(v_1, \dots, \hat{v}_j, \dots, v_{k+1}),$$

where the hat is used to indicate a suppressed argument and the vectors v_j are elements of the corresponding tangent space $T(\Omega) = \mathbb{R}^n$. If $u^1 \in \Lambda^j(\Omega)$ and $u^2 \in \Lambda^k(\Omega)$, then the wedge product, $u^1 \wedge u^2$, is a corresponding form in $\Lambda^{j+k}(\Omega)$ given by

$$(u^1 \wedge u^2)(v_1, \dots, v_{j+k}) = \sum_\sigma (\operatorname{sign} \sigma) u^1(v_{\sigma(1)}, \dots, v_{\sigma(j)}) u^2(v_{\sigma(j+1)}, \dots, v_{\sigma(j+k)}),$$

where the sum is over all permutations σ of $\{1, \dots, j+k\}$, for which $\sigma(1) < \sigma(2) < \dots < \sigma(j)$ and $\sigma(j+1) < \sigma(j+2) < \dots < \sigma(j+k)$. We will use $\lrcorner$ to denote contraction, i.e., if $u \in \Lambda^k(\Omega)$ and $v = v(x)$ is a vector field, then $u \lrcorner v$ denotes the $k-1$ form such that

$$(u \lrcorner v)_x(v_1, \dots, v_{k-1}) = u_x(v(x), v_1, \dots, v_{k-1}).$$

A smooth map $F : \mathcal{M} \to \mathcal{M}'$ between manifolds provides a pullback of a differential form from $\mathcal{M}'$ to $\mathcal{M}$, i.e., a map from $\Lambda^k(\mathcal{M}') \to \Lambda^k(\mathcal{M})$ given by

$$(F^* u)_x(v_1, \dots, v_k) = u_{F(x)}(DF_x(v_1), \dots, DF_x(v_k)).$$

The pullback respects exterior products and differentiation, i.e.,

$$F^*(u^1 \wedge u^2) = F^* u^1 \wedge F^* u^2, \qquad F^*(du) = d(F^* u).$$

In the special case when $\mathcal{M}$ is a submanifold of $\mathcal{M}'$, then the pullback of the inclusion map, $\Lambda^k(\mathcal{M}') \to \Lambda^k(\mathcal{M})$, is the trace map $\operatorname{tr}_{\mathcal{M}}$. We will use $H\Lambda^k(\Omega)$ to denote the Sobolev space given by

$$H\Lambda^k(\Omega) = \{u \in L^2\Lambda^k(\Omega) : du \in L^2\Lambda^{k+1}(\Omega)\},$$

where $L^2\Lambda^k(\Omega)$ is the space of k-forms with values in L^2. As a consequence of the identity $d \circ d = 0$, we obtain the de Rham domain complex given by

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0.$$

We recall from the introduction above that $\Lambda^k(\mathcal{T})$ denotes the corresponding space of piecewise smooth k forms with single valued traces. Then $\Lambda^k(\mathcal{T}) \subset H\Lambda^k(\Omega)$. Furthermore, the piecewise polynomial space $\mathcal{P}_r\Lambda^k(\mathcal{T})$ is the set of elements u of $\Lambda^k(\mathcal{T})$ such that for fixed tangent vectors $v_1, \dots, v_k$, the scalar function

$$u(v_1, \dots, v_k) \in \mathcal{P}_r(T), \quad T \in \Delta_n(\mathcal{T}),$$

where $\mathcal{P}_r(T)$ denote the set of scalar valued polynomials of degree less than or equal to r on T. Finally, the space $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ is the space of functions u in $\mathcal{P}_r\Lambda^k(\mathcal{T})$ such that

$$(u \lrcorner v)(v_1, \dots, v_{k-1}) \in \mathcal{P}_r(T), \quad T \in \Delta_n(\mathcal{T})$$

for any vector field v of the form $v(x) = x - a$, where $a \in \mathbb{R}^n$ is fixed. To summarize the relation between the spaces just introduced, we can state

$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}) \subset \mathcal{P}_r\Lambda^k(\mathcal{T}) \subset \Lambda^k(\mathcal{T}) \subset H\Lambda^k(\Omega).$$

We recall that $\Delta_0(\mathcal{T})$ is the set of simplices of dimension zero, i.e., the set of vertices of $\mathcal{T}$. We will assume that the vertices are numbered by a set of integers $\mathcal{I} = \{0, 1, \dots, N(\mathcal{T})\}$ such that

$$\Delta_0(\mathcal{T}) = \{x_i \,:\, i \in \mathcal{I}\},$$

and leading to an ordering of the vertices. The barycentric coordinate associated to $x_i \in \Delta_0(\mathcal{T})$ is denoted $\lambda_i(x)$. In other words, λ_i is the piecewise linear function equal to one at x_i and zero at all other vertices. Any subset f of $\Delta_0(\mathcal{T})$ corresponds to a set of integers $I(f) \subset \mathcal{I}$. The number of elements in f is denoted $|f|$. In particular, $f \in \Delta_m(\mathcal{T})$ is an ordered subset of $\Delta_0(\mathcal{T})$. We will use the notation $[\cdot, \dots, \cdot]$ to denote convex combinations, such that if $f \in \Delta_m(\mathcal{T})$ with $I(f) = \{\, 0, 1, \dots, m \,\}$ then $f = [x_0, x_1, \dots x_m]$. Furthermore, the statement $g \in \Delta(f)$ means that g is a subcomplex of f with ordering inherited from f. The set $\bar{\Delta}(f)$ contains the emptyset, $\emptyset$, in addition to the elements of $\Delta(f)$, and $\emptyset$ is the single element of $\Delta_{-1}(\mathcal{T})$. If $f \in [x_{j_0}, x_{j_1}, \dots x_{j_m}] \in \Delta_m(\mathcal{T})$ then

$$\sigma_f(x_{j_i}) = i.$$

In other words, $\sigma_f(y)$ gives the internal numbering of y for a vertex y of the simplex f. For any $f \in \bar{\Delta}(\mathcal{T})$ the piecewise linear function $\rho_f = \rho_f(x)$, defined by

$$\rho_f(x) = 1 - \sum_{i \in I(f)} \lambda_i(x),$$

can be seen as a distance function between f and $x \in \Omega$. Note that $0 \le \rho_f(x) \le 1$ and $\rho_f \equiv 1$ if $f = \emptyset$. Recall that for each $f \in \Delta(\mathcal{T})$, the corresponding macroelement Ω_f is defined as the union of all elements of $\Delta_n(\mathcal{T})$ containing f. Alternatively, the interior of Ω_f is the set

$$\{\, x \in \Omega \,:\, \lambda_i(x) > 0,\, i \in I(f) \,\}.$$

As a consequence, if $f \in \Delta_m(\mathcal{T})$ and $g \in \Delta_j(\mathcal{T})$ for $j < m$, then g will not belong to the interior Ω_f. Furthermore, if $g \in \Delta(f)$, then $\Omega_f \subset \Omega_g$. If $f \in \Delta_m(\mathcal{T})$, then ϕ_f will denote the Whitney form associated to f. More precisely, if $f = [x_{j_0}, x_{j_1}, \dots x_{j_m}]$ then ϕ_f is given by

$$\phi_f = \sum_{i=0}^{m} (-1)^i \lambda_{j_i} d\lambda_{j_0} \wedge \dots \wedge \widehat{d\lambda_{j_i}} \wedge \dots \wedge d\lambda_{j_m},$$

and

$$d\phi_f = (m+1) d\lambda_{j_0} \wedge \dots \wedge d\lambda_{j_m}. \tag{2.1}$$

The functions $\{\phi_f\}_{f \in \Delta_m}$ span the space $\mathcal{P}_1^- \Lambda^m(\mathcal{T})$, and they are local with support in Ω_f.

We define a simplex $\mathcal{S} = \mathcal{S}(\mathcal{T})$ by

$$\mathcal{S} = \{\, \lambda = (\lambda_0, \dots, \lambda_N) \in \mathbb{R}^{N+1} \,:\, \sum_{j=0}^{N} \lambda_j = 1, \quad \lambda_j \ge 0 \,\},$$

where $N = N(\mathcal{T})$. The relevance of this simplex can be understood by introducing the barycentric map L given by $L : \Omega \to \mathcal{S}$ by $L(x) = (\lambda_0(x), \lambda_1(x) \dots \lambda_N(x))$. In fact, if $N >> n$ then the range of the barycentric map L will only cover parts of the boundary of the huge simplex $\mathcal{S}$. However, for notational simplicity, we have found it convenient to introduce the simplex $\mathcal{S}$. We let $\mathcal{S}^c$ be the set of all convex combinations between $\mathcal{S}$ and the origin, i.e., $\mathcal{S}^c = [0, \mathcal{S}]$. Alternatively,

$$S^c = \{\, \lambda = (\lambda_0, \ldots, \lambda_N) \in \mathbb{R}^{N+1} \,:\, \sum_{j=0}^{N} \lambda_j \le 1, \quad \lambda_j \ge 0 \,\}.$$

Furthermore, for any $f \in \bar{\Delta}$, the mapping $L_f : \Omega \to \mathcal{S}^c$ is defined by

$$(L_f(x))_i = \lambda_i(x),\ i \in I(f), \quad (L_f(x))_i = 0,\ i \in \mathcal{I} \setminus I(f).$$

Note that for $f = \emptyset$, $(L_f(x))_i = 0$ for all $i \in \mathcal{I}$, while for any $f \in \Delta_m(\mathcal{T})$ the range of the map L_f is a subcomplex of S^c with dimension $m+1$. We will denote this subcomplex of $\mathcal{S}^c$ by $\mathcal{S}_f^c$, and $\mathcal{S}_f = \mathcal{S}_f^c \cap \mathcal{S}$. In fact, $\mathcal{S}_f^c$ is the convex set with the origin and the endpoints of the coordinate vectors $\{e_i,\ i \in I(f)\}$ as extreme points, where e_i denotes a coordinate vector in $\mathbb{R}^{N+1}$. In the construction below, we will frequently use the pullback L_f^* mapping $\Lambda^k(\mathcal{S}_f^c)$ to $\Lambda^k(\mathcal{T})$, i.e., L_f^* maps smooth forms on $\mathcal{S}_f^c$ to piecewise smooth forms on Ω. Similarly, it will also map polynomial forms to piecewise polynomial forms. For $\lambda \in \mathcal{S}^c$ we let

$$b(\lambda) = 1 - \sum_{j=0}^{N} \lambda_j.$$

Hence, $b(\lambda)$ measures the distance from $\lambda \in \mathcal{S}^c$ to $\mathcal{S}$, and $\rho_f(x) = b(L_f(x))$. If $f \in \Delta_m(\mathcal{T})$ and $T \in \Delta_n(\mathcal{T}_f)$, we let $f^*(T) \in \Delta_{n-m-1}(T)$ be the face opposite f. Alternatively,

$$f^*(T) = \{\, x \in T \,:\, \lambda_j(x) = 0,\ j \in I(f) \,\}.$$

We then define

$$f^* = \bigcup_{T \in \Delta_n(\mathcal{T}_f)} f^*(T).$$

The set f^* can be viewed as an $n-m-1$ dimensional manifold composed of the simplexes $f^*(T)$, and all elements of Ω_f can be written uniquely as a convex combination of the points x_i, $i \in I(f)$ and a point $q_f(x) \in f^*$. In fact, if $x \subset T \in \Delta_n(\mathcal{T}_f)$, then

$$q_f(x) = \Big(\sum_{j \in I(f^*(T))} \lambda_j(x) x_j \Big) / \Big(\sum_{j \in I(f^*(T))} \lambda_j(x) \Big),$$

and

$$x = \sum_{i \in I(f)} \lambda_i(x) x_i + \rho_f(x) q_f(x).$$

In the special case when $m = n-1$, the manifold f^* will be reduced to two vertices, or only one close to the boundary, while in the case $f = \emptyset$ we have $\Omega_f = f^* = \Omega$ and $q_f(x) = x$.

2.3. **The average operators.** A key tool for our construction below is a family of average operators, A_f^k, where $f \in \Delta$, which will map elements of $\Lambda^k(\mathcal{T}_f)$ to $\Lambda^k(S_f^c)$. In other words, these operators map piecewise smooth k–forms on Ω_f to smooth k–forms on $\mathcal{S}_f^c$. The operators A_f^k will be defined by a function $G = G(y, \lambda)$ given by

$$G(y, \lambda) = \sum_{i \in \mathcal{I}} \lambda_i x_i + b(\lambda) y,$$

where $y \in \Omega$ and $\lambda \in \mathcal{S}^c$. Note that if $x \in f$ then, since $b(L_f x) = 0$, we have

$$G(y, L_f x) = x, \quad x \in f. \tag{2.2}$$

In fact, we will only consider the function G for $y \in \Omega_f$ and $\lambda \in \mathcal{S}_f^c$ for some simplex $f \in \Delta$. In this case, we will have $G(\lambda, y) \in \Omega_f$, i.e., we can regard G as a map $G : \Omega_f \times \mathcal{S}_f^c \to \Omega_f$. We note that for a fixed y, $G(y, \lambda)$ is linear with respect to λ. The corresponding derivative with respect λ, $DG(y, \cdot)$, is therefore an operator mapping tangent vectors of $\mathcal{S}_f^c$, $T(\mathcal{S}_f^c)$, into $T(\Omega_f)$ which is independent of λ. It is given by

$$DG(y, \cdot) = \sum_{i \in I(f)} (x_i - y) d\lambda_i.$$

For each fixed $y \in \Omega_f$, the map $G(y, \cdot)$ maps $\mathcal{S}_f^c$ to Ω_f. Therefore, the corresponding pullback, $G(y, \cdot)^*$, maps $\Lambda^k(\Omega_f)$ to $\Lambda^k(\mathcal{S}_f^c)$. As a further consequence, the average of these maps over Ω_f with respect to y will also map $\Lambda^k(\Omega_f)$ to $\Lambda^k(\mathcal{S}_f^c)$. The operator A_f^k is defined by

$$A_f^k u = \fint_{\Omega_f} G(y, \cdot)^* u \, dy = \frac{1}{|\Omega_f|} \int_{\Omega_f} G(y, \cdot)^* u \, dy$$

or more precisely,

$$(A_f^k u)_\lambda(v_1, \ldots, v_k) = \fint_{\Omega_f} u_{G(y,\lambda)}(DG(y, \cdot)v_1, \ldots, DG(y, \cdot)v_k) \, dy,$$

where $v_1, \ldots, v_k \in T(\mathcal{S}_f^c)$. Note that since pullbacks commute with the exterior derivative, so do the operators A_f^k, i.e., $dA_f^k u = A_f^{k+1} du$. Other key properties of the operators A_f^k, stated in the lemma below, are that it maps piecewise smooth forms to smooth forms, it maps piecewise polynomial forms to polynomial forms, and it is trace preserving.

Lemma 2.1. *Let $f \in \Delta_m(\mathcal{T})$. The operators A_f^k satisfy*

i) $A_f^k(\Lambda^k(\mathcal{T}_f)) \subset \Lambda^k(\mathcal{S}_f^c)$,
ii) $A_f^k(\mathcal{P}_r\Lambda^k(\mathcal{T}_f)) \subset \mathcal{P}_r\Lambda^k(\mathcal{S}_f^c)$ *and* $A_f^k(\mathcal{P}_r^-\Lambda^k(\mathcal{T}_f)) \subset \mathcal{P}_r^-\Lambda^k(\mathcal{S}_f^c)$,
iii) $\operatorname{tr}_f L_f^* A_f^k u = \operatorname{tr}_f u$ *for* $u \in \Lambda^k(\mathcal{T}_f)$, $k \le m \le n$.

Proof. Assume that $u \in \Lambda^k(\mathcal{T}_f)$. From the definition of the operator A_f^k, we obtain

$$(A_f^k u)_\lambda(v_1, \ldots, v_k) = |\Omega_f|^{-1} \sum_{T \in \Delta_n(\mathcal{T}_f)} \int_T u_{G(y,\lambda)}(DG(y, \cdot)v_1, \ldots, DG(y, \cdot)v_k) \, dy,$$

where $|\Omega_f|$ denote the volume of Ω_f. Also observe that if we fix $y \in \Omega_f$, then the subset of Ω_f given by

$$\{ \, G(y, \lambda) \, : \, \lambda \in \mathcal{S}_f^c \, \}$$

belongs to a single n simplex of Ω_f. Therefore, since $G(y, \cdot)$ is a smooth function of λ and u is piecewise smooth, we can conclude that for each fixed y, the integrand appearing in the definition of $(A_f^k u)_\lambda$ varies smoothly with λ. As a consequence, $(A_f^k u)_\lambda(v_1, \ldots, v_k)$ is a smooth function of λ, and therefore part i) is established.

If $u \in \mathcal{P}_r\Lambda^k(\mathcal{T}_f)$, then the integrand

$$u_{G(y,\lambda)}(DG(y,\cdot)v_1, \ldots, DG(y,\cdot)v_k) \in \mathcal{P}_r(\mathcal{S}_f^c)$$

as a function of λ. The same is true for the integral with respect to y, so $A_f^k u \in \mathcal{P}_r\Lambda^k(\mathcal{S}_f^c)$. To show the corresponding preservation of the $\mathcal{P}_r^-$ spaces, we have to show that $(A_f^k u) \lrcorner \lambda \in \mathcal{P}_r\Lambda^{k-1}(\mathcal{S}_f^c)$ for $u \in \mathcal{P}_r^-\Lambda^k(\mathcal{T}_f)$. However, from the fact that

$$DG(y,\cdot)\lambda = \sum_{i \in I(f)} \lambda_i(x_i - y) = G(y,\lambda) - y,$$

we obtain

$$\begin{aligned}((A_f^k u) \lrcorner \lambda)_\lambda(v_1, \ldots, v_{k-1}) \\ = \fint_{\Omega_f} (u_{G(y,\lambda)} \lrcorner (G(y,\lambda) - y))(DG(y,\cdot)v_1, \ldots, DG(y,\cdot)v_{k-1})\, dy.\end{aligned}$$

If $u \in \mathcal{P}_r^-\Lambda^k(\mathcal{T}_f)$, we have that $u_x \lrcorner (x-y)$ is an element of $\mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_f)$ as a function of x for each fixed y, and therefore, by the linearity of $G(\lambda, y)$ with respect to λ, we can conclude that the integrand above is in $\mathcal{P}_r(\mathcal{S}_f^c)$. In other words, we have established that $A_f^k u \in \mathcal{P}_r^-\Lambda^k(\mathcal{S}_f^c)$.

Finally, we have to show the trace property. However, for each fixed y,

$$L_f^* \circ G(y,\cdot)^* = (G(y,\cdot) \circ L_f)^*,$$

and by (2.2), the function $G(y,\cdot) \circ L_f = G(y, L_f \cdot)$ is the identity on f. We can therefore conclude that

$$\operatorname{tr}_f L_f^* A_f^k u = \operatorname{tr}_f \fint_{\Omega_f} (G(y,\cdot) \circ L_f)^* u \, dy = \operatorname{tr}_f u.$$

This completes the proof of the lemma. □

2.4. **The main results.** We recall that the operators $B_{m,f}^k$ are related to the cut–off operators $C_{m,f}^k$ by the iteration (1.8), i.e.,

$$B_{m,f}^k u = C_{m,f}^k (u - \sum_{j=0}^{m-1} B_j^k u), \quad m = 0, 1, \ldots, n.$$

As a consequence, the operators B_m^k will satisfy

$$B_m^k u = C_m^k (u - \sum_{j=0}^{m-1} B_j^k u), \quad m = 0, 1, \ldots, n, \tag{2.3}$$

where

$$C_m^k = \sum_{\substack{f \in \Delta_{m+j}(\mathcal{T}) \\ 0 \le j \le k}} C_{m,f}^k.$$

The purpose of this section is to show how the desired properties for the operators $\mathcal{B}^k$, given by $\mathcal{B}^k = \{B_{m,f}^k\}$, will follow from corresponding properties of the cut–off operators $C_{m,f}^k$. As a consequence, the rest of the paper will almost entirely be devoted to analysis of the cut–off operators.

Before we state the key results for the operators $C^k_{m,f}$, we will give a precise definition of the local space $\mathring{\Lambda}^k_m(\mathcal{T}, f)$, introduced in the introduction. If $f \in \Delta_m(\mathcal{T})$, we define the space $\Lambda^k_m(\mathcal{T}, f)$ by

$$\Lambda^k_m(\mathcal{T}, f) = \{\, u = \sum_{\substack{g\in\Delta_j(f)\\ 0\le j\le m-1}} \rho_g^{-1} w_g \,:\, w_g \in \Lambda^k(\mathcal{T})\,\}.$$

This space consists of k–forms which can be expressed as a sum of rational functions with possible singularities at the boundary of f. Furthermore, we let $\mathring{\Lambda}^k_m(\mathcal{T}, f)$ be the subspace of functions which are supported on Ω_f, i.e., their trace vanishes on the boundary, $\partial\Omega_f$, and they are identically zero on $\Omega \setminus \Omega_f$.

The results in Lemma 2.2 below provide a summary of results to be established in Lemmas 4.1 and 6.1 and part i) of Proposition 7.1.

Lemma 2.2. *Let $u \in \Lambda^k(\mathcal{T})$ and $f \in \Delta_{m+j}(\mathcal{T})$ for $0 \le m \le n$ and $0 \le j \le k$. Then $C^k_{m,f}u \in \mathring{\Lambda}^k_{m+j}(\mathcal{T}, f)$, while $C^k_m u \in \Lambda^k(\mathcal{T})$. Furthermore, if $k \le m \le n$ and $f \in \Delta_m(\mathcal{T})$, then we also have $\operatorname{tr}_f C^k_{m,f}u = \operatorname{tr}_f u$, which gives*

$$tr_f C^k_m u = \operatorname{tr}_f u, \quad f \in \Delta_m(\mathcal{T}).$$

The first part of the lemma expresses the fact that the operator $C^k_{m,f}$ maps a piecewise smooth form into a rational differential form with local support on Ω_f, and in such a way that when we sum over all $f \in \Delta_{m+j}(\mathcal{T})$, $0 \le j \le k$, we obtain a form which is piecewise smooth. The last statement, that the operator C^k_m preserves the trace of u on all simplexes in $\Delta_m(\mathcal{T})$, follows from the stated trace properties of the local operators $C^k_{m.f}$, since $f \in \Delta_m(\mathcal{T})$ will not belong to the interior of any $\Omega_{f'}$ for $f' \ne f$, $f' \in \Delta_{m+j}(\mathcal{T})$, $0 \le j \le k$. Furthermore, for $f \in \Delta_n(\mathcal{T})$, we have $\Omega_f = f$, and by Lemma 2.2,

$$\operatorname{tr}_f C^k_{n,f}u = \operatorname{tr}_f u, \quad \text{and } C^k_{n,f} \equiv 0 \quad \text{on } \Omega \setminus f.$$

This completely specifies the operators $C^k_{n,f}$.

Remark. If $f \in \Delta_m(\mathcal{T})$ and $g \in \Delta(f)$, $g \ne f$, then $g \subset f \cap \partial\Omega_f$, where $\partial\Omega_f$ denotes the boundary of Ω_f. However, as a consequence of Lemma 2.2, we have $\operatorname{tr}_f C^k_{m,f}u = \operatorname{tr}_f u$, and if $C^k_{m,f}u$ is smooth, we must also have $\operatorname{tr}_{\partial\Omega_f} C^k_{m,f}u = 0$, since $C^k_{m,f}u$ vanishes on the complement of Ω_f. This apparent contradiction is exactly why the space of rational differential forms, $\mathring{\Lambda}^k_m(\mathcal{T}, f)$, appears as part of our construction. Furthermore, the statement $C^k_m u \in \Lambda^k(\mathcal{T})$ has the interpretation that there is a $w \in \Lambda^k(\mathcal{T})$ such that

$$w_x = \sum_{\substack{f\in\Delta_{m+j}(\mathcal{T})\\ 0\le j\le k}} (C^k_{m,f}u)_x, \quad x \in \Omega \setminus \Delta_{m-1}, \quad \Delta_{m-1} = \bigcup_{g\in\Delta_{m-1}(\mathcal{T})} g.$$

In particular, C^k_n is the identity operator.

Proposition 7.1 also contains the result that the operators C^k_m commute with the exterior derivative, i.e.,

$$dC^k_m u = C^{k+1}_m du, \quad u \in \Lambda^k(\mathcal{T}),\, 0 \le k \le n-1.$$

As a consequence of the properties of the cut–off operators C_m^k just stated, we show that the operators B_m^k preserve piecewise smoothness, that they commute with the exterior derivative, that the functions $B_{m,f}^k u$ are rational differential forms with local support, and that these local bubbles define a decomposition of u.

Theorem 2.3. *Let $u \in \Lambda^k(\mathcal{T})$. Then we have*

i) $B_m^k u \in \Lambda^k(\mathcal{T}), \quad 0 \le m \le n,$
ii) $dB_m^k u = B_m^{k+1} du, \quad 0 \le k \le n-1,$
iii) $B_{m,f}^k u \in \mathring{\Lambda}_{m+j}^k(\mathcal{T}, f), \quad f \in \Delta_{m+j}(\mathcal{T}),\ 0 \le m \le n,\ 0 \le j \le k,$
iv) $\operatorname{tr}_f \sum_{j=0}^m B_j^k u = \operatorname{tr}_f u, \quad f \in \Delta_m(\mathcal{T}),\ k \le m \le n,$

and the decomposition

$$u = \sum_{m=0}^n B_m^k u = \sum_{m=0}^n \sum_{\substack{f \in \Delta_{m+j}(\mathcal{T}) \\ 0 \le j \le k}} B_{m,f}^k u.$$

Proof. Property i) is a consequence of a simple induction argument, based on the iteration (2.3), and the corresponding property for the operator C_m^k given in Lemma 2.2. The commuting property follows directly from (2.3) and the corresponding property for the cut–off operators C_m^k, while property iii) follows from i), (1.8), and the corresponding property for the operator $C_{m,f}^k$. Furthermore, for $f \in \Delta_m(\mathcal{T})$, we have from (2.3) and Lemma 2.2 that

$$\operatorname{tr}_f B_m^k u = \operatorname{tr}_f (u - \sum_{j=0}^{m-1} B_j^k u), \quad k \le m \le n,$$

and this implies property iv). Finally, the decomposition of u is a special case of property iv), corresponding to $m = n$. □

We emphasize that we do not claim that that each local operator $B_{m,f}^k$ commutes with the exterior derivative. We have explained above that we need to consider the global operator B_m^k to preserve piecewise smoothness, and in the same way we also need to consider these global operators to obtain the commuting relation.

Recall that the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ are subspaces of $\Lambda^k(\mathcal{T})$. More precisely, these spaces consist of piecewise smooth differential forms which are polynomial forms of class $\mathcal{P}_r$ or $\mathcal{P}_r^-$ on each n simplex in $\Delta_n(\mathcal{T})$. Another key property of the bubble transform is that the operators B_m^k are invariant with respect to the piecewise polynomial spaces, i.e., they map the spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ into themselves. As above, this invariance property is a consequence of a corresponding property for the cut–off operators C_m^k. In Proposition 7.1, it is established that

$$C_m^k(\mathcal{P}_r\Lambda^k(\mathcal{T})) \subset \mathcal{P}_r\Lambda^k(\mathcal{T}), \quad \text{and } C_m^k(\mathcal{P}_r^-\Lambda^k(\mathcal{T})) \subset \mathcal{P}_r^-\Lambda^k(\mathcal{T}).$$

As a consequence, we obtain the following analogous result for the operators B_m^k.

Theorem 2.4. *The operators B_m^k satisfy*

$$B_m^k(\mathcal{P}_r\Lambda^k(\mathcal{T})) \subset \mathcal{P}_r\Lambda^k(\mathcal{T}) \quad \textit{and} \quad B_m^k(\mathcal{P}_r^-\Lambda^k(\mathcal{T})) \subset \mathcal{P}_r^-\Lambda^k(\mathcal{T})$$

for $0 \le m \le n$.

Proof. This follows directly from the iteration (2.3) and the corresponding results for the operators C_m^k. □

Recall that up to now we have only considered the operators $B_{m,f}^k$ and B_m^k applied to functions in $\Lambda^k(\mathcal{T})$, i.e., to piecewise smooth differential forms. However, another desired property of the bubble transform is that both the local operators $B_{m,f}^k$ and the global operators B_m^k are L^2 bounded operators in the sense described in (1.7). In particular, the constant c appearing in (1.7) only depends on the mesh $\mathcal{T}$ through *the shape–regularity constant* $c_{\mathcal{T}}$ defined by

$$c_{\mathcal{T}} = \max_{T\in\mathcal{T}} \frac{\operatorname{diam}(T)}{\operatorname{diam}(\mathfrak{B}_T)}, \tag{2.4}$$

where $\mathfrak{B}_T$ is the largest ball contained in T. As a consequence, since the space of piecewise smooth forms is dense in the corresponding L^2 space, $L^2\Lambda^k(\Omega)$, we can conclude that the operators $B_{m,f}^k$ and B_m^k can be extended to bounded linear operators defined on $L^2\Lambda^k(\Omega)$. Furthermore, as a consequence of the commuting property of the operator B_m^k, we can also conclude that this operator is bounded in $H\Lambda^k(\Omega)$. The precise statements of the various bounds we will obtain will be given in Section 8 below, cf. Theorems 8.3 and 8.4.

3. The case of scalar valued functions

The bubble transform for scalar valued functions, or zero–forms, was introduced in [19]. In this section we will give a review of some of the results from [19]. It was established in [19] that the bubble transform for zero forms is an L^2 bounded map. However, in the present section, we will only discuss the transform in the setting of piecewise smooth scalar valued functions, i.e., for functions in $\Lambda^0(\mathcal{T})$.

As we argued in Section 2.4 above, the main remaining step to define the bubble transform for zero forms is to specify the operators

$$C_m^0 u = \sum_{f\in\Delta_m(\mathcal{T})} C_{m,f}^0 u,$$

where each local operator $C_{m,f}^0$ maps piecewise smooth functions, i.e., functions in $\Lambda^0(\mathcal{T}_f)$, to rational functions in the space $\mathring{\Lambda}_m^0(\mathcal{T}, f)$. The operator $C_{m,f}^0$ is defined by

$$C_{m,f}^0 u = \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} \frac{\rho_f}{\rho_g} L_g^* A_f^0 u.$$

Here we recall that $\bar{\Delta}(f) = \Delta(f) \cup \{\emptyset\}$, i.e., g is allowed to be the empty set in the sum above. The magic property of the operator $C_{m,f}^0$ is that it preserves the trace of u on f, but at the same time the function $C_{m,f}^0 u$ has support on Ω_f. In the simplest case, when $m = 0$, say $f = x_0$, the function $C_{m,f}^0 u$ is given by

$$(C_{m,f}^0 u)_x = (A_f^0 u)_{\lambda_0} - (1-\lambda_0)(A_f^0 u)_0,$$

while if $f = [x_0, x_1] \in \Delta_1(\mathcal{T})$, then

$$(C_{m,f}^0 u)_x = (A_f^0 u)_{\lambda_0,\lambda_1} - \frac{1-\lambda_0-\lambda_1}{1-\lambda_0}(A_f^0 u)_{\lambda_0,0}$$

$$- \frac{1-\lambda_0-\lambda_1}{1-\lambda_1}(A_f^0 u)_{0,\lambda_1} + (1-\lambda_0-\lambda_1)(A_f^0 u)_{0,0},$$

where in all cases $\lambda_i = \lambda_i(x)$. The rational functions ρ_f/ρ_g for $g \in \Delta(f)$ will satisfy

$$\rho_f(x) \le \frac{\rho_f(x)}{\rho_g(x)} \le 1,$$

and if $g \ne f$ then $(\rho_f/\rho_g)|_f = 0$. On the other hand, when $g = f$, then $\rho_f/\rho_g \equiv 1$. We therefore can conclude that

$$\operatorname{tr}_f C_{m,f}^0 u = \operatorname{tr}_f A_f^0 u = \operatorname{tr}_f u,$$

where we have used part iii) of Lemma 2.1 for the final equality. To see that $C_{m,f}^0 u$ has support on the macroelement Ω_f, we consider the function $C_{m,f}^0 u$ on the complement of Ω_f, i.e., where at least one function λ_i for $i \in I(f)$ vanishes. For simplicity, we can assume that $0 \in I(f)$ and we consider a point $x \in \Omega$ such that $\lambda_0(x) = 0$. For any $g \in \bar{\Delta}(f)$ such that $x_0 \notin g$, let $g' \in \Delta(f)$ be such that $g' \setminus g = x_0$. Then, at the point x, $\rho_{g'}(x) = \rho_g(x)$ and $L_{g'}(x) = L_g(x)$, which implies that

$$\Big[\frac{\rho_f}{\rho_g} L_g^* - \frac{\rho_f}{\rho_{g'}} L_{g'}^*\Big] A_f^0 u = 0 \tag{3.1}$$

at x. By summing over all pairs g and g', we can conclude that $C_{m,f}^0 u = 0$ at x, and hence it is identically zero on $\Omega \setminus \Omega_f$. We summarize the results so far in the following lemma.

Lemma 3.1. *Let $u \in \Lambda^0(\mathcal{T}_f)$ and $f \in \Delta_m(\mathcal{T})$ for $0 \le m \le n$. The function $C_{m,f}^0 u$ satisfies $\operatorname{tr}_f C_{m,f}^0 u = \operatorname{tr}_f u$ and $C_{m,f}^0 u \equiv 0$ in $\Omega \setminus \Omega_f$.*

In general, the operator $C_{m,f}^0$ will not map piecewise smooth functions to piecewise smooth functions, due to the singularity of the rational functions ρ_f/ρ_g. On the other hand, if $g \in \Delta(f)$, $g \ne f$, then $g \subset \partial\Omega_f$. The following result shows that if u is piecewise smooth, with $\operatorname{tr}_{\partial f} u = 0$, then $C_{m,f}^0 u$ is piecewise smooth. Furthermore, piecewise polynomials are preserved by the operator $C_{m,f}^0$ in this case.

Lemma 3.2. *If $u \in \Lambda^0(\mathcal{T}_f)$ and $\operatorname{tr}_{\partial f} u = 0$, then $C_{m,f}^0 u \in \Lambda^0(\mathcal{T})$. Furthermore, if in addition $u \in \mathcal{P}_r\Lambda^0(\mathcal{T}_f)$, then $C_{m,f}^0 u \in \mathcal{P}_r\Lambda^0(\mathcal{T})$.*

Proof. It follows from Lemma 2.1 that $A_f^0 u$ is a smooth function on $\mathcal{S}_f^c$. Furthermore, since $L_f : f \to \mathcal{S}_f$ is an isomorphism, mapping ∂f to ∂S_f, it follows from part iii) of Lemma 2.1 that $\operatorname{tr}_{\partial\mathcal{S}_f} A_f u = 0$. In particular, for any $g \in \Delta(f)$, $g \ne f$, we have $\operatorname{tr}_{S_g} u = 0$. Since $\mathcal{S}_g$ has codimension one as a subset of $\mathcal{S}_g^c$, we can conclude that $\operatorname{tr}_{\mathcal{S}_g^c} b^{-1} A_f^0 u$ is a smooth function on $\mathcal{S}_g^c$, and as a consequence,

$$L_g^*(b^{-1}A_f^0 u) = \rho_g^{-1} L_g^* A_f^0 u$$

is a smooth function on Ω. Since this holds for all $g \in \Delta(f)$, $g \ne f$, we can conclude that $C_{m,f}^0 u$ is piecewise smooth. In addition, if $u \in \mathcal{P}_r\Lambda^0(\mathcal{T}_f)$, then $\rho_g^{-1} L_g^* A_f^0 u \in \mathcal{P}_{r-1}\Lambda^0(\mathcal{T}_f)$ by part ii) of Lemma 2.1, and hence $C_{m,f}^0 u \in \mathcal{P}_r\Lambda^0(\mathcal{T})$. $\square$

Remark. The result given in Lemma 3.2 was the key property used in [19] to show that the bubble transform for zero forms preserves piecewise smoothness and piecewise polynomials. Surprisingly, this result will not play a corresponding role for the discussion given in this paper. Instead, we will show below, cf. Section 7, that even if each individual operator $C^0_{m,f}$ maps piecewise smooth functions into rational functions, the complete operator, C^0_m, will indeed map both the space of piecewise smooth functions and piecewise polynomials into themselves.

4. The primal cut off operator

As a first attempt to define the bubble transform for k-forms, in the case $k \ge 0$, we will define local cut-off operators $C^k_{m,f}$ given by

$$C^k_{m,f}u = \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|}\frac{\rho_f}{\rho_g} L^*_g A^k_f u, \quad f \in \Delta_m(\mathcal{T}). \tag{4.1}$$

This is basically the same operator as we use for zero forms, but where we have replaced the average operator A^0_f with the corresponding operator A^k_f. In fact, the discussion leading up to Lemma 3.1 is still true for the case of k-forms. More precisely, assume that $0 \in I(f)$ and consider a subset Γ of Ω such that $\lambda_0 \equiv 0$ on Γ. If $g, g' \in \bar{\Delta}(f)$ are related such that $g' \setminus g = x_0$ then $\rho_{g'} = \rho_g$ and $L^*_{g'} = L^*_g$ on Γ. As a consequence, the cancellation argument used above shows that $C^k_{m,f}u$ is supported on Ω_f. Furthermore, it follows from Lemma 2.1 that $A^k_f u \in \Lambda^k(\mathcal{S}^c_f)$ and that $\operatorname{tr}_f C^k_{m,f}u = \operatorname{tr}_f u$ if $f \in \Delta_m(\mathcal{T})$ for $k \le m \le n$. We summarize these results as follows.

Lemma 4.1. *Let $u \in \Lambda^k(\mathcal{T}_f)$ and $f \in \Delta_m(\mathcal{T})$ for $0 \le k, m \le n$. Then $C^k_{m,f}u \in \mathring{\Lambda}^k_m(\mathcal{T}, f)$ and $\operatorname{tr}_f C^k_{m,f}u = \operatorname{tr}_f u$ for $k \le m \le n$.*

It is also a consequence of this lemma, and by following the path of arguments used for zero forms above, that we can use the operators $C^k_{m,f}$ to produce a decomposition of $u \in \Lambda^k(\mathcal{T}_f)$ into local bubbles. However, in the present case, there seems to be no direct analog of Lemma 3.2. This is due to the fact that a vanishing trace condition for k–forms with respect to a manifold of codimension one, only controls the value of the form applied to tangent vectors, while we have no control of the form when it is applied to vectors normal to the manifold. As a consequence, from a vanishing trace condition we cannot extract a linear factor as we did in the proof of the lemma above.

Another key property we would like to have for the bubble transform is that it should commute with the exterior derivative. However, an identity like $dC^k_{m,f}u = C^{k+1}_{m,f}du$ will in general not be true for the operator introduced above, even in the case $k = 0$. To see this, let us compute $dC^k_{m,f}u$ when the operator $C^k_{m,f}$ is given by (4.1). We have

$$dC^k_{m,f}u = \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|}\Big(\frac{\rho_f}{\rho_g} L^*_g A^{k+1}_f du + d\Big(\frac{\rho_f}{\rho_g}\Big) \wedge L^*_g A^k_f u\Big)$$

$$= C^{k+1}_{m,f} du + \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} d\Big(\frac{\rho_f}{\rho_g}\Big) \wedge L^*_g A^k_f u.$$

To better understand the commutator $dC^k_{m,f}u - C^{k+1}_{m,f}du$, we will use the fact that as long as x is restricted to Ω_f,

$$\frac{\rho_f}{\rho_g} = \frac{\sum_{j\in I(f^*)} \lambda_j}{\sum_{j\in I(f^*)} \lambda_j + \sum_{p \in I(f \cap g^*)} \lambda_p}.$$

As a consequence,

$$d\Big(\frac{\rho_f}{\rho_g}\Big) = \sum_{p \in I(f\cap g^*)} \sum_{j \in I(f^*)} \frac{\phi_{[x_p,x_j]}}{\rho_g^2},$$

where $\phi_{[x_p,x_j]} \in \mathcal{P}^-_1\Lambda^1(\mathcal{T})$ denotes the Whitney form associated to the simplex $[x_p, x_j]$, i.e., $\phi_{[x_p,x_j]} = \lambda_p d\lambda_j - \lambda_j d\lambda_p$. Therefore, the commutator $dC^k_{m,f}u - C^{k+1}_{m,f}du$ can be written as

$$dC^k_{m,f}u - C^{k+1}_{m,f}du = \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} \sum_{p \in I(f\cap g^*)} \sum_{j \in I(f^*)} \frac{\phi_{[x_p,x_j]}}{\rho_g^2} \wedge L^*_g A^k_f u \tag{4.2}$$

for $x \in \Omega_f$. In fact, the identity (4.2) also holds on the complement of Ω_f. To see this, observe that from the properties of $C^k_{m,f}$ and $C^{k+1}_{m,f}$ derived above, we can conclude that the left hand side of the identity is zero on the complement of Ω_f. To show that this is also true for the right hand side, we will use a cancellation property similar to (3.1). Consider a point $x \in \Omega$ where $\lambda_i(x) = 0$ for some $i \in I(f)$. Consider $g, g' \in \bar{\Delta}(f)$ such that $g' \setminus g = \{x_i\}$. When we sum the contributions from these two simplexes on the right hand side of (4.2) we obtain, up to a sign,

$$\sum_{j \in I(f^*)} \Big[\sum_{p \in I(f\cap g^*)} \Big(\frac{\phi_{[x_p,x_j]}}{\rho_g^2} \wedge L^*_g A^0_f u - \frac{\phi_{[x_p,x_j]}}{\rho_{g'}^2} \wedge L^*_{g'} A^0_f u \Big) + \frac{\phi_{[x_i,x_j]}}{\rho_{g'}^2} \wedge L^*_{g'} A^0_f u \Big].$$

However, at points where $\lambda_i(x) = 0$, we have $\phi_{[x_i,x_j]} = 0$. Therefore, the last term can be dropped, and the rest of the terms cancel when $\lambda_i(x) = 0$. We can therefore conclude that (4.2) holds in all of Ω.

By summing the identity (4.2) over all $f \in \Delta_m(\mathcal{T})$, we obtain

$$\sum_{f \in \Delta_m(\mathcal{T})} (dC^k_{m,f}u - C^{k+1}_{m,f}du) = \sum_{g \in \bar{\Delta}(\mathcal{T})} \sum_{\substack{f \in \Delta_m(\mathcal{T}) \\ f \supset g}} (-1)^{|f|-|g|} \sum_{\substack{p \in I(f \cap g^*) \\ j \in I(f^*)}} \frac{\phi_{[x_p,x_j]}}{\rho_g^2} \wedge L^*_g A^k_f u.$$

Note that if $g \in \bar{\Delta}$ is fixed, $f \in \Delta_m(\mathcal{T})$, and x_p, x_j are such that $f \supset g$, $x_p \in f \cap g^*$, and $x_j \in f^*$, then there is a unique element $f' \in \Delta_m(\mathcal{T})$ such that $f \cap f' \in \Delta_{m-1}(\mathcal{T})$ and

$$f' \supset g, \quad x_p \in (f')^*, \quad \text{and } x_j \in f' \cap g^*.$$

In other words, as compared to f, for the simplex f', the role of the vertices x_p and x_j are reversed. Hence, for both the choices (f, p, j) and (f', j, p) in the sum above, the fraction $\phi_{[x_p,x_j]}/\rho_g^2$ will appear, but with different signs. Furthermore, up to a possible reordering, we have that $[f \cap f', x_p, x_j] \in \Delta_{m+1}(\mathcal{T})$. By using this observation, the sum above can be rewritten as

$$\sum_{f \in \Delta_m(\mathcal{T})} (dC^k_{m,f}u - C^{k+1}_{m,f}du) \tag{4.3}$$

$$= \sum_{f\in\Delta_{m+1}(\mathcal{T})} \sum_{e\in\Delta_1(f)} \sum_{g\in\bar{\Delta}(f\cap e^*)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g^2} \wedge L_g^*(\delta A^k u)_{e,f},$$

where

$$(\delta A^k u)_{e,f} = \sum_{j\in I(e)} (-1)^{\sigma_e(x_j)} A^k_{f(\hat{x}_j)} u, \quad e \in \Delta_1(f).$$

Here the hat notation is used to indicate that the vertex x_j should be removed from f, so that $f(\hat{x}_j) \in \Delta_m(\mathcal{T})$, and we recall from Section 2 above that $\sigma_e(x_j)$ denotes the internal numbering of the vertex x_j with respect to the simplex e.

In order to obtain operators C_m^k that commute with the exterior derivative, we have to include the contribution from the triple sum defining the commutator in the definition of these operators. Recall that for $k = 0$, we have already defined the operator $C_m^0 = \sum_{f\in\Delta_m(\mathcal{T})} C^0_{m,f}$. Hence, for $k = 0$, the identity (4.3) can be rewritten as

$$dC_m^0 u - \sum_{f\in\Delta_m(\mathcal{T})} C^1_{m,f} du = \sum_{f\in\Delta_{m+1}(\mathcal{T})} \sum_{e\in\Delta_1(f)} \sum_{g\in\bar{\Delta}(f\cap e^*)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g^2} \wedge L_g^*(\delta A^0 u)_{e,f}.$$

It is easy to see that if u is a constant scalar valued function, then $(\delta A^0 u)_{e,f} = 0$, and as a consequence, we can conclude that $(\delta A^0 u)_{e,f}$ only depends on du. Therefore, if for any $f \in \Delta(\mathcal{T})$ and $e \in \Delta_1(f)$, we can construct operators $R^1_{e,f}$, mapping one forms to zero forms, such that

$$R^1_{e,f} du = \mathrm{tr}_{\mathcal{S}^c_{f\cap e^*}} (\delta A^0 u)_{e,f}, \tag{4.4}$$

then the triple sum above can be expressed in terms of du.

We summarize the discussion so far in the following lemma.

Lemma 4.2. *Assume that for each $f \in \Delta(\mathcal{T})$ and $e \in \Delta_1(f)$, we can construct operators $R^1_{e,f}$ such that relation (4.4) holds. Define the operator C^1_m by*

$$C_m^1 u = \sum_{\substack{f\in\Delta_{m+j}(\mathcal{T})\\ 0\le j\le 1}} C^1_{m,f} u,$$

where $C^1_{m,f}$ is given by (4.1) if $f \in \Delta_m(\mathcal{T})$, and by

$$C^1_{m,f} u = \sum_{e\in\Delta_1(f)} \sum_{g\in\bar{\Delta}(f\cap e^*)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g^2} L_g^* R^1_{e,f} u$$

if $f \in \Delta_{m+1}(\mathcal{T})$. Then the commuting relation $dC_m^0 u = C_m^1 du$ holds.

We will delay the full analysis of the operator C_m^1 until we have defined the operators C_m^k in general. To do that, we will need a general class of *order reduction operators*, $R^k_{e,f}$, mapping k–forms to $(k-j)$–forms, where $f \in \Delta(\mathcal{T})$ and $e \in \Delta_j(f)$. We will construct these operators, with the properties we will need, in the next section.

5. The order reduction operators $R^k_{e,f}$

Above we saw that in order to complete the definition of the operator C^1_m, so that we obtain the commuting relation $dC^0_m u = C^1_m du$, we needed local operators $R^1_{e,f}$, where $f \in \Delta_{m+1}(\mathcal{T})$ and $e \in \Delta_1(f)$, satisfying the identity (4.4). In general, to construct the operators C^k_m, we will utilize a family of local operators $R^k_{e,f}$, where $f \in \Delta(\mathcal{T})$ and $e \in \Delta_j(f)$, $0 \le j \le k$, which maps a k form u to a $k-j$ form $R^k_{e,f}u$. More precisely, for any $f \in \Delta(\mathcal{T})$ and $e \in \Delta_j(f)$, the operators $R^k_{e,f}$ belong to $\mathcal{L}(\Lambda^k(\mathcal{T}), \Lambda^{k-j}(\mathcal{S}^c_{f\cap e^*}))$. In other words, the linear operator $R^k_{e,f}$ maps piecewise smooth k forms defined on Ω to smooth $k-j$ forms defined on $\mathcal{S}^c_{f\cap e^*}$. This section will be devoted to a general discussion of these operators.

5.1. **The general pullback operator G^*.** The function $G(y,\lambda) = \sum_{i\in\mathcal{I}} \lambda_i x_i + b(\lambda)y$, mapping the product spaces $\Omega_f \times \mathcal{S}^c_f$ to Ω_f, will play a key role in the construction. The corresponding pullback, G^*, is a map

$$G^* : \Lambda^k(\Omega_f) \to \Lambda^k(\Omega_f \times \mathcal{S}^c_f).$$

We recall that a space of k–forms on a product space can be expressed by the tensor product as

$$\Lambda^k(\Omega_f \times \mathcal{S}^c_f) = \sum_{j=0}^{k} \Lambda^j(\Omega_f) \otimes \Lambda^{k-j}(\mathcal{S}^c_f).$$

Here the symbol $\otimes$ is the tensor product. In other words, elements $U \in \Lambda^j(\Omega_f) \otimes \Lambda^{k-j}(\mathcal{S}^c_f)$ can be written as a sum of terms of the form

$$a(y,\lambda) dy^j \otimes d\lambda^{k-j},$$

where dy^j and $d\lambda^{k-j}$ run over bases in $\mathrm{Alt}^j(\Omega_f)$ and $\mathrm{Alt}^{k-j}(\mathcal{S}^c_f)$, respectively, and where a is a scalar function on $\Omega_f \times \mathcal{S}^c_f$. Here Alt^k refers to the corresponding space of algebraic k-forms. Furthermore, for each j, $0 \le j \le k$, there is a canonical map $\Pi_j : \Lambda^k(\Omega_f \times \mathcal{S}^c_f) \to \Lambda^j(\Omega_f) \otimes \Lambda^{k-j}(\mathcal{S}^c_f)$ such that

$$U = \sum_{j=0}^{k} \Pi_j U, \quad U \in \Lambda^k(\Omega_f \times \mathcal{S}^c_f).$$

The function $\Pi_j G^* u \in \Lambda^j(\Omega_f) \otimes \Lambda^{k-j}(\mathcal{S}^c_f)$ can be identified as

$$\begin{aligned}(\Pi_j G^* u)_{y,\lambda}&(t_1, \dots t_j, v_1, \dots, v_{k-j}) \\ &= u_{G(y,\lambda)}(D_y G t_1, \dots D_y G t_j, D_\lambda G v_1, \dots, D_\lambda G v_{k-j}),\end{aligned}$$

where the tangent vectors $t_i \in T(\Omega_f)$ and $v_i \in T(\mathcal{S}^c_f)$. For the special function G in our case,

$$D_y G = b(\lambda) I, \quad \text{and } D_\lambda = \sum_{i\in\mathcal{I}} (x_i - y) d\lambda_i,$$

so this can be rewritten as

$$(\Pi_j G^* u)_{y,\lambda}(t_1, \dots t_j, v_1, \dots, v_{k-j}) = b(\lambda)^j u_{G(y,\lambda)}(t_1, \dots t_j, D_\lambda G v_1, \dots, D_\lambda G v_{k-j}).$$

The basic commuting property for pull-backs, namely $dG^* = G^*d$, can be expressed in the present setting as

$$d_\Omega \Pi_{j-1} G^* u + (-1)^j d_S \Pi_j G^* u = \Pi_j dG^* u = \Pi_j G^* du, \quad j = 1, \dots, k, \tag{5.1}$$

where d_Ω and d_S denote the exterior derivative with respect to the spaces Ω and $\mathcal{S}$, respectively.

We recall that in Section 2 we introduced the average operators A_f^k for each $f \in \Delta$ mapping $\Lambda^k(\Omega_f)$ to $\Lambda^k(\mathcal{S}_f^c)$. There we defined the operators A_f^k from an integral with respect to y of the pullbacks $G(y, \cdot)^*$. Alternatively, we can now identify these operators as

$$(A_f^k u)_\lambda = \fint_{\Omega_f} (\Pi_0 G^* u)_\lambda \wedge \mathsf{vol}, \quad \lambda \in \mathcal{S}_f^c,$$

where vol is the volume form on Ω. For any $f \in \Delta$ and $e \in \Delta_j(f)$, the operators $R_{e,f}^k$ will be of the form

$$(R_{e,f}^k u)_\lambda = \int_\Omega (\Pi_j G^* u)_\lambda \wedge z_{e,f}, \quad \lambda \in \mathcal{S}_{f\cap e^*}^c, \tag{5.2}$$

where the weight function $z_{e,f}$ is an $n-j$ form on Ω with local support. This means that $R_{e,f}^k u$ is a $k-j$ form on $\mathcal{S}_{f\cap e^*}^c$ for $0 \le j \le k$, while $R_{e,f}^k u \equiv 0$ for $j > k$.

For any $e \in \Delta_0(f)$, the operator $R_{e,f}^k = -\operatorname{tr}_{\mathcal{S}_{f\cap e^*}^c} A_f^k$, which corresponds to the operator (5.2), where the n form $z_{e,f}$ is given by

$$z_{e,f} = -\frac{\kappa_f}{|\Omega_f|}\mathsf{vol} \equiv -\mathsf{vol}_f, \tag{5.3}$$

where κ_f is the characteristic function of Ω_f. In other words, vol_f is the scaled version of the volume form restricted to Ω_f, such that $\int_{\Omega_f} \mathsf{vol}_f = 1$. To complete the definition of the operators $R_{e,f}^k$, we need to specify the functions $z_{e,f}$ for $e \in \Delta_j(f)$ and $j > 0$.

5.2. **The weight functions $z_{e,f}$.** For each $f \in \Delta$ and $e \in \Delta(f)$, the corresponding functions $z_{e,f}$ will have support on a subdomain of Ω referred to as $\Omega_{e,f}$. The domains $\Omega_{e,f}$ can be defined by a recursive process. If e is the emptyset or $e \in \Delta_0(f)$, then $\Omega_{e,f}$ is taken to be Ω_f. For $j > 0$, we define the domains $\Omega_{e,f}$ recursively by

$$\Omega_{e,f} = \bigcup_{i \in I(e)} \Omega_{e(\hat{x}_i), f(\hat{x}_i)}.$$

An alternative characterization of the domains $\Omega_{e,f}$ is

$$\Omega_{e,f} = \Omega_{f\cap e^*} \cap \Omega_e^E,$$

which can be verified by induction with respect to $|e|$. Here we recall that the extended macroelements Ω_f^E are defined in Section 2.2 above. Note that this characterization gives $\Omega_{\emptyset,f} = \Omega_{x_i,f} = \Omega_f$ and $\Omega_{f,f} = \Omega_f^E$. As a consequence, if $e, g \in \Delta(f)$, $e \subset g$ and $i \in I(e)$ then

$$\Omega_{e(\hat{x}_i),f} \subset \Omega_{e,f} \subset \Omega_{e,g} \subset \Omega_{g,g} = \Omega_g^E \subset \Omega_f^E. \tag{5.4}$$

In particular, we observe that the n simplexes forming $\Omega_{e,f}$ are just a subset of the n simplexes forming $\Omega_{f\cap e^*}$.

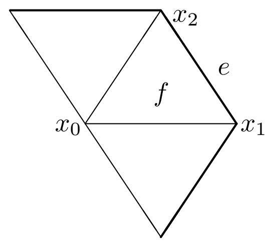

FIGURE 5.1. The domain $\Omega_{e,f}$ for $e = [x_1, x_2]$, $f = [x_0, x_1, x_2]$ and $n = 2$.

Recall that throughout the paper we have made the assumption that the extended macroelements Ω_f^E are contractible. The following result is an immediate consequence.

Lemma 5.1. *All the domains $\Omega_{e,f}$, for $f \in \Delta(\mathcal{T})$ and $e \in \Delta(f)$, are contractible.*

Proof. Since $\Omega_{f,f} = \Omega_f^E$ is assumed to be contractible, it is enough to consider the case $e \in \Delta(f)$, $e \neq f$. But then $f \cap e^*$ is nonempty. Since $\Omega_{e,f}$ is star-shaped with respect to any point in $f \cap e^*$ it follows that $\Omega_{e,f}$ is contractible. □

Since the domain $\Omega_{e,f}$ is contractible, it follows by de Rham's theorem that the complex $(\mathring{\mathcal{P}}_1^- \Lambda^k(\mathcal{T}_{e,f}), d)$ is exact, e.g., see [2, Section 5.5] for further discussion of this fact. This property is crucial for the construction which follows. To define the functions $z_{e,f}$ for $e \in \Delta_j(f)$, $j > 0$, we will introduce two difference operators defined for any set of functions parametrized by pairs (e, f), $e \in \Delta(f)$ and $f \in \Delta(\mathcal{T})$. We define

$$(\delta z)_{e,f} = \sum_{i \in I(e)} (-1)^{\sigma_e(x_i)} z_{e(\hat{x}_i), f(\hat{x}_i)} \quad \text{and} \quad (\delta^+ z)_{e,f} = \sum_{i \in I(e)} (-1)^{\sigma_e(x_i)} z_{e(\hat{x}_i), f}.$$

It follows from standard arguments that these operators satisfy the complex property $\delta^2 = 0$. In fact, we have the following identities.

Lemma 5.2. *The operators δ and δ^+ satisfy*

$$\delta \circ \delta = 0, \quad \delta^+ \circ \delta^+ = 0, \quad \text{and } \delta \circ \delta^+ = -\delta^+ \circ \delta.$$

Proof. The two first properties are standard, so we omit the proofs. To see the third identity, we compute the two expressions as

$$(\delta^+ \delta z)_{e,f} = \sum_{\substack{i,p \in I(e) \\ i \neq p}} (-1)^{\sigma_e(x_i) + \sigma_{e(\hat{x}_i)}(x_p)} z_{e(\hat{x}_i, \hat{x}_p), f(\hat{x}_p)},$$

and

$$(\delta \delta^+ z)_{e,f} = \sum_{\substack{i,p \in I(e) \\ i \neq p}} (-1)^{\sigma_e(x_p) + \sigma_{e(\hat{x}_p)}(x_i)} z_{e(\hat{x}_i, \hat{x}_p), f(\hat{x}_p)}.$$

However, we will always have

$$(-1)^{\sigma_e(x_p) + \sigma_{e(\hat{x}_p)}(x_i)} = -(-1)^{\sigma_e(x_i) + \sigma_{e(\hat{x}_i)}(x_p)},$$

which implies the desired identity. □

Note that if $e = [x_0, x_1] \in \Delta_1(f)$, then

$$(\delta^+ z)_{e,f} = z_{x_1,f} - z_{x_0,f} = \mathsf{vol}_f - \mathsf{vol}_f = 0.$$

We will define all the functions $z_{e,f}$ such that they satisfy $\delta^+ z = 0$. More precisely, if $e \in \Delta_j(f)$, then $z_{e,f} \in \mathring{\mathcal{P}}_1^- \Lambda^{n-j}(\mathcal{T}_{e,f})$ and for $j > 0$, we have

$$dz_{e,f} = (-1)^{j+1}(\delta z)_{e,f}, \quad \text{and } (\delta^+ z)_{e,f} = 0, \quad f \in \Delta(\mathcal{T}), e \in \Delta_j(f). \tag{5.5}$$

In fact, for $j > 0$, the functions $z_{e,f}$ will be of the form $z_{e,f} = (\delta^+ w)_{e,f}$, where the functions $w_{e,f}$ are defined for $f \in \Delta$ and $e \in \bar{\Delta}(f)$. If $e = \emptyset$, we define $w_{e,f} = -\mathsf{vol}_f$. For $e \in \Delta_j(f)$, $j \geq 0$, the functions $w_{e,f}$ will be required to satisfy

$$dw_{e,f} = (-1)^j((\delta - \delta^+)w)_{e,f}. \tag{5.6}$$

In the special case $e = x_i \in \Delta_0(f)$, we will require $w_{x_i,f} \in \mathring{\mathcal{P}}_1^- \Lambda^{n-1}(\mathcal{T}_{f(\hat{x}_i)})$ such that

$$dw_{x_i,f} = ((\delta - \delta^+)w)_{x_i,f} = (w_{\emptyset,f(\hat{x}_i)} - w_{\emptyset,f}) = \mathsf{vol}_f - \mathsf{vol}_{f(\hat{x}_i)}.$$

This is possible since the right hand side has mean value zero on $\Omega_{f(\hat{x}_i)}$. In addition, we make the functions $w_{x_i,f}$ unique by the standard orthogonality condition with respect to $d\mathring{\mathcal{P}}_1^- \Lambda^{n-2}(\mathcal{T}_{f(\hat{x}_i)})$. It now follows by an inductive process, utilizing the exactness of the complexes of the form $(\mathring{\mathcal{P}}_1^- \Lambda(\mathcal{T}_{e,f}), d)$, that we can construct functions $w_{e,f} \in \mathring{\mathcal{P}}_1^- \Lambda^{n-j-1}(\mathcal{T}_{e,f})$ for all $e \in \Delta_j(f)$, $j > 0$, such that (5.6) holds, and with support of $((\delta - \delta^+)w)_{e,f}$ in

$$\bigcup_{i \in I(e)} [\Omega_{e(\hat{x}_i),f(\hat{x}_i)} \cup \Omega_{e(\hat{x}_i),f}] = \bigcup_{i \in I(e)} [\Omega_{e(\hat{x}_i),f(\hat{x}_i)} = \Omega_{e,f}.$$

To see this, just observe that Lemma 5.2 implies that

$$d[((\delta - \delta^+)w)_{e,f}] = ((\delta - \delta^+)dw)_{e,f} = (-1)^{j+1}((\delta\delta^+ + \delta^+\delta)w)_{e,f} = 0.$$

Furthermore, the functions $w_{e,f}$ are uniquely determined if we add the standard orthogonality condition

$$\int_{\Omega_{e,f}} w_{e,f} \wedge \star dq = 0, \quad q \in \mathring{\mathcal{P}}_1^- \Lambda^{n-j-2}(\mathcal{T}_{e,f}), \tag{5.7}$$

where $\star$ is the Hodge star operator. The functions $z_{e,f}$, defined by $z_{e,f} = (\delta^+ w)_{e,f}$, satisfy the following properties.

Lemma 5.3. *Assume that $f \in \Delta(\mathcal{T})$ and $e \in \Delta_j(f)$. The functions $z_{e,f}$, defined above by $z_{e,f} = (\delta^+ w)_{e,f}$, belong to $\mathring{\mathcal{P}}_1^- \Lambda^{n-j}(\mathcal{T}_{e,f})$ and satisfy the two identities* (5.5).

Proof. The support property follows from the support property of the functions $w_{e,f}$, while $\delta^+ z = 0$ follows from the complex property of the operator δ^+. Finally, for $e \in \Delta_j(f)$, we have

$$\begin{aligned} dz_{e,f} = d(\delta^+ w)_{e,f} = (\delta^+ dw)_{e,f} &= (-1)^j((\delta^+ \circ \delta)w)_{e,f} \\ &= (-1)^{j+1}((\delta \circ \delta^+)w)_{e,f} = (-1)^{j+1}(\delta z)_{e,f}, \end{aligned}$$

and this completes the proof. □

5.3. **Properties of the operators $R^k_{e,f}$.** Since $z_{e,f}$ has support on $\Omega_{e,f}$, $R^k_{e,f}u$ only depends on u restricted to $\Omega_{e,f}$ and we can write (5.2) as

$$(R^k_{e,f}u)_\lambda = \int_\Omega (\Pi_j G^* u)_\lambda \wedge z_{e,f} = \int_{\Omega_{e,f}} (\Pi_j G^* u)_\lambda \wedge z_{e,f}, \quad \lambda \in \mathcal{S}^c_{f\cap e^*}.$$

For any $f \in \Delta(\mathcal{T})$ and $e \in \Delta_j(f)$, we define for $\lambda \in \mathcal{S}^c_{f\cap e^*}$,

$$(\delta R^k u)_{e,f} = \sum_{i\in I(e)} (-1)^{\sigma_e(x_i)} R^k_{e(\hat{x}_i),f(\hat{x}_i)}u.$$

Note that for each $i \in I(e)$, we have $f(\hat{x}_i) \cap e(\hat{x}_i)^* = f \cap e^*$. Therefore, $(\delta R^k u)_{e,f}$ is a $k - j + 1$ form on $\mathcal{S}_{f\cap e^*}$. Alternatively, we can represent $(\delta R^k u)_{e,f}$ by

$$((\delta R^k u)_{e,f})_\lambda = \int_\Omega (\Pi_{j-1} G^* u)_\lambda \wedge (\delta z)_{e,f}, \quad \lambda \in \mathcal{S}_{f\cap e^*}. \tag{5.8}$$

We show below that the operators $R^k_{e,f}$ satisfy the relation

$$R^{k+1}_{e,f} du = (-1)^j dR^k_{e,f}u - (\delta R^k u)_{e,f}, \quad e \in \Delta_j(f),\, 0 \le j \le k+1. \tag{5.9}$$

We note that all the three terms appearing here are $k - j + 1$ forms defined on the simplex $\mathcal{S}^c_{f\cap e^*}$, and that the desired formula (4.4) is just a special case corresponding to $k = 0$ and $j = 1$. Furthermore, if we define $R^k_{e,f}$ to be the zero operator when e is the emptyset, then (5.9) with $j = 0$ expresses the commuting property of the operators A^k_f. In addition, we show below that the operators $R^k_{e,f}$ satisfy the identity

$$(\delta^+ R^k u)_{e,f} = 0, \tag{5.10}$$

where

$$(\delta^+ R^k u)_{e,f} = \sum_{i\in I(e)} (-1)^{\sigma_e(x_i)} \operatorname{tr}_{\mathcal{S}^c_{f\cap e^*}} R^k_{e(\hat{x}_i),f}u.$$

The identities (5.9) and (5.10) will be key tools for constructing commuting cut–off operators C^k_m. To derive the identity (5.9), we will use the basic commuting property for pull-backs, $dG^* = G^*d$, which in the present setting is given by (5.1), where $\Pi_j G^* u \in \Lambda^j(\Omega_f)\otimes\Lambda^{k-j}(\mathcal{S}^c_f)$, and the operators d_Ω and d_S denote the exterior derivatives with respect to the spaces Ω and $\mathcal{S}$, respectively.

Proposition 5.4. *The operators $R^k_{e,f}$ satisfy the two identities* (5.9) *and* (5.10).

Proof. For any $e \in \Delta_j(f)$, we have

$$(\delta^+ R^k u)_{e,f} = \operatorname{tr}_{\mathcal{S}^c_{f\cap e^*}} \int_\Omega (\Pi_{j-1} G^* u) \wedge (\delta^+ z)_{e,f},$$

and the relation (5.10) follows directly from the second identity of (5.5). To show (5.9), we use the first relation of (5.5), (5.8), and integration by parts to obtain

$$\begin{aligned} -(\delta R^k u)_{e,f} &= (-1)^j \int_\Omega \Pi_{j-1} G^* u \wedge d_\Omega z_{e,f} \\ &= \int_\Omega d_\Omega \Pi_{j-1} G^* u \wedge z_{e,f} = \int_\Omega \Big[(-1)^{j+1} d_{\mathcal{S}} \Pi_j G^* u + \Pi_j G^* du\Big] \wedge z_{e,f}, \end{aligned}$$

where we have used (5.1) to obtain the last equality. However, since

$$dR^k_{e,f}u = \int_\Omega d_S \Pi_j G^* u \wedge z_{e,f},$$

we see that the right hand side above is exactly equal to

$$(-1)^{j+1} dR^k_{e,f}u + R^{k+1}_{e,f} du,$$

and hence the desired result is obtained. □

We end this section by establishing the polynomial preservation properties of the operators $R^k_{e,f}$. We also show that the operators $R_{e,f}$ map piecewise smooth differential forms to smooth differential forms. In fact, the proposition below can be seen as a generalization of Lemma 2.1, and the two proofs are closely related.

Proposition 5.5. *Assume that $f \in \Delta(\mathcal{T})$, $e \in \Delta_j(f)$.*

i) *If $u \in \Lambda^k(\mathcal{T})$, then $b^{-j} R^k_{e,f}u \in \Lambda^{k-j}(\mathcal{S}^c_{f\cap e^*})$,*
ii) *If $u \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ then $b^{-j} R^k_{e,f}u \in \mathcal{P}_r\Lambda^{k-j}(\mathcal{S}^c_{f\cap e^*})$,*
iii) *if $u \in \mathcal{P}^-_r\Lambda^k(\mathcal{T})$ then $b^{-j} R^k_{e,f}u \in \mathcal{P}^-_r\Lambda^{k-j}(\mathcal{S}^c_{f\cap e^*})$.*

Proof. If $e = f$, then $\mathcal{S}^c_{f\cap e^*}$ consists of a single point, the origin in $\mathbb{R}^{N+1}$, and in this case the conclusion of the proposition is obvious. Therefore, for the rest of the proof, we assume that $e \neq f$, such that $f \cap e^*$ is nonempty. We recall that for $f \in \Delta(\mathcal{T})$ and $e \in \Delta_j(f)$, we have

$$R^k_{e,f}u = \int_{\Omega_{e,f}} \Pi_j G^* u \wedge z_{e,f}.$$

More precisely, $R^k_{e,f}u$ is $k-j$ form on $S^c_{f\cap e^*}$ such that

$$(R^k_{e,f}u)_\lambda(v_1, \dots, v_{k-j}) = \int_{\Omega_{e,f}} (\Pi_j G^* u \lrcorner v_1 \dots \lrcorner v_{k-j})_\lambda \wedge z_{e,f},$$

where $v_i \in T(\mathcal{S}^c_{f\cap e^*})$ and $(\Pi_j G^* u \lrcorner v_1, \dots \lrcorner v_{k-j})_\lambda$ is a j form on Ω. In fact,

$$\begin{aligned} b(\lambda)^{-j}((\Pi_j G^* u \lrcorner v_1 \dots \lrcorner v_{k-j})_\lambda)_y(t_1, \dots, t_j) \\ = u_{G(y,\lambda)}(t_1, \dots, t_j, D_\lambda G v_1, \dots, D_\lambda G v_{k-j}), \end{aligned} \tag{5.11}$$

where $y \in \Omega_{e,f}$ and $t_i \in T(\Omega_{e,f})$. Furthermore, for any fixed $y \in \Omega_{e,f} \subset \Omega_{f\cap e^*}$, the set

$$\{\, G(y,\lambda) \,:\, \lambda \in \mathcal{S}^c_{f\cap e^*} \,\}$$

belongs to a single n simplex of $\Omega_{e,f}$, while the vectors of the form $D_\lambda v$ are independent of λ. This shows that if u is a piecewise smooth k form on $\Omega_{e,f}$, then for each fixed $y \in \Omega_{e,f}$, the right hand side of (5.11) is a smooth function of $\lambda \in \mathcal{S}^c_{f\cap e^*}$. The same must be true for the integral with respect to y, and hence the first statement of the proposition is established.

The second property follows from almost the same argument, since if u is a piecewise polynomial, i.e., $u \in \mathcal{P}_r\Lambda^k(\mathcal{T})$, then the right hand side of (5.11) is a polynomial of degree r with respect to $\lambda \in \mathcal{S}^c_{f\cap e^*}$ for each fixed $y \in \Omega_{e,f}$. Again, the same will hold for the integral with respect to y. To show that the $\mathcal{P}_r^-$ spaces are also preserved, we will consider $R^k_{e,f} u \lrcorner \lambda$, where $\lambda \in \mathcal{S}^c_{f\cap e^*}$. Then

$$(R^k_{e,f} u)_\lambda(\lambda, v_1, \ldots, v_{k-j-1}) = \int_{\Omega_{e,f}} (\Pi_j G^* u \lrcorner \lambda \lrcorner v_1 \ldots \lrcorner v_{k-j-1})_\lambda \wedge z_{e,f},$$

where

$$\begin{aligned} b(\lambda)^{-j}((\Pi_j G^* u \lrcorner \lambda \lrcorner v_1 \ldots \lrcorner v_{k-j-1})_\lambda)_y(t_1, \ldots, t_j) \\ = u_{G(y,\lambda)}(t_1, \ldots, t_j, G(y, \lambda) - y, D_\lambda G v_1, \ldots, D_\lambda G v_{k-j-1}). \end{aligned}$$

However, if $u \in \mathcal{P}_r^- \Lambda^k(\mathcal{T})$, it follows from the linearity of G with respect to λ that for each fixed $y \in \Omega_{e,f}$, the right hand side above is in $\mathcal{P}_r(\mathcal{S}^c_{f\cap e^*})$, and therefore the same holds for the integral with respect to y. As a consequence, we can conclude that $b^{-j} R^k_{e,f} u \in \mathcal{P}_r^- \Lambda^{k-j}(\mathcal{S}^c_{f\cap e^*})$. This completes the proof of the proposition. □

6. The cut off operators $C^k_{m,f}$

Recall that relation (4.4) is just a special case of (5.9). As a consequence of the construction of the order reduction operators $R^k_{e,f}$ in the previous section, we therefore can conclude that the operator C^1_m, specified in Lemma 4.2, satisfies the commuting relation $dC^0_m = C^1_m d$.

In general, for $0 \le k \le n$, we define the operator C^k_m by

$$C^k_m u = \sum_{\substack{f\in\Delta_{m+j}(\mathcal{T})\\ 0\le j\le k}} C^k_{m,f} u,$$

where $C^k_{m,f}$ is given by (4.1) if $f \in \Delta_m(\mathcal{T})$, and by

$$C^k_{m,f} u = j! \sum_{e\in\Delta_j(f)} \sum_{g\in\bar\Delta(f\cap e^*)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L^*_g b^{-j} R^k_{e,f} u \tag{6.1}$$

if $f \in \Delta_{m+j}(\mathcal{T})$, $1 \le j \le k$. Here we recall that ϕ_e is the Whitney form associated to the simplex e and that $\rho_g = L^*_g b$. We now have the following extension of Lemma 4.1.

Lemma 6.1. *Let $u \in \Lambda^k(\mathcal{T}_f)$ and $f \in \Delta_{m+j}(\mathcal{T})$ for $0 \le m \le n$ and $0 \le j \le k$. Then $C^k_{m,f} u \in \mathring{\Lambda}^k_m(\mathcal{T}, f)$ and $\operatorname{tr}_f C^k_m u = \operatorname{tr}_f u$ for $f \in \Delta_m(\mathcal{T})$ and $k \le m \le n$.*

Proof. We only have to consider the case $j > 0$, since the case $j = 0$ is covered by Lemma 4.1. Let $f \in \Delta_{m+j}(\mathcal{T})$, $1 \le j \le k$, be fixed. It is enough to consider each term in the sum of $C^k_{m,f} u$ corresponding to $e \in \Delta_j(f)$ fixed, i.e.,

$$C^k_{m,e,f} u := \sum_{g\in\bar\Delta(f\cap e^*)} (-1)^{|f|-|g|} \frac{\phi_e}{\rho_g} \wedge L^*_g b^{-j} R^k_{e,f} u.$$

By part (i) of Proposition 5.5, $b^{-j} R^k_{e,f} u \in \Lambda^{k-j}(\mathcal{S}^c_{f\cap e^*})$. As a consequence, it follows that $C^k_{m,e,f} u \in \Lambda^k_m(\mathcal{T}, f)$. To show that $C^k_{m,e,f} u$ is supported in Ω_f we will use a variant of the cancellation argument we have used before. Assume that $i \in I(f)$ and let Γ be a subset of Ω such that $\lambda_i \equiv 0$ on Γ. If $i \in I(e)$, then ϕ_e

vanishes on Γ. On the other hand, if $i \notin I(e)$, then $i \in I(f \cap e^*)$ and we can use a cancellation argument to show that $C^k_{m,e,f}u = 0$. We compare two terms in the definition of $C^k_{m,e,f}u$ corresponding to g and g', where $g \subset g'$ and $g' \setminus g = \{x_i\}$. The two terms will cancel on Γ. Therefore we can conclude that $C^k_{m,e,f}u = 0$ on Γ, and this implies the support property of $C^k_{m,e,f}u$. To check the trace property of $C^k_m u$, we recall that if $g \in \Delta_m(\mathcal{T})$ and $f \in \Delta_{m+j}(\mathcal{T})$, $j \geq 0$, where $f \neq g$, then g will not belong to the interior of Ω_f. By combining this observation, the result above, and the trace property given in Lemma 4.1, we can conclude that $\operatorname{tr}_f C^k_m u = \operatorname{tr}_f u$ for $f \in \Delta_m(\mathcal{T})$ and $m \geq k$. ☐

Next we will perform a modest rewriting of the operator $C^k_m u$ which will be useful in the discussion of the next section. We will split the operator $C^k_{m,f}$ for $f \in \Delta_m(\mathcal{T})$ into two terms. For $f \in \Delta_m(\mathcal{T})$ and $g \in \bar{\Delta}(f)$, we have

$$\frac{\rho_f}{\rho_g} L^*_g A^k_f u = (1 + \frac{\rho_f - \rho_g}{\rho_g}) L^*_g A^k_f u = L^*_g A^k_f u + \sum_{e \in \Delta_0(f \cap g^*)} \frac{\phi_e}{\rho_g} \wedge L^*_g R^k_{e,f} u,$$

where we recall that $\phi_e = \lambda_i$ and $R^k_{e,f} = -A^k_f$ for $e = [x_i] \in \Delta_0(f)$. As a consequence, the operator C^k_m can be rewritten as

$$(6.2) \quad C^k_m u = \sum_{f \in \Delta_m(\mathcal{T})} \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} L^*_g A^k_f u + \sum_{\substack{f \in \Delta_{m+j}(\mathcal{T}) \\ 0 \leq j \leq k}} j! \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} \sum_{e \in \Delta_j(f \cap g^*)} \frac{\phi_e}{\rho_g} \wedge L^*_g b^{-j} R^k_{e,f} u.$$

In other words, we have written the operator $C^k_{m,f}$, for $f \in \Delta_m(\mathcal{T})$, as a sum of two operators, where both terms have support on Ω_f, and where the term containing ϕ_e for $e \in \Delta_0(f)$ has the same form as the terms containing ϕ_e, for $|e| > 1$.

Recall that the operator L^*_g maps smooth differential forms to piecewise smooth forms, and that the operators $b^{-j} R^k_{e,f}$ for $e \in \Delta_j(f)$ map piecewise smooth forms to smooth forms, cf. Proposition 5.5. Hence, it appears that all the terms in the second part of (6.2) contain a rational factor $1/\rho_g$. The challenge is to show that this rational factor disappears when we add the terms in the second part of (6.2). This will be a consequence of the discussion given in the next section.

7. Properties of the global operators C^k_m

It will be a consequence of the result of this section that the operator C^k_m commutes with the exterior derivative. Furthermore, we will show that this operator is invariant with respect to the piecewise smooth space $\Lambda^k(\mathcal{T})$, and with respect to the piecewise polynomial spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}^-_r\Lambda^k(\mathcal{T})$. In other words, the operator C^k_m maps these spaces into themselves. In the special case when $m = n$, the operator $\mathcal{C}^k_m$ reduces to the identity, which obviously has the desired properties. Therefore, in the rest of the discussion of this section, we can assume that $0 \leq m \leq n-1$.

We start by recalling the support properties of the operators $C^k_{m,f}$ given in Lemma 6.1. It follows from the fact that $C^k_{m,f}u$ has support on Ω_f that for each n simplex T in $\Delta_n(\mathcal{T})$, we have

$$\operatorname{tr}_T C^k_m u = \sum_{\substack{f\in\Delta_{m+j}(T)\\ 0\le j\le k}} C^k_{m,f}u, \tag{7.1}$$

i.e., we can restrict the sum to the subsimplexes f in $\Delta(T)$. Furthermore, if T_- and T_+ are two n simplexes with a common $n-1$ simplex $T_-\cap T_+ \in \Delta_{n-1}(\mathcal{T})$, then

$$\operatorname{tr}_{T_-\cap T_+}\operatorname{tr}_{T_-} C^k_m u = \operatorname{tr}_{T_-\cap T_+}\operatorname{tr}_{T_+} C^k_m u = \operatorname{tr}_{T_-\cap T_+}\sum_{\substack{f\in\Delta_{m+j}(T_-\cap T_+)\\ 0\le j\le k}} C^k_{m,f}u.$$

This means that for any $u\in\Lambda^k(\mathcal{T})$, the differential form $C^k_m u$ will always have single valued traces on all elements of $\Delta_{n-1}(\mathcal{T})$. As a consequence, to show that the operator C^k_m is invariant with respect to the piecewise smooth space $\Lambda^k(\mathcal{T})$ and the piecewise polynomial spaces, it is enough to consider the restriction of $C^k_m u$ to a single n simplex T, where the restriction is given by (7.1)

7.1. **Restricting to a single n simplex.** We will consider the restriction of $C^k_m u$ to a fixed n simplex T. In fact, in the arguments given below, we can consider the part of $\operatorname{tr}_T C^k_m u$ which corresponds to a fixed simplex $g\in\bar\Delta(T)$. Therefore, for each fixed $T\in\Delta_n(\mathcal{T})$ and $g\in\bar\Delta(T)$, $0\le|g|\le m+1$, we introduce the operator

$$C^k_m(g,T)u = \sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} L^*_g A^k_f u + \sum_{j=0}^k(-1)^j j! \sum_{\substack{f\in\Delta_{m+j}(T)\\ f\supset g}}\ \sum_{e\in\Delta_j(f\cap g^*)} \frac{\phi_e}{\rho_g}\wedge L^*_g b^{-j} R^k_{e,f}u.$$

If $u\in\Lambda^k(\mathcal{T})$, we will view the function $C^k_m(g,T)u$ as a, possibly rational, k form on T. It is a consequence of the characterization of $\operatorname{tr}_T C^k_m$, given by (7.1), that

$$\operatorname{tr}_T C^k_m u = \sum_{g\in\bar\Delta(T)} (-1)^{m+1-|g|} C^k_m(g,T)u, \quad T\in\Delta_n(\mathcal{T}).$$

If we can show that each operator $C^k_m(g,T)$ commutes with the exterior derivative, and that it maps the spaces $\Lambda^k(\mathcal{T})$, $\mathcal{P}_r\Lambda^k(\mathcal{T})$, and $\mathcal{P}^-_r\Lambda^k(\mathcal{T})$ into the corresponding spaces on T, then we can immediately conclude the following fundamental result. In the special case when $m=n$, the operator $\operatorname{tr}_T C^k_m$ reduces to tr_T, which obviously has the desired properties. Therefore, in the rest of the discussion of this section, we can assume that $0\le m\le n-1$.

Proposition 7.1. *The operator C^k_m satisfies the commuting relation*

$$dC^k_m = C^{k+1}_m d, \quad 0\le k\le n-1.$$

Furthermore,

i) *if $u\in\Lambda^k(\mathcal{T})$, then $C^k_m u\in\Lambda^k(\mathcal{T})$,*
ii) *if $u\in\mathcal{P}_r\Lambda^k(\mathcal{T})$, then $C^k_m u\in\mathcal{P}_r\Lambda^k(\mathcal{T})$,*
iii) *if $u\in\mathcal{P}^-_r\Lambda^k(\mathcal{T})$, then $C^k_m u\in\mathcal{P}^-_r\Lambda^k(\mathcal{T})$.*

The desired properties of the operator $C^k_m(g,T)$ will follow from the following decomposition.

Lemma 7.2. *If $u \in \Lambda^k(\mathcal{T})$ and $g \in \Delta_s(T)$, then*

$$C_m^k(g,T)u - \frac{n-m}{n-s} \sum_{\substack{f \in \Delta_m(T) \\ f \supset g}} L_g^* A_f^k u = dQ_m^k u + Q_m^{k+1} du, \tag{7.2}$$

where the operators $Q_m^k = Q_m^k(g,T)$ are given by

$$Q_m^k u = \frac{1}{n-s} \sum_{j=1}^{k} (-1)^j (j-1)! \sum_{\substack{f \in \Delta_{m+j}(T) \\ f \supset g}} \sum_{e \in \Delta_j(f \cap g^*)} (\delta\phi)_e \wedge L_g^* b^{-j} R_{e,f}^k u,$$

with $(\delta\phi)_e = \sum_{i \in I(e)} (-1)^{\sigma(x_i)} \phi_{e(\hat{x}_i)}$. In particular, $Q_m^0 = 0$ and the case $g = \emptyset$ corresponds to $s = -1$.

We will delay the proof of this lemma, and first show how this decomposition immediately leads to a proof of Proposition 7.1.

Proof. (of Proposition 7.1) Recall that we only need to consider $C_m^k(g,T)$ as an operator from $\Lambda^k(\mathcal{T})$ to the space of rational k forms on T. Since the operator A_f^k commutes with d, the commuting property will follow if the right hand side of (7.2) commutes with d. However, this follows since

$$d[dQ_m^k + Q_m^{k+1} d]u = dQ_m^{k+1} du = [dQ_m^{k+1} + Q_m^{k+2} d]du.$$

From the properties of the operator $R_{e,f}^k$ given in Proposition 5.5, we can conclude that the operator Q^k maps the space $\Lambda^k(\mathcal{T})$ to $\Lambda^{k-1}(T)$ and $\mathcal{P}_r\Lambda^k(\mathcal{T})$ to $\mathcal{P}_{r+1}\Lambda^{k-1}(T)$. The desired conclusion, that the operator C_m^k maps the spaces $\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r\Lambda^k(\mathcal{T})$ into themselves, follows directly from the decomposition (7.2). To show the corresponding result for the $\mathcal{P}_r^-$ spaces, we need to show that the operator $C_m^k(g,T)$ preserves these spaces. However, it follows from the definition of the operator $C_m^k(g,T)$, Proposition 5.5, and formula (3.16) of [2] that $C_m^k(g,T)$ maps $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ into $\rho_g^{-1}\mathcal{P}_{r+1}^-\Lambda^k(T)$. Since $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ is a subspace of $\mathcal{P}_r\Lambda^k(\mathcal{T})$ we therefore have that $C_m^k(g,T)$ maps $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ into

$$\mathcal{P}_r\Lambda^k(T) \cap \rho_g^{-1}\mathcal{P}_{r+1}^-\Lambda^k(T).$$

But elements of this space must be in $\mathcal{P}_r^-\Lambda^k(T)$. To see this, let $u \in \mathcal{P}_{r+1}^-\Lambda^k(T)$ be such that $\rho_g^{-1}u \in \mathcal{P}_r\Lambda^k(T)$. For any $x_j \in \Delta_0(T)$, we then have

$$u \lrcorner (x - x_j) \in \mathcal{P}_{r+1}\Lambda^{k-1}(T), \quad \text{and } \rho_g^{-1}(u \lrcorner (x - x_j)) \in \mathcal{P}_{r+1}\Lambda^{k-1}(T).$$

In other words, the polynomial form $u \lrcorner (x - x_j)$ has ρ_g as a linear factor, and as a consequence, $\rho_g^{-1}(u \lrcorner (x - x_j))$ must be in $\mathcal{P}_r\Lambda^{k-1}(T)$. This implies that $\rho_g^{-1}u \in \mathcal{P}_r^-\Lambda^k(T)$. □

Before we prove Lemma 7.2, we will first establish a preliminary result. To simplify the notation in the present setting, where T and g are fixed, we introduce the set $\Delta(m,j)$ given by

$$\Delta(m,j) = \{ (e,f) \,:\, f \in \Delta_{m+j}(T),\, f \supset g,\, e \in \Delta_j(f \cap g^*) \}.$$

Furthermore, in the discussion below, we abbreviate $g^*(T)$ by g^*.

We also introduce the operators $C^k_{m,\ell}(g,T)$ given by

$$C^k_{m,\ell}(g,T)u = \sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} L^*_g A^k_f u + \sum_{j=0}^{\ell}(-1)^j j! \sum_{(e,f)\in\Delta(m,j)} \frac{\phi_e}{\rho_g} \wedge L^*_g b^{-j} R^k_{e,f} u.$$

We note that we have $C^k_{m,k}(g,T) = C^k_m(g,T)$, while the operator $C^k_{m,0}(g,T)$ corresponds to the primal cut off operator studied in Section 4, but rewritten as in (6.2).

Lemma 7.3. *If $g \in \Delta_s(T)$, then*

$$C^k_{m,0}(g,T)u - \frac{n-m}{n-s} \sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} L^*_g A^k_f u = \frac{1}{n-s} \sum_{(f,e)\in\Delta(m,1)} \frac{(\delta\phi)_e}{\rho_g} \wedge L^*_g(\delta R^k u)_{e,f},$$

where the case $g = \emptyset$ corresponds to $s = -1$.

Proof. We observe that the desired identity will follow if we can show that

$$\sum_{(e,f)\in\Delta(m,1)} (\delta\phi)_e \wedge L^*_g(\delta R^k u)_{e,f} - (n-s) \sum_{(e,f)\in\Delta(m,0)} \phi_e \wedge L^*_g R^k_{e,f} u = (m-s)\rho_g \sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} L^*_g A^k_f u. \tag{7.3}$$

By using the special definitions of ϕ_e and $R^k_{e,f}$ for $e \in \Delta_0$, it is straightforward to verify that

$$\begin{aligned}
&\sum_{\substack{f\in\Delta_{m+1}(T)\\ f\supset g}} \sum_{e\in\Delta_1(f\cap g^*)} (\delta\phi)_e \wedge L^*_g(\delta R^k u)_{e,f}\\
&= - \sum_{\substack{f\in\Delta_{m+1}(T)\\ f\supset g}} \sum_{e\in\Delta_1(f\cap g^*)} \sum_{i\in I(e)} (-1)^{\sigma_e(x_i)} \lambda_{e(\hat{x}_i)} \sum_{p\in I(e)} (-1)^{\sigma_e(x_p)} \wedge L^*_g A^k_{f(\hat{x}_p)}\\
&= \sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} \sum_{\substack{p\in I(f^*)\\ i\in I(f\cap g^*)}} (\lambda_p - \lambda_i) \wedge L^*_g A^k_f u = \sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} \sum_{\substack{p\in I(g^*)\\ i\in I(f\cap g^*)}} (\lambda_p - \lambda_i) \wedge L^*_g A^k_f u,
\end{aligned}$$

while

$$-(n-s) \sum_{(e,f)\in\Delta(m,0)} \phi_e \wedge L^*_g R^k_{e,f} u = \sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} \sum_{\substack{p\in I(g^*)\\ i\in I(f\cap g^*)}} \lambda_i \wedge L^*_g A^k_f u.$$

As a consequence, the left hand side of (7.3) is

$$\sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} \sum_{\substack{p\in I(g^*)\\ i\in I(f\cap g^*)}} \lambda_p \wedge L^*_g A^k_f u = (m-s)\rho_g \sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} L^*_g A^k_f u,$$

and this completes the proof. ∎

Note that if $k = 0$, then from (5.9), $R^1_{e,f} du = -(\delta R^0)_{e,f}$. As a consequence, formula (7.2) follows from the result of Lemma 7.3 in the case $k = 0$.

To prove Lemma 7.2, we will also need the following identity.

Lemma 7.4. *The identity*

$$\sum_{(e,f)\in\Delta(m,j)} d\Big(\frac{(\delta\phi)_e}{\rho_g^j}\Big)\wedge L_g^* R_{e,f}^k u + \frac{j}{\rho_g^{j+1}} \sum_{(e,f)\in\Delta(m,j+1)} (\delta\phi)_e \wedge L_g^*(\delta R^k u)_{e,f} = \frac{j}{\rho_g^{j+1}}(n-s)\sum_{(e,f)\in\Delta(m,j)} \phi_e \wedge L_g^* R_{e,f}^k u \tag{7.4}$$

holds for any $0 \le j \le m$.

The proof of this identity is technical, so we delay the proof until we have used it to prove Lemma 7.2.

Proof. (of Lemma 7.2) We introduce the operators

$$Q_{m,\ell}^k u = \frac{1}{n-s}\sum_{j=1}^{\ell}(-1)^j (j-1)! \sum_{(e,f)\in\Delta(m,j)} (\delta\phi)_e \wedge L_g^* b^{-j} R_{e,f}^k u,$$

such that $Q_{m,k}^k = Q_m^k$, and $Q_{m,0}^k = 0$. We will now use induction with respect to ℓ to show that

$$C_{m,\ell}^k(g,T)u - \frac{n-m}{n-s}\sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} L_g^* A_f^k u = dQ_{m,\ell}^k u + Q_{m,\ell}^{k+1} du + \frac{(-1)^\ell \ell!}{n-s}\sum_{(e,f)\in\Delta(m,\ell+1)} \frac{(\delta\phi)_e}{\rho_g^{\ell+1}} \wedge L_g^*(\delta R^k u)_{e,f}, \quad \ell = 0,1,\dots k. \tag{7.5}$$

For $\ell = 0$, this is exactly the identity given in Lemma 7.3. On the other hand, for $\ell = k$, we have that

$$Q_{m,k}^{k+1} du + \frac{(-1)^k k!}{n-s}\sum_{(e,f)\in\Delta(m,k+1)} \frac{(\delta\phi)_e}{\rho_g^{k+1}} \wedge L_g^*(\delta R^k u)_{e,f} = Q_m^{k+1} du,$$

where we have used the facts that $\rho_g = L_g^* b$ and $(\delta R^k u)_{e,f} = -R_{e,f}^{k+1} du$ for $e \in \Delta_{k+1}(f)$, cf. (5.9). So the desired identity, (7.2) , follows from (7.5) with $\ell = k$.

If we assume that (7.5) holds for $\ell - 1$, then

$$\begin{aligned} C_{m,\ell}^k(g,T)u - \frac{n-m}{n-s}\sum_{\substack{f\in\Delta_m(T)\\ f\supset g}} L_g^* A_f^k u &= (-1)^\ell \ell! \sum_{(e,f)\in\Delta(m,\ell)} \frac{\phi_e}{\rho_g^{\ell+1}} \wedge L_g^* R_{e,f}^k u \\ &+ dQ_{m,\ell-1}^k u + Q_{m,\ell-1}^{k+1} du - \frac{(-1)^\ell(\ell-1)!}{n-s}\sum_{(e,f)\in\Delta(m,\ell)} \frac{(\delta\phi)_e}{\rho_g^\ell} \wedge L_g^*(\delta R^k u)_{e,f} \\ &= dQ_{m,\ell-1}^k u + Q_{m,\ell}^{k+1} du \\ &+ \sum_{(e,f)\in\Delta(m,\ell)} \Big[(-1)^\ell \ell! \frac{\phi_e}{\rho_g^{\ell+1}} \wedge L_g^* R_{e,f}^k u - \frac{(\ell-1)!}{n-s}\frac{(\delta\phi)_e}{\rho_g^\ell} \wedge dL_g^* R_{e,f}^k u\Big], \end{aligned}$$

where we have used (5.9) for the last equality. However, by (7.4), the last sum can rewritten as

$$\frac{(-1)^\ell(\ell-1)!}{n-s}\Big[\sum_{(e,f)\in\Delta(m,\ell)} d\Big(\frac{(\delta\phi)_e}{\rho_g^\ell}\wedge L_g^* R_{e,f}^k u\Big)+\ell\sum_{(e,f)\in\Delta(m,\ell+1)}\frac{(\delta\phi)_e}{\rho_g^{\ell+1}}\wedge L_g^*(\delta R^k u)_{e,f}\Big],$$

and hence we obtain the identity (7.5) at level ℓ. This completes the induction argument, and hence the proof of Lemma 7.2.

□

To complete the discussion of this section, leading to Proposition 7.1, we need to establish the identity (7.4).

Proof. (of Lemma 7.4) We observe that if $e\in\Delta_j(f\cap g^*)$, it follows from (2.1) and the identity $\rho_g=\sum_{p\in I(g^*)}\lambda_p$ that

$$d\Big(\frac{(\delta\phi)_e}{\rho_g^j}\Big)=\frac{j}{\rho_g^{j+1}}\sum_{i\in I(e)}(-1)^{\sigma_e(x_i)}\sum_{p\in I(g^*)}\phi_{[x_p,e(\hat{x}_i)]}$$
$$=\frac{j}{\rho_g^{j+1}}\Big[(j+1)\phi_e+\sum_{i\in I(e)}(-1)^{\sigma_e(x_i)}\sum_{p\in I(g^*\setminus e)}\phi_{[x_p,e(\hat{x}_i)]}\Big]. \tag{7.6}$$

To proceed, we will treat the sum with respect to p above in the two cases $p\in I((g^*\setminus e)\cap f)$ and $p\in I((g^*\setminus e)\cap f^*=I(f^*)$ separately. In the first case, for any fixed $f\in\Delta_{m+j}(T)$, consider

$$\sum_{e\in\Delta_j(f\cap g^*)}\sum_{i\in I(e)}(-1)^{\sigma_e(x_i)}\sum_{p\in I((g^*\setminus e)\cap f)}\phi_{[x_p,e(\hat{x}_i)]}\wedge L_g^*R_{e,f}^k u$$
$$=\sum_{e\in\Delta_{j+1}(f\cap g^*)}\sum_{p\in I(e)}\sum_{i\in I(e(\hat{x}_p))}(-1)^{\sigma_{e(\hat{x}_p)}(x_i)+\sigma_{e(\hat{x}_i)}(x_p)}\phi_{e(\hat{x}_i)}\wedge L_g^*R_{e(\hat{x}_p),f}^k u,$$

where the identity is obtained by introducing $e'\in\Delta_{j+1}$ as the ordered version of the simplex $[x_p,e]$, i.e., $(-1)^{\sigma_{e'}(x_p)}e'=[x_p,e]$, and then dropping primes. However, it is easy to show that

$$\sigma_{e(\hat{x}_p)}(x_i)+\sigma_{e(\hat{x}_i)}(x_p)=\sigma_e(x_i)+\sigma_e(x_p)-1. \tag{7.7}$$

As a consequence, we can express the sum above as

$$\sum_{e\in\Delta_j(f\cap g^*)}\sum_{i\in I(e)}(-1)^{\sigma_e(x_i)}\sum_{p\in I((g^*\setminus e)\cap f)}\phi_{[x_p,e(\hat{x}_i)]}\wedge L_g^*R_{e,f}^k u$$
$$=-\sum_{e\in\Delta_{j+1}(f\cap g^*)}\sum_{p\in I(e)}\sum_{i\in I(e)}(-1)^{\sigma_e(x_i)+\sigma_e(x_p)}\phi_{e(\hat{x}_i)}\wedge L_g^*R_{e(\hat{x}_p),f}^k u$$
$$+\sum_{e\in\Delta_{j+1}(f\cap g^*)}\sum_{p\in I(e)}\phi_{e(\hat{x}_p)}\wedge L_g^*R_{e(\hat{x}_p),f}^k u$$
$$=-\sum_{e\in\Delta_{j+1}(f\cap g^*)}(\delta\phi)_e\wedge L_g^*(\delta^+R^k u)_{e,f}+(m-s-1)\sum_{e\in\Delta_j(f\cap g^*)}\phi_e\wedge L_g^*R_{e,f}^k u,$$

where we have used the fact that for $f\in\Delta_{m+j}(T)$ and $g\in\Delta_s(f)$, $|f\cap g^*|=m+j-s$. Choosing an $e\in\Delta_j(f\cap g^*)$ leaves $m+j-s-j-1=m-s-1$ vertices that can be deleted from an $e' \in \Delta_{j+1}(f \cap g^*)$ to produce that same e. However, the first term on the right hand side vanishes since $(\delta^+ Ru)_{e,f} = 0$ by Proposition 5.4. Therefore, we can conclude that

$$\sum_{e\in\Delta_j(f\cap g^*)} \sum_{i\in I(e)} (-1)^{\sigma_e(x_i)} \sum_{p\in I((g^*\setminus e)\cap f)} \phi_{[x_p,e(\hat{x}_i)]} \wedge L_g^* R_{e,f}^k u = (m-s-1) \sum_{e\in\Delta_j(f\cap g^*)} \phi_e \wedge L_g^* R_{e,f}^k u. \tag{7.8}$$

In an analogous manner, and by using the identity (7.7) as above, we obtain

$$\begin{aligned}\sum_{(e,f)\in\Delta(m,j)} \sum_{i\in I(e)} (-1)^{\sigma_e(x_i)} \sum_{p\in I(f^*)} \phi_{[x_p,e(\hat{x}_i)]} \wedge L_g^* R_{e,f}^k u \\ = - \sum_{(e,f)\in\Delta(m,j+1)} \sum_{p\in I(e)} \sum_{i\in I(e)} (-1)^{\sigma_e(x_i)+\sigma_e(x_p)} \phi_{e(\hat{x}_i)} \wedge L_g^* R_{e(\hat{x}_p),f(\hat{x}_p)}^k u \\ + \sum_{(e,f)\in\Delta(m,j+1)} \sum_{p\in I(e)} \phi_{e(\hat{x}_p)} \wedge L_g^* R_{e(\hat{x}_p),f(\hat{x}_p)}^k u,\end{aligned}$$

where as above we have introduced $(-1)^{\sigma_{e'}(x_p)} e' = [x_p, e]$, and the corresponding extension of f to $f' \in \Delta_{m+j+1}$ by including x_p. However, the final right hand side above can rewritten as

$$- \sum_{(e,f)\in\Delta(m,j+1)} (\delta\phi)_e \wedge L_g^* (\delta R^k u)_{e,f} + (n-m-j) \sum_{(e,f)\in\Delta(m,j)} \phi_e \wedge L_g^* R_{e,f}^k u.$$

In this case, for each $f \in \Delta_{m+j}(T)$, there are $n-m-j$ vertices that can be deleted from $f' \in \Delta_{m+j+1}(T)$ to produce the same f. Deleting this same vertex from $e' \in \Delta_{j+1}(f' \cap g^*)$ produces the above result.

By combining this result with (7.6) and (7.8), we obtain

$$\begin{aligned}\sum_{(e,f)\in\Delta(m,j)} d\Big(\frac{(\delta\phi)_e}{\rho_g^j}\Big) \wedge L_g^* R_{e,f}^k u \\ = \frac{j}{\rho_g^{j+1}} \Big[(n-s) \sum_{(e,f)\in\Delta(m,j)} \phi_e \wedge L_g^* R_{e,f}^k u - \sum_{(e,f)\in\Delta(m,j+1)} (\delta\phi)_e \wedge L_g^* (\delta R^k u)_{e,f}\Big],\end{aligned}$$

which is exactly the desired identity. □

Remark. By a careful inspection of the proofs of Lemmas 6.1 and 7.2, we will discover that all properties of the operators A_f^k and $R_{e,f}^k$ are used, except for the trace preserving property given by statement iii) of Lemma 2.1, i.e., that $\operatorname{tr}_f A_f^k u = \operatorname{tr}_f u$. In fact, this property is only used to establish the identity (1.4). In future work, we will consider the possibility of constructing approximations of a form u by using its decomposition by the bubble transform, cf. (1.4). One direct way to construct such an approximation is to approximate the operator C_m^k, studied above, by an operator of the form

$$\tilde{C}_m^k u = \sum_{f\in\Delta_m(\mathcal{T})} \sum_{g\in\bar{\Delta}(f)} (-1)^{|f|-|g|} L_g^* \tilde{A}_f^k u$$

$$+ \sum_{\substack{f \in \Delta_{m+j}(\mathcal{T}) \\ 0 \le j \le k}} j! \sum_{g \in \bar{\Delta}(f)} (-1)^{|f|-|g|} \sum_{e \in \Delta_j(f \cap g^*)} \frac{\phi_e}{\rho_g} \wedge L_g^* b^{-j} \tilde{R}_{e,f}^k u,$$

i.e., we have replaced the operators A_f^k and $R_{e,f}^k$ by corresponding approximations $\tilde{A}_f^k$ and $\tilde{R}_{e,f}^k$. By the observation above, we can conclude that if these operators satisfy the two relations (5.9) and (5.10), then the operator $\tilde{C}_m^k$ commutes with the exterior derivative. Furthermore, piecewise polynomial properties of the functions $\tilde{C}_m^k u$ and the support properties of the corresponding operators $\tilde{C}_{m,f}^k u$ can be derived from similar properties of the operators $\tilde{A}_f^k$ and $\tilde{R}_{e,f}^k$

8. Bounding the operator norms

The constructions above are derived under the assumptions given in Section 2.1. However, to give rigorous proofs of the estimates stated below, we will in this final section make the additional assumption that the manifold x_i^* is connected for each $x_i \in \Delta_0(\mathcal{T})$. We note this will be the case if Ω is a Lipschitz domain.

The various constants that appear in the bounds below only depend on the mesh $\mathcal{T}$ through the shape–regularity constant $c_{\mathcal{T}}$, defined by (2.4). The consequence of this is that if we consider a family of meshes, $\{\mathcal{T}^h\}$, parametrized by a real parameter $h \in (0, 1]$, typically obtained by mesh refinements, the bounds will be uniform with respect to h as long as we restrict to a family with a uniform bound on the constants $\{c_{\mathcal{T}^h}\}$. In the bounds we derive below, the various constants that appear will depend on the space dimension n and the domain Ω, in addition to the dependence explicitly stated. Throughout this section we will assume that the operators under investigation are applied to piecewise smooth differential forms. However, since the space $\Lambda^k(\mathcal{T})$ is dense in $L^2\Lambda^k(\Omega)$, it a consequence of the bound obtained in Theorem 8.3 that all the operators $B_{m,f}^k$ and B_m^k can be extended to bounded operators mapping $L^2\Lambda^k(\Omega)$ to itself.

8.1. **The main bounds.**

If u is a k form, we let $|u_x|$ be defined by

$$|u_x| = \sup u_x(t_1, \ldots, t_k),$$

where the sup is taken over all collections of unit tangent vectors. As a consequence,

$$\|u\|_{L^2(\Omega)} = \Big(\int_\Omega |u_x|^2\, dx\Big)^{1/2}.$$

Our estimates will use the domains $\Omega_{e,f}$, defined in Section 5.2 above, consisting of finite unions of n simplexes in $\Delta_n(\mathcal{T})$ and the extended macroelements Ω_f^E, consisting of the union of the macroelements associated to the vertices of f. The bounds for the operators $B_{m,f}^k$ and B_m^k will be obtained from the following bound for the cut–off operator $C_{m,f}^k$.

Lemma 8.1. *There exists a constant c, depending on the mesh $\mathcal{T}$ only through the shape-regularity constant $c_{\mathcal{T}}$, such that for $f \in \Delta_{m+j}(\mathcal{T})$ and $e \in \Delta_j(f)$, we have*

$$\|C_{m,f}^k u\|_{L^2(\Omega_f)} \le c\|u\|_{L^2(\Omega_f^E)}, \tag{8.1}$$

where $0 \le m \le n$ and $0 \le j \le k$.

In addition to this result, the proof of the desired bounds will depend on bounds for the overlap of the sets $\{\Omega_f\}$ and $\{\Omega_f^E\}$. In the present setting, the overlap of a set of subdomains can be defined as the smallest upper bound for the number of domains which will contain any fixed element $T \in \Delta_n(\mathcal{T})$. Alternatively, the overlap of the set is the L^∞ norm of the sum of the characteristic functions of the set. The overlap of the set of macroelements, $\{\Omega_f\}_{f\in\Delta_m(\mathcal{T})}$, will only depend on m and the space dimension n, while the overlap for the sets $\{\Omega_f^E\}$ will depend on the mesh $\mathcal{T}$, as established in the following result.

Lemma 8.2. *The overlap of the domains $\{\Omega_f^E\}_{f\in\Delta(\mathcal{T})}$ can be bounded by a constant which depends on the mesh $\mathcal{T}$ only through the shape regularity constant $c_{\mathcal{T}}$.*

We will defer the proof of the two lemmas above until after the proof of the main results given in this section.

Theorem 8.3. *There exists a constant c, depending on the shape-regularity constant $c_{\mathcal{T}}$, such that for $0 \le m \le n$, we have*

$$\|B_m^k u\|_{L^2(\Omega)}, \Big(\sum_{j=0}^{k} \sum_{f\in\Delta_{m+j}(\mathcal{T})} \|B_{m,f}^k u\|_{L^2(\Omega)}^2\Big)^{1/2} \le c\|u\|_{L^2(\Omega)}.$$

Proof. We recall that the the operator C_m^k is defined by

$$C_m^k u = \sum_{f\in\Delta[m,k]} C_{m,f}^k u,$$

where, to simplify notation, we have introduced the set $\Delta[m,k] = \{f \in \Delta_{m+j}(\mathcal{T}) : 0 \le j \le k\}$. We will first show that

$$\|C_m^k u\|_{L^2(\Omega)} \le c_1\|u\|_{L^2(\Omega)}, \tag{8.2}$$

where the constant c_1 depends on $c_{\mathcal{T}}$. To see this, let κ_f be the characteristic function of the set Ω_f. Since the functions $C_{m,f}^k u$ have support in Ω_f, cf. Lemma 6.1, we have by repeated use of the Cauchy-Schwarz inequality, that

$$\begin{aligned}
\|C_m^k u\|_{L^2(\Omega)}^2 &= \sum_{f,g\in\Delta[m,k]} \int_\Omega \kappa_f\kappa_g |C_{m,f}^k u|\,|C_{m,g}^k u|\,dx \\
&\le \sum_{f,g\in\Delta[m,k]} (\int_\Omega \kappa_f\kappa_g |C_{m,f}^k u|^2\,dx)^{1/2} (\int_\Omega \kappa_f\kappa_g |C_{m,g}^k u|^2\,dx)^{1/2} \\
&\le \Big(\sum_{f,g\in\Delta[m,k]} \int_\Omega \kappa_f\kappa_g |C_{m,f}^k u|^2\,dx\Big)^{1/2} \Big(\sum_{f,g\in\Delta[m,k]} \int_\Omega \kappa_f\kappa_g |C_{m,g}^k u|^2\,dx\Big)^{1/2} \\
&\le \alpha_0 \sum_{f\in\Delta[m,k]} \|C_{m,f}^k u\|_{L^2(\Omega_f)}^2,
\end{aligned}$$

where α_0 is the overlap of set $\{\Omega_f\}_{f\in\Delta[m,k]}$. However, by the bound (8.1), we have

$$\sum_{f\in\Delta[m,k]} \|C_{m,f}^k u\|_{L^2(\Omega_f)}^2 \le c^2 \sum_{f\in\Delta[m,k]} \|u\|_{L^2(\Omega_f^E)}^2 \le \alpha_1 c^2 \|u\|_{L^2(\Omega)}^2, \tag{8.3}$$

where α_1 is the overlap of the set $\{\Omega_f^E\}_{f\in\Delta[m,k]}$, cf. Lemma 8.2. Hence, we have verified the bound (8.2). The desired bound for the functions $B_m^k u$, $0 \le m \le n$, now follows from this bound, the iteration (2.3), and a simple induction argument with respect to m. Finally, the L^2 bound the functions $B^k_{m,f}u$ follows from the bound on the functions $B^k_m u$, (1.8), and (8.3). □

Combining Theorem 8.3 with the fact that the operators B^k_m commute with the exterior derivative, cf. Theorem 2.3, we also obtain a bound on the operators B^k_m in the norm $\|\cdot\|_{H\Lambda(\Omega)}$, where

$$\|u\|_{H\Lambda(\Omega)} = (\|u\|^2_{L^2(\Omega)} + (\|du\|^2_{L^2(\Omega)})^{1/2}.$$

Theorem 8.4. *There exists a constant c, depending on the shape regularity constant $c_{\mathcal{T}}$, such that*

$$\|B^k_m u\|_{H\Lambda(\Omega)} \le c\|u\|_{H\Lambda(\Omega)}, \quad 0 \le m \le n.$$

Proof. Since $dB^k_m u = B^{k+1}_m du$, this is a direct consequence of the L^2 bounds given in Theorem 8.3. □

8.2. **Deriving the bounds.** To complete the proofs of the main results above, we need to prove Lemmas 8.1 and 8.2. We will first present the proof of Lemma 8.2.

Proof. (of Lemma 8.2) For each $x \in \Delta_0(\mathcal{T})$, we let N_x be the number of n simplices containing the vertex x. We will show that the number N_x can be bounded from above by a constant which only depends on $\mathcal{T}$ though the shape-regularity constant $c_{\mathcal{T}}$. In fact, for any vertex x_0 we have

$$N_{x_0} = \sum_{T\in\Delta_n(\mathcal{T}_{x_0})} \le \sum_{T\in\Delta_n(\mathcal{T}_{x_0})} \frac{|T|}{|\mathfrak{B}_T|} = \sum_{T\in\Delta_n(\mathcal{T}_{x_0})} \frac{h_T^n}{|\mathfrak{B}_T|} h_T^{-n}|T|,$$

where h_T is the diameter of the n simplex T and $\mathfrak{B}_T$ is the largest ball contained in T. Next we use the fact that $|\mathfrak{B}_T| = \beta_n(\mathrm{diam}(\mathfrak{B}_T)/2)^n$, where β_n is the volume of the unit ball in $\mathbb{R}^n$ to obtain

$$N_{x_0} \le \beta_n^{-1}2^n \sum_{T\in\Delta_n(\mathcal{T}_{x_0})} \frac{h_T^n}{\mathrm{diam}(\mathfrak{B}_T)^n} h_T^{-n}|T| \le \beta_n^{-1}(2c_{\mathcal{T}})^n \sum_{T\in\Delta_n(\mathcal{T}_{x_0})} h_T^{-n}|T|,$$

where we have used the definition of $c_{\mathcal{T}}$ for the last inequality. However, by substituting $\theta(x) = (x - x_0)/h_T$ for $x \in T$, we obtain

$$\sum_{T\in\Delta_n(\mathcal{T}_{x_0})} h_T^{-n}|T| \le \int_{|\theta|\le 1} d\theta = \beta_n.$$

Hence, we can conclude that

$$N_{x_0} \le (2c_{\mathcal{T}})^n. \tag{8.4}$$

Note that it follows from (5.4) that if $g \subset f$ then $\Omega^E_g \subset \Omega^E_f$. Therefore, to derive an upper bound for the overlap of the set $\{\Omega^E_f\}$, it is enough to consider the sets $\{\Omega^E_f\}_{f\in\Delta_n(\mathcal{T})}$. However, if T is any fixed n simplex, then T is a subset of Ω^E_f if and only if $T \cap f$ contains at least one vertex. As a consequence, T belongs to at most $(n+1)\max_{x\in\Delta_0(\mathcal{T})} N_x$ domains of the set $\{\Omega^E_f\}_{f\in\Delta_n(\mathcal{T})}$, and therefore the desired bound follows from (8.4). □

It remains to prove Lemma 8.1. To do so, will require several preliminary results. We begin with a discussion of some further consequences of shape-regularity. By using the fact that the volume of $\mathfrak{B}_T$, $|\mathfrak{B}_T|$, is less than $|T|$, we obtain the estimate

$$h_T^n \le \beta_n^{-1}(2c_{\mathcal{T}})^n|\mathfrak{B}_T| \le \beta_n^{-1}(2c_{\mathcal{T}})^n|T|,$$

where the constant β_n is the same constant as in the proof above. In fact, if $f \in \Delta_m(T)$, then we can utilize the natural projection from T to f, given by

$$\sum_{i\in I(T)} \lambda_i(x)x_i \mapsto \sum_{i\in I(f)} \lambda_i(x)x_i/[\sum_{i\in I(f)} \lambda_i(x)],$$

to obtain the more general estimate

$$h_T^m \le \beta_m^{-1}(2c_{\mathcal{T}})^m|f|, \tag{8.5}$$

where $|f|$ is the m dimensional volume of f. A further consequence of shape-regularity is local quasi-uniformity of the mesh. In particular, we have the following result for the macroelements $\Omega_{e,f}$.

Lemma 8.5. *There is a constant c, depending on $\mathcal{T}$ only through the shape-regularity constant $c_{\mathcal{T}}$, such that*

$$\max_{T\in\Delta_n(\mathcal{T}_{e,f})} h_T \le c \min_{T\in\Delta_n(\mathcal{T}_{e,f})} h_T, \quad f \in \Delta(\mathcal{T}),\ e \in \Delta(f). \tag{8.6}$$

Proof. We first prove that

$$\max_{T\in\Delta_n(\mathcal{T}_{x_i})} h_T \le c \min_{T\in\Delta_n(\mathcal{T}_{x_i})} h_T, \quad x_i \in \Delta_0(\mathcal{T}).$$

To do so, let T_- and T_+ be two n-simplices in Ω_{x_i}, and assume that there is a finite sequence of n simplexes $\{T_j\}_{j=0}^s$ in Ω_{x_i} such that $T_- = T_0$, $T_s = T_+$ and $T_j \cap T_{j+1}$ contains at least one element $e \in \Delta_1(\mathcal{T})$ containing x_i. By repeated use of the inequality (8.5) with $m = 1$, we then obtain

$$\max(h_{T_-}, h_{T_+}) \le (2c_{\mathcal{T}})^s \min(h_{T_-}, h_{T_+}).$$

However, since we have assumed that x_i^* is connected, any two n simplexes T_- and T_+ in Ω_{x_i} can be connected by a sequence of the form above. Furthermore, as a consequence of Lemma 8.2, the number s can be bounded by a constant which only depends on $\mathcal{T}$ through the shape-regularity constant.

Since $\Omega_{e,f} \subset \Omega_{f,f} = \Omega_f^E$, to prove (8.6), it is enough to prove the result for $\Omega_{f,f}$. Now for each $f \in \Delta(\mathcal{T})$, we have

$$\bigcup_{i\in I(f)} \Omega_{x_i} = \Omega_f^E, \quad \text{and} \quad \bigcap_{i\in I(f)} \Omega_{x_i} = \Omega_f \ne \emptyset.$$

Suppose $\max_{T\in\Delta_n(\mathcal{T}_{f,f})} h_T$ occurs for $T \in \mathcal{T}_{x_i}$ and $\min_{T\in\Delta_n(\mathcal{T}_{f,f})} h_T$ occurs for $T \in \mathcal{T}_{x_j}$. Then by the result above for Ω_{x_i},

$$\begin{aligned}\max_{T\in\Delta_n(\mathcal{T}_{f,f})} h_T = \max_{T\in\Delta_n(\mathcal{T}_{x_i})} h_T \le c \min_{T\in\Delta_n(\mathcal{T}_{x_i})} h_T \le c \min_{T\in\Delta_n(\mathcal{T}_f)} h_T \\ \le c \max_{T\in\Delta_n(\mathcal{T}_f)} h_T \le c \max_{T\in\Delta_n(\mathcal{T}_{x_j})} h_T \le c^2 \min_{T\in\Delta_n(\mathcal{T}_{x_j})} h_T = c^2 \min_{T\in\Delta_n(\mathcal{T}_{f,f})} h_T.\end{aligned}$$

This completes the proof of the lemma. □

Next, recall that the operator $L : \Omega \to \mathcal{S}$ is defined by

$$Lx = \{\lambda(x_i)\}_{i\in\mathcal{I}}.$$

If we apply the map L to an n simplex $T \in \Delta_n(\mathcal{T})$, we obtain a corresponding n simplex $L(T) \subset \mathcal{S}$. More precisely, if $T = [x_{j_0}, \dots, x_{j_n}]$ then $L(T) = [e_{j_0}, \dots, e_{j_n}]$, where $e_i = Lx_i$ corresponds to unit vectors in R^{N+1}, where $N+1$ is the number of elements in $\Delta_0(\mathcal{T})$. The operator L restricted to T, L_T, has an inverse $F = F_T$. More precisely,

$$L_T x = \sum_{i\in I(T)} \lambda_i(x)e_i, \quad \text{and } F_T\lambda = \sum_{i\in I(T)} \lambda_i x_i.$$

Furthermore, $DL_T = D_x L_T$ satisfies $DL_T(x_i - x_j) = (e_i - e_j)$ for $i, j \in I(T)$. The shape regularity constant $c_\mathcal{T}$ can be used to bound DL_T. More precisely, we can easily derive the bound

$$\|DL_T\| \le c_\mathcal{T} h_{L(T)} h_T^{-1} \le 2c_\mathcal{T} h_T^{-1}, \tag{8.7}$$

where $\|\cdot\|$ is the operator norm corresponding to the Euclidean vector norm, and where h_T and $h_{L(T)}$ denote the diameter of T and $L(T)$, respectively. This bound can, for example, be found in [9, Theorem 3.1.3]. For each $f \in \Delta(\mathcal{T})$ and $e \in \bar{\Delta}(f)$, we define $\mathcal{S}_{e,f} \subset \mathcal{S}$ by

$$\mathcal{S}_{e,f} = \bigcup_{\substack{T\in\Delta_n(\mathcal{T})\\ T\subset\Omega_{e,f}}} L(T).$$

Hence, $\mathcal{S}_{e,f}$ is an n dimensional manifold such that all n simplexes of $\mathcal{S}_{e,f}$ contain $\mathcal{S}_{f\cap e^*}$ as a subcomplex. Furthermore, restricted to $\mathcal{S}_{e,f}$, the map L can be inverted, with an inverse $F_{e,f} : \mathcal{S}_{e,f} \to \Omega_{e,f}$ given by

$$F_{e,f}\lambda = F_T\lambda, \quad \lambda \in L(T).$$

In order to establish Lemma 8.1, we will need a bound for the functions $z_{e,f}$, constructed in Section 5.2 to define the order reduction operators $R^k_{e,f}$.

Lemma 8.6. *There exists a constant c, depending on the mesh $\mathcal{T}$ only through the shape regularity constant $c_\mathcal{T}$, such that*

$$\|z_{e,f}\|_{L^\infty(\Omega_{e,f})} \le c h_{e,f}^{j-n}, \qquad e \in \Delta_j(f),$$

where $h_{e,f} = \max_{T\subset\Delta_n(\mathcal{T}_{e,f})} h_T$.

Proof. Recall that the functions $z_{e,f}$ are defined by $z_{e,f} = (\delta^+ w)_{e,f}$, where $w_{e,f} \in \mathring{\mathcal{P}}_1^-\Lambda^{n-j-1}(\mathcal{T}_{e,f})$ for $e \in \Delta_j(f)$, $j \ge 0$. The desired bound on the functions $z_{e,f}$ will be derived from a corresponding bound on the functions $w_{e,f}$, and to obtain this bound, we will use a scaling argument. For each $e \in \bar{\Delta}(f)$, we define $\tilde{w}_{e,f} = F^*_{e,f} w_{e,f}$, such that $w_{e,f} = L^*\tilde{w}_{e,f}$. From the process defining the functions $w_{e,f}$, we obtain that the functions $\tilde{w}_{e,f}$ are uniquely specified by a corresponding process on $\mathcal{S}$. In particular, the initial functions $\tilde{w}_{\emptyset,f}$ are piecewise constants with integral equal to minus one,

$$d\tilde{w}_{e,f} = (-1)^j((\delta - \delta^+)\tilde{w})_{e,f},$$

and condition (5.7) translates to the corresponding relation

$$\int_{\mathcal{S}_{e,f}} \tilde{w}_{e,f} \wedge \star dq = 0, \quad q \in \mathring{\mathcal{P}}_1^-\Lambda^{n-j-2}(\mathcal{S}_{e,f}).$$

Since the simplex $\mathcal{S}$ is of unit size, and since the number of n simplexes belonging to the manifolds $\mathcal{S}_{e,f}$ is bounded by the shape regularity constant, we can conclude that

$$\|\tilde{w}_{e,f}\|_{L^\infty(\mathcal{S})} \le c, \quad f \in \Delta(\mathcal{T}), e \in \Delta(f), \tag{8.8}$$

where the constant c depends on $c_\mathcal{T}$. Finally, we use the fact that for $e \in \Delta_j(f)$, the $n-j$ form $z_{e,f}$ satisfies the relation

$$z_{e,f} = L^*(\delta^+ \tilde{w})_{e,f}.$$

By the definition of the pullback L^*, we then obtain from (8.7) and (8.8) that

$$\|z_{e,f}\|_{L^\infty(\Omega_{e,f})} \le c[\min_{T\subset\Delta_n(\mathcal{T}_{e,f})} h_T]^{j-n} \le c[\max_{T\subset\Delta_n(\mathcal{T}_{e,f})} h_T]^{j-n},$$

where we have used the inequality (8.6) in the last step. □

To prove Lemma 8.1, we first recall some notation and formulas developed in [19]. If $f \in \Delta_m(\mathcal{T})$ and $0 \le m \le n-1$, then we can write $x \in \Omega_f$ in the form

$$x = \sum_{i\in I(f)} \lambda_i(x)x_i + \rho_f(x)q_f(x), \qquad q_f(x) \in f^*,$$

where f^* is a piecewise flat manifold of dimension $n-m-1$, see also Section 2 above. As it was done in [19, Section 5], we can use the mapping $x \mapsto (L_f(x), q_f(x))$ to express integrals over Ω_f as integrals over $\mathcal{S}_f^c \times f^*$. In particular, if $\Omega_f' \subset \Omega_f$ is a union of n simplexes belonging to Ω_f, then we have

$$\int_{\Omega_f'} \phi(L_f(x), q_f(x))\, dx = \int_{\mathcal{S}_f^c} \int_{f^*\cap\Omega_f'} \phi(\lambda, q) J(f,q)\, dq\, b(\lambda)^{n-m-1}\, d\lambda, \tag{8.9}$$

for any sufficiently regular and real-valued function ϕ defined on $\mathcal{S}_f^c \times f^*$. Here dq means integration with respect to the standard Lebesgue measure derived from the imbedding of the tangent space of f^* into $\mathbb{R}^{n-m-1}$. The determinant $J(f,q)$ is a real valued piecewise constant function with respect to q. If $f = [x_0, x_1, \dots, x_m]$, then

$$J(f,q) = \det([x_0 - \hat{q}, x_1 - \hat{q}, \dots, x_m - \hat{q}, t_{m+1}, \dots, t_{n-1}]),$$

where $\hat{q} = \hat{q}(q)$ is the barycenter of $f^* \cap T$ for $q \in f^* \cap T$ and any n simplex $T \subset \Omega_f$. Furthermore, $t_{m+1}, \dots, t_{n-1} \in \mathbb{R}^n$ is an orthonormal basis for the tangent space of $f^* \cap T$. It follows from (8.9), with $\phi \equiv 1$, that if $T \in \Delta_n(\mathcal{T}_f)$, that

$$\frac{|T|}{|f^* \cap T|} = \Big(\int_{\mathcal{S}_f^c} b(\lambda)^{n-m-1}\, d\lambda\Big) J(f,q)|_T.$$

However, the estimate (8.5) implies that the fraction $|T|/|f^* \cap T|$ can be bounded, above and below, by h_T^{m+1} times constants which depend on $c_\mathcal{T}$. As a consequence of the bound (8.6), we can therefore conclude that there exist constants c_1 and c_2, depending on the shape–regularity constant $c_\mathcal{T}$, such that

$$c_1 h_f^{m+1} \le J(f,q) \le c_2 h_f^{m+1}, \quad q \in f^*, \tag{8.10}$$

where $h_f = \max_{T\in\Delta_n(\mathcal{T}_f)} h_T$.

Proof. (of Lemma 8.1) Recall that the operator $C^k_{m,f}$ is defined by

$$C^k_{m,f}u = \sum_{g\in\bar\Delta(f)} (-1)^{|f|-|g|}\frac{\rho_f}{\rho_g}\wedge L^*_g b^{-j}A^k_f u,$$

if $f \in \Delta_m(\mathcal{T})$, and by

$$C^k_{m,f}u = j!\sum_{e\in\Delta_j(f)}\ \sum_{g\in\bar\Delta(f\cap e^*)} (-1)^{|f|-|g|}\frac{\phi_e}{\rho_g}\wedge L^*_g b^{-j}R^k_{e,f} u,$$

if $f \in \Delta_{m+j}(\mathcal{T})$, $1 \le j \le k$. If $m = n$, such that f is an n simplex, then $\mathrm{tr}_f\, C^k_{m,f} = \mathrm{tr}_f$ and the conclusion of the lemma obviously holds. Therefore, we can assume that $0 \le m \le n-1$ in the rest of the proof.

The function $C^k_{m,f}u$ has support on Ω_f, and for $x \in \Omega_f$ and $g \in \bar\Delta(f)$, we have $\rho_f/\rho_g \le 1$. Furthermore, it is a consequence of (8.6) that

$$|\phi_e/\rho_g| \le ch_f^{-j}, \quad e \in \Delta_j(f\cap g^*),$$

where the constant c depends on the shape-regularity constant. Therefore, since $\Omega_f \subset \Omega_{e,f}$, to prove an inequality of the form (8.1) for the case $f \in \Delta_{m+j}(\mathcal{T})$, it will be sufficient to show that

$$\|L^*_g[b^{-j}R^k_{e,f}u]\|_{L^2(\Omega_f)} \le ch^j_{e,f}\|u\|_{L^2(\Omega_{e,f})}, \qquad e \in \Delta_j(f), \quad g \in \bar\Delta(f\cap e^*), \tag{8.11}$$

where $h_{e,f} = \max_{T\subset\mathcal{T}_{e,f}} h_T$. Here we recall from Section 5 that the operator $R^k_{e,f}$ is defined by

$$(R^k_{e,f}u)_\lambda = \int_{\Omega_{e,f}} (\Pi_j G^* u)_\lambda \wedge z_{e,f}.$$

However, for any $e \in \Delta_0(f)$, $R^k_{e,f}u$ corresponds to the operator $A^k_f u$, so the desired bound, (8.1), for the case $f \in \Delta_m(\mathcal{T})$, will follow from (8.11) with $j = 0$.

To show the bound (8.11), we assume that $f \in \Delta_{m+j}(\mathcal{T})$, $e \in \Delta_j(f)$ such that $f\cap e^* \in \Delta_{m-1}(\mathcal{T})$ and $g \in \Delta_s(f\cap e^*)$ for $0 \le s \le m-1$. We also need to treat the case $g = \emptyset$, but this will be done as a special case below. We will use formula (8.9) with f replaced by g. In this case, it follows from (8.10) that the determinant $J(g,q) = O(h^{s+1})$, where here, and in the rest of this proof $h = h_{e,f}$. Furthermore, g^* is an $n-s-1$ dimensional manifold of size h, so its volume, $|g^*| = O(h^{n-s-1})$. Therefore, since $\Omega_f \subset \Omega_g$, and noting that $b^{-j}R^k_{e.f}u$ only depends on λ, we have from (8.9) that

$$\|L^*_g[b^{-j}R^k_{e.f}u]\|_{L^2(\Omega_f)} \le c\Big[h^n\int_{\mathcal{S}^c_g} b(\lambda)^{n-s-1}\Big(b(\lambda)^{-j}|(R^k_{e,f}u)_\lambda|\Big)^2\, d\lambda\Big]^{1/2}, \tag{8.12}$$

where the constant c only depends on $\mathcal{T}$ through the shape regularity constant $c_{\mathcal{T}}$. By using the fact that

$$D_\lambda G = \sum_{i\in I(g)} (x_i - y)d\lambda_i$$

is uniformly bounded for $y \in \Omega_{e,f}$, and that $D_y G$ is $b(\lambda)$ times the identity, we obtain

$$b(\lambda)^{-j}|(R^k_{e,f}u)_\lambda| \le c\int_{\Omega_{e,f}} |u_{G(y,\lambda)}|\,|(z_{e,f})_y|\, dy \le ch^{j-n}\int_{\Omega_{e,f}} |u_{G(y,\lambda)}|\, dy,$$

where we have used the result of Lemma 8.6 for the final inequality. Furthermore, since $\Omega_{e,f} \subset \Omega_{f\cap e^*} \subset \Omega_g$, we have from (8.9) and (8.10) that

$$b(\lambda)^{-j}|(R^k_{e,f}u)_\lambda| \le ch^{j+s+1-n}\int_{\mathcal{S}_g^c} b(\mu)^{n-s-1}\int_{g^*\cap\Omega_{e,f}} |u_{G(G(q,\mu),\lambda)}|\,dq\,d\mu.$$

By inserting this inequality into (8.12) and using Minkowski's integral inequality, we obtain

$$\begin{aligned}
&\|L^*_g[b^{-j}R^k_{e.f}u]\|_{L^2(\Omega_f)}\\
&\le c\Big[h^n\int_{\mathcal{S}_g^c} b(\lambda)^{n-s-1}\Big(h^{j+s+1-n}\int_{\mathcal{S}_g^c} b(\mu)^{n-s-1}\int_{g^*\cap\Omega_{e,f}}|u_{G(G(q,\mu),\lambda)}|\,dq\,d\mu\Big)^2 d\lambda\Big]^{1/2}\\
&\le ch^{j+s+1-n/2}\int_{\mathcal{S}_g^c} b(\mu)^{n-s-1}\Big[\int_{\mathcal{S}_g^c} b(\lambda)^{n-s-1}\Big(\int_{g^*\cap\Omega_{e,f}}|u_{G(q,\lambda'(\lambda,\mu))}|\,dq\Big)^2 d\lambda\Big]^{1/2} d\mu.
\end{aligned}$$

Here we have used the fact that

$$G(G(q,\mu),\lambda) = G(q,\lambda'), \quad \text{where } \lambda'(\lambda,\mu) = \lambda + b(\lambda)\mu.$$

Next we introduce the change of variables $\lambda \to \lambda'$, where $\det(d\lambda'/d\lambda) = b(\mu)$ and $b(\lambda') = b(\lambda)b(\mu)$. We obtain

$$\begin{aligned}
&\|L^*_g[b^{-j}R^k_{e.f}u]\|_{L^2(\Omega_f)}\\
&\le ch^{j+s+1-n/2}\int_{\mathcal{S}_g^c} b(\mu)^{(n-s-2)/2}\Big[\int_{\mathcal{S}_g^c} b(\lambda')^{n-s-1}\Big(\int_{g^*\cap\Omega_{e,f}}|u_{G(q,\lambda')}|\,dq\Big)^2 d\lambda'\Big]^{1/2} d\mu\\
&\le ch^{j+s+1-n/2}\Big[\int_{\mathcal{S}_g^c} b(\lambda')^{n-s-1}\Big(\int_{g^*\cap\Omega_{e,f}}|u_{G(q,\lambda')}|\,dq\Big)^2 d\lambda'\Big]^{1/2},
\end{aligned}$$

where we used that for $s < m \le n$, $(n-s-2)/2 \ge -1/2$, and hence the integral with respect to μ is finite. To complete the argument, we apply the Cauchy-Schwarz inequality to the integral over $g^*\cap\Omega_{e,f}$. Since the volume of $g^*\cap\Omega_{e,f}$ is $\mathrm{O}(h^{n-s-1})$, we obtain

$$\begin{aligned}
\|L^*_g[b^{-j}R^k_{e.f}u]\|_{L^2(\Omega_f)} &\le ch^j\Big[h^{s+1}\int_{\mathcal{S}_g^c} b(\lambda)^{n-s-1}\int_{g^*\cap\Omega_{e,f}}|u_{G(q,\lambda}|^2\,dq\,d\lambda\Big]^{1/2}\\
&\le ch^j\|u\|_{L^2(\Omega_{e,f})} \le ch^j\|u\|_{L(\Omega_f^E)}.
\end{aligned}$$

This complete the verification of (8.11) when $g \ne \emptyset$.

When $g = \emptyset$, then $L^*_g[b^{-j}R^k_{e,f}u] = 0$ for $j < k$. When $j = k$, we have

$$L^*_g[b^{-j}R^k_{e,f}u] = (R^k_{e,f}u)_0 = \int_{\Omega_{e,f}}(\Pi_j G^*u)_0\wedge z_{e,f} = \int_{\Omega_{e,f}} u_y\wedge z_{e,f}.$$

Hence, by the bound on $z_{e,f}$ given in Lemma 8.6, we have

$$\|L^*_\emptyset[b^{-j}R^k_{e.f}u]\|_{L^2(\Omega_f)} \le ch^{n/2}\Big|\int_{\Omega_{e,f}} u_y\wedge z_{e,f}\Big| \le ch^k\|u\|_{L^2(\Omega_{e,f})},$$

which shows that (8.11) also holds in this case. As a consequence, we have established the bound (8.1). □

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