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Unit-Linked Insurance Policies in the Presence of Credit Risk

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In this thesis we derive a new version of Thiele's partial differential equation for computing the insurance reserves of unit-linked policies based on stochastic yields of bonds in a defaultable bond market. This is based on the Bielecki-Rutkowski credit model which can be considered as an extension of the Heat-Jarrow-Morton framework for modelling the term structure of interest rates ([6]). The equation is something that, to our knowledge, has not previously been presented in the existing literature.

This thesis consists of six chapters. The first chapter introduces the topic of interest, as well as some elementary theory. The second chapter provides a general introduction to the case of classic life insurance, and includes an introduction to probability theory, stochastic calculus and credit risk modelling. In the third chapter, we further discuss the case of life insurance, and we introduce the case of unit-linked policies and fundamentals of stochastic interest rates. We also include a known version of Thiele's partial differential equation for stochastic interest rates and provide an example under the Vasicek interest rate model using a direct approach for the numerical scheme. Chapter 4 introduces the Bielecki-Rutkowski framework ([3]) and provides a summary of parts of the doctoral thesis of Christodoulou ([5]) which includes some interesting aspects of the model.

The main result of the thesis is presented in Chapter 5. We provide a framework for the reserves based on stochastic yields of bonds in a defaultable bond market, before we extend it to the unit-linked case. This is followed by the derivation of the new version of Thiele's partial differential equation.

Chapter 6 provides a short summary of the thesis, as well as discussion of some interesting extensions.

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Notation

Some useful abbreviations:

- *a.s.* refers to "almost surely" with respect to a probability measure $\mathbb{P}$,
- a similar abbreviation exists for *almost everywhere*, shortened to *a.e.*
- *r.v.* will be used as a short-hand notation for *random variable(s)*.
- PDE/ SDE often replaces *partial* or *stochastic differential equations*.

We denote the indicator function as

$$\mathbf{1}_A = \mathbf{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

We will use both e^x and $\exp(x)$ to signify the *exponential function of* x.

In set-notation, we will denote the inclusive set as $[s, t]$, i.e., that both s and t are included in the set, whereas (s, t) denotes the exclusive set, dismissing both s and t. The half-open intervals are thus denoted by $(s, t]$ and $[s, t)$.

The conditional expectation is denoted as

$$\mathbb{E}[X|Y].$$

Furthermore, we often specify in which probability measure the expectation is operating, e.g.,

$$\mathbb{E}_{\mathbb{P}}[X|Y].$$

We will by $\sigma(\mathcal{A})$ for $\mathcal{A} \in \Omega$ denote the smallest σ-algebra which contains $\mathcal{A}$. Furthermore, we make the assumption that a σ-algebra constructed this way also contains the null-sets of $\mathcal{A}$.

We will refer to equations per chapter, that is: equation (1.2), refers to the second equation in the first chapter. Similarly, we will refer to definitions etc. as, e.g., 4.1.3 meaning that it is situated in Chapter 4, Section 1 and the third element appearing. References are enclosed in [], meaning that [1] is the first element listed in the bibliography.

Contents

CHAPTER 1

Introduction

In financial markets, there are three fundamental factors of risk that needs to be managed: liquidity risk, market risk and credit risk. The latter risk is a phenomenon that is still not well understood, even though the lending of money has been one of the basic activities of banks for centuries. This is mainly due to the fact that the impact that credit risk carries on price processes in different markets, is subject to the complex interplay of many factors, like recovery risk or default-correlation risk.

In short, credit risk describes the exposure of losses due to changes in the credit quality of the issuer of a corporate bond. That is, credit risk deals with whether or not a bond defaults before maturity, as well as the price of said bond. This includes the credit rating of the bond-seller, and it is well known that the price of the bond $(B(t,T))$ depends on this rating. For example, a bond sold by a company with low credit rating is usually priced lower than one sold by a company with a better credit rating. This is because there is a higher risk of default associated with a low rating.

The urgent need to further understand this source of risk and to develop appropriate quantitative models through risk analysis and management tools, was especially made clear by the global financial crisis of 2008, which was significantly caused by the sudden occurrence of illiquidity in the credit markets. Nevertheless, credit risk is commonly induced by corporations and governments aiming to increase capital on a short-term basis by selling bonds.

A bond is a contract where the buyer lends money to the seller (e.g., a company or a government), in exchange for regular interest payments. We will focus on the most basic bond in financial markets, the zero-coupon bond. Here, the value of the contract follows the short-term interest rate and no interest payments are paid during the contract period. That is, the value of the bond is defined as

$$B(t,T) = e^{-\int_t^T r_s ds}, \tag{1.1}$$

where r_s represents the interest rate at time s, t denotes the initial time and T the maturity time, i.e., the time at which the contract is executed.

In the above expression the interest rates was considered deterministic, meaning that we can determine, at each point in the future, what the interest rate will be. However, more realistically, we cannot predict the future value of interest rates, and thus we are required to use the risk-neutral probability measure (or pricing measure) to determine the price of, e.g., bonds. The risk-neutral price of a zero-coupon bond is defined as

$$B(t,T) = \mathbb{E}_{\mathbb{P}^*}[e^{-\int_t^T r(s)ds} | \mathcal{F}_t], \tag{1.2}$$

which is the conditional expectation of the previously defined value, given market information ($\mathcal{F}_t$) with respect to the risk-neutral probability measure $\mathbb{P}^*$. The theory behind these types of expressions is introduced in Section 2.2.

In this thesis, we will present a framework for the reserves of life insurance policies where the benefit in case of the insured event is linked to a defaultable bond. Pricing and reserve calculations based on $B(t,T)$ are dependent on two bond-classifications: corporate and Treasury (governmental) bonds. We define them as defaultable and default-free bonds respectively, due to companies being considered prone to default and governments generally not. In other words, we say that a bond is default-free if it pays the agreed-upon amount at maturity T with probability one. However, more often, a bond is not certain to pay the agreed-upon amount, and we will refer to these as defaultable bonds. The interested reader may also see Section 1.1 in [3] for a more detailed introduction of corporate (defaultable) bonds.

To cover the present cost of future liabilities, insurance companies hold a specific amount of money called the *reserve*. This represents the amount of money required in order to remain solvent. Reserves are subject to strict rules and regulations, like the Solvency II directive. Nevertheless, the reserve is dependent on the policy functions describing the insurance policy as well as the mortality rates and survival probability of the insured. The (prospective) mathematical reserve is typically given by

$$V_i(t,A) = \frac{1}{v(t)} \int_t^T v(u) \sum_{j \in \mathcal{S}} p_{i,j}(t,u) \left(a_j(u) + \sum_{k \neq j} \mu_{jk}(u) a_{jk}(u) \right) du, \tag{1.3}$$

where i denotes the state of the insured, $p_{i,j}(t,u)$ represents the probability of transitioning from state i to state j, $i, j \in S$, during the time period $[t,u]$, for $t \leq u$. $v(t)$ is the discount factor of interest rates and A denotes the cash-flow of the policy. More details will be provided in Section 2.1 and Section 3.1. Furthermore, see [7] for a detailed description of the insurance mathematics applied in this thesis.

As mentioned, the benefits provided to the insured in case of the insured event is occasionally dependent on the value of an underlying unit, e.g., a bond. If this is the case, the payoff is considered stochastic (see Section 2.3) and the contract is referred to as an *unit-linked* insurance policy. The reserve calculation of these policies are more complex, as you deal with the (conditional) expected value of the underlying unit, and thus need to consider the new risk-neutral probability measure previously mentioned.

Additionally, one may consider insurance policies with respect to stochastic interest rates. It is very similar to the unit-linked case, except that the payoff is not stochastic. Instead, one may consider interest rates with dynamics like

$$dr_t = \alpha(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{1.4}$$

which is a stochastic differential equation. These will be introduced in Section 2.3. Furthermore, see Chapter 3 for more details on the unit-linked case as well as the case of stochastic interest rates.

The purpose of this thesis is to apply the formula for the reserves, combined with the risk-neutral price of bonds in a defaultable bond market following the Bielecki-Rutkowski model of credit risk to develop a version of the reserves given stochastic interest rates based on yields of defaultable bonds. The Bielecki-Rutkowski model is an extension of the Heat-Jarrow-Morthon methodology for modelling the term structure of interest rates. See [6], [4] and [3] for details.

More specifically, this will be done for a unit-linked term insurance in a two-state model based on the zero-recovery case, meaning that the value of the defaultable bond drops to zero after default. Furthermore, we derive a version of Thiele's differential equation for the above setting. To do so, we will use some of the results in [5], to ensure the absence of arbitrage.

To achieve this in a comprehensive manner, the thesis is structured as following:

2. In Chapter 2 we give a general introduction to case of life insurance where we present different types of insurances, before introducing relevant applications of probability theory and stochastic calculus. Finally, we introduce credit risk modelling and provide the standing assumptions of this thesis.

3. Chapter 3 deals with the insurance mathematical prerequisites on reserves. We first introduce some general theory and then explore case of unit-linked policies and stochastic interest rates. Then, we present Thiele's partial differential equation with stochastic interest rates and include an example based on the well-known Vasicek interest rate-model.

4. Chapter 4 pertains to the Bielecki-Rutkowski model of credit risk. It also works as a summary of Chapter 4 in the doctoral thesis of P. Christodoulou ([5]), which contains some interesting aspects and difficulties of the aforementioned model.

5. In Chapter 5 we present the main result of this thesis. We first introduce mathematical reserves for unit-linked policies based on defaultable bonds, and using this, we develop a new version of Thiele's differential equation. This is something that, as far as we know, does not exist in the existing literature.

6. We conclude with a short summary of the thesis and some discussion on further explorations of the topic. This is provided in Chapter 6.

CHAPTER 2

Theoretical background

This chapter introduces important aspects of the thesis. In the *preliminaries*, the case general of classic life insurance (in continuous time) is introduced. This is followed by an introduction to probability theory and stochastic calculus, providing important definitions and relevant theory. We then introduce credit risk in some detail and conclude with a list of standing assumptions.

2.1 Preliminaries

In insurance, the reserve denotes the present value of the contract at some given time. It is, in other words, the specific amount of money the insurance company needs to set aside in order to remain solvent, and is thus a very important aspect of insurance.

Furthermore, the reserve is obviously highly dependent on the functions describing the insurance policy. These are called policy functions and are assumed to be of finite variation, ensuring that they behave nicely and that we are able to perform the necessary computations. Additionally, we occasionally allow for a jump in the maturity time T. This will become more clear later.

The policy functions are defined as:

Definition 2.1.1 Let $a_j, a_{jk} : [0, \infty) \to \mathbb{R}, j, k \in S, j \neq k$ be continuous functions of bounded variation. We then have that:

- a_j, represent the cash flow of staying in state $j \in S$
- a_{jk}, represent the cash flow of transitioning from j to k, $j, k \in S$

Remark 2.1.2 We also assume that the policy function $a_j(t)$ is at least *a.e.* differentiable, allowing for at most one jump at maturity time T. This is denoted by $\dot{a}_j(t)$.

In life insurance, it is common to operate in the state space $S = \{*, \diamond, \dagger\}$, corresponding to the actuarial notation for the *active*, *disabled* and *deceased* state, respectively.

- The active state is when the insured typically works and thus pays premiums to the insurance company. The active state is most important in a *pure endowment insurance*.

- The disabled state is when the insured is not paying premiums, but also is not dead. If the contract permits, e.g., a *disability insurance*, the insured receives periodic payments.
- The deceased state represents the death of the insured. This is important in the case of a *term insurance*.

The types of insurance policies typically considered are, as alluded to above, *pure endowment*, *disability* and *term* insurance. They all represent different types of covers, as clarified below.

Definition 2.1.3 (Pure endowment insurance) A pure endowment insurance considers a contract providing a payment to the insured if the insured *survives* the contract period, i.e., is *active* at maturity time T. This policy is thus determined by the policy function:

$$a_*(t) = \begin{cases} 0, & t \in [0, T), \\ E, & t \in [T, \infty), \end{cases}$$

where E represents the benefit of surviving the contract period.

Definition 2.1.4 (Disability insurance) A disability insurance deals with the cases where, e.g., the insured no longer is able to work, so payments are periodic and provided within the contract period. The policy function is described by:

$$a_\diamond(t) = \begin{cases} Dt, & t \in [0, T], \\ DT, & t > T, \end{cases}$$

where D is the benefit received periodically.

Remark 2.1.5 Please note that the policy function a_i represent accumulated payments during the policy time, whereas $a_{i,j}$ represent punctual payments triggered by a change of state of the insured.

Definition 2.1.6 (Term insurance) A term insurance yields a one-time payout in the case of death of the insured. This is provided that the insured dies within the contract period. The policy functions are described by:

$$a_{*,\dagger}(t) = \begin{cases} DB, & t \in [0, T), \\ 0, & t \geq T, \end{cases}$$

where DB denotes the death benefit.

The endowment, a combination of the pure endowment and the term insurance, is typically considered as the classic life insurance policy. This insurance is described by the combination of the policy functions $a_*(t)$ and $a_{*,\dagger}(t)$ as above.

Remark 2.1.7 There is a common practice regarding the signs of the above policy functions. Negative values are considered payments from the insured to the insurance company, whereas positive values are considered as income for the insured. This becomes especially relevant if one considers *premiums*[1] in the

[1] That is, the cost of the insurance for the insured. Could be considered as a one time payment or periodic payments. See [7] Section 1.2.2 for details.

policy functions. We will not consider premiums in this thesis. Despite this, a mention of the single premium required to finance the policy will be provided in Example 3.3.4.

The preliminaries of life insurance will be continued in Chapter 3, after some relevant background information on probability theory and stochastic calculus.

2.2 Introduction to Probability Theory

This section will focus on introducing, and if appropriate also contextualising, the probability theory relevant to the thesis. A similar section regarding stochastic calculus follows in Section 2.3.

This section is based on [10], and the interested reader is referred there for details and proofs.

We begin by defining the probability space in which we will be operating, and then diving into the elements of said space.

Definition 2.2.1 A probability space, denoted $(\Omega, \mathcal{F}, \mathbb{P})$, is a triple consisting of a set Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω and a probability measure $\mathbb{P}$.

The state space Ω represents the set of all possible outcomes of a random mechanism, e.g., paths of a stock's value in a financial market. The events/ paths are usually denoted using the lower-case omega, so that we have paths $\omega \in \Omega$.

The next element of the triple, the σ-algebra (sometimes referred to as σ-field in the literature), deserves its own definition.

Definition 2.2.2 Let X be a set. Then, a σ-algebra $\mathcal{F}$ is a collection of subsets of X such that the following three properties hold:

2.2.2.1. $X \in \mathcal{F}$

2.2.2.2. If $B \in \mathcal{F}$, then $B'(:= X \setminus B) \in \mathcal{F}$

2.2.2.3. If B_n is a sequence of elements in $\mathcal{F}$, then $\bigcup_n B_n \in \mathcal{F}$

Remark 2.2.3 If you replace X with Ω as defined above (set of all possible paths of a stock's value), $\mathcal{F}$ represents the market information generated by the stock process. However, we are often just interested in the information available at certain points in time. This is represented by sub-σ-algebras denoted $\mathcal{F}_t$, $t \in \mathbb{R}^+$. A collection of sub-σ-algebras is called a filtration $\mathbb{F}$ (to be formally defined later), which then represents a collection of information.

Remark 2.2.4 Note that since $\Omega \in \mathcal{F}$, then $\emptyset \in \mathcal{F}$. This follows directly from 2.2.2.2, which states that the σ-algebra $\mathcal{F}$ is closed under complementation.

Finally, the last part not yet disclosed from Definition 2.2.1 is the probability measure $\mathbb{P}$.

Definition 2.2.5 Let $\mathcal{F}$ be a σ-algebra as defined above. Then, a probability measure $\mathbb{P}$ with respect to $(\Omega, \mathcal{F})$ is a real-valued function defined on $\mathcal{F}$ such that the following is true

2.2.5.1. If $B \in \mathcal{F}$, then $\mathbb{P}(B) \geq 0$

2.2.5.2. $\mathbb{P}(\Omega) = 1$

2.2.5.3. If B_n is a finite (or countable infinite) sequence of disjoint elements of $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_n B_n\right) = \sum_n \mathbb{P}(B_n).$$

Stochastic processes

The word *stochastic* is unfamiliar to many and a short explanation thus follows. It originates from Greek, and it's original meaning was *to guess* or *conjecture*. It was later given another meaning in German; *randomly determined* (stochastik).

Thus, when dealing with stochastic processes, we are often interested in making educated guesses on future values of randomly generated processes. This gives rise to the following, more formal definition:

Definition 2.2.6 A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a function $X : \Omega \to \mathbb{R}$ which is $\mathcal{F}$-measurable.

That is, we have

$$\{\omega \in \Omega : X(\omega) < \alpha\} \in \mathcal{F},$$

for all $\alpha \in \mathbb{R}$.

The r.v. X could, e.g., represent the value of a risky asset. However, often we are more interested in modelling a portfolio of risky assets. This provides inspiration to the following definition:

Definition 2.2.7 A stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of stochastic random variables $X_t, t \in \mathbb{R}^+$ taking values in a measurable space.

In the context of credit risk theory, we will replace Ω with the description of the credit-rating classes, $\mathcal{K}$. More details will be provided in Section 2.4 and Chapter 4.

We move on to a formal definition of the previously mentioned collection of information (sub-σ-algebras):

Definition 2.2.8 A filtration $\mathbb{F}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of sub-σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_s \subset \mathcal{F}_t$, for $s < t$.

Remark 2.2.9 A filtration is increasing in time, i.e., the amount of available information increases in time, assuming that no information is lost.

We also need the following definition to work with risk-neutral prices of risky assets given by conditional expectations.

Definition 2.2.10 We say that a process $X = (X_t)_{t \in \mathbb{R}^+}$ is $\mathbb{F}$-adapted if X_t is $\mathcal{F}_t$-measurable for all t.

In option pricing theory, it is common to model stock prices as adapted processes. In general, an observer cannot look into the future and determine the future price of the stock. In fact, trying to determine future prices provides inspiration to a very important class of processes in applications of insurance and mathematical finance, *martingales*. These processes will be defined in the next section, in Definition 2.3.1.

2.3 Introduction to Stochastic Calculus

We begin by defining the martingales previously mentioned. Then, we move on to the Brownian motion, stochastic differential equations and some important formulas. The interested reader may see [1] for a more detailed introduction to stochastic calculus, [8] for an introduction to stochastic calculus applied to finance and [11] regarding stochastic differential equations.

Definition 2.3.1 Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. An adapted family $(M_t)_{t\geq 0}$ of integrable random variables, i.e., $\mathbb{E}[|M_t|] < +\infty$ for all t, is

- a martingale if, for any $s \leq t$, $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$,
- a super-martingale if, for any $s \leq t$, $\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s$,
- a sub-martingale if, for any $s \leq t$, $\mathbb{E}(M_t|F_s) \geq M_s$.

Thus, it follows that, if $(M_t)_{t\geq 0}$ is a martingale, then $\mathbb{E}(M_t) = \mathbb{E}(M_0)$ for any t.

Remark 2.3.2 One can think of martingales as fair bets, i.e., that the value of your payout corresponds to the expected value of the process. Similarly, a sub-martingale is a more-than-fair bet, and a super-martingale is a less-than-fair bet.

In [5], the notion of local martingales is often mentioned. These are defined as:

Definition 2.3.3 A process $\{X_t, \mathcal{F}_t, t \in T\}$ is said to be a local martingale if there exists an increasing sequence $(\tau_n)_n$ of stopping times[2] such that

2.3.3.1. $\lim_{n\to\infty} \tau_n = +\infty$ a.s.

2.3.3.2. $(X_{t\wedge\tau_n})_t$ is a $(\mathcal{F}_t)_t$-martingale for every n.

Remark 2.3.4 Also, every martingale is a local martingale. It can be proved that a bounded local martingale is a martingale, and that a lower bounded local martingale is a super-martingale

Remark 2.3.5 It follows that if local martingales are used in pricing theory, the prices obtained are in reality not necessarily correct. This is because a natural condition on the amount of money available to borrow imposes a lower bound, which results in the process being a super-martingale.

A well-known example of a stochastic process, is the standard Brownian motion. It is defined as follows:

[2]A random time τ is defined as a stopping time, if for any time t, one can determine whether the event $\{\tau \leq t\}$ (and thus also the complementary event) has occurred or not, e.g., the first time the value of a stock reaches a predetermined value.

Definition 2.3.6 The standard Brownian motion W_t is a continuous, stochastic, real-valued process attaining the properties

2.3.6.1. $W_0 = 0, \mathbb{P} - a.s.$

2.3.6.2. For every $s \in [0, t]$, the r.v. $W_t - W_s$ is independent of events in $\mathcal{F}_s$.

2.3.6.3. The law of $W_t - W_s$ is $\mathcal{N}(0, t-s)$, for every $s \in [0, t]$.

Remark 2.3.7 If W_t is a standard $\mathcal{F}_t$-Brownian motion, then W_t is an $\mathcal{F}_t$-martingale.

In fact, a multi-dimensional variant of this process will be used as noise (uncertainty) in the price processes discussed in Chapter 4. This is defined as:

Definition 2.3.8 A $\mathbb{R}^m$-valued process $W = (W_t)_{t \in \mathbb{R}^+}$ on a filtered probability space is a m-dimensional Brownian motion if

2.3.8.1. $W_0 = 0, \mathbb{P} - a.s.$,

2.3.8.2. For every $s \in [0, t]$, the r.v. $W_t - W_s$ is independent of events in $\mathcal{F}_s$,

2.3.8.3. For every $s \in [0, t]$, $W_t - W_s$ is $\mathcal{N}(0, (t-s)I_m)$-distributed, with I_m being the $m \times m$ identity matrix.

In view of the definition of the concept of a stochastic integral for integrand processes X_s, $s \geq 0$, we also need the following definition, which can be found in [11].

Definition 2.3.9

- $M^p_{\text{loc}}[a, b]$ is defined as the space of equivalence classes of real-valued progressively measurable processes $X = (X_t)_{t \in [a,b]}$ in a filtered probability space, such that
$$\int_a^b |X_s|^p ds < +\infty \qquad \mathbb{P}\text{-a.s.}, \tag{2.1}$$
- $M^p[a, b]$ is the space of equivalent classes of progressively measurable processes such that
$$\mathbb{E}\left[\int_a^b |X_s|^p ds\right] < \infty \tag{2.2}$$

Remark 2.3.10 By *progressively measurable process*, we mean that for all t $X : \Omega \times [0, t] \to \mathbb{R}^d$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable, where $\mathcal{B}([0, t])$ is the Borel-σ-algebra on $[0, t]$. This is satisfied for all right-continuous adapted functions.

Remark 2.3.11 By equivalence classes, we mean that we identify two processes X and Y whenever they are indistinguishable, that is:

$$\int_a^b |X_s - Y_s| ds = 0.$$

Stochastic Differential Equations

Stochastic differential equations are commonly used to model stochastic processes that evolve in time, e.g., an interest rate processes. In the following, we make a general introduction to SDEs and present two important formulae. The interested reader should see [11] for a thorough introduction.

The stochastic differential equation (SDE)

$$dX(t,T) = \alpha(t, X(t,T), T)dt + \sigma(t, X(t,T), T)dW_t,$$

can be rewritten as

$$X(t,T) = X(0,T) + \int_0^T \alpha(t, X(t,T), T)dt + \int_0^T \sigma(t, X(t,T), T)dW_t,$$

if $\alpha(t, X(t,T), T) \in M^1_{\text{loc}}([0,T])$ and $\sigma(t, X(t,T), T) \in M^2_{\text{loc}}([0,T])$.

Furthermore, the above SDE is an Itô-process, i.e., the sum of a finite variation process and of a local martingale. Solutions to these are called diffusions, and they uniquely exist if $\alpha(\cdot, T), \sigma(\cdot, T) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions of linear growth. If they exist, they are often found using numerical methods (see Chapter 9 of [8]).

However, given a suitable function $f(X_t, t)$, the stochastic differential of f can be found using an analytical approach with Itô's formula, which is stated in the following theorem.

Theorem 2.3.12 *(Itô's formula) Let X be a process with stochastic differential*

$$dX_t = \alpha(t)dt + \sigma(t)dW_t,$$

and let $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a continuous function in (x,t), once continuously differentiable in t and twice in x. Then

$$df(X_t, t) = \frac{\partial f}{\partial t}(X_t, t)dt + \frac{\partial f}{\partial x}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t, t)\sigma(t)^2 dt.$$

This can also be written as:

$$df(X_t, t) = \left(\frac{\partial f}{\partial t}(X_t, t) + \frac{\partial f}{\partial x}(X_t, t)\alpha(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t, t)\sigma(t)^2\right) dt$$
$$+ \frac{\partial f}{\partial x}(X_t, t)\sigma(t)dW_t.$$

Throughout Chapter 4, we are in need of changing the probability measure from the physical measure $\mathbb{P}$ to a more suitable risk-neutral probability measure (or equivalent martingale measure) $\mathbb{Q}^*$ under which the price processes of bonds are (local) martingales.

To obtain the martingale-property needed to deal with conditional expectations, we have to adjust for this change of measure, so that we have a Brownian motion under the new probability measure. This is done using Girsanov's theorem.

Theorem 2.3.13 *(Girsanov's theorem) Let W_t be a d-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $\gamma \in M^2_{loc}([0,T])$. Assume that the process*

$$X_t = \exp\left(\int_0^t \gamma_s dW_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right),$$

is a martingale on $[0, T]$ *and let* $\mathbb{Q}^*$ *be the probability on* $(\Omega, \mathcal{F})$ *having the Radon-Nikodym density* X_T *with respect to* $\mathbb{P}$. *Then, the process*

$$\tilde{W}_t = W_t - \int_0^t \gamma_s ds$$

is a $(\mathcal{F}_t)_t$*-Brownian motion on* $[0, T]$ *with respect to* $\mathbb{Q}^*$.

Remark 2.3.14 Note that the process γ has the meaning of a *market price of risk* in option pricing theory.

2.4 Introduction to Credit Risk

By credit risk, we refer to all risks associated with any kind of a credit-linked event. These events may consist of changes in the credit quality of a bond, variations of credit spreads or the default of a counterparty in a contract (sometimes referred to as counterparty risk in the literature). This section is based on [3].

Now, the credit-linked events can be explained as:

- The credit quality of a bond describes the classification of corporate debt. Modelling of changes in the credit quality (rating) of a bond can be done using a *reduced-form approach*. Here, the asset's of a corporation is taken into account, whereas the capital structure is not modelled at all (this is considered in the structural approach, see [3] Section 1.4.1). We also distinguish between an intensity-based approach and the credit migration models. In the intensity-based approach we are only interested in modelling the random time of default, whereas the alternative deals with models of migration between credit rating classes (and thus default implicitly through the migration process).

- A corporation's credit rating is a measure of the likelihood to default. The ratings are often categorised on a finite set, the set of credit classes. For example, the best rating could be given by, e.g., AAA and C may denote the worst active rating, whereas D typically denotes the default event, in the case that a corporation already has failed in some payment. This notation is used by, e.g., *Standard & Poor's*[3] and *Fitch Ratings*[4].

- A credit spread measures the excess return on a corporate (defaultable) bond over the return on an equivalent Treasury (default-free) bond, i.e., a bond assumed to be free of credit risk. We will denote the spreads as the difference between respective forward rates, and distinguish between the fundamental spreads and the inter-rating spreads. More details will be provided in Chapter 4.

Furthermore, we will operate on the credit rating set $\mathcal{K} = \{1, \ldots, K\}$, where 1 denotes the best rating, $K - 1$ the worst active rating, and K the default

[3] https://www.spglobal.com/ratings/en/
[4] https://www.fitchratings.com/

event, as above. Furthermore, in the spirit of [3], we shall use the term *credit rating* to describe any classification of corporate debt that can be justified for specific purposes.

Within the reduced-form framework, (conditionally) Markov chains, C, are used to model the credit migrations between different rating classes. This is done using transition probabilities (discrete time) or transition intensities (continuous time). We will be focusing on the continuous time models, specifically that of Bielecki and Rutkowski (see [3] for details). Additionally, some interesting results will be disclosed based on Chapter 4 of the doctoral thesis of P. Christodoulou [5]. Additional details regarding the credit migration modelling can be found in Chapter 4.

The model of Bielecki and Rutkowski lie inside the context of the Heat-Jarrow-Morton (HJM)-methodology (see [6]) for modelling the defaultable term-structure of interest rates. The idea of this framework is to model the entire forward rate curve directly. This is because the short-rate models are not always flexible enough to be calibrated to the observed initial term-structure.

Furthermore, in the reduced-form approach, the credit migration process is given endogenously, resulting in the conditionally Markov model of credit risk. A conditionally Markov-chain is defined as follows

Definition 2.4.1 A process C is called a conditionally $\mathbb{G}$-Markov chain relative to $\mathbb{F}$ and under $\mathbb{Q}^*$, if for every $0 \leq t \leq s$ and any function $h : \mathcal{K} \to \mathbb{R}$ we have

$$\mathbb{E}_{\mathbb{Q}^*}[h(C_s)|\mathcal{G}_t] = \mathbb{E}_{\mathbb{Q}^*}[h(C_s)|\mathcal{F}_t \vee \sigma(C_t)]. \tag{2.3}$$

Remark 2.4.2 Please note that $\mathcal{F}_t \vee \sigma(C_t)$ denotes the joint filtration.

In the intensity-based approach, the default time is modelled by a random variable τ. A feature of the intensity-based models, is that τ may not be a stopping time with respect to the default-free filtration $\mathbb{G} = (\mathcal{G}_t)$. Instead, these models use the hazard rate, λ, the rate of occurrence of default, to characterise the default-time, which in this case is a exogenous, random time.

A direct result of this is that default may occur as a surprise. This is contrary to the structural approach, where the default time is modelled in such a way so that it does not provide any element of surprise[5].

Now, the hazard rate can also be understood as the transition intensity of a continuous-time Markov chain, and is a function satisfying

$$\mathbb{P}(\tau > t) = e^{-\int_0^t \lambda(s)ds}, \quad \forall t > 0 \tag{2.4}$$

for a random variable τ satisfying

1. $\mathbb{P}(\tau > 0) = 1$,
2. $\mathbb{P}(\tau > t) > 0$,

for all $t > 0$, if the cumulative distribution function of τ is continuous and differentiable.

The information related to the default-free market is denoted by the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}}$ in a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. The filtration consists typically

[5] That is, the resulting random times are predictable with respect to the underlying filtrations.

of information regarding interest rates, which means that the instantaneous interest rate process is $\mathbb{G}$-adapted. Implicitly, as the price process of the risk-free asset is given by

$$B(0,t) = \exp(\int_0^t r(s)ds),$$

it is also $\mathbb{G}$-adapted.

At time t, observers in the market know if default has occurred, so the total market information at time t is given by the σ-algebra $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$, constructed as

$$\mathcal{F}_t = \mathcal{G}_t \vee \sigma(\{\tau \leq s\}, s \leq t),$$

where $\sigma(\{\tau \leq s\}, s \leq t)$ is the natural σ-algebra generated by default events $\{\tau \leq s\}, s \in [0,t]$. A direct result of this is that, although τ is not necessarily a stopping time with respect to $\mathbb{G}$, it is with respect to $\mathbb{F} = (\mathcal{F}_t)$ a.s.

2.5 Standing Assumptions

We make the same standard, technical assumptions as in [3]:

- all reference probability spaces are assumed to be complete, with respect to the reference probability measure,

- all filtrations satisfy the conditions of right-continuity and completeness,

- the sample pahts of all stochastic processes are assumed to be càdlàg[6],

- all random variables and stochastic processes satisfy suitable integrability conditions, ensuring the existence of considered conditional expectations, deterministic or stochastic integrals, etc.

[6]That is, right-continuous with existing left limits.

CHAPTER 3

Insurance Mathematical Prerequisites

This chapter is dedicated to the review of some concepts from classic life insurance, which we will use in Chapter 5 in connection with our main result. After having recalled some basic probability theory in Section 2.2, we are now ready to build directly on Section 2.1.

An insurance policy may be considered as a bet between the insured and the insurance company, explaining the relevance of probability theory. A result of this is that insurance companies are interested in predicting the likely outcome of their portfolio. One very important aspect of this, is the state of the insured. The main idea is that by using a Markov chain, one can model the transitions between the state of the insured, enabling the derivation of transition probabilities. As described in the introduction (equation 1.3), these transition rates and transition probabilities are used to calculate the required reserves for the insurance company.

The results presented in this chapter can be found in [7].

3.1 Common Terms and Notation

Recall the aspects from Section 2.1. Then, as mentioned above, the idea is to model the state of the insured using a Markov chain:

Definition 3.1.1 (Markov chain) Let $X_t \in S$, $t \in J \subseteq \mathbb{R}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then $X_t, t \in J$ is called a Markov chain, if

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} | X_{t_1} = i_1, \ldots, X_{t_n} = i_n) = \mathbb{P}(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n), \quad (3.1)$$

for all $t_1 < t_2 < \cdots < t_{n+1} \in J$, $i_1, \ldots, i_{n+1} \in S$ with $\mathbb{P}(X_{t_1} = i_1, \ldots, X_{t_n} = i_n) \neq 0$.

Remark 3.1.2 That is, the process only remembers its last position, i.e., it is a memory-less process.

Then, the transition probability $p_{i,j}(s,t)$ represents the probability of transitioning between states i and j for times $0 \leq s \leq t$. It is defined as:

Definition 3.1.3 (Transition probabilities) Let X_t be a stochastic process (Markov chain) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the transition probability $p_{i,j}(s,t)$ is defined as:

$$p_{i,j}(s,t) \coloneqq \mathbb{P}(X_t = j | X_s = i), \tag{3.2}$$

where $s \leq t$ and $i, j \in S$, being a finite state space.

Remark 3.1.4 It is common in life insurance that the transition probabilities are dependent on the age of the insured. Thus, the above notation slightly abuses this, as it really should be denoted, e.g., $p^x_{i,j}(t,s) = p_{i,j}(x+t, x+s)$, for an x year old individual. This is omitted for ease of notation. Please note that the times t and s thus refers to the contract age. The same applies for the transition rates.

Equally important as the transition probabilities, are the transition rates $\mu_i(t)$ and $\mu_{i,j}(t)$, representing the rate at which transitions occur[1].

Definition 3.1.5 (Transition rates) Let $X = \{X_t, t \in J\}$ be a Markov process with finite state space S. The transition rates $\mu_i, \mu_{i,j}, i, j \in S, j \neq i$ are the functions defined by:

$$\mu_i(t) \coloneqq \lim_{\substack{h \to 0 \\ h > 0}} \frac{1 - p_{i,i}(t, t+h)}{h}, \quad t \in J, \quad i \in S, \tag{3.3}$$

and

$$\mu_{i,j}(t) \coloneqq \lim_{\substack{h \to 0 \\ h > 0}} \frac{p_{i,j}(t, t+h)}{h}, \quad t \in J, \quad i, j \in S, \quad j \neq i, \tag{3.4}$$

whenever they exist and are finite.

Remark 3.1.6 Note that by using either of *Kolmogorov's equations* (See [7], Theorem 2.3.4), one may find expressions for the transition probabilities using the transition rates.

Then, the behaviour of the Markov chain modelling the state space $S = \{*, \diamond, \dagger\}$ is given by the matrix of transition rates:

$$\Lambda(t) = \begin{pmatrix} \mu_{*,*}(t) & \mu_{*,\diamond}(t) & \mu_{*,\dagger}(t) \\ \mu_{\diamond,*}(t) & \mu_{\diamond,\diamond}(t) & \mu_{\diamond,\dagger}(t) \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

where the sum of each row equals to zero.

Remark 3.1.7 Please note that the last row of the matrix being zero corresponds to the fact that once deceased, it is impossible to transition out of the state. This is known as an *absorbing* state in the literature.

Remark 3.1.8 There is also a common notation of $\mu_i(t) = -\mu_{i,i}(t)$.

We are mostly interested in knowing the value of the reserve in *today's* money, meaning that we need to discount the value of the portfolio according to the interest rate levels. This is done using the *discount factor*:

[1]That is, one may consider the transition rates as the derivatives of the transition probabilities. This follows directly from the definition of the derivative and using that $p_{i,j}(t,t) = 0$.

Definition 3.1.9 (Discount factor) A function $v : [0, \infty) \to [0, \infty)$ defined as

$$v(t) := \exp\left(- \int_0^t r(s) ds \right), \tag{3.6}$$

for a (deterministic) integrable function $r(t)$, is called the discount factor if $r(t)$ is modelling the interest rate.

3.1.1 Insurance reserves

By combining the elements of Section 2.1 and the above definitions we can formally define the reserve. The reserve is given by

Definition 3.1.10 (Reserves) Assume that $a_i(t)$ and $a_{i,j}(t)$ are policy functions, and that $p_{i,j}(t, s)$ and $\mu_{i,j}(t)$ are the transition probabilities and transition rates, respectively. Then, the discounted (prospective) reserves are given by:

$$\begin{aligned} V_i(t) = \frac{1}{v(t)} \Bigg(& \sum_{j \in S} \int_t^\infty v(s) p_{i,j}(t, s) da_j(s) \\ & + \sum_{\substack{i,j \in S \\ j \neq i}} \int_t^\infty v(s) p_{i,j}(t, s) \mu_{j,k}(s) a_{j,k}(s) ds \Bigg), \end{aligned} \tag{3.7}$$

Remark 3.1.11 Note that the above equation technically is the prospective reserves, which account for all states, whereas the mathematical reserve is given by an expectation over each state. See Section 4.6 of [7] for details.

Remark 3.1.12 Note that the first integral is given in the Riemann-Stieltjes sense. See Chapter 4 of [7] for details on how to deal with this.

3.2 Unit-linked Policies

The aim of this thesis is to explore reserves of unit-linked policies exposed to credit risk, more specifically: the case when the interest rates follow the term-structure of a bond in a defaultable bond market.

Unit-linked policies are policies where the value of the contract is dependent on the value of a underlying unit, e.g., the value of a stock or a bond, as illustrated in the following example.

Example 3.2.1 Assume the case of a term insurance. Let $B(t, s)$ denote the value of a bond, representing the payout to the insured. That is, $a_{*,\dagger}(s) = B(t, s)$ for $t \leq s \leq T$. The reserves of a unit-linked term insurance, given deterministic interest rate, is then defined by

$$V_*(t, B) = \frac{1}{v(t)} \mathbb{E}_{\mathbb{P}^*} \left[\int_t^T v(s) p_{*,*}(t, s) \mu_\dagger(s) B(t, s) ds \middle| \mathcal{F}_t \right], \tag{3.8}$$

where $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}}$ is the filtration consisting of market information and $\mathbb{P}^*$ is an equivalent martingale measure. This follows directly from the definition of the reserves.

However, we consider both the discount factor, transition probabilities and mortality rate as deterministic, meaning that the above expression can be rewritten as

$$V_*(t,B) = \frac{1}{v(t)} \int_t^T v(s) p_{*,*}(t,s) \mu_\dagger(s) \mathbb{E}_{\mathbb{P}^*}\left[B(t,s)|\mathcal{F}_t\right] ds, \tag{3.9}$$

using Fubini's theorem[2] to interchange the expectation and the integral.

A quite similar case to the unit-linked, is the case of stochastic interest rates. This will be explored more in-depth in the following section. We will also explore the case of numerical approximation to the reserves, using the finite-difference method on Thiele's partial differential equation, and provide an example of the resulting reserve surface when the stochastic interest rates are given by the Vasicek model.

3.3 Stochastic Interest Rates

As illustrated by the discount factor, the level of interest rates is highly important when considering reserves. In this section, we will explore the aspect of *stochastic* interest rates, that is: interest rates that do not follow a deterministic function, but rather a SDE.

From now on, we require that the flow of information generated by the stochastic interest rates and the information generated by the state of the insured are independent. Furthermore, we will assume a an arbitrage-free market, allowing us to find an equivalent martingale measure. Then, one may consider a bond that models the value of the policy. The price of this bond is given by the (conditional) expectation with respect to the new measure.

Now, let r be the interest rate generating the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$. We then consider the dynamics:

$$dr_t = \alpha(t, r_t)dt + \sigma(t, r_t)dW_t, \quad r_0 \in \mathbb{R}, \quad t \in [0,T], \tag{3.10}$$

where W is the standard Brownian motion and α, σ are measurable functions such that a unique global strong solution[3] to the above equation exists.

3.3.1 Mathematical reserves and stochastic interest rates

To make things simpler, we will henceforth consider a term insurance in a two-state model, only considering that the insured is either active ($*$) or deceased ($\dagger$). Thus, the state in which benefits are provided to the insured is an absorbing state, meaning that no transitions out of the benefit-providing state may occur. Furthermore, we do not consider payments provided to the insured in the case of survival up to maturity T, as in the case of a pure endowment.

Given the above setting, we let $\mathbb{P}^*$ denote the risk-neutral probability measure and $\mathcal{F}_t$ the information generated by the interest rate process. Finally, we denote by $DB(t)$ the death benefit provided to the insured in case of death,

[2]See e.g. [1] Theorem 1.2.

[3]That is, they are progressively measurable functionals of the driving noise, i.e., the Wiener process W_t, $t \geq 0$.

given that death occurs at time $t < T$. Then, we have that the mathematical reserves is given by

$$V_*(t, r_t) = \int_t^T \mathbb{E}_{\mathbb{P}^*}[e^{-\int_t^s r_u du}|\mathcal{F}_t] p_{*,*}(t,s)\mu_\dagger(s)DB(s)ds, \tag{3.11}$$

where r_u is the interest rate process. This is done following similar steps as for the unit-linked case.

Remark 3.3.1 Note that we include r_t in the function V^+ to emphasise the additional dependency of interest rates.

3.3.2 Thiele's partial differential equation

Given the complex nature of reserves, especially those based on stochastic interest rates, we often require a faster way of finding them. This is often solved using numerical methods.

Thiele's differential equation provides such a method. It relies on the fact that reserves are dependent on their previous value in a recursive way, and that one knows what the reserve should be a maturity, i.e., the terminal condition. Furthermore, it attains the benefit of excluding survival probabilities.

For a general insurance policy, with stochastic interest rates, it is given by:

Definition 3.3.2 (Thiele's partial differential equation)

$$\frac{\partial}{\partial t} V_i(t, r_t) = r_t V_i(t, r_t) - \dot{a}_i(t) - \sum_{\substack{j \in S \\ j \neq i}} \mu_{i,j}(t)(a_{i,j}(t) + V_j(t, r_t) - V_i(t, r_t)) - LV_i(t, r_t), \tag{3.12}$$

where $LV_i(t, r_t)$ is the differential operator defined by

$$LV_i(t, r_t) = (\alpha(t, r_t) + \gamma(t, r_t)\sigma(t, r_t)) \frac{\partial V_i(t, r_t)}{\partial r_t} + \frac{1}{2}\sigma^2(t, r_t) \frac{\partial^2 V_i(t, r_t)}{\partial r_t^2}, \tag{3.13}$$

where $\gamma(t, r_t)$ denotes the *market price of risk*.

Proof. See Section 9.5 in [7], especially Theorem 9.5.1. ■

Remark 3.3.3 In the above formula, you may distinguish three parts:

3.3.3.1. The classical part:

$$r_t V_i(t, r_t) - \dot{a}_i(t) - \sum_{\substack{j \in S \\ j \neq i}} \mu_{i,j}(t)(a_{i,j}(t) + V_j(t, r_t) - V_i(t, r_t)),$$

3.3.3.2. The part corresponding to stochastic interest rates:

$$(\alpha(t, r_t) + \gamma(t, r_t)\sigma(t, r_t)) \frac{\partial V_i(t, r_t)}{\partial r_t} + \frac{1}{2}\sigma^2(t, r_t) \frac{\partial^2 V_i(t, r_t)}{\partial r_t^2},$$

3.3.3.3. The part $\gamma(t, r_t)\sigma(t, r_t)$ corresponding to the use of an equivalent martingale measure. The term $\gamma(t, r_t)$, often referred to as the market price of risk, is dependent on the dynamics of the interest rates. Nevertheless, by operating with respect to the physical measure $\mathbb{P}$, this term disappears.

Example 3.3.4 In this example, we will show an application of Thiele's partial differential equation given a term insurance under the Vasicek interest rate model[4]. We assume no arbitrage, meaning that we may obtain an equivalent martingale measure, $\mathbb{P}^*$.

Recall that a term (life) insurance provides a death benefit, $DB(t)$, given that the insured dies before maturity T. We assume that this is constant, so that $DB(t) = DB$ for all t. Thus, this insurance is specified by

$$\dot{a}_*(t) = 0$$

and

$$a_{*,\dagger}(t) = \begin{cases} 0, & t \geq T, \\ DB, & t \in [0, T). \end{cases}$$

Furthermore, the Vasicek model has the following dynamics:

$$dr_t = a(b - r_t)dt + \sigma dW_t, \quad t \in [0, T], \quad r_0 \in \mathbb{R}, \tag{3.14}$$

where $a, b, \sigma \in \mathbb{R}, \sigma > 0$ are fixed parameters. In the literature, this model is referred to as a *mean reverting model*[5], with mean speed a and mean reversion level b.

A feature of the model is that the market price of risk, $\gamma(t, r_t)$ is constant, so we introduce the notation $\gamma \in \mathbb{R}$. Then, under the equivalent martingale measure $\mathbb{P}^*$, we have, by Girsanov's theorem:

$$dr_t = a(b + \frac{\gamma\sigma}{a} - r_t)dt + \sigma dW_t^{\mathbb{P}^*}, \tag{3.15}$$

where $W^{\mathbb{P}^*}$ denotes the $\mathbb{P}^*$-Brownian motion obtained from Girsanov's theorem.

Remark 3.3.5 In fact, this shows that the model is invariant under a measure change to an equivalent martingale measure, obtaining a new mean reversion level.

Now, using the equation for the reserves obtained previously, (3.11), we have that the price of the bond (under the equivalent martingale measure) is given by

$$P(t, T) = \mathbb{E}_{\mathbb{P}^*}\left[\exp\left(-\int_t^T r_s ds\right) \middle| \mathcal{F}_t\right]. \tag{3.16}$$

The choice of Vasicek is based on the fact that one may compute the price of the bond explicitly. It can be shown that:

$$P(t, T) = e^{-A(T-t)r_t + B(T-t)}, \quad 0 \leq t \leq T, \tag{3.17}$$

[4] A version of the Ornstein-Uhlenbeck process.

[5] This is a term explaining the behaviour of some models that tend to converge to the assets average value over time.

where

$$A(x) := \frac{1-e^{-ax}}{a}, \quad B(x) := \left(b + \frac{\gamma\sigma}{a} - \frac{\sigma^2}{2a^2}\right)(A(x) - x) - \frac{\sigma^2}{4a}A(x)^2.$$

Proof. The proof can be found in [8]. ■

Thus, it follows that the formula for the reserves in the active state becomes

$$V_*(t, r_t) = \int_t^T e^{-A(s-t)r_t + B(s-t)} p_{*,\dagger}(t,s)\mu_{*,\dagger}(s) B \, ds. \tag{3.18}$$

Furthermore, Thiele's equation under the Vasicek model can be written as

$$\begin{aligned}\frac{\partial}{\partial t}V_*(t,r_t) &= r_t V_*(t,r_t) - \mu_{*,\dagger}(t)(B - V_*(t,r_t)) \\ &\quad - (a(b - r_t) + \gamma\sigma)\frac{\partial V_*(t,r_t)}{\partial r_t} - \frac{1}{2}\sigma^2\frac{\partial^2 V_*(t,r_t)}{\partial r_t^2},\end{aligned} \tag{3.19}$$

with terminal condition $V_*(T, r_t) = 0$.

Now, as you we see, the above equation contains partial derivatives. We will deal with these using incremental approximations, more precisely the finite difference method (see [2] Chapter 5.2). Specifically, we will use

$$\begin{aligned}\frac{\partial V(t,x)}{\partial t} &\approx \frac{V(t+\Delta t, x) - V(t,x)}{\Delta t}, \\ \frac{\partial V(t,x)}{\partial x} &\approx \frac{V(t, x+\Delta x) - V(t,x)}{\Delta x}, \\ \frac{\partial^2 V(t,x)}{\partial x^2} &\approx \frac{V(t, x+\Delta x) - 2V(t,x) + V(t, x-\Delta x)}{(\Delta x)^2}.\end{aligned}$$

Now, substituting the approximations into Thiele's PDE derived above. Furthermore, we make the notation that $r_t = x_j$, as we evaluate different levels of interest rate in the *space*, and using that $t_i + \Delta t = t_{i+1}$ and $x_j \pm \Delta x = x_{j\pm 1}$, we evaluate at (t_{i+1}, x_j)

$$\begin{aligned}\frac{V_*(t_{i+1},x_j) - V_*(t_i,x_j)}{\Delta t} &\approx x_j V_*(t_{i+1},x_j) - \mu_{*,\dagger}(t_{i+1})(DB - V_*(t_{i+1},x_j)) \\ &\quad - (a(b-x_j)+\gamma\sigma)\frac{V(t_{i+1},x_{j+1}) - V(t_{i+1},x_j)}{\Delta x} \\ &\quad - \frac{1}{2}\sigma^2\frac{V(t_{i+1},x_{j+1}) - 2V(t_{i+1},x_j) + V(t_{i+1},x_{j-1})}{(\Delta x)^2}.\end{aligned}$$

Then, by rearranging so that we have an expression for $V(t_i, x_j)$, we obtain the following numerical scheme

$$\begin{aligned}V_*(t_i,x_j) \approx\; & V_*(t_{i+1},x_j) \\ & - \Delta t\Bigg(V_*(t_{i+1},x_j) - \mu_{*,\dagger}(t_{i+1})(DB - V_*(t_{i+1},x_j)) \\ & - (a(b-x_j)+\gamma\sigma)\frac{V(t_{i+1},x_{j+1}) - V(t_{i+1},x_j)}{\Delta x} \\ & - \frac{1}{2}\sigma^2\frac{V(t_{i+1},x_{j+1}) - 2V(t_{i+1},x_j) + V(t_{i+1},x_{j-1})}{(\Delta x)^2}\Bigg).\end{aligned} \tag{3.20}$$

To be able to implement the above scheme, using an explicit method[6], we need to specify the boundary conditions $V(t,0)$ and $V(t,X)$, X representing a very high interest rate.

It makes sense to choose the boundary condition $V_i(t,0)$ the reserve corresponding to V_i^+ in the case of zero interest, and that $V(t,X) = 0$ for a very large interest rate X. This follows from that in the case of infinite interest rate, we have zero price. Furthermore, as in the ordinary Thiele sense, we need to specify a terminal condition $V(T,x)$. In our case, we let $V(T,x) = 0$, as we are dealing with a term insurance which pays nothing if the insured is active at maturity, as the insured event has not occurred.

Then, our lower boundary condition ($V(t,0)$) is given by

$$V(t,0) = \int_t^T e^{B(s-t)} p_{*,*}(t,s)\mu_{*,\dagger}(s) DB ds,$$

where $\exp(B(s-t))$ represents the solution to the Vasicek model given that $r_t = 0$ and DB is the death benefit. The upper boundary condition is, on the other hand, given by

$$V(t,X) = 0,$$

as previously stated.

We will solve the numerical scheme for the following parameters:

- Initial age of the insured is $x = 24$ years. This is not strictly needed for the survival probability in the lower boundary condition as we assume constant mortality.
- Contract end at $T = 50$ and $t = 0$. We consider a fixed mortality ($\mu = 0.009$) and thus constant survival probability ($p_{*,*}(t+x, s+x) = \exp(-\mu(s-t))$, for $s > t$).
- Vasicek specifications are $a = 0.05$, $b = 0.03$, $\sigma = 0.02$ and $\gamma = 0$ (for simplicity).
- The step sizes in time Δt and space Δx are equal and constant at $\Delta t = \Delta x = 0.01$.
- The maximum interest rate considered is $r = 20\%$.

The scheme is implemented in R, and the relevant code can be found in Appendix A. The results are presented the next subsection.

[6] A method in which the we can recursively find the previous reserve (V_i^j), given the the "current" (V_{i+1}^j) using a direct formula: $V_i^j = f(V_{i+1}^{j-1}, V_{i+1}^j, V_{i+1}^{j+1})$.

3.3.3 Plotting the reserve

In Figure 3.1, one can see the reserves iterated backwards from the terminal condition $V(T, r_T) = 0$. Obviously, the value is higher at time $t = 0$ for lower interest rate, and thus it is the highest when the rate is zero. It is, however, not very easy to differentiate anything else from the plot, and thus, Figure 3.2, is produced to provide further detail.

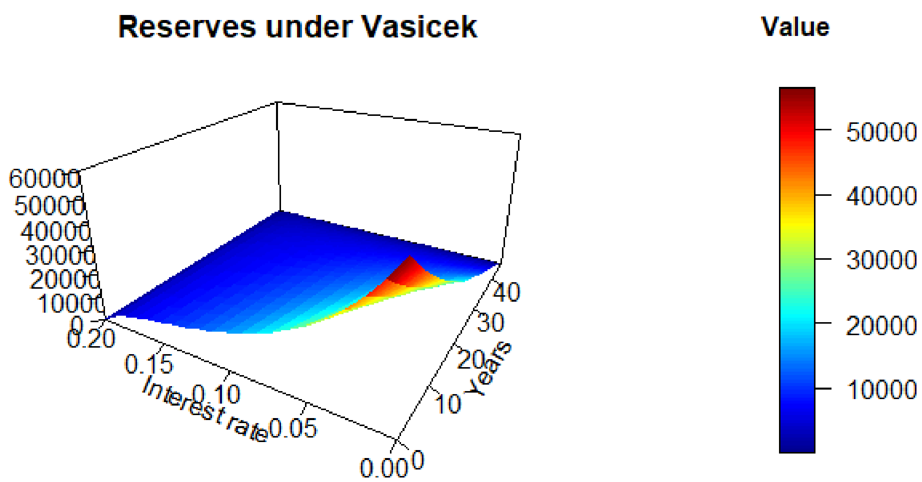

Figure 3.1: Reserves for a term-insurance for a contract period of 50 years, with stochastic interest rates following the Vasicek model, for a death benefit of NOK 100000, constant mortality rate $\mu = 0.009$ and survival probability $p_{*,*}(t, s) = \exp(-\mu(s - t))$. Vasicek parameters are $a = 0.05$, $b = 0.03$, $\sigma = 0.02$ and $\gamma = 0$.

In Figure 3.2, a similar picture to that of Figure 3.1 is presented, but now it is more clear exactly how the reserves behave. Combining the information from both figures, we see that there is a bigger difference for the lower-valued interest rates, and the higher the rates go, the less difference between the previous level, and the next.

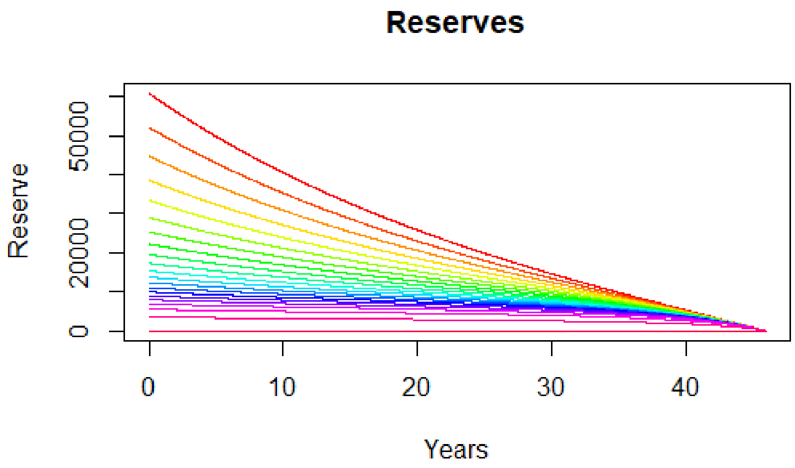

Figure 3.2: Reserves over time for each level of interest rate as found using the Thiele approach. The highest line represents $r = 0\%$ and the lowest represents $r = 20\%$.

Especially note that for the case of zero interest, we see a almost linear line. Actually, the only reason that it is not completely linear, is that it is also affected by the case of $r = 1\%$, as seen in Equation 3.20.

As mentioned in Chapter 2, premiums are used to finance the contract. From the literature, we know that the single premium can be found by looking at $V(0, r)$ for different interest rates. This is illustrated in Figure 3.3.

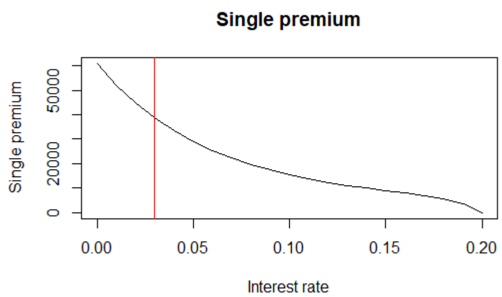

Figure 3.3: Different values of the single premium required to finance the contract. The red line represents the case of 3% interest, resulting in a single premium π_0 of NOK 38 564.

It should be mentioned that the simulation can be improved by including more realistic mortality rates, and survival probabilitiy when specifying the lower boundary condition. This would drastically change the plots, but also introduces more complexity to the simulation. This is because the unlikely scenario of death at a young age, which is not represented in the case of constant mortality rates.

Furthermore, by replacing the upper boundary condition with the reserve corresponding to maximum interest rate (20%) similar to the lower boundary condition, we would obtain smoother plots. This can especially be seen in Figure 3.3, in the interval $15\% - 20\%$.

CHAPTER 4

The Bielecki-Rutkowski Credit Risk Model

The Bielecki-Rutkowski model is a case of the previously mentioned intensity-based credit risk models. It provides an advantage by modelling the forward curve directly, which avoids the problem of calibrating the initial term structure that is present in many short-rate models. It allows for easy integration of the entire forward curve, and also for negative interest rates.

This chapter is dedicated towards introducing the model (See [3] Sections 13.1 and 13.2) and summarising the findings in the doctoral thesis by P. Christodoulou (See [5] Chapter 4). The interested reader is referred there for proof of the statements in this thesis. We will include direct references for the main results.

4.1 HJM-model with Credit Migrations

The following conditions are presented in [3] for a HJM-model with credit migrations:

- **(BR.1)** Let T^* denote maximum maturity. We are given a d-dimensional Brownian motion W, defined on the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with $\mathbb{P}$ representing the real-world probability measure, and $\mathbb{F}$ being the filtration generated by the process W.

- **(BR.2)** For any fixed maturity $T \leq T^*$, the *default-free instantaneous forward rate* $f(t,T)$ satisfies

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \tag{4.1}$$

  for the $\mathbb{F}$-adapted stochastic processes $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$ attaining values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

- **(BR.3)** For any $T \leq T^*$, the *defaultable instantaneous forward rate* $g_i(t,T)$, corresponding to the rating class $i = 1, \ldots, K$ satisfies, under $\mathbb{P}$

$$dg_i(t,T) = \alpha_i(t,T)dt + \sigma_i(t,T)dW_t, \tag{4.2}$$

  for the $\mathbb{F}$-adapted stochastic processes $\alpha_i(\cdot,T)$ and $\sigma_i(\cdot,T)$ attaining values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

- **(BR.4)** There exists an adapted $\mathbb{R}^d$-valued process γ such that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^{T^*}\gamma_s dW_s - \frac{1}{2}\int_0^{T^*}|\gamma_s|^2 ds\right)\right] = 1. \tag{4.3}$$

Remark 4.1.1 There is also an assumption regarding that the processes in (4.1) and (4.2) follow some technical conditions ensuring the existence of strong solutions. These can be found in [5] section 4.2.1.

Remark 4.1.2 Also, the volatilities in (4.1) and (4.2) may depend on the rates themselves, hence the notation $\sigma(t,T) = \sigma(f(t,T),t,T)$ and $\sigma_i(t,T) = \sigma(g_i(t,T),t,T)$.

Remark 4.1.3 The condition **(BR.4)** avoids arbitrage opportunities for all bonds with maturity $T \leq T^*$ in the default-free setting by assuming that the adapted $\mathbb{R}^d$-valued process γ exists.

We then fix the notation of the Euclidean inner product, denoted by:

$$\int_0^t \beta_s dW_s = \sum_{i=1}^d \int_0^t \beta_s^i dW_s^i, \qquad \alpha_t\beta_t = \sum_{i=1}^d \alpha_t^i\beta_t^i, \tag{4.4}$$

for processes α_t and β_t. Furthermore, for $x \in \mathbb{R}^d$, $|x|$ denotes the ordinary Euclidean norm.

In the the set of credit rating classes, we have the set $\mathcal{K} = \{1, 2, \ldots, K\}$, where K denotes the number of available rating classes and the default event.

For each rating class, except for the default event, we say that $\delta_i \in [0, 1)$ is the deterministic recovery rate. That is, the fraction of the face value of the bond the owed party receives in case of default. It is assumed that the face value of a bond equals $L = 1$.

Continuing, the price of a T-maturity default-free zero-coupon bond is defined as:

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right), \tag{4.5}$$

for $0 \leq t \leq T$. Similarly, given that a bond trades in rating class i, the bond price at time $0 \leq t \leq T$, is defined as:

$$D_i(t,T) = \exp\left(-\int_t^T g_i(t,u)du\right), \tag{4.6}$$

and is referred to as the conditional rating bond price, for $i \neq K$.

Based on the above mentioned processes $B(t,T)$ and $D_i(t,T)$, we have the following two lemmas that provide definitions for the dynamics of the bond price processes.

Lemma 4.1.4 *The default-free bond price dynamics for B(t,T) under $\mathbb{P}$ satisfy*

$$dB(t,T) = B(t,T)(a(t,T)dt + b(t,T)dW_t), \tag{4.7}$$

where

$$a(t,T) = \frac{1}{2}\left|\int_t^T \sigma(t,u)du\right|^2 - \int_t^T \alpha(t,u)du + f(t,t),$$

and

$$b(t,T) = -\int_t^T \sigma(t,u)du.$$

Lemma 4.1.5 *The conditional rating-based bond price dynamics for $D_i(t,T)$ under $\mathbb{P}$ satisfy*

$$dD_i(t,T) = D_i(t,T)(a_i(t,T)dt + b_i(t,T)dW_t), \tag{4.8}$$

where

$$a_i(t,T) = \frac{1}{2}\left|\int_t^T \sigma_i(t,u)du\right|^2 - \int_t^T \alpha_i(t,u)du + g_i(t,t),$$

and

$$b_i(t,T) = -\int_t^T \sigma_i(t,u)du.$$

It is in [3] also assumed a possibility of investing in a savings account. Define the short-term interest rate as $r(t) := f(t,t)$. This means that the risk-free savings account yields returns at this rate. It is denoted as

$$B(t) = \exp\left(\int_0^t r(s)ds\right). \tag{4.9}$$

The following lemma provide equivalent definitions as in 4.1.4 and 4.1.5, but in the discounted case:

Lemma 4.1.6 *Given $Z(t,T) := B(t)^{-1}B(t,T)$, we have*

$$dZ(t,T) = Z(t,T)\left(\left(\frac{1}{2}\left|\int_t^T \sigma(t,u)du\right|^2 - \int_t^T \alpha(t,u)du\right)dt + b(t,T)dW_t\right), \tag{4.10}$$

and

Lemma 4.1.7 *Given $Z_i(t,T) := B(t)^{-1}D_i(t,T)$, we have*

$$dZ_i(t,T) = Z_i(t,T)(\left(\frac{1}{2}\left|\int_t^T \sigma_i(t,u)du\right|^2 - \int_t^T \alpha_i(t,u)du + g_i(t,t) - r(t)\right)dt$$
$$+b_i(t,T)dW_t), \tag{4.11}$$

By assuming no arbitrage opportunities, and using the process from **(BR.4)**, for maturities $T \leq T^*$ we have:

$$\gamma_t \int_t^T \sigma(t,u)du = \frac{1}{2}\left|\int_t^T \sigma(t,u)du\right|^2 - \int_t^T \alpha(t,u)du. \tag{4.12}$$

It is also assumed that the drift condition of the risk-free forward rate $f(t,T)$ satisfies

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du - \gamma_t\sigma(t,T), \tag{4.13}$$

for $0 \leq t \leq T$.

Now, using γ_t as the market price of risk, a probability measure $\mathbb{P}^*$, the spot martingale measure, is defined $\mathbb{P}$-a.s. through applying Girsanov's theorem.

Let

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(\int_0^{T^*} \gamma_s dW_s - \frac{1}{2}\int_0^{T*} |\gamma_s|^2 ds\right). \tag{4.14}$$

Then, define the corresponding Brownian motion W_t^* under $\mathbb{P}^*$ by

$$W_t^* = W_t - \int_0^t \gamma_s ds, \tag{4.15}$$

for $t \in [0, T^*]$. Then, for any fixed maturity $T \leq T^*$, the discounted default-free bond price process satisfies, under $\mathbb{P}^*$, that

$$dZ(t,T) = Z(t,T)b(t,T)dW_t^*, \tag{4.16}$$

implying that it is a $\mathbb{P}^*$-martingale.

The conditional rating-based bond price dynamics satisfy for $i \in \mathcal{K} \setminus K$ and for all $0 \leq t \leq T$:

$$dZ_i(t,T) = Z_i(t,T)(\eta_i(t,T)dt + b_i(t,T)dW_t^*), \tag{4.17}$$

under $\mathbb{P}^*$, where

$$\eta_i(t,T) = a_i(t,T) - r(t) + b_i(t,T)\gamma_t. \tag{4.18}$$

Remark 4.1.8 Note that the processes $Z_i(t,T)$ do not correspond to prices of traded assets, so they do not need to be $\mathbb{P}^*$-martingales in order to exclude arbitrage opportunities.

4.1.1 Credit migration process

The model allows for migration between the credit rating classes, and thus, the credit rating migration process is introduced. The necessity of a framework where both discounted bond price processes are martingales leads to the introduction of an enlarged probability space, based on the underlying probability space $(\Omega, \mathbb{F}, \mathbb{P}^*)$. A new probability measure $\mathbb{Q}^*$ is constructed to allow for the credit rating migration process, with an appropriate infinitesimal generator. The following definition can be found in [3].

Definition 4.1.9 An $\mathbb{F}$-progressively measurable bounded, matrix-valued process Λ^* is called an $\mathbb{F}$-conditional infinitesimal generator for a $\mathcal{K}$-valued $\mathbb{F}$-conditional $\mathbb{G}$-Markov chain C under $\mathbb{Q}^*$ if for any function $h : \mathcal{K} \to \mathbb{R}$, the process M^h, given as

$$M_t^h = h(C_t) - h(C_0) - \int_0^t \Lambda_u^* h(C_u) du,$$

follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$ for all $t \in \mathbb{R}_+$.

Remark 4.1.10 Note that the infinitesimal generator may *almost* be considered as a transition rate matrix, which is commonly used in life insurance to model transitions between states of an insured, e.g., *active, disabled, dead.* And thus, it is commonly referred to as the *matrix of stochastic intensities* for C under $\mathbb{Q}^*$.

As previously mentioned, the credit migrations are modelled by a conditionally Markov chain as defined in Definition 2.4.1. This is because modelling credit migrations in terms of an $\mathbb{F}$-conditional $\mathbb{G}$-Markov chain is considered more appropriate than in terms of a $\mathbb{G}$-Markov chain, when considering information provided in $\mathbb{F}$.

We denote the credit migration process $C = (C_t^1, C_t^2)$, where C_t^1 is the current rating at time t, and C_t^2 is the previous rating before the current rating.

Remark 4.1.11 All stochastic processes maintain their names on the enlarged probability space, which is done for the sake of keeping the notation uniform. For example, W^*, the Brownian motion under $\mathbb{P}^*$, follows again a standard Brownian motion under $\mathbb{Q}^*$ with respect to the enlarged filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T^*]}$.

The construction is done using a *canonical filtration-enlargement argument* starting in $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbb{P}^*)$ and going to $(\tilde{\Omega}, \tilde{\mathbb{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T^*]}, \mathbb{Q}^*)$. Details can be found in [5], Section 4.2, and [3], Section 11.3.1.

The matrix of stochastic intensities of C^1 at time t is $\mathbb{F}$-conditional under $\mathbb{Q}^*$ and given by

$$\Lambda_t^* = \begin{pmatrix} \lambda_{1,1}^*(t) & \dots & \lambda_{1,K}^*(t) \\ \vdots & \dots & \vdots \\ \lambda_{K-1,1}^*(t) & \dots & \lambda_{K-1,K}^*(t) \\ 0 & \dots & 0 \end{pmatrix}, \tag{4.19}$$

where $\lambda_{i,j}^*(t)$ are $\mathbb{F}$-adapted, non-negative processes and $\mathbb{Q}^*$-a.s. integrable on every interval $[0,t]$ such that for $i = 1, \dots, K-1$

$$\lambda_{i,i}^*(t) = -\sum_{j\in\mathcal{K}\setminus\{i\}} \lambda_{i,j}^*(t). \tag{4.20}$$

Also, the fact that the last row of the matrix is zero follows from K being an absorbing state, as bankruptcy occurs in the defaulted firm.

The T-maturity defaultable bond price process $D_{C_t}(t,T)$ is then defined by

$$D_{C_t}(t,T) \coloneqq \mathbf{1}_{\{C_t^1\neq K\}} D_{C_t^1}(t,T) + \mathbf{1}_{\{C_t^1=K\}} \delta_{C_t^2} B(t,T), \tag{4.21}$$

which for all $0 \le t \le T$ is equivalent to

$$D_{C_t}(t,T) = \sum_{i=1}^{K-1} H_i(t) D_i(t,T) + \delta_i H_{i,K}(t) B(t,T), \tag{4.22}$$

where $H_i(t) \coloneqq \mathbf{1}_{s\geq 0:\, C_s^1=i}(t)$, for $i \in \mathcal{K}$ and $H_{i,j} \coloneqq \sum_{0\leq t} H_i(u-)H_j(u)$, for $i \neq j$. That is, $H_{i,j}(t)$ is the number of transitions from rating i to j in $[0,t]$ for $i \neq j$. It is stressed that $D_i(t,T)$ is not the process of a traded defaultable bond, but rather $D_{C_t}(t,T)$ is a traded bond.

The T-maturity discounted defaultable bond price process is

$$\hat{Z}(t,T) \coloneqq B(t)^{-1} D_{C_t}(t,T) = \mathbf{1}_{\{C_t^1\neq K\}} Z_{C_t^1}(t,T) + \mathbf{1}_{\{C_t^1=K\}} \delta_{C_t^2} Z(t,T) \tag{4.23}$$

for all $0 \le t \le T$. This means that the process switches its dynamics between the various $\hat{Z}_i(t,T)$ according to the states of the credit migration process C.

It is also assumed that the initial value is given by

$$\hat{Z}(0,T) := \sum_{i=1}^{K-1} H_i(0) Z_i(0,T), \tag{4.24}$$

meaning that at time $t = 0$, no bankruptcy has occurred, that is $C_0^1 \neq K$.

The bond default time $\tau \colon \tilde{\Omega} \to \mathbb{R}_+$, is defined as the $\tilde{\mathcal{F}}_t$-stopping time given by

$$\tau := \inf\{t \geq 0 \colon C_t^1 = K\}, \tag{4.25}$$

meaning that τ is the first point in time where the bond defaults.

4.2 Consistency Conditions

In the literature, consistency conditions are given as a set of equations ensuring absence of arbitrage. The conditions relates the class-specific price dynamics and the migrations intensities, and depending on the type, they either *imply* no arbitrage or are *equivalent* to no arbitrage.

The class-conditional bond prices and the migration process are in a credit migration bond model connected by a no-arbitrage argument. This is due to the fact that the defaultable price dynamics depend upon both these elements.

In [5], the author refers to the *weak* and the *strong* condition. We explore both conditions.

The following is included in [5]:

- Highlight and demonstration of possible problems and restrictions under the different arbitrage conditions,
- Exploration of explosions of the spreads,
- Construction of a non-explosive, admissible model, through a transformation of the spreads.

4.2.1 The weak consistency condition

The weak consistency condition is equivalent to absence of arbitrage. Equivalence means that it is both necessary and sufficient for a no-arbitrage argument to be made. The condition may potentially provide constraints on the model specifications, as it introduces a connection between model components. This may result in problems if the constraints provided are not met by the coefficients of the affected components. In fact, in [5], it is shown that the fundamental spread of the last credit rating class explodes in finite time with positive probability. However, as only the current rating class is considered *active*, last-class-explosions imply current-class-explosions, according to the author.

The following lemmas are required for the proof of martingale property of the discounted defaultable bond:

Lemma 4.2.1 *For all $i = 1, \ldots, K-1$ and $0 \leq t \leq T$ we have*

$$H_i(t) = H_i(0) - \sum_{j=1, j\neq i}^{K-1} H_{i,j}(t) + \sum_{j=1, j\neq i}^{K-1} Hj, i(t) - H_{i,K}(t). \tag{4.26}$$

Lemma 4.2.2 *For all $i, j = 1, \dots, K-1$ with $i \neq j$ and $0 \leq t \leq T$, define the adapted process,*

$$M_{i,j}(t) := H_{i,j}(t) - \int_0^t \lambda^*_{i,j}(u) H_i(u) du, \tag{4.27}$$

with respect to the filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T^*]}$. Then $M_{i,j}(t)$ is a $\mathbb{Q}^*$-martingale.

The weak consistency condition establishes a direct connection between the conditional credit state bond price processes, the defaultable bond and the migration intensities. It is stated as follows:

Definition 4.2.3 (The weak consistency condition) Assume that the entries of the transition rate matrix for all $0 \leq t \leq T, \Lambda^*$ on the set $\{C^1_t \neq K\}$ satisfy

$$\sum_{\substack{j=1 \\ j\neq C^1_t}} \lambda^*_{C^1_t,j}(t)(Z_j(t,T) - Z_{C^1_t}(t,T)) + \lambda_{C^1_t,K}(t)(\delta_{C^1_t} Z(t,T) - Z_{C^1_t}(t,T)) \tag{4.28}$$

$$= -\eta_{C^1_t}(t,T) Z_{C^1_t}(t,T).$$

The above condition is a necessary and sufficient condition for the (local) martingale property of the discounted defaultable bond $\hat{Z}(t,T)$ under $\mathbb{Q}^*$. This follows from equations (4.26) and (4.27).

Extended HJM no-arbitrage drift condition

In [5], a reformulation of (4.28) is proposed. This reformulation uses the instantaneous inter-rating forward spreads, defined as

$$s_{i,j}(t,T) := g_i(t,T) - g_j(t,T), \tag{4.29}$$

for $i, j = 1, \dots, K-1, i \neq j$, as well as the instantaneous fundamental spreads defined as

$$s^f_i(t,T) := g_i(t,T) - f(t,T), \tag{4.30}$$

for $i = 1, \dots, K-1$. Also, define

$$s_i(t,T) := g_i(t,T) - g_{i-1}(t,T), \tag{4.31}$$

for $i = 2, \dots, K-1$.

The reformulation is as follows:

Proposition 4.2.4 *Assume that (4.28) holds, and fix some maturity $T \leq T^*$. We have that (4.28) is equivalent to the following:*

For all $0 \leq t \leq T$ the drift condition

$$\alpha_{C^1_t}(t,T) = \sigma_{C^1_t}(t,T) \int_t^T \sigma_{C^1_t}(t,u) du - \gamma_t \sigma_{C^1_t}(t,T) \tag{4.32}$$

$$+ \sum_{\substack{j=1 \\ j\neq C^1_t}}^{K-1} \lambda^*_{C^1_t,j}(t) s_{C^1_t,j}(t,T) \exp\left(\int_t^T s_{C^1_t,j}(t,u) du\right)$$

$$+\lambda^*_{C_t^1,K}(t)\delta_{C_t^1} s^f_{C_t^1}(t,T)\exp\left(\int_t^T s^f_{C_t^1}(t,u)du\right),$$

of the current forward rate $g_{C_t^1}(t,T)$ *holds, together with that*

$$s^f_{C_t^1}(t,t) = \lambda^*_{C_t^1,K}(t)(1-\delta_{C_t^1}), \tag{4.33}$$

on the set $\{C_t^1 \neq K\}$.

Proof. See proof of Proposition 4.3.6 in [5]. ∎

From Proposition 4.2.4, it is derived that the drifts of all forward rates are determined by the active drift $\alpha_{C_t^1}(t,T)$ (4.32), the active defaultable intensity $\lambda^*_{C_t^1,K}$ (4.33) and the spread structure of drifts of forward rates $g_i(t,T)$. This means that given the volatilites $\sigma_i(t,T)$, the migration intensities $\lambda^*_{i,j}(t)$, the recovery rates δ_i, the spread structure, condition 4.33 and the no-arbitrage drift condition (4.32) the model is fully specified on $t \leq \tau$, where τ is the stopping time of default. Note that the drifts of the risky forward rates can be chosen freely after default.

The following example is identical to example 4.3.1 in [5]. We will from this point onwards denote the weak consistency condition (4.28) as Condition N.1.

Example 4.2.5 Consider a model where $K \geq 3$, with zero recovery rate, i.e., $\delta_i = 0$ for $i = 1,\dots,K-1$. Furthermore, assume $\lambda^*_{i,j}(t) = \lambda^*$ for all $i,j = 1,\dots,K-1$ with $i \neq j$. Let a 1-dimensional Brownian motion drive all the forward rates and assume constant volatilities, such that $\sigma(t,T) = \sigma$ and $\sigma_i(t,T) = \sigma_1$ for all $i = 1,\dots,K-1$. Then, we have:

$$df(t,T) = \alpha(t,T)dt + \sigma dW_t^*, \tag{4.34}$$

$$dg_i(t,T) = \alpha_i(t,T)dt + \sigma_1 dW_t^*, \qquad i = 1,\dots,K-1,$$

$$ds_i(t,T) = (\alpha_i(t,T) - \alpha_{i-1}(t,T))dt, \qquad i = 2,\dots,K-1.$$

Note that inter-rating spreads become constant if $\alpha_i(t,T) = \alpha_j(t,T)$ for all $i,j = 1,\dots,K-1$.

To specify constant inter-rating spreads, we require that for all $0 \leq t \leq T$, one has:

$$s_i(t,T) = c, \quad \text{for all } i = 2,\dots,K-1. \tag{4.35}$$

Then, by the no-arbitrage drift consistency condition N.1, we have from Proposition 4.2.4, for almost all $0 \leq t \leq T$ the drift condition:

$$\alpha_{C_t^1}(t,T) = \sigma_1^2(T-t) - \gamma_t\sigma_1 + (K-2)\lambda^* \cdot c\exp(c(T-t)), \tag{4.36}$$

$$\alpha(t,T) = \sigma^2(T-t) - \gamma_t\sigma, \tag{4.37}$$

of the current forward rate $g_{C_t^1}(t,T)$ on the set $\{C_t^1 \neq K\}$ and of the risk-free forward rate $f(t,T)$, respectively.

So, by setting all the drifts equal to the active one, that is for almost all $t \leq T$

$$\alpha_i(t,T) = \alpha_{C_t^1}(t,T), \quad \text{for } t < \tau, \tag{4.38}$$

$$\alpha_i(t,T) = 0, \quad \text{for } t \geq \tau, \tag{4.39}$$

for all $i = 1, \dots, K-1$, where τ is the time of default. Then, one can have a model with inter-rating spreads under the no-arbitrage consistency condition N.1.

By making this choice of the drift $\alpha_i(t,T)$, then by (4.38), the forward rates $g_i(t,T)$ are given by

$$\begin{aligned} g_i(t,T) = g_i(0,T) + \sigma_1^2(tT - \frac{1}{2}t^2) - \sigma_1 \int_0^t \gamma_s ds + \lambda^* \exp(cT)[1 - \exp(-ct)] \\ + \sigma_1 W_t^*, \quad i = 1, 2, \end{aligned} \tag{4.40}$$

for $t < \tau$.

It is also quite clear that one can now derive and determine the fundamental spreads $s_i^f(t,T) = g_i(t,T) - f(t,T)$ for $i = 1, \dots, K-1$ as well as the defaultable intensity parameters $\lambda^*_{C_t^1,K}(t)$ from (4.33). Finally, specifying $f(0,T)$ and $g_i(0,T)$ for some $i \in \{1, \dots, K-1\}$, then all the other initial values are derived from (4.35).

Explosions under the weak consistency

It can be shown that the current fundamental spread, $s^f_{C_t^1}(t,T)$, explodes in finite time with positive probability on the set $C_t^1 \neq K$ for the non-zero recovery case, given that the weak consistency condition holds. In [5], this is done by including an economical assumption on the forward rates, reflecting that the price of a bond must decrease if the default-risk increases. It is referred to as the *ordering condition*, and is included below:

Lemma 4.2.6 *Assume:*

$$f(t,T) < g_1(t,T) < \dots < g_{K-2}(t,T) < g_{K-1}(t,T). \tag{4.41}$$

The above condition is often assumed in the literature, but it is not necessary for the model framework and there is no direct link to no-arbitrage. Furthermore, another change of measure is performed. The physical measure $\mathbb{Q}$ on the extended probability space is constructed such that it entails no market price of credit risk when changing from $\mathbb{Q}$ to $\mathbb{Q}^*$.

Remark 4.2.7 The probability measures $\mathbb{P}, \mathbb{P}^*, \mathbb{Q}^*$ and $\mathbb{Q}$ are all equivalent measures.

The measure $\mathbb{Q}$ is constructed as:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{Q}^*}\right|_{\tilde{\mathcal{F}}_t} = L_t, \tag{4.42}$$

where

$$dL_t = -Lt\gamma_t dW_t^*. \tag{4.43}$$

This follows from equation (4.27), and by defining the $\mathbb{Q}^*$-local martingale M as:

$$dM_t = \sum_{i \neq j} (\varphi_{i,j}(t) - 1) dM_{i,j}(t), \tag{4.44}$$

for an arbitrary, non-negative, $\mathbb{F}$-predictable process $\varphi_{i,j}$, satisfying for all $i \neq j$:

$$\mathbb{Q}^* \left(\int_0^{T^*} \varphi_{i,j}(t) \lambda^*_{i,j}(t) dt < \infty \right) = 1. \tag{4.45}$$

Furthermore, set the $\mathbb{Q}^*$-local positive martingale L to be:

$$dL_t = -L_t \gamma_t dW_t^* + L_{t^-} dM_t, \tag{4.46}$$

and define $\mathbb{Q}$ as above.

The following dynamics of the fundamental spread $s^f_{C^1_t}(t,T)$ is obtained:

Corollary 4.2.8 *For any fixed maturity $T \leq T^*$ and under the consistency condition N.1, we have the dynamics:*

$$\begin{aligned} ds^f_{C^1_t}(t,T) = \Bigg\{ & \sigma_{C^1_t}(t,T) \int_t^T \sigma_{C^1_t}(t,u) du - \sigma(t,T) \int_t^T \sigma(t,u) du \\ & + \sum_{j=1, j \neq C^1_t}^{K-1} \lambda^*_{C^1_t,j}(t) s_{C^1_t,j}(t,T) \exp\left(\int_t^T s_{C^1_t,j}(t,u) du \right) \\ & + \lambda^*_{C^1_t,K}(t) \delta_{C^1_t} s^f_{C^1_t}(t,T) \exp\left(\int_t^T s^f_{C^1_t,j}(t,u) du \right) \Bigg\} dt \\ & + (\sigma_{C^1_t}(t,T) - \sigma(t,T)) dW_t, \end{aligned} \tag{4.47}$$

for $t < \tau_1$, where $\tau_1 := \inf\{t > 0 : C^1_t \neq C^1_0\}$ is the first time where a jump in another class occurs.

Proof. See proof of Corollary 4.3.9 in [5]. ∎

To present the main result regarding the weak consistency condition, the following notation is fixed:
For $i = 1, \ldots, K-1$:

$$N_i(t,T) := \tilde{N}_i(t,T) + \sum_{j=1, j \neq i}^{K-1} \int_0^t \lambda^*_{i,j}(s) s_{i,j}(s,T) \exp\left(\int_s^T s_{i,j}(s,u) du \right) ds, \tag{4.48}$$

where

$$\begin{aligned} \tilde{N}_i(t,T) := s^f_i(0,T) & + \int_0^t \int_s^T \{\sigma_i(s,T)\sigma_i(s,u) - \sigma(s,T)\sigma(s,u)\} du ds \\ & + \int_0^t (\sigma_i(s,T) - \sigma(s,T)) dW_s, \end{aligned} \tag{4.49}$$

and for $\lambda^*_{i,K}(t)\delta_i > 0$ define the sets

- $A_i^{R,S,a} := \Big\{\omega : \tilde{N}_i(t,T) \geq \frac{2a}{(a-S)^2}\frac{(a-R^2)}{(a-RT)^2}\frac{1}{\hat{\lambda}^*_K(t)\delta_i}$ *for all* $R \leq t \leq S\Big\}$,
- $B_i^S := \{\omega : C_t^1 = i$ *for all* $0 \leq t \leq S\}$,
- $\hat{A}_i^{R,S,a} := A_i^{R,S,a} \cap B_i^S$,
- $\hat{K}^{R,S} := \{(t,T) : R \leq t \leq S$ and $t \leq T \leq T^*\}$,

where a, R, S are positive constants with $a > S^2$, $a > RT$, $R \leq S \leq T$ and $\hat{\lambda}^*_{i,K}(t) := \inf_\omega \lambda^*_{i,K}$.

Theorem 4.2.9 *Assume Condition N.1 and the ordering condition. Also, assume* $\delta_{K-1} > 0$ *and that* $\lambda^*_{K-1,K}(t)$ *is uniformly bounded from below in* ω*. Furthermore, for all* $T \leq T^*$ *it holds* $\mathbb{Q}(A_\infty) > 0$ *with*

$$A_\infty = \hat{A}_{K-1}^{R,S,a}, \;\; R = \frac{T}{4}, \;\; S = \frac{T}{2}, \;\; a = \frac{T^2}{2}.$$

Then, if a solution to (4.47) exists, we have for all $T \leq T^*$ *that* $\lim_{t\to\frac{T}{2}} s^f_{K-1}(t,T) = +\infty$ *with positive probability under* $\mathbb{Q}$ *on* B^S_{K-1}*. In particular,*

$$\lim_{t\to\frac{T}{2}} s^f_{C_t^1}(t,T) = +\infty, \tag{4.50}$$

with positive probability under $\mathbb{Q}$ *prior to default.*

Proof. See proof of Theorem 4.3.14 in [5]. ∎

Note that since the probability measure $\mathbb{Q}, \mathbb{Q}^*$ and $\mathbb{P}^*$ are all equivalent measures, the corresponding statements of the above theorem holds under any of these measures. For more details and an example when $K = 2$, see [5] Remark 4.3.16 and onwards. An important feature of this example is that the ordering condition is omitted, showing that the explosions are mostly due to the consistency condition.

A non-exploding, non-zero recovery model

The author of [5] presents a transformation from a zero-recovery model to a model with non-zero recovery rates attaining the non-exploding property. This is performed by a transformation based on no-arbitrage arguments.

By assuming a zero-recovery model ($\delta_i = 0$ for all $i = 1, \ldots, K-1$), and defining the zero-recovery fundamental and non-zero-recovery fundamental spreads by $s^{f,0}_{C_t^1}(t,T)$ and $s^{f,\delta}_{C_t^1}(t,T)$, respectively, one finds the relations:

$$\int_t^T s^{f,\delta}_{C_t^1}(t,u)du = -\log\left(\frac{\hat{Z}(t,T)}{Z(t,T)}\right) = -\log\left(\frac{\hat{D}(t,T)}{B(t,T)}\right), \tag{4.51}$$

and

$$\int_t^T s^{f,0}_{C_t^1}(t,u)du = -\log\left(\frac{\hat{Z}^0(t,T)}{Z^0(t,T)}\right) = -\log\left(\frac{\hat{D}^0(t,T)}{B(t,T)}\right), \tag{4.52}$$

where

$$\hat{D}^0(t,T) = B_t \hat{Z}^0(t,T), \tag{4.53}$$

by recalling the discounted defaultable bond price process $\hat{Z}(t,T)$ and denoting

$$\hat{Z}^0(t,T) := \mathbf{1}_{\{C_t^1 \neq K\}} Z_{C_t^1}(t,T). \tag{4.54}$$

Then, for any fixed maturity $T \leq T^*$, the following spread relation is obtained on the set $\{C_t^1 \neq K\}$ where $\mathbb{E}_T^*$ denotes the expectation with respect to the terminal measure $\mathbb{Q}_T^*$ using $B(t,T)$ as the numéraire:

$$s^{f,\delta}_{C_t^1}(t,T) = \frac{\exp\left(-\int_t^T s^{f,0}_{C_t^1}(t,u)du\right) s^{f,0}_{C_t^1}(t,u) - \frac{\partial}{\partial T}\mathbb{E}_T^*[\delta_{C_T^2}\mathbf{1}_{\{\tau \leq T\}}|\tilde{\mathcal{F}}_t]}{\exp\left(-\int_t^T s^{f,0}_{C_t^1}(t,u)du\right) + \mathbb{E}_T^*[\delta_{C_T^2}\mathbf{1}_{\{\tau \leq T\}}|\tilde{\mathcal{F}}_t]}. \tag{4.55}$$

Then, for constant recoveries, $\delta_i = \delta$ for all $i = 1, \ldots, K-1$, for some constant $\delta \in [0,1)$, the following holds:

$$s^{f,\delta}_{C_t^1}(t,T) = \frac{(1-\delta)\exp\left(-\int_t^T s^{f,0}_{C_t^1}(t,u)du\right)}{(1-\delta)\exp\left(-\int_t^T s^{f,0}_{C_t^1}(t,u)du\right) + \delta} s^{f,0}_{C_t^1}(t,T). \tag{4.56}$$

The proof of both results as well as the justification of the transformation can be found in [5] Proposition 4.3.22., specifically equation 4.3.75 (see (4.59) below) in accordance with Remark 4.3.23.

The essence of the proof of the non-exploding property of the resulting model is captured below. It stems from the choice of numéraire and change of measure from $\mathbb{Q}^*$ to the terminal measure which together with the non-zero recovery defaultable bond price $\hat{D}(t,T)$ yields the relation:

$$\exp\left(-\int_t^T s^{f,\delta}_{C_t^1}(t,u)du\right) = \exp\left(-\int_t^T s^{f,0}_{C_t^1}(t,u)du\right) + \mathbb{E}_T^*[\delta_{C_T^2}\mathbf{1}_{\{\tau \leq T\}}] \tag{4.57}$$

Remark 4.2.10 Let

$$x = \int_t^T s^{f,0}_{C_t^1}(t,u)du \text{ and } y = \int_t^T s^{f,\delta}_{C_t^1}(t,u)du. \tag{4.58}$$

Then,

$$y = -\log\left(\exp(-x) + \mathbb{E}_T^*[\delta_{C_T^2}\mathbf{1}_{\{\tau \leq T\}}]\right). \tag{4.59}$$

Now, as

$$\lim_{x\to\infty} y = -\log\left(\mathbb{E}_T^*[\delta_{C_T^2}\mathbf{1}_{\{\tau \leq T\}}]\right) \tag{4.60}$$

and as (4.59) is increasing in x, we have that

$$\int_t^T s^{f,\delta}_{C_t^1}(t,u)du \tag{4.61}$$

cannot explode, which proves the non-exploding property of

$$s^{f,\delta}_{C_t^1}(t,T). \tag{4.62}$$

Note that for the case of constant recoveries, $\lim_{x\to\infty} y = -\log(\delta)$.

Now, following the proof of the above results, a condition on the relationship between the integrated forward rates is obtained. That is, a requirement of

$$\int_t^T g_{C_t^1}(t,u)du > \int_t^T f(t,u)du, \tag{4.63}$$

which follows from the fact that

$$\frac{\hat{D}^0(t,T)}{B(t,T)} = \exp\left(-\int_t^T g_i(t,u)du - f(t,u)du\right), \tag{4.64}$$

on $\{C_t^1 = i\}$ and that $\frac{\hat{D}^0(t,T)}{B(t,T)} = \mathbb{Q}_T^*(\tau > T|\tilde{\mathcal{F}}_t)$, which the authors states as seemingly necessary. Furthermore, the integral condition implies that

$$g_{C_t^1}(t,T) > f(t,T), \tag{4.65}$$

resembling the ordering condition from before. Furthermore, the transformation provides a class of admissible non-zero recovery models where the volatility has the feature of *vanishing at zero* when the spread goes to infinity, which kills the explosion.

A final mentioned is then dedicated to the fact that the defaultable bond price with constant recovery is a convex combination of the zero-recovery bond and the default-free bond price. That is, for the special case of constant recovery, $\delta_i = \delta$, one has

$$\hat{D}(t,T) = (1-\delta)\hat{D}^0(t,T) + \delta B(t,T). \tag{4.66}$$

Remark 4.2.11 In fact, a similar result can be found in [3], Equation 13.46.

4.2.2 The strong consistency condition

In contrast to N.1 (4.28), the no-arbitrage drift condition under the strong consistency (4.67, henceforth N.2) requires that an extended HJM no-arbitrage drift condition holds for the current forward rate, as well as for all other forward rates, i.e., that they are all *active*. Despite this, similar explosions are obtained as shown for N.1. Furthermore, it is also disclosed in [5], that, in a multiple issuer model, the conditions N.1 and N.2 align. The strong consistency condition is defined as:

Definition 4.2.12 Assume that the entries of Λ^* satisfy for all $0 \leq t \leq T$ and for all $i = 1, \ldots, K-1$:

$$\sum_{j=1, j\neq i}^{K-1} \lambda_{i,j}^*(t)(Z_j(t,T) - Z_i(t,T)) + \lambda_{i,K}^*(t)(\delta_i Z(t,T) - Z_i(t,T)) \tag{4.67}$$

$$= -\eta_i(t,T)Z_i(t,T).$$

Remark 4.2.13 It is visible that N.2 is a condition over each rating class at any time $t \in [0,T]$. This means that condition N.2 implies N.1. Also, if N.2 holds, then the discounted defaultable bond $\hat{Z}(\cdot,T)$ is a local martingale under $\mathbb{Q}^*$ (see [5], Corollary 4.4.1 and Theorem 4.3.5).

Remark 4.2.14 Due to the requirement of all forward rates being active, this condition allows for different issuers of possibly different classes, making it more suitable in a multiple-issuer migration model.

Remark 4.2.15 For positive recovery rates, explosions under N.2 is consequential of explosions under N.1.

No-arbitrage drift condition on all forward rates

Recall the spreads from earlier, that is:

1. the instantaneous forward inter-rating spread
$$s_{i,j}(t,T) = g_i(t,T) - g_j(t,T),$$
2. the instantaneous fundamental spread
$$s_i^f(t,T) = g_i(t,T) - f(t,T).$$

The no-arbitrage drift condition on all forward rates, following from 4.67 is stated as follows

Proposition 4.2.16 *Assume that N.2 holds and fix maturity $T \leq T^*$. Then, N.2 is equivalent to that the drift condition:*

$$\alpha_i(t,T) = \sigma_i(t,T) \int_t^T \sigma_i(t,u)du - \gamma_t \sigma_i(t,T) \tag{4.68}$$
$$+ \sum_{j=1, j\neq i}^{K-1} \lambda_{i,j}^*(t) s_{i,j}(t,T) \exp\left(\int_t^T s_{i,j}(t,u)du \right)$$
$$+\lambda_{i,K}^*(t)\delta_i s_i^f(t,T) \exp\left(\int_t^T s_i^f(t,u)du \right),$$

holds for the forward rate $g_i(t,T)$, for all $0 \leq t \leq T$ and $i = 1,\ldots,K-1$. Also, the condition

$$s_i^f(t,t) = \lambda_{i,K}^*(t)(1-\delta_i), \tag{4.69}$$

must hold for all $i = 1,\ldots,K-1$.

Proof. See [5], proof of Proposition 4.4.2 and Proposition 4.3.6. ■

Explosions under the strong consistency condition

By making the standing assumption of zero-recovery, as well as recalling that N.2 implies N.1, the author of [5] makes sure that the spreads do not explode. Still, explosions in N.2 are obtained as a consequence of explosions in N.1, due to the previously mentioned implication.

In [5], Section 4.4.2, it is shown that by deriving the no-arbitrage inter-rating spread dynamics, the inter-rating spread $s_{K-1,1}(t,T)$ explodes in finite time with positive probability for a general K, in the zero-recovery case, under N.2. This results from the requirements of all rating classes being active, so that both $g_i(t,T)$ and $g_j(t,T)$ must satisfy the drift condition.

It should also be mentioned that the aforementioned ordering condition is also part of the explosion proof. However, for the special case of $K = 3$, explosion occurs even without assuming the ordering condition of forward rates. This is shown in Corollary 4.4.9 in [5].

The following corollary provides the dynamics of the inter-rating spread under the equivalent (physical) measure $\mathbb{Q}$.

Corollary 4.2.17 *Assume zero-recovery for all classes. For any fixed maturity $T \leq T^*$ and under N.2, we have the dynamics*

$$\begin{aligned} ds_{i,j}(t,T) = \Bigg\{ & \sigma_i(t,T) \int_t^T \sigma_i(t,u)du - \sigma_j(t,T) \int_t^T \sigma_j(t,u)du \\ & + \sum_{l=1, l \neq i,j}^{K-1} \Bigg[\lambda^*_{i,l}(t) s_{i,l}(t,T) \exp\left(\int_t^T s_{i,l}(t,u)du \right) \\ & - \lambda^*_{j,l}(t) s_{j,l}(t,T) \exp\left(\int_t^T s_{j,l}(t,u)du \right) \Bigg] \\ & + \lambda^*_{i,j}(t) s_{i,j}(t,T) \exp\left(\int_t^T s_{i,j}(t,u)du \right) \\ & + \lambda^*_{j,i}(t) s_{i,j}(t,T) \exp\left(- \int_t^T s_{i,j}(t,u)du \right) \Bigg\} dt \\ & + (\sigma_i(t,T) - \sigma_j(t,T)) dW_t, \end{aligned} \tag{4.70}$$

for $0 \leq t \leq T$, for the inter-rating spread $s_{i,j}(t,T)$ for $i,j = 1, \dots K-1$ with $i \neq j$.

Proof. See [5] Corollary 4.4.3. ■

It is shown that the inter-rating spread explodes in finite time with positive probability. It is very similar to that under N.1, so recall the notation of $N_{i,j}(t,T), \tilde{N}_{i,j}(t,T)$ and $A^{R,S,a}_{i,j}$ from before.

Then, one of the main results from [5], regarding the inter-rating spreads is as follows:

Theorem 4.2.18 *Assume the strong consistency condition and the ordering condition. Let $K \geq 3$, $\delta_i = 0$ for all $i = 1, \dots, K-1$ and let*

- *$\lambda^*_{K-1,1}(t)$ be positive, continuous and uniformly bounded from below in ω,*
- *$\mathbb{Q}(A_\infty) > 0$ for all $T \leq T^*$, with*

$$A_\infty = A^{R,S,a}_{K-1,1}, \; R = \frac{T}{4}, \; S = \frac{T}{2}, \; a = \frac{T^2}{2}. \tag{4.71}$$

If a solution to 4.70 exists, then for all $T \leq T^*$ we have

$$\lim_{t \to \frac{T}{2}} s_{K-1,1}(t,T) = +\infty \tag{4.72}$$

with positive probability under $\mathbb{Q}$.

Proof. The proof can be found in [5]. Theorem 4.3.14 and Theorem 4.4.4. ■

Remark 4.2.19 Note that since the measures $\mathbb{Q}, \mathbb{Q}^*$ and $\mathbb{P}^*$ are all equivalent measures, the corresponding statements of Theorem 4.2.18 above hold under any of these measures.

It is also stressed by the author of [5] that specification of the volatilities of the forward rates is essential in achieving a model without explosions. Furthermore, under N.1, $\lambda^*_{K-1,K}(t)$ must satisfy $s^f_{C^1_t}(t,t) = \lambda^*_{C^1_t,K}(t)(1-\delta_{C^1_t})$ on the set $\{C^1_t = K-1\}$, whereas under N.2, it is a free parameter.

The author then proceeds to show the construction of an example where the inter-rating spreads explodes prior to default. This is done under assumptions of deterministic intensity parameters that are continuous functions, as well as volatilities specified such that $\mathbb{Q}\left(A^{R,S,a}_{K-1,K}\right) > 0$ for all maturities. An corollary for the special case of $K = 3$, showing explosions without assuming the ordering condition, is also included. See Corollary 4.4.9 in [5]. This corollary is applied in Example 4.4.1 in [**5**] in a zero-recovery setup.

A multiple-issuer migration model

The author of [5] proceeds with the construction of a model, based on the Markov chain C from the previous sections, that allows for multiple raters. It makes use of the fact that N.2 is more suitable in a model with multiple issuers, because it requires that all forward rates are active.

The construction is based on zero-recovery, with M different issuers such that $M > K-1$. That is, it is based on M independent copies of a Markov chain C^m_t for $m = 1, \ldots, M$, with $\lambda^m_{i,j}(t) = \lambda_{i,j}(t)$, i.e., the same migration intensity parameters. The result is the credit migration process $C^m = (C^{m,1}, C^{m,2})$, where $C^{m,1}_t$ is the current rating of issuer m at time t and $C^{m,2}_t$ the previous rating, before the current rating. It is summarised below:

Definition 4.2.20 For all rating classes $i \in \{1, \ldots, K-1\}$ there exists some issuer $m_i \in \{1, \ldots, M\}$ such that $C^{m_i,1}_t = i$.

Define, for each issuer $m \in \{1, \ldots, m\}$, defaultable bonds

$$\hat{D}_{C^m_t}(t,T) := D_{C^{m,1}_t}(t,T)\mathbf{1}_{\{C^{m,1}_t \neq K\}} = \sum_{i=1}^{K-1} \mathbf{1}_{\{C^{m,1}_t = i\}} D_i(t,T). \tag{4.73}$$

It then follows that the model attains M discounted bonds

$$\hat{Z}_{C^m_t}(t,T) := Z_{C^{m,1}_t}(t,T)\mathbf{1}_{\{C^{m,1}_t \neq K\}} = \sum_{i=1}^{K-1} \mathbf{1}_{\{C^{m,1}_t = i\}} Z_i(t,T), \tag{4.74}$$

that are tradeable assets, which under N.1 are local martingales for each m. For more details, see Section 4.4.3 and Theorem 4.3.5 in [5]. Furthermore, another consistency condition is introduced, N.3, that entertains the property of being equivalent to N.2 (4.67) in a model with multiple issuers. It is given as:

Definition 4.2.21 Assume that the entries of $\Lambda^{*,m} (= \Lambda^*)$ satisfy for almost all $0 \leq t \leq T$ on the set $\{C_t^{m,1} \neq K\}$:

$$\eta_{C_t^{m,1}}(t,T) = \lambda^*_{C_t^{m,1},K}(t) + \sum_{\substack{j=1 \\ j \neq C_t^{m,1}}}^{K-1} \lambda^*_{C_t^{m,1},j}(t) \left(1 - \exp\left\{\int_t^T s_{C_t^{m,1},j}(t,u)du\right\}\right), \tag{4.75}$$

for each $m = 1, \ldots, M$.

The final result of this section is that if N.3 holds, the discounted defaultable bonds are local martingales under $\mathbb{Q}^*$ for each $m = 1, \ldots, M$. The equivalency of N.3 and N.2 is stated as a consequence of the condition that if at each time t and for each class $i = 1, \ldots, K-1$, there is at least one issuer m in the class i, then N.3 *becomes* N.2.

4.3 Dynamics of the Equal Volatility Specification

This section deals with the previously discovered fact (See Remark 4.3.16 in [5]) that to avoid explosive model specifications, the volatility structure of the forward rates is highly important. The author considers the special case of equal volatility under the strong consistency condition (4.67) and describes the bounded properties of the initial spread structure. That is, the author provides closed-form solutions in the deterministic case[1] and derives the requirement of a *vanishing at zero* property, i.e., the initial spread value tends to zero as T tends to infinity.

Furthermore, the case of the non-vanishing, equal volatility setup is explored, and it is found in that explosions occur if this is the case.

4.3.1 Closed-form solutions for the spreads

It is stated that despite the explosions shown previously under N.2, meaningful, consistent simple models can still be specified. Furthermore, in some special, deterministic cases a closed-form solution of the inter-rating spread exists. That is, by considering N.2, the following:

$$K = 3, \quad \delta_1 = \delta_2 = 0, \quad \sigma_1(t,T) = \sigma_2(t,T),$$

as well as recalling that the above is a setting with only the deterministic inter-rating spread $s_2(t,T)$, the author provides the dynamics:

$$\begin{aligned} ds_2(t,T) = {} & \lambda^*_{2,1}(t) s_2(t,T) \exp\left(\int_t^T s_2(t,u)du\right) dt \\ & + \lambda^*_{1,2} s_2(t,T) \exp\left(-\int_t^T s_2(t,u)du\right) dt, \end{aligned} \tag{4.76}$$

for $0 \leq t \leq T$.

[1] That is, no Brownian motion is involved, so the spread dynamics is an ordinary differential equation (ODE).

The closed-form solutions of (4.76) are then presented for some different cases. With the above model specification, the author considers non-negative, constant, real-valued migration intensities, i.e., $\lambda^*_{2,1} = \lambda_2$ and $\lambda^*_{1,2} = \lambda_1$.

Then, for $\lambda_1 = 0, \lambda_2 > 0$ the solution is given by:

$$s_2(t,T) = \lambda_2 \frac{c(1+c)}{1+c-\exp(-c\lambda_2(T-t))}, \tag{4.77}$$

for $0 \leq t \leq T$ and a constant $c \in [-1,\infty)$. It is stressed that for each non-negative initial value $s_2(0,T)$ there exists some $c \in [-1,\infty)$ such that $s_2(t,T)$ from (4.77) solves (4.76).

Secondly, consider $\lambda_1 > 0, \lambda_2 = 0$, the solution is then given by

$$s_2(t,T) = \lambda_1 \frac{c(c-1)}{c-1+\exp(c\lambda_1(T-t))}, \tag{4.78}$$

for $0 \leq t \leq T$ and for $c \in [1,\infty)$.

Proof. See proof of Proposition 4.5.2 in [5]. ■

In the proof of (4.77) and (4.78), initial and final conditions are given by

$$s_2(0,T) = \frac{c(c-1)}{c-1+\exp(cT)}, \quad s_2(T,T) = c-1, \tag{4.79}$$

and the author specifies that the initial condition converges to zero as maturity converges to infinity, and explains that models without this property have explosive dynamics and are thus problematic. That is, it is stated that the spread $s_2(t,T)$ explodes in finite time with positive probability prior to default, if the initial condition is bounded from below uniformly. This is explored in Section 4.5.2 in [5] and is summarised in the following section (4.3.2).

Another example is then presented, for the case when $K = 2$ and by assuming $C^1_t = 1$ as well as looking at a result from the transformation from zero recovery to non-zero recovery (4.56), for the constant zero-recovery fundamental spreads $s^{f,0}_1(t,T) = 1$, the following closed-form solution is presented:

$$s^{f,\delta}_1(t,T) = \frac{(1-\delta_1)\exp(-(T-t))}{(1-\delta_1)\exp(-(T-t))+\delta_1}, \tag{4.80}$$

for the non-zero recovery fundamental spread, solving:

$$ds^{f,\delta}_1(t,T) = \delta_1 s^{f,\delta}_1(t,T) \exp\left(\int_t^T s^{f,\delta}_1(t,u)du\right), \tag{4.81}$$

for $\delta_1 \in (0,1]$. In fact, for $K = 3$ and for $\lambda_1 \in (0,1], \lambda_2 = 0$, by choosing $c = \frac{1}{\lambda_1}$, Christodoulou presents the following solution:

$$s_2(t,T) = \frac{1}{1+\frac{\exp((T-t))}{\frac{1}{\lambda_1}-1}}, \tag{4.82}$$

and concludes that for $\lambda_1 = \delta_1$ the solutions (4.80) and (4.82) are equal. It is also emphasised that for $\delta_1 = \lambda_1 = 1$, one has the zero solution, and that either $s_2(0,T)$ or λ_1 is a free parameter.

Finally, the author discloses that that by setting the two intensity parameters equal to zero in (4.76), closed-form solutions are on the form $s_2(t,T) = k(-(T-t))$, for a function $k : [-T, 0] \to \mathbb{R}_{\geq 0}$. It is also mentioned that this occurs if both parameters are positive. See Proposition 4.5.5 in [5] and it's proof for more details.

4.3.2 The vanishing property on the initial spread value

This section deals with the aforementioned explosions if the vanishing property of the initial spread value is not satisfied, that is, if the initial value $s_2(0,T)$ does not go to zero as $T \to \infty$.

The statement is as follows:

Theorem 4.3.1 *Assume N.2 as well as the ordering condition. Furthermore, assume that $K = 3$ and $\delta_i = 0$ for all $i = 1, \ldots, K-1$. Moreover, assume that $\sigma_{K-1}(t,T) = \sigma_1(t,T)$ and $\lambda^*_{K-1,K}(t)$ is continuous. Assume also that $s_{K-1,1}(0,T) \geq M_1$ and $\lambda^*_{K-1,K}(t) \geq M_2$ where M_1, M_2 are positive constants independent of T and t. Then, there exists some $T^0 \in (0,\infty)$ so that if $T^* > T^0$ then $\lim_{t \to \frac{T}{2}} s_{K-1,1}(t,T) = +\infty$ for all $T^0 < T < T^*$ with positive probability under $\mathbb{Q}$ and any other equivalent measure prior to default.*

Proof. See proof of Theorem 4.5.10, as well as Lemma 4.5.9 in [5]. ■

The author makes note of the assumption that the initial condition needs to be bounded from below by a positive constant, and mentions that models defined in this manner are not admissible. It is also stressed that the ordering condition is not required for the case of $K = 3$. Furthermore, it is stated that it is possible to prove that the non-zero recovery, deterministic fundamental spread will explode in finite time with positive probability prior to default, under similar conditions as in Theorem 4.3.1.

The discussion is followed by an extension of the above theorem, to all maturities $0 < T < T^0$ that are not originally covered. It is stated as:

Lemma 4.3.2 *Assume condition N.2. Furthermore, assume that $K \geq 3$ and $\delta_i = 0$ for all $i = 1, \ldots, K-1$. Then, for all $\hat{a} > 0$ with $\hat{t} := \hat{a}t$, $\hat{T} := \hat{a}T$, there exists a spread $\hat{s}_{i,j}(\hat{t},\hat{T})$ with:*

$$\hat{s}_{i,j}(\hat{t},\hat{T}) = \frac{1}{\hat{a}} s_{i,j}(t,T), \tag{4.83}$$

which satisfies

$$d\hat{s}_{i,j}(\hat{t},\hat{T}) = d\frac{1}{\hat{a}} s_{i,j}(t,T), \tag{4.84}$$

for $0 \leq t \leq T$ and $i, j = 1, \ldots, K-1$ with $i \neq j$.

Proof. See proof of Lemma 4.5.14 in [5]. ■

The section is then concluded by further extending Theorem 4.3.1 to show that if one can show explosions for one maturity $T^0 \in (0, T^*)$, then explosions occur for all maturities $T \in (0, T^*)$. Formally:

Corollary 4.3.3 *Let the assumptions of Theorem 4.3.1 hold true. Then, if there exists some $T^0 \in (0, T^*)$ with*

$$\lim_{t \to \frac{T^0}{2}} s_{K-1,1}(t,T) = +\infty,$$

with positive probability under $\mathbb{Q}$, then for all $0 < T < T^*$ we have

$$\lim_{t \to \frac{T}{2}} s_{K-1,1}(t,T) = +\infty,$$

with positive probability under $\mathbb{Q}$ and any other equivalent measure, prior to default.

Proof. See proof of Corollary 4.5.15 in [5]. ■

4.4 Proportional Volatility Spread Models

Proportional volatility denotes the cases where the difference in volatility between to rating classes i and j, may be expressed as:

$$\sigma_i(t,T) - \sigma_j(t,T) = \sigma^s_{i,j}(t,T) s_{i,j}(t,T), \tag{4.85}$$

for $0 \le t \le T$ and for all $i, j = 1, \ldots, K-1$ where $\sigma^s_{i,j}(t,T)$ is an $\mathbb{F}$-adapted stochastic process attaining values in $\mathbb{R}^d$.

A feature of the HJM forward rate models with proportional volatility, is that they result in a forward rate with positive dynamics. Furthermore, the author of [5] states that one can find in the literature that such models explode in finite time with positive probability. That is, there is no global solution to the SDE in such a case. However, by assuming bounded volatility, existence results for the solution may be obtained.

The author also states that by assuming N.1 or N.2, additional exponential terms appear into the forward rate dynamics, and the above existence result are not applicable, even for constant volatilites. Furthermore, [5] investigates the zero-recovery inter-rating spread under N.2 and for $K = 3$, and states that also for the fundamental spreads and under N.1, one may achieve similar results.

It is also mentioned that to the author's knowledge, no example exist for a HJM migration model where the forward rates are positive and ordered, as it seems difficult to construct such an example due to the inclusion of additional exponential terms of the forward rates, as well as since comparison theorems are only available for SDEs with equal volatilities.

Inter-rating spread dynamics for the proportional volatility spread structure is provided in [5] for the general case, but omitted from this thesis as it is not necessary for discussion. Only the special case of $K = 3$ is included, as this is the one used in the explosion-argument. See [5] Equation 4.6.2 for the complete spread dynamics.

Corollary 4.4.1 *Assume a spread volatility structure of the form 4.85. Furthermore, assume N.2 as well as zero-recovery for all states. For any fixed maturity* $T \leq T^*$ *and for the special case of* $K = 3$ *and* $\sigma_1(t,T) = 0, \sigma^s_{2,1}(t,T) = \sigma^s > 0$, *one has:*

$$\begin{aligned} s_2(t,T) = s_2(0,T)\exp\Bigg\{ &|\sigma^s|^2 \int_0^t \int_s^T s_2(s,u)duds \\ &+ \int_0^t \lambda^*_{2,1}(s)\exp\left(\int_s^T s_2(s,u)du\right)ds \\ &+ \int_0^t \lambda^*_{1,2}(s)\exp\left(-\int_s^T s_2(s,u)du\right)ds \\ &+ \sigma^s W_t - \frac{|\sigma^s|^2}{2}t\Bigg\}, \end{aligned} \tag{4.86}$$

for $0 \leq t \leq T$ *for the inter-rating spread for* $i,j = 1,\dots K-1$ *with* $i \neq j$ *when* $g_i(t,T) \neq g_j(t,T)$ *for all* i,j. *Furthermore,*

$$\begin{aligned} ds_2(t,T) = s_2(t,T)\Bigg(\Bigg\{ &|\sigma^s|^2 \int_t^T s_2(t,u)du \\ &+ \lambda^*_{2,1}(t)\exp\left(\int_t^T s_2(t,u)du\right) \\ &+ \lambda^*_{1,2}(t)\exp\left(-\int_t^T s_2(t,u)du\right)\Bigg\}dt \\ &+ \sigma^s dW_t\Bigg). \end{aligned} \tag{4.87}$$

In [5], it is stressed that (4.86) implies a model with positive spreads with ordered forward rates. The ordering is implied by the assumption of $g_i(t,T) \neq g_j(t,T)$. However, the spread admits no solution since it explodes in finite time. That is,

Proposition 4.4.2 *Assume condition N.2. Furthermore, assume* $K = 3$, $\delta_1 = \delta_2 = 0$ *and a spread volatility structure of the form (4.85) with* $\sigma_1(t,T) = 0$ *and* $\sigma^s_{2,1}(t,T) = \sigma^s$, *where* σ^s *is a positive real constant. Then, for all* $T > 0$, $\lim_{t\to\frac{T}{2}} s_2(t,T) = +\infty$ *with positive probability under* $\mathbb{Q}$ *and any other equivalent measure.*

Proof. See [5]Proposition 4.6.3. ∎

4.5 Vanishing Migration Intensities

The final section promotes independence on the maturity time of the risk premium processes for the HJM forward rates structure and introduces a condition (M.2) regarding this. The author stresses that it is a rather optional condition for the development of the model, as stated in [3], but also that it is required for the derivation of the risk-neutral valuation formula for the defaultable bond.

The condition, M.2, may be combined with both N.1 and N.2, but nevertheless leads to model complications as illustrated in Proposition 4.7.2 in [5]. Both the condition M.2 and the complications are included below.

Definition 4.5.1 (M.2) Let γ be a stochastic process. For $i = 1, \dots K-1$, the process η_i does not depend on the maturity T.

Remark 4.5.2 Recall that $\eta_i(t, T)$ is defined as:

$$\eta_i(t,T) \coloneqq a_i(t,T) - r(t) + b_i(t,T)\gamma_t. \tag{4.88}$$

See (4.18) for details, or 4.2.19 in [5]. For the definition of γ, see BR.4.

An implication of M.2 is then presented:

Corollary 4.5.3 *Assume conditions M.2 and N.1. Furthermore, fix some maturity $T \le T^*$. N.1 is equivalent to the following:*

For all $0 \le t \le T$, the drift condition

$$\alpha_{C_t^1}(t,T) = \sigma_{C_t^1}(t,T)\int_t^T \sigma_{C_t^1}(t,u)du - \gamma_t\sigma_{C_t^1}(t,T), \tag{4.89}$$

of the current forward rate $g_{C_t^1}(t,T)$ holds, together with the condition

$$s^f_{C_t^1}(t,t) = \lambda^*_{C_t^1,K}(t)(1-\delta_{C_t^1}), \tag{4.90}$$

on the set $\{C_t^1 \neq K\}$.

It is then visible, as stated in [5], that the active risky forward rate $g_{C_t^1}(t,T)$ has the classic HJM drift condition. Furthermore, it is stressed that this imposes restrictions on the intensity parameters of the matrix Λ_t^*, and thus complications for the model. As stated by the author: *under some mild conditions on the current bond forward rate process, imposing M.2 trivialises the intensity matrix structures.* This is presented in the following proposition.

Proposition 4.5.4 *Assume conditions M.2 and N.1. Furthermore, assume that $g_{C_t^1} \neq f(t,T)$ and $g_{C_t^1}(t,T) \neq g_j(t,T)$ for all $j = 1, \dots, K-1$ with $j \neq C_t^1$.*

Then, for almost all $0 \le t \le T$ and on the set $\{C_t^1 \neq K\}$ we have $\lambda^*_{C_t^1,j}(t) = 0$ and either $\lambda^*_{C_t^1,K}(t) = 0$ or $\delta_{C_t^1} = 0$. This implies no migration between the classes.

In particular, for the non-zero recovery case where $\delta_i \neq 0$ for all $i = 1, \dots, K-1$, there is no migration nor default for the active class, i.e., $C_0^1 = C_t^1$ for all $0 \le t \le T$.

It is further stressed by the author that the ordering condition is *not* assumed, providing evidence of how much the condition M.2 affects the model. Finally, by recalling that N.2 implies N.1, we have that a similar result holds also under the combination of N.2 and M.2. See Remark 4.7.4 in [5].

CHAPTER 5

Unit-linked Policies with Stochastic Interest Rates

In this chapter we will provide the necessary tools and assumptions needed to present the main result of this thesis, a Thiele partial differential equation for unit-linked insurance policies exposed to credit risk in a defaultable bond market. This will be done for a term insurance.

For this purpose, we recall some notions of Chapter 3 where we presented the case of classic life insurance, as well as the unit-linked case and fundamentals regarding insurance policies affected by stochastic interest rates. Especially, recall that unit-linked policies are insurance policies where the payout in case of the insured event is linked to the performance of an underlying unit.

We will extend these types of policies, so that we have stochastic yields based on prices of defaultable bonds, meaning that we combine the theory of unit-linked policies and stochastic interest rates.

More information regarding the above-mentioned theory can be found in [7], in the setting of classic life insurance. See especially Chapter 9.5 in [7] for the derivation of Thiele's equation in the case of stochastic interest rates.

5.1 Mathematical Reserves

In this section, we provide a version of the mathematical reserves, where we have included stochastic yields based on the price of defaultable bonds in a defaultable bond market. The bond market consists of:

1. The risk-less asset:
$$B(t) = \exp\left(\int_0^t r(s)ds\right), \tag{5.1}$$
2. The T-maturity defaultable bond price process:
$$D_{C_t}(t,T) = \sum_{i=1}^{K-1} H_i(t)D_i(t,T) + \delta_i H_{i,K}(t)B(t,T), \tag{5.2}$$
for $0 \leq t \leq T$. See Chapter 4 for details.

Additionally, we require in this chapter the assumptions BR.1 through BR.4 from Chapter 4. Also, $\Lambda^* = (\lambda^*_{ij})_{i,j\in\mathcal{K}}$ is assumed to satisfy the weak consistency condition (N.1, 4.28).

The mathematical reserve is the amount of money an insurance company has to keep for the expected liabilities in order to remain solvent. Formally, it is defined as:

Definition 5.1.1

$$V_i(t) = \int_t^T F_i(t,u)du, \tag{5.3}$$

where $i \in \mathcal{K}$ denotes the state of the insured, t denotes the initial time, T the maturity time and

$$F_i(t,u) = B(t,u) \times P_i(t,u), \tag{5.4}$$

$$P_i(t,u) = \sum_{i\in\mathcal{K}} p_{i,j}(t,u) \left(a_i(u) + \sum_{\substack{k\in\mathcal{K} \\ k\neq i}} \mu_{i,j}(u) a_{i,j}(u) \right), \tag{5.5}$$

with $B(t,u)$ being the risk-neutral price of a zero-coupon bond in a defaultable bond market. Furthermore, from classical life insurance, we have the policy functions $a_i(t)$, $a_{i,j}(t)$, the transition rates $\mu_{i,j}(u)$ as well as the transition probability $p_{i,j}(t,s)$.

The above bond price is given by

$$B(t,u) = \mathbb{E}_{\mathbb{Q}^*}\left[\exp\left(-\int_t^u r(s)ds\right)\middle|\tilde{\mathcal{F}}_t\right], \tag{5.6}$$

where $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\in\mathbb{R}}$ is the σ-algebra of market information such that the default-time is a stopping time with respect to $\tilde{\mathbb{F}}$ (recall the definition of $\mathbb{G}$ from Section 2.4), and $r(s)$ denotes the short-rate interest rate.

There is, however, no general solution to the above conditional expectation, so instead, we would require a numerical approximation given the above setting. This will not be investigated in this thesis.

Remark 5.1.2 In the definition of the mathematical reserves based on stochastic interest rates, it is also conceivable to consider bond values of the form

$$\mathbb{E}_{\mathbb{Q}^*}\left[\exp\left(-\int_t^u r(s)ds\right)\mathbf{1}_{\{t<\tau\}}\middle|\tilde{\mathcal{F}}_t\right], \tag{5.7}$$

where τ is the default-time, instead of the variant given in (5.6). In this version, interest rates are only paid until the company becomes bankrupt.

So, if we assume that the migration process $C_t, t \geq 0$ is independent of events in $\mathcal{F}_t, t \geq 0$ under $\mathbb{Q}^*$, then we find that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^*}\left[\exp\left(-\int_t^u r(s)ds\right)\mathbf{1}_{\{t<\tau\}}\middle|\tilde{\mathcal{F}}_t\right] \\ &= \mathbb{E}_{\mathbb{P}^*}\left[\exp\left(-\int_t^u r(s)ds\right)\middle|\tilde{\mathcal{F}}_t\right]\mathbb{P}(\tau > t), \end{aligned} \tag{5.8}$$

where $\mathbb{Q}^* = \mathbb{P}^* \times \mathbb{P}$, and $\mathbb{P}$ is the probability measure with respect to $C_t, t \geq 0$.

5.2 Mathematical Reserves for Unit-linked Policies based on Defaultable Bonds

In this section we aim at extending the concept of mathematical reserves for unit-linked policies. We will focus on extending the setting of stock prices in a Black-Scholes market, to the case of interest rates derivatives in a defaultable bond market, as described in Section 5.1.

This is based on Chapter 3, but especially note the following:

Assumption 5.2.1

1. *The filtration* $\{\mathcal{F}_t^D\}_{t\geq 0}$ *is constituted by information coming from the the defaultable bond prices (under N.1) in the time interval* $[0,t]$ *and the insurance events. That is:*

$$\mathcal{F}_t^D := \sigma(\mathcal{G}_t, \mathcal{H}_t), \tag{5.9}$$

   where

$$\mathcal{G}_t := \tilde{\mathcal{F}}_t,$$

   with $\tilde{\mathcal{F}}_t$ *being as in the previous section, and*

$$\mathcal{H}_t := \sigma(X_s, 0 \leq s \leq t),$$

   where X_s, $s \geq 0$ *represents the state process of the insured.*

2. *Insurance events are independent of events on the defaultable bond market, i.e., events in* $\sigma(\mathcal{H}_t), t \geq 0$ *are independent of events in* $\sigma(\mathcal{G}_t), t \geq 0$.

Remark 5.2.2 Note that we include the null-sets, $\mathcal{N}$, in the definition of $\mathcal{G}_t$. By doing this, we ensure that the σ-algebra is complete, by considering elements in the appropriate probability space, i.e., the physical one.

Now, inspired by Chapter 3, we can define the prospective reserves, $V^+_{\mathcal{F}^D}(t,A)$, of unit-linked policies based on defaultable bonds as follows:

Define

$$\begin{aligned} V^+(t,A) = &\sum_{i\in S}\int_t^\infty \mathbf{1}_{\{X_j=i\}}\pi_t^i(s)ds \\ &+ \sum_{\substack{i,j\in S\\ i\neq j}}\int_t^\infty \pi_t^{i,j}(s)dN_{i,j}(s), \end{aligned} \tag{5.10}$$

where $N_{i,j}(s)$ is the number of transitions from state i to state j in the interval $(0,s)$. Furthermore, $\pi_t^i(s)$ and $\pi_t^{i,j}(s)$ are the fair values of pension payments $\dot{a}_i(s)$ and benefit payments $a_{i,j}(s)$, respectively, at time $t, t<s$. The payments are given by:

$$\begin{aligned} \dot{a}_i(s) &= f_i(s, D_{C_s}(s,T)), \\ a_{i,j}(s) &= f_{i,j}(s, D_{C_s}(s,T)), \end{aligned}$$

for Borel-measurable payoff functions $f_i, f_{i,j} : [0,\infty)\times[0,\infty)\to[0,\infty)$, $i,j\in S$, that is

$$\pi_t^i(s) = \mathbb{E}_{\mathbb{Q}^*}\left[\exp\left(-\int_t^s r(u)du\right)\dot{a}_i(s)|\tilde{\mathcal{F}}_t\right], \tag{5.11}$$

and

$$\pi_t^{i,j}(s) = \mathbb{E}_{\mathbb{Q}^*}\left[\exp\left(-\int_t^s r(u)du\right) a_{i,j}(s)|\tilde{\mathcal{F}}_t\right], \tag{5.12}$$

where $\mathbb{Q}^*$ is given as in the previous section, and $r(s)$ is the short-rate interest rate.

Now, using the Markov property of $(X_t)_{t\geq 0}$, we can finally define $V_{\mathcal{F}}^+(t, A)$ by

$$V_{\mathcal{F}}^+(t, A) = \mathbb{E}_X[V^+(t, A)|X_t], \tag{5.13}$$

where $\mathbb{E}_X$ denotes the expectation in the direction of $X_s, s > t$, i.e., on a separate sample space Ω_X.

5.3 Thiele's Differential Equation for Defaultable Unit-linked Policies

In this section, we will obtain the main result of this thesis. That is, a version of Thiele's partial differential equation for unit-linked policies based on stochastic yields of bonds in a defaultable bond market. We will consider a term insurance (see 2.1.6) and stochastic interest rates based on yields of defaultable bonds.

We are going to need Kolmogorov's backward equation (see Theorem 2.3.4 in [7]), given as

$$\frac{\partial}{\partial t} p_{i,j}(t, s) = -\mu_{i,i}(t) p_{i,j}(t, s) - \sum_{\substack{k\in S \\ k\neq i}} \mu_{i,k} p_{k,j}(t, s). \tag{5.14}$$

In our case (term insurance) we only consider the states active $(*)$ and deceased $(\dagger)$. Thus, we find

$$\begin{aligned}\frac{\partial}{\partial t} p_{*,*}(t, s) &= -\mu_{*,*}(t) p_{*,*}(t, s) - \mu_{*,\dagger}(t) p_{\dagger,*}(t, s) \\ &= \mu_{*,\dagger}(t) p_{*,*}(t, s),\end{aligned} \tag{5.15}$$

because $p_{\dagger,*}(t, s) = 0$, as transitions out of the state cannot occur, and $-\mu_{*,*}(t) = \mu_{*,\dagger}(t)$.

We are also going to consider a differential operator $\mathcal{A}$, which is similar, but not identical to the one considered in Section 3.3.2. Both operators are used in the Feynman-Kac representation[1] and thus indirectly follow from Itô's formula (2.3.12). See [11], Theorem 8.2.1 and the following remark regarding *killing a diffusion* for details.

Now, in order to derive the equation, we are required to make the following assumptions (in addition to the conditions in Section 4.1 and the condition N.1).

[1] A relation that relates conditional expectations of functionals of Markov processes (e.g. the price of a bond) with solutions of second order PDEs

Assumption 5.3.1

1. *Let*
$$B(t,T) = B_T(t, r_t^x), \tag{5.16}$$
and
$$D_i(t,T) = D_{T,i}(t, r_t^{i,x}), \;\; i = 1, \ldots, K-1, \tag{5.17}$$
for all $0 \le t \le T$*, where* $B_T, D_{T,i} \in C^{1,2}([0,T] \times \mathbb{R})$.

2. *Furthermore, let the short-rate processes* $r_t^x, r_t^{i,x}, 0 \le t \le T, i = 1, \ldots, K-1$ *satisfy the SDEs:*
$$r_t^x = x + \int_0^t \alpha(s, r_s^x)ds + \int_0^t \sigma(s, r_s^x)dW_s^*, \tag{5.18}$$
$$r_t^{i,x} = x + \int_0^t \alpha_i(s, r_s^{i,x})ds + \int_0^t \sigma_i(s, r_s^{i,x})dW_s^*, \tag{5.19}$$
for $i = 1, \ldots, K-1$*, where* $W_t^*, 0 \le t \le T$ *is the Brownian motion under the equivalent martingale measure* $\mathbb{Q}^*$ *and where* $\alpha, \sigma, \alpha_i, \sigma_i$ *for* $i = 1, \ldots, K-1$ *are globally Lipschitz continuous functions of linear growth.*

Remark 5.3.2 By $C^{1,2}([0,T] \times \mathbb{R})$, we mean that the function has existing continuous partial derivatives, of order 1 with respect to time and up to order 2 with respect to space, as well as continuous extensions of the derivatives to $([0,T] \times \mathbb{R})$.

Remark 5.3.3 The condition that $\alpha, \sigma, \alpha_i, \sigma_i, i = 1, \ldots, K-1$ are globally Lipschitz continuous functions of linear growth, ensures the existence of unique strong[2] solutions of the SDEs.

Now, we know that the mathematical reserve of a term insurance based on defaultable bonds is given by
$$V^+_{\mathcal{F}^D}(t, A) = \int_t^\infty \pi_t^{*,\dagger}(s) p_{*,*}(t,s) \mu_{*,\dagger}(s) ds, \tag{5.20}$$
given that $X_t = *$. Here, $\pi_t^{*,\dagger}(s)$ is given by
$$\pi_t^{*,\dagger}(s) = \mathbb{E}_{\mathbb{Q}^*}\left[\exp\left(-\int_t^s r^x(u)du\right) f(s, D_{C_s}(s,T)) | \tilde{\mathcal{F}}_t\right], \tag{5.21}$$
for a Borel-measurable payoff function f.

Assume that the recovery rates $\delta_i = 0$ for all $i = 1, \ldots, K-1$. Then, we see that
$$\begin{aligned} f(s, D_{C_s}(s,T)) &= \sum_{i=1}^{K-1} \mathbf{1}_{\{C_s = i\}} f(s, D_i(s,T)) \\ &= \sum_{i=1}^{K-1} \mathbf{1}_{\{C_s = i\}} f(s, D_{T,i}(s, r_s^{i,x})), \end{aligned} \tag{5.22}$$

[2] A term referring to unique, cádlág, adapted solutions.

having used relation (5.17). Hence, we get the expression

$$\pi_t^{*,\dagger}(s) = \sum_{i=1}^{K-1} \mathbb{E}_{\mathbb{Q}^*}\left[e^{-\int_t^s r^x(u)du} f(s, D_{T,i}(s, r_s^{i,x}))\mathbf{1}_{\{C_s=i\}}|\tilde{\mathcal{F}}_t\right]. \tag{5.23}$$

Now, require that the intensity parameters, $\lambda_{i,j}^*(t)$ for $0 \le t \le T$, of the process C_t for $0 \le t \le T$ are deterministic. By doing this, we can obtain a migration process independent of events in $\mathcal{F}_t$, $0 \le t \le T$ under $\mathbb{Q}^*$.

Then, by using the definition of conditional expectations, we find that

$$\pi_t^{*,\dagger}(s) = \sum_{i=1}^{K-1} \mathbb{E}_{\mathbb{P}^*}\left[e^{-\int_t^s r^x(u)du} f(s, D_{T,i}(s, r_s^{i,x}))|\mathcal{F}_t\right] \mathbb{P}(C_s = i), \tag{5.24}$$

where $\mathbb{Q}^* = \mathbb{P}^* \times \mathbb{P}$ and $C_s, 0 \le t \le T$ is a Markov process under $\mathbb{P}$, and $\mathbb{P}(C_s = i) := P_i(s)$.

Hence, we rewrite the above expression as

$$\pi_t^{*,\dagger}(s) = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\int_t^s r^x(u)du} f^*(z_s^{\tilde{x}})|\tilde{\mathcal{F}}_t\right], \tag{5.25}$$

where

$$f^*(x_0, x_1, \ldots, x_K) := \sum_{i=1}^{K-1} P_i(s) f(x_0, D_{T,i}(x_0, x_{i+1})), \tag{5.26}$$

$$z_s^{\tilde{x}} := (s, r_s^x, r_s^{1,x}, \ldots, r_s^{K-1,x})^T, \tag{5.27}$$

for $\tilde{x} = (0, x, \ldots, x) \in \mathbb{R}^{K+1}$, where T denotes the transpose.

Then, using the Markov property of the process z_s^y, $0 \le t \le T$, we get that

$$\pi_t^{*,\dagger}(s) = \phi_{s,t}(z_t^{\tilde{x}}), \tag{5.28}$$

where

$$\phi_{s,t}(x_0, x_1, \ldots, x_K) := \mathbb{E}_{\mathbb{P}^*}\left[e^{-\int_t^s r^{t,x}(l)dl} f^*(z_s^{t,x_0,x_1,\ldots,x_K})\right],$$

with

$$z_s^{t,x_0,x_1,\ldots,x_K} = (x_0 + s - t, r_s^{t,x_1}, r_s^{1,t,x_2}, \ldots, r_s^{K-1,t,x_K})^T.$$

Here, $r_s^{t,x_1}, r_s^{i,t,x_i}$ are the short-rate processes satisfying the SDEs:

$$r_s^{t,x_1} = x_1 + \int_t^s \alpha(u, r_u^{t,x_1})du + \int_t^s \sigma(u, r_u^{t,x_1})dW_u^*, \tag{5.29}$$

$$r^{i,t,x_{i+1}} = x_{i+1} + \int_t^s \alpha_i(u, r_u^{i,t,x_{i+1}})du + \int_t^s \sigma_i(u, r_u^{i,t,x_{i+1}})dW_u^*. \tag{5.30}$$

See, e.g., [11].

Now, for smooth functions $f : \mathbb{R}^{K+1} \to \mathbb{R}$, define the differential operator $\mathcal{A}$ given by

$$\begin{aligned}\mathcal{A}f(x_0, x_1, \ldots, x_K) = \frac{1}{2}\sum_{i=1}^{K+1}\sum_{j=1}^{K+1} b_{ij}(x_0, x_1, \ldots, x_K)\frac{\partial^2}{\partial x_i \partial x_j} f(x_0, x_1, \ldots, x_K)\\ + \frac{\partial}{\partial x_0} f(x_0, x_1, \ldots, x_K) + \mu(x_0, x_1, \ldots, x_K)\frac{\partial}{\partial x_1} f(x_0, x_1, \ldots, x_K)\\ + \sum_{i=1}^{K-1} \mu_1(x_0, x_1, \ldots, x_K)\frac{\partial}{\partial x_{i+1}} f(x_0, x_1, \ldots, x_K),\end{aligned} \tag{5.31}$$

where $b_{i,j}(x_0, x_1, \ldots, x_K)$ are the matrix entries of $\Sigma \cdot \Sigma^T$, where

$$\sum = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \sigma(x_0, x_1) & 0 & \ldots & 0 \\ \sigma_1(x_0, x_1) & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \sigma_{K-1}(x_0, x_K) & 0 & \ldots & 0 \end{pmatrix}.$$

In the following, let us assume that f^* is continuous. Furthermore, suppose that there exist a function $v_s : [0, s] \times \mathbb{R}^{K+1} \to \mathbb{R} \in C^{1,2}([0, s) \times \mathbb{R}^{K+1})$ such that v_s solves the PDE

$$\begin{cases} -\frac{\partial}{\partial t} v_s(t, x_0, \ldots, x_K) + x_1 \cdot v_s(t, x_0, \ldots, x_K) = \mathcal{A}v_s(t, x_0, \ldots, x_K) \\ v_s(s, x_0, \ldots, x_K) = f^*(x_0, \ldots, x_K) \end{cases} \tag{5.32}$$

and that

$$\max_{0 \le t \le s} |v_s(t, x_0, \ldots, x_K)| \le M_s(1 + (\sum_{j=0}^{K} x_j^2)^{\mu_s}),$$

for constants $M_s > 0, \mu_s \ge 1$.

Then, it follows from the Feynman-Kac representation, see [11], that

$$v_s(t, x_0, \ldots, x_K) = \phi_{s,t}(x_0, \ldots, x_K), \tag{5.33}$$

for all $x_0, \ldots, x_K$. And so,

$$V^+_{\mathcal{F}^D}(t, A) = \int_t^\infty v_s(t, t, r_t^x, r_t^{1,x}, \ldots, r_t^{K-1,x}) p_{*,*}(t, s)\mu_{*,\dagger}(s)ds, \tag{5.34}$$

where we have used $x_0 = t$.

Now, define

$$V(t, x_0, \ldots, x_K) = \int_t^\infty v_s(t, x_0, \ldots, x_K) p_{*,*}(t, s)\mu_{*,\dagger}(s)ds. \tag{5.35}$$

Then,

$$V^+_{\mathcal{F}^D}(t, A) = V(t, t, r_t^x, r_t^{1,x}, \ldots, r_t^{K-1,x}). \tag{5.36}$$

Using the chain rule combined with dominated convergence, we observe that

$$\begin{aligned}\frac{\partial}{\partial t}V(t,x_0,\dots,x_K) &= -v_t(t,x_0,\dots,x_K)p_{*,*}(t,t)\mu_{*,\dagger}(t) \\ &\quad + \int_t^\infty \frac{\partial}{\partial t}v_s(t,x_0,\dots,x_K)p_{*,*}(t,s)\mu_{*,\dagger}(s)ds \\ &\quad + \int_t^\infty v_s(t,x_0,\dots,x_K)\frac{\partial}{\partial t}p_{*,*}(t,s)\mu_{*,\dagger}(s)ds.\end{aligned} \tag{5.37}$$

Furthermore, using that $v_t(t,x_0,\dots,x_K) = f^*(t,x_0,\dots,x_K)$ from (5.32) and Kolmogorov's backward equation which states that

$$\frac{\partial}{\partial t}p_{*,*}(t,s) = \mu_{*,\dagger}(t)p_{*,*}(t,s), \tag{5.38}$$

we get

$$\begin{aligned}\frac{\partial}{\partial t}V(t,x_0,\dots,x_K) &= -f^*(x_0,\dots,x_K)p_{*,*}(t,t)\mu_{*,\dagger}(t) \\ &\quad + \mu_{*,\dagger}(t)V(t,x_0,\dots,x_K) \\ &\quad + \int_t^\infty \frac{\partial}{\partial t}v_s(t,x_0,\dots,x_K)p_{*,*}(t,s)\mu_{*,\dagger}(s)ds,\end{aligned} \tag{5.39}$$

where we also applied (5.35).

Finally, we make use of the definition of the differential operator $\mathcal{A}$, to rewrite the above integral as

$$\begin{aligned}&\int_t^\infty \frac{\partial}{\partial t}v_s(t,x_0,\dots,x_K)p_{*,*}(t,s)\mu_{*,\dagger}(s)ds \\ &= \int_t^\infty (-\mathcal{A}v_s(t,x_0,\dots,x_K) + x_1 v_s(t,x_0,\dots,x_K))p_{*,*}(t,s)\mu_{*,\dagger}(s)ds \\ &= -\mathcal{A}V(t,x_0,\dots,x_K) + x_1 V(t,x_0,\dots,x_K).\end{aligned} \tag{5.40}$$

Thus, we obtain for

$$V^+_{\mathcal{F}^D}(t,A) = V(t,t,r_t^x,r_t^{1,x},\dots,r_t^{K-1,x}),$$

the Thiele equation

$$\begin{aligned}\frac{\partial}{\partial t}V(t,x_0,\dots,x_K) &= -f^*(t,x_0,\dots,x_K)\mu_{*,\dagger}(t) \\ &\quad + (x_1 + \mu_{*,\dagger}(t))V(t,x_0,\dots,x_K) \\ &\quad - \mathcal{A}V(t,x_0,\dots,x_K),\end{aligned} \tag{5.41}$$

having used that $p_{*,*}(t,t) = 1$.

Note that this can be rewritten to resemble Thiele's partial differential equation for stochastic interest rates presented in Chapter 3. That is,

$$\begin{aligned}\frac{\partial}{\partial t}V(t,x_0,\ldots,x_K) = {} & x_1 V(t,x_0,\ldots,x_K) \\ & - \mu_{*,\dagger}(t)\left[f^*(t,x_0,\ldots,x_K) - V(t,x_0,\ldots,x_K)\right] \\ & - \mathcal{A}V(t,x_0,\ldots,x_K),\end{aligned}$$

where

- $x_1 V(t,x_0,\ldots,x_K) - \mu_{*,\dagger}(t)\left[f^*(t,x_0,\ldots,x_K) - V(t,x_0,\ldots,x_K)\right]$, represents the classical part,
- $-\mathcal{A}V(t,x_0,\ldots,x_K)$, represents the component corresponding to the stochastic yields.

CHAPTER 6

Conclusion

6.1 Summary

This thesis can be summarised as follows.

- In Chapter 2 we gave a general introduction to the case of classic life insurance, as well as probability theory, stochastic calculus and credit risk modelling. We also listed the standing assumptions.

- In Chapter 3 we further introduced the mathematical prerequisites for the insurance reserves in the case of unit-linked policies and the fundamentals of stochastic interest rates. We discussed an example of Thiele's partial differential equation with stochastic interest rates under the Vasicek model, and included some plots of the resulting reserves.

- In Chapter 4, we introduced the Bielecki-Rutkowski model ([3]) and summarised the findings of Christodoulou ([5]). Amongst other elements, this included two consistency conditions and the price process of defaultable bonds, providing the necessary framework for a no-arbitrage argument to be made in Chapter 5.

- The main result of the thesis was then presented in Chapter 5. We first laid the foundations for mathematical reserves based on stochastic yields and then extended it to allow for unit-linked policies based on defaultable bonds. Finally, we derived a new version of Thiele's partial differential equation.

6.2 Extensions

Regarding possible extensions, an immediate and rather unproblematic extension would be the case of non-zero recovery rates. This, as well as restructuring of the defaulted firm, is explored in, e.g., [9]. Also, the case of correlation between the migration process and the forward rates would be interesting to further explore.

Additionally, one could extend this theory to the class of Lévy processes. This would allow for jumps in the price process of the defaultable bonds, but also in the credit migration process. This would perhaps provide a more general version of the model, and would thus be more realistic in some real-life scenarios, for positive recovery rates.

Furthermore, it would be interesting to implement a numerical scheme of the new equation derived in the Chapter 5. One could either try to implement a similar method as to what was done in Chapter 3, or investigate other methods. This would be quite challenging, as one would need to make the necessary specifications regarding the elements in the obtained Thiele partial differential equation.

Appendices

APPENDIX A

R-code for Thiele example

```
#Purpose: plotting reserves using Thiele's partial
#differential equation with interest rates
#under the Vasicek model for a two-state term insurance

require(plot3D)

#-------------------------------
#Contract specifications
x0 = 24
T = 50 #contract length in years

DB = 100000 #death benefit

#Constant mortality
mort <- 0.009

#Survival probability, from t to s
surv_prob = function(t,s) exp(-mort*(s-t))

#-------------------------------
#Vasicek specifications
a = 0.05
b = 0.03
sigma = 0.02
gamma = 0


#Other specifications
#Step size
dt <- 0.01
dx <- 0.01

#maximum value of interest rate
x.max <- 0.20
```

```
#--------------------------------
#Boundaries

#case of zero interest
boundary.down <- function(t){
  B <- function(x){
    return( (b + gamma*sigma/a - sigma^2/(2*a^2) ) *
            ( ((1-exp(-a*x))/a) - x ) -
             (sigma^2 / 4*a) * ((1-exp(-a*x))/a)^2 )}

  integrand <- function(s){
    return( exp(B(s-t))*surv_prob(x0+t, x0+s)*mort*DB )}

  integral <- integrate(integrand, t, T)$value
  return(integral)
}


#Grid
time <- seq(0,T,by=dt) #time segment
space <- seq(0,x.max,by=dx) #interest rates
Nt <- length(time) - 1
Nx <- length(space) - 1


#--------------------------------
#Matrix definitions
V <- matrix(rep(0,(Nt+1)*(Nx+1)), nrow=Nt+1)

V[Nt+1, ] <- 0
V[ ,1] <- sapply(time,boundary.down)
V[ ,Nx+1] <- 0 #upper boundary condition


#--------------------------------
#Thiele scheme

for(i in Nt:1){

  for(j in Nx:2){
    V[i,j] <- V[i+1,j] - dt*(
                space[j]*V[i+1,j] - mort*(DB-V[i+1,j])
              - (a*(b-space[j]) - gamma*sigma)*(((V[i+1,j+1])
              - V[i+1,j]) / dx)
              - 0.5*(sigma^2)*(V[i+1,j+1] - 2*V[i+1,j] + V[i+1,j-1])
              / (dx^2) )
  }
}
```

```
#-------------------------------
#3D plot
persp3D(time,space,V, theta=300, phi=25, expand=0.5,
        ticktype="detailed", clab = c("Value"),
        xlab="Years", ylab="Interest rate", zlab="", axes=TRUE,
        main = "Reserves under Vasicek")



#Lines plot
colors <- c(rainbow(length(space)))

plot(time, V[,1], type = "l", ylim = c(0,max(V)),
            ylab = "Reserve", xlab = "Years",
            col = colors[1], main = "Reserves")

for(i in 2:length(space)){
  lines(time, V[ ,i], type="l", col = colors[i])
}


#Single premium plot
plot(space, V[1, ], type = "l", xlab = "Interest rate",
            ylab = "Single premium", main = "Single premium")
abline(v = 0.03, col ="red")
```

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