



# Planes in cubic fourfolds

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## ABSTRACT

We show that the maximal number of planes in a complex smooth cubic fourfold in  $\mathbb{P}^5$  is 405, realized by the Fermat cubic only; the maximal number of real planes in a real smooth cubic fourfold is 357, realized by the so-called Clebsch–Segre cubic. Altogether, there are but three (up to projective equivalence) cubics with more than 350 planes.

## 1. Introduction

The study of linear spaces in projective hypersurfaces is a classical problem in algebraic geometry. The 27 lines on a smooth cubic surface in  $\mathbb{P}^3$ , going back to A. Cayley and G. Salmon in the 19th century [Cay49], and at most 64 lines on quartic surfaces, going back to B. Segre [Seg43], are two of the most famous examples. In the last decade, there has been a substantial progress in studying and counting lines and other low-degree rational curves on polarized  $K3$ -surfaces; see [RS15, Deg19, Deg22a, Deg21, Deg22b, DIS17]. Moreover, thanks to the global Torelli theorem for  $K3$ -surfaces [PŠ71] and the surjectivity of the period map [Kul77], it has also been possible to obtain a complete description of the surfaces with large configurations of lines or, sometimes, conics.

While varieties of lines play an important role in the geometry of hypersurfaces (especially cubic threefolds [CG72] and cubic fourfolds [Has00, Voi86]), much less is known about linear spaces of higher dimension.

In this paper, we study 2-planes in smooth cubic fourfolds  $X \subset \mathbb{P}^5$ . Planes in cubic fourfolds have already appeared in many contexts: they are a key ingredient in C. Voisin’s proof of the global Torelli theorem [Voi86], they define one of the so-called Hassett divisors in the moduli space of cubics [Has00], and they serve as important examples in connection with the rationality problem for cubic fourfolds.

As in the case of lines on  $K3$ -surfaces, the plane-counting problem for smooth cubic fourfolds is not strictly enumerative in the sense that a generic cubic contains no planes at all. Therefore, one looks for estimates on the maximal number of planes in a smooth cubic and, if possible, a description of the cubics realizing this maximum. In fact, the similarities between  $K3$ -surfaces and cubic fourfolds are much deeper, as cubic fourfolds have their own version of the global

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Torelli theorem [Voi86] and surjectivity of the period map [Laz09]. This can be used to recast the original geometric problem in purely lattice-theoretic terms and eventually obtain a complete characterisation of extremal cubics. Formally, our characterisation is in terms of the periods. However, thanks to the injectivity of the period map, we can identify the extremal cubics found by comparing their configurations of planes with those of known explicit examples.

Our main result is the following theorem.

**THEOREM 1.1** (see §9.1). *Let  $X \subset \mathbb{P}^5$  be a complex smooth cubic fourfold. Then, either  $X$  has at most 350 planes, or, up to projective equivalence,  $X$  is*

- the Fermat cubic (with 405 planes), see §2.5 and (8.4); or
- the Clebsch–Segre cubic (with 357 planes), see §2.6 and (8.5); or
- the 351-cubic (with 351 planes); see §2.7 and (8.6).

We also consider a similar problem for real planes in real cubics. Recall that a *real algebraic variety* is a complex algebraic variety  $X$  equipped with a *real structure*, that is, anti-holomorphic involution  $c: X \rightarrow X$ . A subvariety  $P \subset X$  is said to be *real* if  $c(P) = P$ . One can show (cf. Theorem 2.1 and its proof) that any real structure on a cubic  $X \subset \mathbb{P}^5$  in appropriate coordinates in  $\mathbb{P}^5$  is induced from the coordinatewise complex conjugation. In these coordinates, both  $X$  and a real plane  $P \subset X$  can be given by equations with real coefficients.

For the number of real planes, we have the following stronger bound.

**THEOREM 1.2** (see §9.2). *The number of real planes in a real smooth cubic  $Y \subset \mathbb{P}^5$  is at most 357, the equality holding if and only if  $Y$  is projectively equivalent over  $\mathbb{R}$  to the Clebsch–Segre cubic (see §2.6).*

Note that the Clebsch–Segre cubic (2.10) can be regarded as a 4-dimensional analogue of the Clebsch cubic surface, which also has all of its 27 lines real.

*Remark 1.3.* Theorem 1.1 provides a sharp upper bound on the total number of planes. Another interesting question is that on the possible values that can be taken by the plane count. It appears that for smooth cubics, the list is much more sparse than those counting lines on polarized  $K3$ -surfaces. The values observed in our computation are

$$0 .. 225, 227, 229, 231, 233, 235, 237, 239, 241, 243, 245, 247, 249, \\ 255, 257, 259, 261, 267, 273, 285, 297, 351, 357, 405,$$

but we do not assert that this list is complete.

Given the conclusion of Theorem 1.1, one can also ask about the maximal number of planes on singular cubic fourfolds. There are easy examples of such cubics where the number of planes is infinite; however, if one restricts to nodal cubic fourfolds, the number is finite. Furthermore, we show that, similarly to several known results on rational curves on  $K3$ -surfaces, the presence of nodes reduces the maximal number of planes.

**PROPOSITION 1.4** (see §9.3). *The number of planes in a nodal cubic fourfold is at most 302.*

The best example known to us has 291 planes; see Example 9.3. Conjecturally, 291 is the sharp upper bound in the presence of nodes.

As mentioned above, our main approach is lattice-theoretic. More precisely, given an abstract graph, there is a way to decide whether it is realized by the configuration of planes in a smooth

cubic fourfold. However, unlike the case of lines on a polarized  $K3$ -surface, we lack geometric intuition (for example, elliptic pencils) which would narrow the search down to a sufficiently small collection of graphs. For this reason, we take a detour and replant a (modified) abstract lattice of algebraic cycles to a Niemeier lattice (that is, one of the 24 positive definite even unimodular lattices of rank 24), the planes mapping to certain vectors of square 4. This approach has a number of benefits. First, instead of dealing with abstract graphs of a priori unbounded complexity, we merely consider subsets of several finite sets known in advance; in particular, this implies the (not immediately obvious) fact that the number of planes is uniformly bounded. Second, these finite sets have a rich intrinsic structure that can be used in the construction of large realizable subsets. Finally, working with known sets, all symmetry groups can be expressed in terms of permutations, which makes the computation in GAP [GAP19] very effective.

The idea of using Niemeier lattices is not new (cf. [Kon98, Nik15, Nis96]). The novelty of our treatment is in the fact that the original lattice is odd and, therefore, needs to be modified. Of course, one could have used embeddings to odd unimodular definite lattices of rang 24, but their number is overwhelming.

The paper is organized as follows. In §2, we fix the terminology and recall a few basic facts related to integral lattices and cubic fourfolds. Towards the end, in §2.5 and §2.6, we describe the geometric configurations of planes in the two extremal cubics, namely Fermat and Clebsch–Segre. In §3, we replant the lattice of algebraic cycles of a cubic fourfold to a Niemeier lattice, thus reducing the original geometric problem to an arithmetic one; the algorithms used to solve the latter are outlined in §4. In §5–§8, the 24 Niemeier lattices are treated one by one; this is followed by the proofs of Theorems 1.1 and 1.2 in §9.

## 2. Preliminaries

The principal goal of this section is to fix the terminology and notation as well as to cite, in a convenient form, a few fundamental results used further on.

### 2.1 Lattices (see [Nik79])

A *lattice* is a free abelian group  $L$  of finite rank equipped with a symmetric bilinear form  $b: L \otimes L \rightarrow \mathbb{Z}$ . Since the form  $b$  is assumed fixed (and omitted from the notation), we use the abbreviation  $x \cdot y := b(x, y)$  and  $x^2 := b(x, x)$ . The *determinant*  $\det L \in \mathbb{Z}$  is the determinant of the Gram matrix of  $b$  in any integral basis;  $L$  is called *nondegenerate* (respectively, *unimodular*) if  $\det L \neq 0$  (respectively,  $\det L = \pm 1$ ). The *inertia indices*  $\sigma_{\pm} L$  are those of the quadratic space  $L \otimes \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \otimes r \mapsto r^2 x^2$ .

A sublattice  $S \subset L$  is called *primitive* if the quotient  $L/S$  is torsion free. A *d-polarized lattice* is a lattice equipped with a distinguished element  $h$ ,  $h^2 = d$ . Morphisms in the category of (polarized) lattices are called *isometries*.

A lattice  $L$  is called *even* if  $x^2 = 0 \pmod{2}$  for all  $x \in L$ ; otherwise,  $L$  is *odd*. A *characteristic vector* is an element  $v \in L$  such that  $x^2 = x \cdot v \pmod{2}$  for all  $x \in L$ . If  $L$  is unimodular, a characteristic vector exists and is unique  $\pmod{2L}$ . If  $v \in L$  is characteristic, the orthogonal complement  $v^{\perp} \subset L$  is even.

For lattices of rank 1 and 2, we use the abbreviations

- $[a] := \mathbb{Z}x$ ,  $x^2 = a$ ;
- $[a, b, c] := \mathbb{Z}x + \mathbb{Z}y$ ,  $x^2 = a$ ,  $x \cdot y = b$ ,  $y^2 = c$ .

The *hyperbolic plane*  $\mathbf{U} := [0, 1, 0]$  is the unique unimodular even lattice of rank 2.

A nondegenerate lattice  $L$  admits a canonical inclusion

$$L \hookrightarrow L^\vee := \{x \in L \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in L\}$$

to the dual group  $L^\vee$ . The finite abelian group  $\mathcal{L} := \text{discr } L := L^\vee/L$  (denoted by  $q_L$  in [Nik79]) is called the *discriminant group* of  $L$ . Clearly,  $|\mathcal{L}| = (-1)^{\sigma-L} \det L$ . This group is equipped with the nondegenerate symmetric bilinear form

$$\mathcal{L} \otimes \mathcal{L} \rightarrow \mathbb{Q}/\mathbb{Z}, \quad (x \bmod L) \otimes (y \bmod L) \mapsto (x \cdot y) \bmod \mathbb{Z},$$

and if  $L$  is even, with its quadratic extension

$$\mathcal{L} \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad x \bmod L \mapsto x^2 \bmod 2\mathbb{Z}.$$

We denote by  $\mathcal{L}_p := \text{discr}_p L := \mathcal{L} \otimes \mathbb{Z}_p$  the  $p$ -primary components of  $\text{discr } L$ . The 2-primary component  $\mathcal{L}_2$  is called *even* if  $x^2 \in \mathbb{Z}$  for all order 2 elements  $x \in \mathcal{L}_2$ ; otherwise,  $\mathcal{L}_2$  is *odd*. The *determinant*  $\det \mathcal{L}_p$  is the determinant of the ‘‘Gram matrix’’ of the quadratic form in any minimal set of generators. (This is equivalent to the alternative definition given in [Nik79].) Unless  $p = 2$  and  $\mathcal{L}_2$  is odd (in which case the determinant is not defined or used), we have  $\det \mathcal{L}_p = u_p/|\mathcal{L}_p|$ , where  $u_p$  is a well-defined element of  $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ .

The *length*  $\ell(\mathcal{A})$  of a finite abelian group  $\mathcal{A}$  is the minimal number of generators of  $\mathcal{A}$ . We use the abbreviation  $\ell_p(\mathcal{A}) := \ell(\mathcal{A} \otimes \mathbb{Z}_p)$  for a prime  $p$ .

Given a lattice  $L$  and  $q \in \mathbb{Q}$ , we use the notation  $L(q)$  for the same abelian group with the form  $x \otimes y \mapsto q(x \cdot y)$ , assuming that it is still a lattice. We use the abbreviation  $-L := L(-1)$ , and this notation applies to discriminant forms as well. The notation  $nL$ ,  $n \in \mathbb{N}$ , is used for the orthogonal direct sum of  $n$  copies of  $L$ .

A cyclic group  $\mathbb{Z}/b$  equipped with a quadratic form  $1 \mapsto a \bmod 2\mathbb{Z}$  is denoted by

$$\left[ \frac{a}{b} \right]; \quad \text{we assume } a, b \in \mathbb{Z}, \gcd(a, b) = 1, ab = 0 \bmod 2.$$

Another notation used in the description of the discriminants is  $\mathcal{U} := \text{discr } \mathbf{U}(2)$ .

A *root* in an even lattice  $L$  is a vector of square  $\pm 2$ . A *root system* is a positive definite lattice generated by roots. Any root system has a unique decomposition into an orthogonal direct sum of irreducible components, which are of types  $\mathbf{A}_n$  ( $n \geq 1$ ),  $\mathbf{D}_n$  ( $n \geq 4$ ),  $\mathbf{E}_6$ ,  $\mathbf{E}_7$  or  $\mathbf{E}_8$  (see, for example, [Bou02]), according to their *Dynkin diagrams*.

A *Niemeyer lattice* is a positive definite unimodular even lattice of rank 24. Up to isomorphism, there are 24 Niemeier lattices (see [Nie73]): the *Leech lattice*  $\Lambda$ , which is root free, and 23 lattices *rationaly* generated by roots. In the latter case, the isomorphism class of a lattice  $N := N(D)$  is uniquely determined by that of its maximal root system  $D$ . For more details, see [CS88].

## 2.2 Cubic fourfolds (see [Has00, Laz09, Voi86])

Let  $X$  be a smooth cubic fourfold in  $\mathbb{P}^5$ . The middle Hodge numbers of  $X$  are as follows:

$$h^{0,4} = h^{4,0} = 0, \quad h^{1,3} = h^{3,1} = 1, \quad h^{2,2} = 21.$$

We are concerned with the middle integral cohomology group  $H^4(X) := H^4(X; \mathbb{Z})$ ; via the Poincaré duality isomorphism, this group is canonically identified with  $H_4(X; \mathbb{Z})$ . (From now on, unless stated otherwise, all homology and cohomology groups are with coefficients in  $\mathbb{Z}$ .) With respect to the intersection form,  $H^4(X)$  is the unique (up to isomorphism) odd unimodular lattice of signature  $(21, 2)$ . This lattice is canonically 3-polarized, and the distinguished class  $h_X$ ,

namely the square of the hyperplane divisor of  $X$ , is characteristic. There is a lattice isomorphism

$$H^4(X) \simeq \mathbf{L} := 21[+1] \oplus 2[-1], \quad h_X \mapsto h := (1, \dots, 1, 3, 3).$$

Alternatively,

$$\mathbf{L} \simeq 3[+1] \oplus 2\mathbf{U} \oplus 2\mathbf{E}_8, \quad h \mapsto (1, 1, 1) \in 3[+1].$$

In particular, the sublattice  $\mathbf{L}^0$  of primitive classes (that is, the orthogonal complement of  $h$  in  $\mathbf{L}$ ) decomposes as

$$\mathbf{L}^0 \simeq \mathbf{A}_2 \oplus 2\mathbf{U} \oplus 2\mathbf{E}_8.$$

The choice of a polarized lattice isomorphism  $\phi: H^4(X) \rightarrow \mathbf{L}$  is called a *marking* of the cubic fourfold  $X$ , and we call  $(X, \phi)$  a *marked cubic fourfold*.

By definition, the sublattice  $M_X := H^{2,2}(X; \mathbb{C}) \cap H^4(X)$  of integral Hodge classes is primitive in  $H^4(X)$ . The Hodge–Riemann relations imply that  $M_X$  is positive definite. By [Voi07], the integral Hodge conjecture holds for  $X$ , so that  $M_X$  is generated (over  $\mathbb{Z}$ ) by the classes of algebraic surfaces in  $X$ .

We denote by  $T_X := M_X^\perp$  the *transcendental lattice* of  $X$ .

### 2.3 The global Torelli theorem

The *period* of a marked cubic fourfold  $(X, \phi)$  is defined as the line  $\omega_X = \phi(H^{3,1}(X; \mathbb{C})) \subset \mathbf{L}^0 \otimes \mathbb{C}$ . Thus, denoting by  $\mathcal{M}$  the moduli space of marked cubic fourfolds  $(X, \phi)$ , we may define the *period map*

$$\mathcal{P}: \mathcal{M} \rightarrow \mathcal{D} \subset \mathbb{P}(\mathbf{L}^0 \otimes \mathbb{C}), \quad (X, \phi) \mapsto [\omega_X],$$

where  $\mathcal{D}$  is the *period domain*, that is, a distinguished connected component of

$$\{x \in \mathbb{P}(\mathbf{L}^0 \otimes \mathbb{C}) \mid x^2 = 0, x \cdot \bar{x} < 0\};$$

see [Has16]. The component is distinguished by the so-called *positive sign structure*, that is, a coherent choice of orientations of the maximal negative definite subspaces of  $\mathbf{L} \otimes \mathbb{R}$ . More generally, we can consider cubic fourfolds  $X$  polarized by a positive definite polarized sublattice  $h \in K \subset \mathbf{L}$ . This gives us a moduli space  $\mathcal{M}_K$  of dimension  $21 - \text{rk } K$  and an associated period domain  $\mathcal{D}_K$ , which is a connected component of

$$\{x \in \mathbb{P}(K^\perp \otimes \mathbb{C}) \mid x^2 = 0, x \cdot \bar{x} < 0\}.$$

The following result is a version of the global Torelli theorem for cubic fourfolds, which is due to Voisin [Voi86].

**THEOREM 2.1.** *Let  $X, Y \subset \mathbb{P}^5$  be two smooth cubic fourfolds, with the respective classes  $h_X, h_Y$  and periods  $\omega_X, \omega_Y$  as above. Then, an isometry*

$$f^*: H^4(Y) \rightarrow H^4(X)$$

*is induced by a holomorphic (respectively, anti-holomorphic) projective isomorphism  $f: X \rightarrow Y$  if and only if*

- (i)  $f^*$  is polarized, that is,  $f^*(h_Y) = h_X$ ;
- (ii)  $f^*(\omega_Y) = \omega_X$  (respectively,  $f^*(\omega_Y) = \bar{\omega}_X$ ).

*If an isomorphism  $f$  as above exists, it is unique.*

*Proof.* The holomorphic statement is essentially found in [Voi86] (see also [FK09]), and the anti-holomorphic counterpart is immediately obtained by composing with complex conjugation. The uniqueness follows from the well-known fact that an automorphism  $X \rightarrow X$  which acts as the identity on  $H^4(X)$  must be the identity (which can be proved using the Lefschetz fixed-point theorem).  $\square$

**COROLLARY 2.2** (cf. [DIS17, Lemma 3.8] or [FK08]). *A smooth cubic  $X \subset \mathbb{P}^5$  admits a real structure identical on  $M_X$  if and only if  $T_X$  contains a sublattice  $-\mathbf{A}_1$  or  $\mathbf{U}(2)$ .*

A major consequence of Theorem 2.1 is the fact that the period map  $\mathcal{P}: \mathcal{M} \rightarrow \mathcal{D}$  is injective; its image was computed by R. Laza [Laz09] and E. Looijenga [Loo09].

**THEOREM 2.3** (Surjectivity of the period map, see [Laz09, Theorem 1.1]). *A 3-polarized lattice  $M \ni h$  admits an isometry  $\varphi$  onto  $M_X \ni h_X$  for a smooth cubic  $X \subset \mathbb{P}^5$  if and only if*

- (i)  *$M$  is positive definite and  $h$  is a characteristic vector in  $M$ ;*
- (ii)  *$M$  admits a primitive embedding into  $\mathbf{L}$  such that  $M^\perp$  is even;*
- (iii) *there is no element  $e \in M$  such that  $e^2 = e \cdot h = 1$ ;*
- (iv) *there is no element  $e \in M$  such that  $e^2 = 2$  and  $e \cdot h = 0$ .*

*Under these assumptions,  $\mathcal{M}_M \subset \mathcal{M}$  has codimension  $\text{rk}(M) - 1$ .*

## 2.4 Planes in cubic fourfolds

Our main result is an upper bound on the number of planes in a smooth cubic fourfold. The following lemma appears in J. M. Starr's appendix to [BH06], where it is attributed to O. Debarre.

**LEMMA 2.4.** *Any smooth cubic fourfold contains but finitely many planes.*

Let  $X$  be a smooth cubic fourfold,  $P \subset X$  a plane and  $p := [P] \in M_X$  its class. Clearly,  $h_X \cdot p = 1$ , and using the normal bundle sequence, we find that  $p^2 = c_2(N_{P|X}) = 3$ . Furthermore, given two distinct planes  $P_1$  and  $P_2$  with classes  $p_1$  and  $p_2$ , one has the following trichotomy:

- $p_1 \cdot p_2 = 0$  if  $P_1$  and  $P_2$  are disjoint;
- $p_1 \cdot p_2 = 1$  if  $P_1$  and  $P_2$  intersect at a point;
- $p_1 \cdot p_2 = -1$  if  $P_1$  and  $P_2$  intersect in a line.

This has the following important consequence.

**LEMMA 2.5.** *Each class  $p \in M_X$  is represented by at most one plane.*

A configuration of planes in a smooth cubic  $X \subset \mathbb{P}^5$  is described by means of its *graph of planes*  $\text{Fn } X$ : the vertices of  $\text{Fn } X$  are planes  $P \subset X$ , and two vertices  $P_1$  and  $P_2$  are connected by a solid (respectively, dotted) edge whenever  $P_1$  and  $P_2$  intersect at a point (respectively, in a line). By  $|\text{Fn } X|$ , we denote the number of vertices of this graph, that is, the number of planes in  $X$ .

The next proposition gives us a precise relationship between classes in  $M_X$  and planes in  $X$ .

**PROPOSITION 2.6.** *Given a smooth cubic fourfold  $X \subset \mathbb{P}^5$ , the map  $P \mapsto [P]$  establishes a bijection between  $\text{Fn } X$  and the set of classes  $p \in M_X$  such that  $p^2 = 3$  and  $h_X \cdot p = 1$ .*

*Proof.* The map  $P \mapsto [P]$  is injective by Lemma 2.5, whereas the surjectivity is essentially stated in the first paragraph of [Voi86, § 3]. Here is a more direct argument.

Let  $\pi: \mathcal{X} \rightarrow B$  be a local universal family of marked cubic fourfolds, with  $X$  as the fibre over  $b_0 \in B$ . The marking allows us to identify each lattice  $M_{X_b}$ ,  $b \in B$ , with a sublattice of  $\mathbf{L}$ . Let  $p \in M_X \subset \mathbf{L}$  be a class with  $p^2 = 3$  and  $h_X \cdot p = 1$ , and let  $B' \subset B$  denote the Hodge locus of  $p$ , parameterizing the fibres  $\mathcal{X}_b$  for which the class  $p$  stays Hodge, that is,  $p \in M_{\mathcal{X}_b}$ . Note that

$$\text{the sublattice } K := \mathbb{Z}h + \mathbb{Z}p \simeq [3, 1, 3] \subset M_{\mathcal{X}_b} \text{ is necessarily primitive} \quad (2.7)$$

(as its only proper finite-index extension contains a vector as in Theorem 2.3(iii)); hence, the closed codimension 1 subset  $B' \subset B$  is irreducible (see [Has00, Theorem 3.2.3]). On the other hand, a simple dimension count shows that the cubic fourfolds containing a plane form an irreducible divisor in the parameter space of cubics.

Therefore, if  $b \in B'$  is a very general point, then both

- the cubic  $X' = \mathcal{X}_b$  contains a plane  $P'$ , and
- the lattice  $M_{X'}$  has rank 2; hence  $M_{X'} = K$  by (2.7).

Since also

- $p \in M_{X'} = K$  is the only class such that  $h_{X'} \cdot p = 1$  and  $p^2 = 3$ ,

we conclude that  $[P'] = p$ . Then, by specialization, in  $X$  the class  $p$  is also represented by an effective cycle of degree 1 in  $\mathbb{P}^5$ , hence a plane  $P \subset X$ .  $\square$

**COROLLARY 2.8.** *Given a real structure  $c: X \rightarrow X$  on a smooth cubic fourfold  $X$ , a class  $p \in M_X$  is represented by a real plane  $P \subset X$  if and only if*

- (i)  $h_X \cdot p = 1$  and  $p^2 = 3$ ;
- (ii)  $c^*p = p$ .

*Proof.* If part (i) holds, then  $p$  is represented by a unique plane  $P$  by Proposition 2.6, and  $P$  satisfies  $c(P) = P$  by part (ii). The converse is immediate.  $\square$

## 2.5 The Fermat cubic

Let  $X \subset \mathbb{P}^5$  be the *Fermat cubic*, defined by the equation

$$x_0^3 + x_1^3 + \cdots + x_5^3 = 0. \quad (2.9)$$

One can easily see that  $X$  contains at least 405 planes. Indeed, consider one of the  $5 \times 3$  splittings of the index set  $\{0, \dots, 5\}$  into three pairs, for example,  $\{0, 1\}$ ,  $\{2, 3\}$ ,  $\{4, 5\}$ , and pick three cubic roots  $\xi_1, \xi_2, \xi_3$  of  $-1$ . Then, each of the planes

$$x_0 = \xi_1 x_1, \quad x_2 = \xi_2 x_3, \quad x_4 = \xi_3 x_5$$

clearly lies in  $X$ . The number of planes obtained in this way is  $15 \times 3^3 = 405$ .

A direct calculation (for example, following the argument of Segre [Seg44, § 2]) shows that  $X$  does not contain any other plane. Alternatively, this statement is an immediate corollary of Theorem 1.1; moreover, we assert that  $X$  is the only (up to projective equivalence) smooth cubic with 405 planes.

### 2.6 The Clebsch–Segre cubic

The *Clebsch–Segre cubic* is the cubic fourfold  $Y$  defined by the following equations in  $\mathbb{P}^6$ :

$$x_0 + x_1 + \cdots + x_6 = x_0^3 + x_1^3 + \cdots + x_6^3 = 0. \quad (2.10)$$

Note that the full symmetric group  $\mathbb{S}_7$  acts on  $Y$  by permuting the coordinates.

According to K. Hulek and M. Schütt (private communication),  $Y$  contains at least 357 planes, constituting two  $\mathbb{S}_7$ -orbits. One orbit consists of the 105 Fermat-type planes

$$x_i + x_j = x_k + x_l = x_m + x_n = x_o = 0,$$

where  $i, j, k, l, m, n, o$  is a permutation of  $0, 1, 2, 3, 4, 5, 6$ . To describe the other orbit, we fix an ordered sequence of three pairwise disjoint couples in the index set, for example  $(1, 2), (3, 4), (5, 6)$ . (Other sequences, as well as all related objects described below, are obtained by permuting the coordinates via  $\mathbb{S}_7$ ; we choose to avoid cryptic multi-index constructs in the description.) Consider a vector

$$(0, 1, -1, \varphi, -\varphi, 0, 0) \in \mathbb{C}^7,$$

where  $\varphi^2 + \varphi - 1 = 0$ , and denote by  $O$  its orbit under the dihedral group  $\mathbb{D}_{10} \subset \mathbb{S}_7$  generated by the simultaneous transposition  $1 \leftrightarrow 2, 3 \leftrightarrow 4$  and the 5-cycle

$$0 \mapsto 1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 0.$$

It is straightforward to check that the image  $[O] \subset \mathbb{P}^6$  of  $O$  consists of five collinear points and the plane spanned by  $[O]$  and  $[0 : 0 : 0 : 0 : 0 : 1 : -1]$  lies in  $Y$ . This plane is stabilized by  $\mathbb{D}_{10} \times \langle 5 \leftrightarrow 6 \rangle$ ; hence, its  $\mathbb{S}_7$ -orbit is of size 252, resulting in the total of 357 planes in  $Y$ .

By a direct computation, or as a consequence of Lemma 9.2, we conclude that the cubic  $Y$  given by (2.10) contains no other planes. Also observe that, immediately by the construction, all 357 planes in  $Y$  are real (as they are spanned by real points).

### 2.7 The 351-cubic

The cubic fourfold  $Y \subset \mathbb{P}^5$  with 351 planes is given by the equation

$$g(x_0, x_1, x_2) = g(x_3, x_4, x_5), \quad g(t_0, t_1, t_2) := t_0 t_1^2 - t_2^3 - t_0^2 t_2; \quad (2.11)$$

this cubic appears in the recent paper of K. Koike [Koi22]. To describe the planes in  $Y$ , observe that  $g = 0$  defines smooth cubic curves  $C_1$  in the  $[x_0 : x_1 : x_2]$ -plane  $P_1$  and  $C_2$  in the  $[x_3 : x_4 : x_5]$ -plane  $P_2$ . For each pair of flex tangents  $L_1 \subset P_1$  and  $L_2 \subset P_2$ , the span of  $L_1$  and  $L_2$  is a space  $\mathbb{P}^3 \subset \mathbb{P}^5$ , and the intersection  $\mathbb{P}^3 \cap Y$  is a union of three planes. Since each cubic curve has 9 inflection points, this gives us  $3 \times 9 \times 9 = 243$  planes, which are all pairwise distinct.

The reason for the choice of  $g$  is that  $C_1 \cong C_2$  have many automorphisms: Koike shows that  $Y$  contains 108 additional planes, one for each automorphism of  $C_1$  (see [Koi22, Lemma 2.2]). This results in a total of  $243 + 108 = 351$  planes in  $Y$ .

*Remark 2.12.* Note that the Fermat cubic fourfold, considered in § 2.5, can also be represented by (2.11), with  $g(t_0, t_1, t_2) = t_0^3 + t_1^3 + t_2^3$ , so that the respective cubic  $C_1$  has even more automorphisms.

## 3. Reduction to the Niemeier lattices

The principal goal of this section is replanting a lattice  $M$  as in Theorem 2.3 to an appropriate Niemeier lattice. Then, in §§ 3.3, 3.4, we outline the strategy of our proof of Theorems 1.1 and 1.2.



### 3.1 Replanting to a Niemeier lattice

Consider a positive definite 3-polarized lattice  $M \ni h$ . Assume that  $h$  is characteristic in  $M$  and  $M$  contains at least one  $h$ -plane, that is, a vector  $l$  such that  $l^2 = 3$  and  $l \cdot h = 1$ .

We denote by  $S := S(M)$  the index 3 extension of the lattice  $h^\perp \oplus \mathbb{Z}h$ ,  $\bar{h}^2 = 12$ , by the vector  $\frac{1}{3}(3l - h + \bar{h})$ ; this extension does not depend on the choice of an  $h$ -plane  $l$ . A plane in  $S$  is a vector  $l \in S$  such that  $l^2 = l \cdot \bar{h} = 4$ .

The following statement is immediate.

PROPOSITION 3.1. *The lattice  $S$  constructed above has the following properties:*

- (i) *the lattice  $S$  is even and positive definite;*
- (ii)  *$\bar{h} \in 4S^\vee$ ;*
- (iii) *the orthogonal complements  $h^\perp \subset M$  and  $\bar{h}^\perp \subset S$  are identical; hence, in particular, there is a canonical bijection between their sets of roots.*

Furthermore, the map  $x \mapsto \frac{1}{3}(3x - h + \bar{h})$  establishes bijections between:

- (iv)  *$h$ -planes in  $M$  and planes in  $S$ ,*
- (v) *vectors as in Theorem 2.3(iii) and  $e \in S$  such that  $e^2 = 2$  and  $e \cdot \bar{h} = 4$ .*

*Remark 3.2.* In view of item (ii) in Proposition 3.1, for any root  $e \in S$ , we have  $e \cdot \bar{h} \in \{0, \pm 4\}$ ; hence, either  $e \in \bar{h}^\perp$  or  $\pm e$  is as in item (v). We are mainly interested in lattices  $S$  not containing either of these two classes of vectors; it follows that this condition is equivalent to the requirement that  $S$  should be root free.

PROPOSITION 3.3. *If  $M$  admits a primitive embedding to  $\mathbf{L}$  with even orthogonal complement, then  $S$  admits an embedding to a Niemeier lattice  $N$  such that the torsion of  $N/S$  is a 2-group.*

*Proof.* The proof relies upon V. V. Nikulin's theory of discriminant forms (see [Nik79]). Let  $\rho = \text{rk } M$  and  $T = M^\perp \subset \mathbf{L}$ . Then, since  $h^\perp \subset M$  is the orthogonal complement of  $\mathbb{Z}h \oplus T$  in the unimodular lattice  $\mathbf{L}$ , we have

$$\text{discr } h^\perp \simeq \langle \eta \rangle \oplus \text{discr}(-T),$$

where  $3\eta = 0$  and  $\eta^2 = \frac{2}{3} \pmod{2\mathbb{Z}}$ ; we use the assumption that  $M$  contains an  $h$ -plane, so that  $h \notin 3M^\vee$  and, hence, the sublattice  $\mathbb{Z}h \oplus T$  is primitive in  $\mathbf{L}$ . The passage from  $h^\perp$  to  $S$  changes the discriminant to

$$\text{discr } S = \langle \frac{1}{4}\bar{h} \rangle \oplus \text{discr}(-T), \quad (\frac{1}{4}\bar{h})^2 = \frac{3}{4} \pmod{2\mathbb{Z}},$$

which almost satisfies the hypotheses of [Nik79, Theorem 1.12.2]. The only difficulty is in item (4) of loc. cit.: if  $\ell(\text{discr}_2 T) = \text{rk } T$  and  $\text{discr}_2 T$  is even, then  $\text{discr}_2 S$  is also even and has a wrong determinant

$$\pm 3|\text{discr } S| \pmod{(\mathbb{Q}_2^\times)^2}$$

instead of  $\pm|\text{discr } S| \pmod{(\mathbb{Q}_2^\times)^2}$ . However, since  $\ell(\text{discr}_2 S) = 24 - \rho \geq 3$ , we may pass to an appropriate iterated index 2 extension and either make  $\text{discr } S$  odd or reduce its length. Indeed, the classification of discriminant 2-forms (see, for example, [Nik79]) implies that, with the exception of  $\pm 3[\frac{1}{2}]$  and a few forms of length  $\ell \leq 2$ , any such form contains an isotropic element and, hence, can be reduced to a smaller one.  $\square$

The construction above is invertible: starting from a pair  $S \ni \bar{h}$ , where  $S$  is a positive definite even lattice and  $\bar{h}^2 = 12$ , and assuming that  $\bar{h} \in 4S^\vee$ , one can construct a unique 3-polarized

lattice  $M \ni h$  such that  $S = S(M)$ . However, the converse of Proposition 3.3 does not hold, and we state it as an extra restriction.

PROPOSITION 3.4. *Let  $S$  be a positive definite even lattice, and let  $\tilde{h} \in S \cap 4S^\vee$  be a vector of square 12. Then, the 3-polarized lattice  $M \ni h$  obtained from  $S \ni \tilde{h}$  by the inverse construction admits a primitive embedding to  $\mathbf{L}$  with even orthogonal complement if and only if:*

(i)  $\text{rk } S \leq 21$  and

the discriminant  $\mathcal{S} := \text{discr } S$  has the following properties at each prime  $p$ , where we use the notation  $\delta := 23 - \text{rk } S \geq 2$ :

- (ii) if  $p > 2$ , then either  $\ell(\mathcal{S}_p) < \delta$ , or  $\ell(\mathcal{S}_p) = \delta$  and  $\det \mathcal{S}_p = |\mathcal{S}| \pmod{(\mathbb{Q}_p^\times)^2}$ ;
- (iii) if  $p = 2$ , then  $\ell(\mathcal{S}_2) \leq \delta + 1$  and, in the case of equality  $\ell(\mathcal{S}_2) = \delta + 1$ , either  $\mathcal{S}$  is odd, or  $\det \mathcal{S}_2 = \pm 3|\mathcal{S}| \pmod{(\mathbb{Q}_2^\times)^2}$ .

*Proof.* We use [Nik79, Theorem 1.12.2]. Under the assumption that  $\tilde{h} \in 4S^\vee$ , there is a splitting  $S = \langle \frac{1}{4}\tilde{h} \rangle \oplus \mathcal{T}$ , and we merely restate the restrictions on  $\mathcal{T} \cong \text{discr } M$  in terms of  $\mathcal{S}$ .  $\square$

### 3.2 Admissible and geometric sets

In view of Propositions 3.1 and 3.3, we can replace the lattice  $M_X$  of a smooth cubic  $X \subset \mathbb{P}^5$  with the corresponding lattice  $S(M_X)$  and construct the latter directly in an appropriate Niemeier lattice, as the span of its set of planes.

Thus, we fix a Niemeier lattice  $N$  and a vector  $\tilde{h} \in N$ ,  $\tilde{h}^2 = 12$ , and consider the set of *planes*

$$\mathfrak{F}(\tilde{h}) = \{l \in N \mid l^2 = 4, l \cdot \tilde{h} = 4\}.$$

For any subset  $\mathfrak{L} \subset \mathfrak{F}(\tilde{h})$ , we define its *span*

$$\text{span}_2 \mathfrak{L} = (\mathbb{Z}_2 \mathfrak{L} + \mathbb{Z}_2 \tilde{h}) \cap N \subset N$$

(where the intersection is in  $N \otimes \mathbb{Z}_2$ ). The *rank* of  $\mathfrak{L}$  is  $\text{rk } \mathfrak{L} := \text{rk span}_2 \mathfrak{L}$ , and we say that  $\mathfrak{L}$  is *generated* by a subset  $\mathfrak{L}' \subset \mathfrak{L}$  if  $\mathfrak{L} = \mathfrak{F}(\tilde{h}) \cap \text{span}_2 \mathfrak{L}'$ .

By definition, the torsion of  $N/\text{span}_2 \mathfrak{L}$  is a 2-group and  $\tilde{h} \in 4(\text{span}_2 \mathfrak{L})^\vee$ . A finite-index extension  $S \supset \text{span}_2 \mathfrak{L}$  in  $N$  is called *mild* if  $\tilde{h} \in 4S^\vee$  and  $S$  is root free; cf. Remark 3.2.

A subset  $\mathfrak{L} \subset \mathfrak{F}(\tilde{h})$  is called *complete* if  $\mathfrak{L} = \mathfrak{F}(\tilde{h}) \cap \text{span}_2 \mathfrak{L}$ ; it is called *saturated* if the identity  $\mathfrak{L} = \mathfrak{F}(\tilde{h}) \cap S$  holds for any mild extension  $S$  of  $\text{span}_2 \mathfrak{L}$ .

DEFINITION 3.5. A subset  $\mathfrak{L} \subset \mathfrak{F}(\tilde{h})$  is called *admissible* if  $\text{span}_2 \mathfrak{L}$  is root free; cf. Remark 3.2. A complete admissible subset  $\mathfrak{L}$  is *pseudo-geometric* if  $S = \text{span}_2 \mathfrak{L}$  satisfies conditions (i) and (ii) in Proposition 3.4; it is called *geometric* if  $\text{span}_2 \mathfrak{L}$  admits a mild extension  $S$  satisfying all hypotheses of Proposition 3.4 and such that  $\mathfrak{L} = \mathfrak{F}(\tilde{h}) \cap S$ .

Since the lattice  $N$  is positive definite, we have  $-1 \leq l_1 \cdot l_2 \leq 3$  for any two distinct planes  $l_1$  and  $l_2$ . If  $\mathfrak{L} \subset \mathfrak{F}(\tilde{h})$  is admissible, then

$$l_1 \cdot l_2 \in \{0, 1, 2\} \text{ for any distinct planes } l_1, l_2 \in \mathfrak{L}. \tag{3.6}$$

(Indeed, if  $l_1 \cdot l_2 = 3$  or  $-1$ , then, respectively,  $e := l_1 - l_2$  or  $\tilde{h} - l_1 - l_2$  is a root.) Thus, we can regard  $\mathfrak{L}$  as a graph, connecting two vertices  $l_1 \neq l_2$  by an edge of multiplicity  $l_1 \cdot l_2 - 1$  (cf. the description of the graph of planes in §2.4).

Two 12-polarized Niemeier lattices  $N_i \ni \tilde{h}_i$ ,  $i = 1, 2$ , are called *equivalent* if the corresponding polarized lattices  $\text{span}_2 \mathfrak{F}(\tilde{h}_i) \ni \tilde{h}_i$  are isomorphic and both torsions  $\text{Tors}(N_i/\mathfrak{F}(\tilde{h}_i))$

are 2-groups. Clearly, equivalent polarized lattices share the same collections of admissible and pseudo-geometric sets of planes. A priori, this is not true for geometric sets.

### 3.3 Orbits, counts and bounds

Let  $N \ni \hbar$  be a 12-polarized Niemeier lattice as in §3.2, and let  $O(N) \supset R(N)$  be the full orthogonal group of  $N$  and its subgroup generated by reflections, respectively. We denote by  $O_{\hbar}(N) \supset R_{\hbar}(N)$  the stabilizers of  $\hbar$  in these two groups. The stabilizers act on  $\mathfrak{F}(\hbar)$ , and hence  $\mathfrak{F}(\hbar)$  splits into  $O_{\hbar}(N)$ -orbits  $\bar{\mathfrak{o}}_n$ , each orbit splitting into  $R_{\hbar}(N)$ -orbits  $\mathfrak{o} \subset \bar{\mathfrak{o}}_n$ , called *combinatorial orbits*. The number of combinatorial orbits in an orbit  $\bar{\mathfrak{o}}_n$  is denoted by  $m(\bar{\mathfrak{o}}_n)$ , and the set of all combinatorial orbits is denoted by  $\mathfrak{D} := \mathfrak{D}(\hbar)$ . This set inherits a natural action of the group

$$\text{stab } \hbar := O_{\hbar}(N)/R_{\hbar}(N),$$

which preserves each orbit  $\bar{\mathfrak{o}}_n$ . (By an obvious abuse of notation, occasionally  $\bar{\mathfrak{o}}_n$  is treated as a subset of  $\mathfrak{D}$ , whereas subsets of  $\mathfrak{D}$  are treated as sets of planes.)

For a subset  $\mathfrak{C} \subset \mathfrak{D}$ , let

$$\mathfrak{B}(\mathfrak{C}) := \{\mathfrak{L} \cap \mathfrak{C} \mid \mathfrak{L} \subset \mathfrak{F}(\hbar) \text{ is pseudo-geometric}\}, \quad \mathfrak{b}(\mathfrak{C}) := \{|\mathfrak{L}| \mid \mathfrak{L} \in \mathfrak{B}(\mathfrak{C})\}.$$

Since the pseudo-geometric property is obviously inherited by complete subsets, in the computation we can confine ourselves to the pseudo-geometric sets  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$  generated by  $\mathfrak{L} \cap \mathfrak{C}$ .

Following [Deg22b], define the *count*  $c(\mathfrak{o})$  and *bound*  $b(\mathfrak{o})$  of a single combinatorial orbit  $\mathfrak{o} \in \mathfrak{D}$  via

$$c(\mathfrak{o}) := |\mathfrak{o}|, \quad b(\mathfrak{o}) := \max \mathfrak{b}(\mathfrak{o}).$$

Usually, the bound  $b(\mathfrak{o})$  is replaced by its rough estimate computed as explained in Theorem 4.1 below. Clearly,  $c$  and  $b$  are constant within each orbit  $\bar{\mathfrak{o}}_n$ . We extend these notions to subsets  $\mathfrak{C} \subset \mathfrak{D}$  by *additivity*:

$$c(\mathfrak{C}) := \sum_{\mathfrak{o} \in \mathfrak{C}} c(\mathfrak{o}), \quad b(\mathfrak{C}) := \sum_{\mathfrak{o} \in \mathfrak{C}} b(\mathfrak{o}).$$

Thus, we have a naïve a priori bound

$$|\mathfrak{L}| \leq b(\mathfrak{D}) = \sum m(\bar{\mathfrak{o}}_n) b(\mathfrak{o}), \quad \mathfrak{o} \subset \bar{\mathfrak{o}}_n. \tag{3.7}$$

Clearly, the true count  $|\mathfrak{L} \cap \mathfrak{C}|$  is genuinely additive, whereas the true sharp bound  $\max \mathfrak{b}(\mathfrak{C}) \leq b(\mathfrak{C})$  is only subadditive; thus, our proof of Theorem 1.1 will essentially consist in reducing (3.7) down to a certain preset goal  $\gamma$ . To this end, we will consider the set

$$\mathcal{B} = \mathcal{B}(\mathfrak{F}(\hbar)) := \{\mathfrak{L} \subset \mathfrak{F}(\hbar) \mid \mathfrak{L} \text{ is geometric}\} / O_{\hbar}(N)$$

and, for a collection of orbits  $\mathfrak{C} = \bar{\mathfrak{o}}_1 \cup \dots$  and integer  $d \in \mathbb{N}$ , let

$$\mathcal{B}_d(\mathfrak{C}) := \{[\mathfrak{L}] \in \mathcal{B} \mid \mathfrak{L} \text{ is generated by } \mathfrak{L} \cap \mathfrak{C} \text{ and } |\mathfrak{L} \cap \mathfrak{C}| \geq b(\mathfrak{C}) - d\}.$$

We will also consider the oversets  $\tilde{\mathcal{B}}_d(\mathfrak{C}) \supset \mathcal{B}_d(\mathfrak{C})$  consisting of pseudo-geometric (rather than geometric) sets. The computation of these sets is discussed in §4.5.

### 3.4 Idea of the proof

Fix a 12-polarized Niemeier lattice  $N \ni \hbar$  and a goal

$$|\mathfrak{L}| \geq \gamma \quad (\text{typically, } \gamma = 351). \tag{3.8}$$

We need to list all geometric sets  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$  satisfying this inequality. Clearly, such sets may exist only if  $b(\mathfrak{D}) \geq \gamma$ , where  $b(\mathfrak{D})$  is the naïve bound given by (3.7); otherwise, the pair  $N \ni \hbar$  does not need to be considered.

In the few remaining cases, we use brute force and, for each combinatorial orbit  $\mathfrak{o}$ , compute the  $R_{\hbar}(N)$ -orbits on the set  $\mathfrak{B}(\mathfrak{o})$ . (Obviously, it suffices to consider one representative of each orbit  $\bar{\mathfrak{o}}_n$ ; the rest is obtained by translation.) In particular, we obtain sharp bounds  $b(\mathfrak{o})$  and sets of values

$$\mathfrak{b}(\mathfrak{o}) = \{b(\mathfrak{o}) > b'(\mathfrak{o}) > \dots\}. \quad (3.9)$$

This may yield a better bound  $b(\mathfrak{D})$  given by (3.7); this improved bound is used in the subsequent computation. If still  $b(\mathfrak{D}) \geq \gamma$ , we choose a collection of pairwise disjoint unions of orbits  $\mathfrak{C}_1, \dots, \mathfrak{C}_m \subset \mathfrak{D}$  and integers  $d_1, \dots, d_m \geq 0$  such that

$$d_1 + \dots + d_m + m > b(\mathfrak{D}) - \gamma.$$

Then, clearly, any geometric set  $\mathfrak{L}$  satisfying (3.8) is in the union

$$\mathcal{E} := \mathcal{B}_{d_1}(\mathfrak{C}_1) \cup \dots \cup \mathcal{B}_{d_m}(\mathfrak{C}_m),$$

and the same assertion holds for pseudo-geometric sets, with  $\mathcal{B}_d$  replaced by  $\tilde{\mathcal{B}}_d$ . We try to fix the choices so that the union  $\mathcal{E}$  consists of relatively few sufficiently large sets; then, these exceptional sets are analyzed one by one using one of the arguments described below.

**3.4.1 Maximal sets** (see [Deg22b]). If a set  $\mathfrak{L} \in \mathcal{E}$  is saturated and the rank  $\text{rk } \mathfrak{L} = 21$  is maximal, cf. Proposition 3.4(i), then  $\mathfrak{L}$  has no proper geometric extension; hence, this set can be either disregarded if  $|\mathfrak{L}| < \gamma$ , or listed as an exception in the respective statement.

**3.4.2 Extension by a maximal orbit** (see [Deg22b]). In many cases, a set  $\mathfrak{L} \in \mathcal{B}_d(\mathfrak{C})$  has the property that

$$\sum (b(\mathfrak{o}) - b'(\mathfrak{o})) \geq b(\mathfrak{D}) - \gamma, \quad \mathfrak{o} \in \mathfrak{D}_{\delta} := \{\mathfrak{o} \in \mathfrak{D} \mid |\mathfrak{L} \cap \mathfrak{o}| < b(\mathfrak{o})\};$$

see (3.9). This implies that any (pseudo-)geometric extension  $\mathfrak{L}' \supset \mathfrak{L}$  satisfying (3.8) must have maximal intersection,  $|\mathfrak{L}' \cap \mathfrak{o}| = b(\mathfrak{o})$ , with at least one orbit  $\mathfrak{o} \in \mathfrak{D}_{\delta}$ . Trying these orbits one by one, we obtain larger sets, which are usually maximal; see §3.4.1. Note that here we always assume  $\mathfrak{L} \cap \mathfrak{C}$  fixed; that is, we accept only those extensions  $\mathfrak{L}' \supset \mathfrak{L}$  that have the property

$$\mathfrak{L}' \supset \mathfrak{L} \text{ pseudo-geometric and } \mathfrak{L}' \cap \mathfrak{C} = \mathfrak{L} \cap \mathfrak{C}. \quad (3.10)$$

Indeed, otherwise, we would have started with a larger set  $\mathfrak{L}'' \in \mathcal{B}_d(\mathfrak{C})$ , namely the one generated by  $\mathfrak{L}' \cap \mathfrak{C}$ . Thus, for the computation, we merely extend the restricted pattern

$$\pi_{\mathfrak{L}}: \mathfrak{C} \rightarrow \mathbb{N}, \quad \mathfrak{o} \mapsto |\mathfrak{L} \cap \mathfrak{o}|$$

(see §4.5.1 below), by a single extra value  $\mathfrak{o}' \mapsto b(\mathfrak{o}')$  for some orbit  $\mathfrak{o}' \in \mathfrak{D}_{\delta} \setminus \mathfrak{C}$  and perform one extra step of the algorithm. We make use of the symmetry of  $\mathfrak{L}$ , trying for  $\mathfrak{o}'$  a single representative of each orbit of the action on  $\mathfrak{D}_{\delta} \setminus \mathfrak{C}$  of the  $(\text{stab } \hbar)$ -stabilizer of  $\pi_{\mathfrak{L}}$ .

**3.4.3 Maximal orbit count.** For smaller sets  $\mathfrak{L} \in \mathcal{B}_d(\mathfrak{C})$ , we choose an appropriate *test set*  $\mathfrak{T} \subset \mathfrak{D} \setminus \mathfrak{C}$  (typically, also a union of orbits) and use the same techniques as in §3.4.2 to compute the set

$$\mathfrak{T}_{\mu} := \{\mathfrak{o} \in \mathfrak{T} \mid \mathfrak{L}' \cap \mathfrak{o} = b(\mathfrak{o}) \text{ for some extension } \mathfrak{L}' \supset \mathfrak{L} \text{ satisfying (3.10)}\}.$$

Then, if

$$(b(\mathfrak{C}) - |\mathfrak{L} \cap \mathfrak{C}|) + \sum (b(\mathfrak{o}) - b'(\mathfrak{o})) \geq b(\mathfrak{D}) - \gamma, \quad \mathfrak{o} \in \mathfrak{T} \setminus \mathfrak{T}_\mu,$$

see (3.9), we conclude that  $\mathfrak{L}$  has no extensions satisfying (3.8) and (3.10).

#### 4. Algorithms

In the first four subsections, we describe a rough estimate on the bounds  $b(\mathfrak{o})$  in the Niemeier lattices with many roots (and, hence, large combinatorial orbits). Then, in §4.5, we explain the algorithms used to compute the sets  $\mathcal{B}_d(\mathfrak{C})$  in those few cases where the rough estimates do not suffice.

##### 4.1 Bounds via blocks (see [Deg22b])

Let  $N := N(D) = N(\bigoplus_k D_k)$  be a Niemeier lattice rationally generated by roots, where the *blocks*  $D_k$ ,  $k \in \Omega$ , are the irreducible components of the *maximal* root system  $D \subset N$  and  $\Omega$  is the index set. Thus, we have  $N \subset D^\vee = \bigoplus_k D_k^\vee$ ; the vectors in  $\text{discr } D = D^\vee/D = \bigoplus_k \text{discr } D_k$  that are declared “integral” are described in [CS88]. (We also use the convention of [CS88] for the numbering of the discriminant classes of irreducible root systems.)

Let  $|_k: N \rightarrow D_k^\vee$  be the orthogonal projection. For a vector  $v \in N$ , we often use the abbreviation  $v_k := v|_k$ , so that  $v = \sum_k v_k$ ,  $v_k \in D_k^\vee$ . Define the *support*

$$\text{supp } v := \{k \in \Omega \mid v_k \neq 0\}.$$

The group  $R_{\bar{h}}(N)$  preserves each block  $D_k$ , and hence we can also speak about the *support*  $\text{supp } \mathfrak{o} \subset \Omega$  of a combinatorial orbit  $\mathfrak{o}$ . (It is worth mentioning that, for each  $k \in \Omega$ , the squares  $l^2, l_k^2 \in \mathbb{Q}$ , products  $l \cdot \bar{h}, l_k \cdot \bar{h}_k \in \mathbb{Q}$  and respective discriminant classes  $l \bmod D \in \text{discr } D$  and  $l_k \bmod D_k \in \text{discr } D_k$ ,  $l \in \mathfrak{o}$ , are also constant within each combinatorial orbit  $\mathfrak{o}$ .)

Fix a combinatorial orbit  $\mathfrak{o}$ , and define the *count* and *bound* of a block  $D_k$  via

$$c(D_k) := |\mathfrak{o}|_k, \quad b(D_k) := \max|\mathfrak{R}|,$$

where  $\mathfrak{R} \subset \mathfrak{o}|_k$  is a subset satisfying the condition

$$\text{for } l', l'' \in \mathfrak{R}, \text{ one has } l'^2 - l' \cdot l'' = 0 \text{ (if and only if } l' = l''), \text{ 2, 3 or 4.} \quad (4.1)$$

In other words, we bound the cardinality of subsets  $\mathfrak{L} \subset \mathfrak{o}$  satisfying (3.6) and such that all planes  $l \in \mathfrak{L}$  have the same fixed restriction to all other blocks  $D_s \neq D_k$ . Then, we have (cf. [Deg22b])

$$c(\mathfrak{o}) = \prod_k c(D_k), \quad b(\mathfrak{o}) \leq c(\mathfrak{o}) \min_k \frac{b(D_k)}{c(D_k)}. \quad (4.2)$$

For smaller blocks ( $\mathbf{A}_{\leq 7}$ ,  $\mathbf{D}_{\leq 6}$  and most  $\mathbf{E}$ -type blocks), the individual counts  $c(D_k)$  and bounds  $b(D_k)$  are computed by brute force, and the resulting estimates (4.2) suit most of our needs. For larger blocks, we use even rougher estimates, based on the standard representation of the  $\mathbf{A}$ - and  $\mathbf{D}$ -type root systems as sublattices of the odd unimodular lattice

$$\mathbf{H}_n := \bigoplus \mathbb{Z}e_i, \quad e_i^2 = 1, \quad i \in \mathcal{I} := \{1, \dots, n\}.$$

(When working with this lattice, we let  $\bar{1}_o := \sum_{i \in o} e_i$  for a subset  $o \subset \mathcal{I}$ .) Then, given a vector  $\bar{h}_k = \sum_i \alpha_i e_i \in \mathbf{H}_n \otimes \mathbb{Q}$ , we subdivide the block  $D_k^\vee \subset \mathbf{H}_n \otimes \mathbb{Q}$  into “subblocks”

$$D_k(\alpha) := \{\sum_i \beta_i e_i \in \mathbf{H}_n \otimes \mathbb{Q} \mid i \in \text{supp}(\alpha)\}, \quad \text{supp}(\alpha) := \{i \in \mathcal{I} \mid \alpha_i = \alpha\}$$

on which  $\hbar_k$  is constant. We obtain combinatorial counts and bounds, in the sense of (4.1), for each subblock and use an obvious analogue of (4.2) to estimate  $b(D_k)$ . The technical details are outlined in the next few subsections. We use, without further references, the following simple observation.

LEMMA 4.3. *If  $l \in \mathfrak{F}(\hbar)$  and  $l \bmod D \neq 0 \in \text{discr } D$ , then each  $l_k$ ,  $k \in \Omega$ , is a shortest vector in its discriminant class  $l_k \bmod D_k \in \text{discr } D_k$ .*

*Proof.* Otherwise, the *nontrivial* discriminant class  $l \bmod D$  would contain a shorter vector  $v$ , necessarily of square  $l^2 - 2 = 2$ , contradicting the assumption that all roots in  $N$  are in  $D$ .  $\square$

#### 4.2 Root systems $\mathbf{A}_n$

A block  $D_k$  of type  $\mathbf{A}_n$  is  $\bar{1}_{\mathcal{I}} \subset \mathbf{H}_{n+1}$ :

$$\mathbf{A}_n = \{ \sum_i \alpha_i e_i \in \mathbf{H}_{n+1} \mid \sum_i \alpha_i = 0 \}.$$

One has  $\text{discr } \mathbf{A}_n = \mathbb{Z}/(n+1)$ , with a generator of square  $n/(n+1) \bmod 2\mathbb{Z}$ , and the shortest representatives of the discriminant classes are vectors of the form

$$\bar{e}_o := \frac{1}{n+1} (|\bar{o}|\bar{1}_o - |o|\bar{1}_{\bar{o}}), \quad \bar{e}_o^2 = \frac{|o||\bar{o}|}{n+1},$$

where  $o \subset \mathcal{I}$  and  $\bar{o}$  is the complement. We have  $\bar{e}_{\bar{o}} = -\bar{e}_o$  and

$$\bar{e}_r \cdot \bar{e}_s = |r \cap s| - \frac{|r||s|}{n+1}.$$

In particular, if  $|r| = |s|$  or, equivalently,  $e_r$  and  $e_s$  are in the same discriminant class, then

$$\bar{e}_r^2 - \bar{e}_r \cdot \bar{e}_s = \frac{1}{2} |r \Delta s|, \tag{4.4}$$

where  $\Delta$  is the symmetric difference. Hence in the case when  $l_k$  is a shortest vector in its (nonzero) discriminant class, the bound  $b(D_k(\alpha))$  can be estimated by the following lemma, applied to  $\Sigma = \text{supp}(\alpha)$ .

LEMMA 4.5. *Consider a finite set  $\Sigma$ ,  $|\Sigma| = n$ , and let  $\mathfrak{S}$  be a collection of subsets  $s \subset \Sigma$  with the following properties:*

- (i) *all subsets  $s \in \mathfrak{S}$  have the same fixed cardinality  $m$ ;*
- (ii) *if  $r, s \in \mathfrak{S}$ , then  $|r \Delta s| \in \{0, 4, 6, 8\}$ .*

*Then, for small  $(n, m)$ , the maximum  $\mathcal{A}_{m,n} := \max|\mathfrak{S}|$  is as follows:*

$$\begin{array}{cccccccc} (n, m): & (n, 1) & (n, 2) & (6, 3) & (7, 3) & (8, 3) & (9, 3) & (8, 4) \\ \mathcal{A}_{m,n}: & 1 & \lfloor n/2 \rfloor & 4 & 7 & 8 & 12 & 14 \end{array}$$

*More generally,*

$$\mathcal{A}_{3,n} \leq \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor, \quad \mathcal{A}_{m,n} \leq \left\lfloor \frac{1}{m} \binom{n}{m-1} \right\rfloor \quad \text{for } m \geq 1.$$

Note that if a collection  $\mathfrak{S}$  is as in Lemma 4.5, then so is the collection  $\{\bar{s} \mid s \in \mathfrak{S}\}$ . Hence  $\mathcal{A}_{m,n} = \mathcal{A}_{n-m,n}$ , and we can always assume  $2m \leq n$ .

*Proof.* The first two values are obvious; the others are obtained by listing all admissible collections.

The general estimate for  $m = 3$  follows from the observation that any two subsets in  $\mathfrak{S}$  have at most one common point, and hence each point of  $\Sigma$  is contained in at most  $\lfloor \frac{1}{2}(n-1) \rfloor$  such subsets.

For the last bound, we merely observe that each  $(m-1)$ -element set  $r \subset \Sigma$  is contained in at most one set  $s \in \mathfrak{S}$ .  $\square$

There remains to consider a subblock  $D_k(\alpha)$  of a block  $D_k$  containing vectors of the form  $l_k = \bar{1}_r - \bar{1}_s$ , where  $r, s \subset \mathcal{I}$ ,  $r \cap s = \emptyset$ , and  $|r| = |s| = 1$  or  $2$ . In the latter case, one must have  $l_k \cdot \bar{h}_k = 4$ , and it follows that  $|(r \cup s) \cap \text{supp}(\alpha)| \leq 3$  for each  $\alpha \in \mathbb{Q}$ . Letting  $r_\alpha := r \cap \text{supp}(\alpha)$  and  $s_\alpha := s \cap \text{supp}(\alpha)$ , the bounds are as follows:

- (i) if  $|r_\alpha| + |s_\alpha| = 1$ , then, obviously,  $b(D_k(\alpha)) = 1$ ;
- (ii) if  $(|r_\alpha|, |s_\alpha|) = (2, 0)$  or  $(0, 2)$ , then distinct sets  $r_\alpha$  (respectively,  $s_\alpha$ ) must be pairwise disjoint, and hence  $b(D_k(\alpha)) = \lfloor \frac{1}{2}|\text{supp}(\alpha)| \rfloor$ ;
- (iii) if  $|r_\alpha| = |s_\alpha| = 1$ , then distinct sets  $r_\alpha$  must also be pairwise disjoint, and hence  $b(D_k(\alpha)) = |\text{supp}(\alpha)|$ ;
- (iv) if  $(|r_\alpha|, |s_\alpha|) = (2, 1)$  or  $(1, 2)$ , then  $b(D_k(\alpha)) = |\text{supp}(\alpha)| = 3$ .

### 4.3 Root systems $\mathbf{D}_n$

A block  $D_k$  of type  $\mathbf{D}_n$  can be defined as the maximal even sublattice in  $\mathbf{H}_n$ :

$$\mathbf{D}_n = \left\{ \sum_i \alpha_i e_i \in \mathbf{H}_n \mid \sum_i \alpha_i = 0 \pmod{2} \right\}.$$

If  $n \geq 5$ , the group  $O(\mathbf{D}_n)$  is an index 2 extension of  $R(\mathbf{D}_n)$ : it is generated by the reflection against the hyperplane orthogonal to any of the vectors  $e_i$ . Hence, up to  $O(\mathbf{D}_n)$ , we can assume that in the expression  $\bar{h}_k = \sum_i \alpha_i e_i$ , all coefficients satisfy  $\alpha_i \geq 0$ . We always make this assumption (and adjust the results afterwards) when describing the orbits and computing counts and bounds, as otherwise the description of combinatorial orbits is not quite combinatorial.

One has  $\text{discr } \mathbf{D}_n = \mathbb{Z}/2 \oplus \mathbb{Z}/2$  (if  $n$  is even) or  $\mathbb{Z}/4$  (if  $n$  is odd); the shortest vectors are

$$e_i, i \in \mathcal{I}, \quad \text{and} \quad \bar{e}_o := \frac{1}{2}(\bar{1}_o - \bar{1}_{\bar{o}}), \quad o \subset \mathcal{I}, \quad \bar{e}_o^2 = \frac{n}{4}$$

(the class  $\bar{e}_o \pmod{\mathbf{D}_n}$  depends on the parity of  $|o|$ ), and we have a literal analogue of (4.4) for any pair  $r, s \subset \mathcal{I}$ . Thus, if  $D_k \ni \bar{e}_o$ , the bounds  $b(D_k(\alpha))$  are estimated by Lemma 4.5 (if  $\alpha \neq 0$ ) or Lemma 4.6 below (if  $\alpha = 0$ ), applied to  $\Sigma = \text{supp}(\alpha)$ .

LEMMA 4.6. *The maximal cardinality of a collection  $\mathfrak{S}$  satisfying condition (ii) of Lemma 4.5 is bounded via*

$$|\mathfrak{S}| \leq \max_{m \geq 0} (\mathcal{A}_{m,n} + \mathcal{A}_{m+2,n} + \mathcal{A}_{m+4,n} + \mathcal{A}_{m+6,n} + \mathcal{A}_{m+8,n}),$$

where  $\mathcal{A}_{m,n}$  is as in Lemma 4.5 and we let  $\mathcal{A}_{m,n} = 0$  unless  $0 \leq m \leq n$ .

*Proof.* It suffices to observe that all sets  $s \in \mathfrak{S}$  have cardinality of the same parity and that  $||s| - |r|| \leq 8$  for any pair  $r, s \in \mathfrak{S}$ .  $\square$

The few remaining cases are listed below.

- (i) If  $D_k(\alpha) \ni \pm 2e_i$ ,  $i \in \text{supp}(\alpha)$ , then  $b(D_k(\alpha)) = |\text{supp}(\alpha)|$ .

Assume  $l_k = \sum(\pm e_i)$ ,  $i \in o \subset \mathcal{S}$ ,  $|o| \leq 4$ . If  $\alpha = 0$ , then

- (ii)  $|o \cap \text{supp}(\alpha)| = 0, 1, 2$  and  $b(D_k(\alpha)) \leq 1, 2, 4 \lfloor \frac{1}{2}|\text{supp}(\alpha)| \rfloor$ , respectively,

similarly to the results of § 4.2. (Here, the last number is a bound on the size of a union of (affine) Dynkin diagrams admitting an isometry to  $\mathbf{D}_{|\text{supp}(\alpha)|}$ ; cf. § 4.4.1 below.)

If  $\alpha \neq 0$ , the numbers of signs  $\pm$  within  $\text{supp}(\alpha)$  are also fixed, and the options are as follows:

- (iii)  $m := |\sigma \cap \text{supp}(\alpha)| \leq 3$ , and all signs are the same: by an analogue of (4.4), a bound on  $b(D_k(\alpha))$  is given by Lemma 4.5 applied to  $\Sigma = \text{supp}(\alpha)$ ;
- (iv)  $|\sigma \cap \text{supp}(\alpha)| = 2$ , and the signs differ:  $b(D_k(\alpha)) = |\text{supp}(\alpha)|$  as in § 4.2(iii);
- (v)  $|\sigma \cap \text{supp}(\alpha)| = 3$ , and the signs differ:  $b(D_k(\alpha)) = |\text{supp}(\alpha)| = 3$ .

#### 4.4 Other root lattices

We use the description of § 4.2 for the blocks  $\mathbf{A}_n$ ,  $n \geq 8$ , and that of § 4.3 for  $\mathbf{D}_n$ ,  $n \geq 7$ . Below we outline a few more tricks that simplify the computations for other blocks, mainly those of type  $\mathbf{E}_8$ .

4.4.1 *Integral roots.* Assume  $\bar{h}_k = 0$  and that  $l_k$  is an integral root; that is,  $l_k^2 = 2$  and  $l_k \in D_k$ . By (4.1), the pairwise products of roots take values in  $\{0, -1, -2\}$ ; that is, an admissible subset in  $D_k$  is a union  $\Gamma$  of (affine) Dynkin diagrams (including those of type  $\tilde{\mathbf{A}}_1$ ) that is mapped isometrically to  $D_k$ . If  $D_k \cong \mathbf{E}_8, \mathbf{E}_7$  or  $\mathbf{D}_{\text{even}}$ , cf. § 4.3(ii), then  $D_k$  is rationally generated by pairwise orthogonal roots, and hence the maximal union  $\Gamma$  as above is  $n\tilde{\mathbf{A}}_1$ ,  $n = \text{rk } D_k$ . Thus, we have  $b(D_k) = 2n$  in this case.

If  $D_k \cong \mathbf{E}_8$  and  $\bar{h}_k, l_k \in D_k$  are orthogonal integral roots,  $\bar{h}_k^2 = l_k^2 = 2$ ,  $l_k \cdot \bar{h}_k = 0$ , we can apply the above argument to  $\bar{h}_k^\perp \cong \mathbf{E}_7$  to obtain  $b(D_k) = 14$ . If  $l_k \cdot \bar{h}_k = 1$ , the bound is  $b(D_k) \leq 4$ , which is the maximal valency of a vertex in (any) affine Dynkin diagram. If  $l_k \cdot \bar{h}_k = 2$ , then  $l_k = \bar{h}_k$  and  $c(D_k) = b(D_k) = 1$ .

4.4.2 *Planes contained in  $\mathbf{E}_8$ .* Assume  $D_k \cong \mathbf{E}_8$  and  $l_k^2 = 4$ , so that necessarily  $l = l_k$  and  $l_k \cdot \bar{h}_k = 4$  (and hence  $\bar{h}_k^2 \geq 4$ ). Below we consider two cases where a long computation can be simplified.

If  $\bar{h}_k^2 = 12$ , then  $\bar{h} = \bar{h}_k$  and all planes  $l_k$  are in the index 2 sublattice

$$\{x \in D_k \mid x \cdot \bar{h}_k = 0 \pmod{2}\} \cong \mathbf{D}_8.$$

Up to an automorphism of  $\mathbf{D}_8$ , we have  $\bar{h}_k = 2(\bar{1}_o)$ ,  $o = \{1, 2, 3\}$  and  $l_k = \bar{1}_r \pm e_i \pm e_j$ , where  $r \subset o$ ,  $|r| = 2$  and  $i, j \in \mathcal{I} \setminus o$ . Arguing as in § 4.3, we obtain  $b(D_k) \leq 24$ .

If  $\bar{h}_k^2 = 8$  and  $\bar{h}_k \in 2D_k$ , then each plane has the form  $l_k = \frac{1}{2}\bar{h}_k + r$ , where  $r$  is a root in  $\bar{h}_k^\perp \cong \mathbf{E}_7$ . Hence we have  $b(D_k) = 14$  as in § 4.4.1.

#### 4.5 The computation of $\mathcal{B}_d(\mathfrak{C})$

Most sets  $\mathcal{B}_d(\mathfrak{C})$  or  $\tilde{\mathcal{B}}_d(\mathfrak{C})$  introduced in § 3.3 are computed using so-called *patterns*; this computation is explained in this subsection.

4.5.1 *Patterns* (see [Deg22b]). The *pattern* of a pseudo-geometric set  $\mathfrak{L}$  is the function

$$\pi_{\mathfrak{L}}: \mathfrak{D} \rightarrow \mathbb{N}, \quad \mathfrak{o} \mapsto |\mathfrak{L} \cap \mathfrak{o}| \in \mathfrak{b}(\mathfrak{o}).$$

To compute a set  $\mathcal{B}_d(\mathfrak{C})$ , we start with listing, up to the action of the group  $\text{stab } \bar{h}$ , the abstract restricted patterns  $\pi: \mathfrak{C} \rightarrow \mathbb{N}$  satisfying the conditions

$$\pi(\mathfrak{o}) \in \mathfrak{b}(\mathfrak{o}), \quad \sum \pi(\mathfrak{o}) \geq b(\mathfrak{C}) - d, \quad \mathfrak{o} \in \mathfrak{C}.$$



Then, using the precomputed collections  $\mathfrak{B}(\mathfrak{o})$ , we build a (pseudo-)geometric set  $\mathfrak{L}$  orbit by orbit, as the completion of a union  $\bigcup_{\mathfrak{o} \in \mathfrak{C}} \mathfrak{L}_{\mathfrak{o}}$ ,  $\mathfrak{L}_{\mathfrak{o}} \in \mathfrak{B}(\mathfrak{o})$ ,  $|\mathfrak{L}_{\mathfrak{o}} \cap \mathfrak{o}| = \pi(\mathfrak{o})$ . In this computation, we add combinatorial orbits  $\mathfrak{o} \in \mathfrak{C}$  one by one, use the group  $R_{\hbar}(N)$  to reduce the overcounting and, at each step, make sure that the partial set  $\mathfrak{L}$  constructed is (pseudo-)geometric and that  $\pi_{\mathfrak{L}}|_{\mathfrak{C}} = \pi$ . Further details are found in [Deg22b]; we reuse the code developed there.

4.5.2 *Clusters* (see [Deg22b]). If the number  $m(\bar{\mathfrak{o}}_n)$  of combinatorial orbits in an orbit  $\bar{\mathfrak{o}}_n = \mathfrak{C}$  is large, listing all abstract patterns in § 4.5.1 is difficult. In this case, we try to subdivide  $\bar{\mathfrak{o}}_n$  into *clusters*  $\mathfrak{c}_k$ , not necessarily disjoint, consisting of whole combinatorial orbits and such that  $\text{stab } \hbar$  induces an at least 1-transitive action on the set of clusters. (The construction of clusters, usually “natural”, is described in the respective proof sections case by case.) Then, we can compute abstract patterns and construct (pseudo-)geometric sets cluster by cluster, assuming the latter ordered by the decreasing of their “complexity” (see [Deg22b] for more details). Within each cluster, we still use patterns (compatible with the clusters already considered) and extend the (pseudo-)geometric set orbit by orbit, as in § 4.5.1.

4.5.3 *Meta-patterns*. In some cases, the number  $u$  of clusters is still large (requiring too many steps in § 4.5.2), whereas each cluster is relatively small, so that we can easily compute the set  $\mathcal{B}_*(\mathfrak{c}_*)$  of  $O(\hbar)$ -orbits of geometric sets pseudo-generated by a single cluster. In these cases, we construct a geometric set

$$\mathfrak{L} = \mathfrak{L}_u \supset \mathfrak{L}_{u-1} \supset \cdots \supset \mathfrak{L}_0 = \emptyset$$

cluster by cluster, using the full symmetry group  $O_{\hbar}(N)$  and a *meta-pattern*, that is, a set of values

$$\tilde{\pi} := \{\mathfrak{T}_1 \geq \cdots \geq \mathfrak{T}_u\}, \quad \mathfrak{T}_i \in \mathcal{B}_*(\mathfrak{c}_*),$$

ordered lexicographically by the decreasing of the pair  $(|\mathfrak{t}_i|, |\mathfrak{T}_i|)$ ,  $\mathfrak{t}_i \in \mathfrak{T}_i$ , and such that  $\sum |\mathfrak{t}_i| \geq b(\mathfrak{C}) - d$ . At each step  $k$ , we pass from  $\mathfrak{L}_{k-1}$  to the sets

$$\mathfrak{L}_k := \mathfrak{F}(\hbar) \cap \text{span}_2(\mathfrak{L}_{k-1} \cup \mathfrak{t}_k),$$

taking for  $\mathfrak{t}_k \in \mathfrak{T}_k$  a single representative of each orbit of the  $O_{\hbar}(N)$ -stabilizer of  $\mathfrak{L}_{k-1}$ ; then, we select those sets  $\mathfrak{L}_k$  that are pseudo-geometric and satisfy the condition  $|\mathfrak{L}_k \cap \mathfrak{c}_i| = |\mathfrak{t}_i|$  for all  $i = 1, \dots, k$ . This algorithm is similar to § 4.5.1 (cf. [Deg22b]), except that clusters are used instead of combinatorial orbits and the full symmetry group  $O_{\hbar}(N)$  is used instead of  $R(N)$ : at the beginning, preparing the meta-patterns, we do not associate the values  $\mathfrak{T}_k$  with particular clusters  $\mathfrak{c}_k$ .

4.5.4 *Iterated maximal subsets*. Let  $\mathfrak{C}$  be a complete *admissible* set. (Typically, we take for  $\mathfrak{C}$  a union of orbits.) Clearly, any subset  $\mathfrak{L} \subset \mathfrak{C}$  is also admissible. If  $\mathfrak{L}$  is also complete, then either  $\text{span}_2 \mathfrak{L} \subset \text{span}_2 \mathfrak{C}$  has positive corank, or  $\text{span}_2 \mathfrak{C} / \text{span}_2 \mathfrak{L}$  is a 2-group, nontrivial if  $\mathfrak{L}$  is proper. It follows that any maximal (with respect to inclusion) proper complete subset  $\mathfrak{L} \subset \mathfrak{C}$  is of the form

$$\mathfrak{L} = \mathfrak{C} \cap \text{Ker } v, \quad v \in \text{Hom}(\text{span}_2 \mathfrak{C}, \mathbb{F}_2) = \text{span}_2 \mathfrak{C} / 2 \text{span}_2 \mathfrak{C},$$

and these sets can easily be listed. (Usually, we also take into account the action of the  $O_{\hbar}(N)$ -stabilizer of  $\mathfrak{C}$  on  $\text{span}_2 \mathfrak{C} / 2 \text{span}_2 \mathfrak{C}$ . Note that we do not assert that  $\text{span}_2 \mathfrak{C} / \text{span}_2 \mathfrak{L} = \mathbb{Z}/2$ .) Iterating this procedure (and eliminating repetitions at the intermediate steps), we can list all complete subsets  $\mathfrak{L} \subset \mathfrak{C}$  and, in particular, compute the sets  $\mathcal{B}_d(\mathfrak{C}) \subset \tilde{\mathcal{B}}_d(\mathfrak{C})$ .

TABLE 1. The lattice  $N(6\mathbf{D}_4)$  (cf. Convention 5.2).

<b>1:</b>	$[2]_{\circ}$	$[2]_{\circ}$	$[2]_{\circ}$	$[2]_{\circ}$	$[2]_{\circ}$	$[2]_{\circ}$	735	<b>375</b>	
1:	$[2]_{\circ}$	$[2]_{\circ}$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	15	1	<u>1</u>
2:	$[1]_3$	$[1]_3$	$[1]_3$	$[1]_3$	$[0]_0$	$[0]_0$	45	16	<u>8</u>

### 5. Lattices with few components

In this section we consider the Niemeier lattices with at most eight irreducible root components and prove the following theorem.

**THEOREM 5.1.** *Let  $N$  be a Niemeier lattice other than  $N(12\mathbf{A}_2)$ ,  $N(24\mathbf{A}_1)$  or  $\Lambda$ . Then, for any  $\hbar \in N$ ,  $\hbar^2 = 12$ , and any geometric set  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ , one has  $|\mathfrak{L}| \leq 350$ .*

*Proof.* For each 12-polarized Niemeier lattice  $N \ni \hbar$ , we list all  $O_{\hbar}(N)$ -orbits  $\bar{\sigma}_n$  and compute the number  $m(\bar{\sigma}_n)$  of combinatorial orbits  $\sigma \subset \bar{\sigma}_n$ , the count  $c(\sigma)$  and the naïve bound on  $|\mathfrak{L} \cap \sigma|$  given by (4.2). Sometimes, this bound is improved by a brute force computation; the best bound obtained is denoted by  $b(\sigma)$ . The results are listed in several tables below, where the bounds  $b(\sigma)$  confirmed by brute force are underlined.

**CONVENTION 5.2.** Listed in the tables are the isomorphism classes of vectors  $\hbar$  (the rows with bold index in the first column) and, for each  $\hbar$ , the orbits of conics. The columns are as follows:

- index for further references,
- the vector  $\hbar$  or conic, one entry for each indecomposable component of the maximal root system (see the next paragraph for the notation),
- the number of combinatorial orbits in each orbit,
- the line count, total (for  $\hbar$ ) or per single combinatorial orbit, and
- the naïve bound (also total or per single combinatorial orbit).

For the components  $\hbar_k \in D_k^{\vee}$  of  $\hbar$ , we use the notation  $[[\hbar_k^2]]_d$ , where  $d$  is either the discriminant class of  $\hbar_k^2$  (in the notation of [CS88]) or, if  $\hbar_k \in D_k$ , the symbol

$$0 \text{ (if } \hbar_k = 0), \quad \circ \text{ (if } \hbar_k^2 = 2), \quad \bullet \text{ (if } \hbar_k^2 = 4), \quad * \text{ (if } \hbar_k^2 = 6).$$

For the components  $l_k$  of a plane  $l$ , we use the notation  $[l_k \cdot \hbar_k]_d$ , where  $d$  has the same meaning as for  $\hbar$ . (By Lemma 4.3, the value of  $l_k^2$  is uniquely determined by the subscript.) *In the cases considered*, these data determine  $(\hbar_k, l_k)$  up to  $R(D_k)$ . (The occasional superscripts are artefacts left over from the complete set of tables.)

Also shown in the tables is the naïve a priori estimate  $b(\mathfrak{D})$  given by (3.7). For the vast majority of lattices  $N \ni \hbar$ , we have  $b(\mathfrak{D}) \leq \gamma$ , and these pairs are omitted. The few cases where  $b(\mathfrak{D}) \geq \gamma$  are shown in bold, and we treat them separately. In the “trivial” cases equivalent to those already considered in another lattice  $N$ , we usually also omit the list of orbits  $\bar{\sigma}_n$ .

#### 5.1 The lattice $N(6\mathbf{D}_4)$

There are 36 pairs  $N \ni \hbar$ , and  $b(\mathfrak{F}) \leq 375$  (see Table 1).

TABLE 2. The lattice  $N(8A_3)$  (cf. Convention 5.2).

<b>1:</b>	$[5]_2^+$	$[1]_2$	$[1]_2$	$[1]_2$	$[1]_2$	$[1]_2$	$[1]_2$	$[1]_2$	799	<b>407</b>	
1:	$[4]_\bullet$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	1	1	<u>1</u>
2:	$[3]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	7	4	<u>2</u>
3:	$[2]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	21	4	<u>2</u>
4:	$[2]_2$	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[\frac{1}{2}]_3$	$[0]_0$	$[0]_0$	$[\frac{1}{2}]_1$	14	16	<u>8</u>
5:	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[0]_0$	7	1	<u>1</u>
6:	$[\frac{3}{2}]_3$	$[1]_2$	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{2}]_3$	56	8	<u>4</u>
7:	$[0]_0$	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[1]_2$	$[0]_0$	$[0]_0$	7	1	<u>1</u>
<b>2:</b>	$[1]_2$	$[1]_2$	$[1]_2$	$[2]_\circ$	$[2]_\circ$	$[2]_\circ$	$[1]_2$	$[2]_\circ$	743	<b>375</b>	
1:	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[0]_0$	1	1	<u>1</u>
2:	$[1]_2$	$[1]_2$	$[0]_0$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	12	4	<u>2</u>
3:	$[1]_2$	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[1]_3$	$[0]_0$	$[0]_0$	$[1]_1$	48	4	<u>2</u>
4:	$[1]_2$	$[0]_0$	$[0]_0$	$[1]_3$	$[1]_1$	$[1]_1$	$[0]_0$	$[0]_1$	32	2	<u>1</u>
5:	$[0]_2$	$[0]_0$	$[0]_0$	$[1]_3$	$[1]_1$	$[1]_1$	$[0]_0$	$[1]_1$	8	4	<u>2</u>
6:	$[\frac{1}{2}]_3$	$[\frac{1}{2}]_3$	$[0]_0$	$[1]_2$	$[0]_0$	$[1]_3$	$[0]_0$	$[1]_1$	48	8	<u>4</u>
7:	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_\circ$	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	6	1	<u>1</u>
8:	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[1]_2$	1	16	<u>8</u>

5.1.1 *Configuration 1.* We subdivide the orbit  $\bar{\mathfrak{o}}_2$  into six clusters (not disjoint)

$$\mathfrak{c}_k := \{ \mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid k \notin \text{supp } \mathfrak{o} \}, \quad k \in \Omega,$$

and use these clusters (see § 4.5.2) to compute  $\tilde{\mathcal{B}}_{24}(\bar{\mathfrak{o}}_2) = \emptyset$ .

## 5.2 The lattice $N(8A_3)$

There are 110 pairs  $N \ni \mathfrak{h}$ , and  $b(\mathfrak{F}) \leq 407$  (see Table 2).

5.2.1 *Configuration 1.* Let  $\mathcal{K} := \{ k \in \Omega \mid \mathfrak{h}_k^2 = 1 \}$ , and subdivide  $\bar{\mathfrak{o}}_6$  into seven pairwise disjoint clusters

$$\mathfrak{c}_k := \{ \mathfrak{o} \subset \bar{\mathfrak{o}}_6 \mid l_k^2 = 1 \text{ for } l \in \mathfrak{o} \}, \quad k \in \mathcal{K}.$$

Using these clusters (see § 4.5.2), we show that  $\tilde{\mathcal{B}}_{56}(\bar{\mathfrak{o}}_6) = \emptyset$ .

5.2.2 *Configuration 2.* Let  $\mathcal{K} := \{ k \in \Omega \mid \mathfrak{h}_k \text{ is a root} \}$ , and subdivide  $\bar{\mathfrak{o}}_6$  into four pairwise disjoint clusters

$$\mathfrak{c}_k := \{ \mathfrak{o} \subset \bar{\mathfrak{o}}_6 \mid k \notin \text{supp } \mathfrak{o} \}, \quad k \in \mathcal{K}.$$

Using these clusters (see § 4.5.2), we show that  $\tilde{\mathcal{B}}_{24}(\bar{\mathfrak{o}}_6) = \emptyset$ . □

## 6. The lattice $N(12\mathbf{A}_2)$

The goal of this section is the following theorem.

**THEOREM 6.1.** *For any  $\hbar \in N(12\mathbf{A}_2)$ ,  $\hbar^2 = 12$  and any geometric set  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ , one has  $|\mathfrak{L}| \leq 350$ .*

*Proof.* We proceed as in the previous section, considering  $O(N)$ -orbits of square 12 vectors  $\hbar \in N(12\mathbf{A}_2)$  one by one. There are 29 orbits;  $b(\mathfrak{F}) \leq 407$  (see Table 3).

### 6.1 Configuration 1

Let  $\mathcal{K} := \{k \in \Omega \mid \hbar_k^2 = \frac{2}{3}\}$ , and subdivide  $\bar{\mathfrak{o}}_3$  into eleven clusters (not disjoint)

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid l_k \cdot \hbar_k = \frac{2}{3}\}, \quad k \in \mathcal{K},$$

to show (see § 4.5.2) that  $\tilde{\mathcal{B}}_{56}(\bar{\mathfrak{o}}_3) = \emptyset$ .

### 6.2 Configuration 3

Let  $\mathcal{K} := \{k \in \Omega \mid \hbar_k \text{ is a root}\}$ , and subdivide  $\bar{\mathfrak{o}}_2$  into three pairwise disjoint clusters

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid k \notin \text{supp } \mathfrak{o}\}, \quad k \in \mathcal{K}.$$

Using these clusters (see § 4.5.2), we show that  $\tilde{\mathcal{B}}_{20}(\bar{\mathfrak{o}}_2) = \emptyset$ . More precisely, we find eight sets  $\mathfrak{L} \subset \tilde{\mathcal{B}}_6(\mathfrak{c}_1)$ , all of size  $|\mathfrak{L}| = 72$ . Seven sets are maximal; see § 3.4.1. The eighth one is of rank 20, and adding a single maximal orbit (see § 3.4.2) from another cluster produces a single maximal set of size 156.

A direct computation using patterns (see § 4.5.1) shows that there are four sets  $\mathfrak{L} \in \tilde{\mathcal{B}}_7(\bar{\mathfrak{o}}_5)$ ; they are all maximal (see § 3.4.1) and of size  $|\mathfrak{L}| = 44$ .

Finally, let  $\mathfrak{C} := \bar{\mathfrak{o}}_4 \cup \bar{\mathfrak{o}}_6$ ; this set is complete and admissible. Listing iterated maximal subsets (see § 4.5.4), we find 19 sets  $\mathfrak{L} \in \mathcal{B}_{13}(\mathfrak{C})$ . All but one are ruled out by counting maximal orbits (see § 3.4.3) with the test set  $\mathfrak{T} = \bar{\mathfrak{o}}_2$ . The exception is  $\mathfrak{L} = \bar{\mathfrak{o}}_4$ , and in this case we use patterns (see § 4.5.1) to show that, for any extension  $\mathfrak{L}' \supset \mathfrak{L}$  satisfying (3.10), we have

- $|\mathfrak{L}' \cap \bar{\mathfrak{o}}| \leq b(\bar{\mathfrak{o}}) - 4$  for  $\bar{\mathfrak{o}} = \bar{\mathfrak{o}}_1, \bar{\mathfrak{o}}_3$  or any of the three clusters  $\mathfrak{c}_k \subset \bar{\mathfrak{o}}_2$  and
- $|\mathfrak{L}' \cap \bar{\mathfrak{o}}_5| \leq b(\bar{\mathfrak{o}}_5) - 33$ .

*Remark 6.2.* The sublattice  $\text{span}_2 \mathfrak{C} \subset N$  has no proper mild extensions, and its 2-discriminant violates the condition stated in Proposition 3.4(iii). Therefore, when computing  $\mathcal{B}_{13}(\mathfrak{C})$ , we confine ourselves to subsets  $\mathfrak{L} \subset \mathfrak{C}$  of corank at least 1. The rest of the argument applies to pseudo-geometric sets as well.

### 6.3 Configuration 4

Let  $\mathcal{K} := \{k \in \Omega \mid \hbar_k^2 = \frac{8}{3}\}$ , and subdivide  $\bar{\mathfrak{o}}_5$  into two disjoint clusters

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_5 \mid l_k \cdot \hbar_k = \frac{4}{3} \text{ for } l \in \mathfrak{o}\}, \quad k \in \mathcal{K},$$

to show (see § 4.5.2) that  $\mathcal{B}_{25}(\bar{\mathfrak{o}}_5) = \emptyset$ . On the other hand, one has

$$\mathcal{B}_0(\bar{\mathfrak{o}}_1 \cup \bar{\mathfrak{o}}_3 \cup \bar{\mathfrak{o}}_4 \cup \bar{\mathfrak{o}}_7 \cup \bar{\mathfrak{o}}_8 \cup \bar{\mathfrak{o}}_9) = \mathcal{B}_0(\bar{\mathfrak{o}}_2) = \mathcal{B}_0(\bar{\mathfrak{o}}_6) = \emptyset$$

directly as in § 4.5.1. (Here, we do use condition (iii) in Proposition 3.4.)

TABLE 3. The lattice  $N(12A_2)$  (cf. Convention 5.2).

<b>1:</b>	$[\frac{14}{3}]_2^+$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	<b>803</b>	<b>407</b>	
1:	$[3]_0$	$[1]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	11	2	<u>1</u>
2:	$[\frac{5}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	55	2	<u>1</u>
3:	$[\frac{5}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	55	8	<u>4</u>
4:	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	55	4	<u>2</u>
5:	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	11	1	<u>1</u>
<b>3:</b>	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_1$	$[2]_0$	$[2]_0$	$[2]_0$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	<b>747</b>	<b>393</b>	
1:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{1}{3}]_2$	$[0]_0$	$[0]_0$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	54	2	<u>1</u>
2:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[1]_2$	$[0]_0$	$[1]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	108	4	<u>2</u>
3:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_1$	$[1]_1$	$[1]_2$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	36	2	<u>1</u>
4:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_1$	$[1]_1$	$[0]_2$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	$[0]_0$	36	1	<u>1</u>
5:	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_2$	$[1]_2$	$[1]_1$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_2$	$[\frac{1}{3}]_2$	$[\frac{1}{3}]_2$	$[0]_0$	12	8	<u>4</u>
6:	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_0$	$[2]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	3	1	<u>1</u>
<b>4:</b>	$[\frac{8}{3}]_2^-$	$[\frac{8}{3}]_2^-$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	<b>743</b>	<b>379</b>	
1:	$[2]_0$	$[2]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	1	4	<u>2</u>
2:	$[\frac{4}{3}]_1$	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_2$	$[\frac{1}{3}]_2$	$[\frac{1}{3}]_2$	$[0]_0$	30	4	<u>2</u>
3:	$[\frac{4}{3}]_1$	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_1$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	10	1	<u>1</u>
4:	$[\frac{4}{3}]_1$	$[\frac{4}{3}]_1$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	5	16	<u>8</u>
5:	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	30	8	<u>4</u>
6:	$[\frac{4}{3}]_1$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	60	4	<u>2</u>
7:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	5	4	<u>2</u>
8:	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	12	2	<u>1</u>
9:	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	5	1	<u>1</u>
<b>5:</b>	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_1$	$[6]_*^+$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	$[\frac{2}{3}]_1$	<b>855</b>	<b>351</b>	
1:	$[1]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[3]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	9	4	<u>2</u>
2:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[1]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	9	4	<u>2</u>
3:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[2]_2$	$[0]_0$	$[0]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	36	12	<u>4</u>
4:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[2]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	9	1	<u>1</u>
5:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[2]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_2$	$[\frac{1}{3}]_2$	$[\frac{1}{3}]_2$	$[\frac{1}{3}]_2$	$[0]_0$	36	4	<u>2</u>
6:	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_2$	$[0]_2$	$[0]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	6	9	<u>3</u>
7:	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_2$	$[2]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	9	16	<u>8</u>

### 6.4 Configuration 5

We have  $\mathcal{B}_0(\mathfrak{D}) = \emptyset$  directly as in §4.5.1. □

## 7. The lattice $N(24\mathbf{A}_1)$

The goal of this section is the following theorem.

**THEOREM 7.1.** *Let  $\hbar \in N(24\mathbf{A}_1)$ ,  $\hbar^2 = 12$ , and let  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$  be a geometric set. Then, unless*

$$- |\mathfrak{L}| = 357 \text{ and } \mathfrak{L} = \mathfrak{G}_{357}^i, \text{ see (7.2),}$$

*one has  $|\mathfrak{L}| \leq 350$ .*

*Proof.* We proceed as in the previous sections. Each component  $v_k \in D_k^\vee$ ,  $k \in \Omega$ , of a vector  $v \in N$  is a multiple of the generator  $r_k \in D_k$ . To save space, we use the following notation for the coefficient  $\alpha$  in  $v_k = \alpha r_k$ :

$$\cdot (\alpha = 0), \quad - \text{ or } = (\alpha = \pm \frac{1}{2}), \quad \circ (\alpha = \pm 1), \quad + (\alpha = \pm \frac{3}{2}), \quad \bullet (\alpha = \pm 2).$$

Here,  $=$  is used only for  $l_k$  and only if  $\hbar_k \cdot l_k < 0$ ; in all other cases, the signs of  $l_k$  and  $\hbar_k$  agree, so that we have  $\hbar_k \cdot l_k \geq 0$ .

There are 13 orbits  $\hbar \in N(24\mathbf{A}_1)$ , and  $b(\mathfrak{F}) \leq 759$  (see Table 4).

Fix a basis  $\{r_k\}$ ,  $k \in \Omega$ , for  $24\mathbf{A}_1$  consisting of roots. The kernel

$$N \bmod 24\mathbf{A}_2 \subset \text{discr } 24\mathbf{A}_1 \cong (\mathbb{Z}/2)^{24}$$

of the extension is the Golay code  $\mathcal{C}_{24}$  (see [CS88]). The map  $\text{supp}$  identifies codewords with subsets of  $\Omega$ ; then,  $\mathcal{C}_{24}$  is invariant under complement and, in addition to  $\emptyset$  and  $\Omega$ , it consists of 759 octads, 759 complements thereof and 2576 dodecads.

To simplify the notation, we identify the basis vectors  $r_k$  (assumed fixed) with their indices  $k \in \Omega$ . For a subset  $\mathcal{S} \subset \Omega$ , we let  $\bar{1}_{\mathcal{S}} := \sum_{r \in \mathcal{S}} r$ , and use the abbreviation  $[\mathcal{S}] := \frac{1}{2}\bar{1}_{\mathcal{S}} \in N$  if  $\mathcal{S} \in \mathcal{C}_{24}$  is a codeword.

### 7.1 Configuration 1

We have  $\text{stab } \hbar = M_{24}$  (the Mathieu group, which is the symmetry group of the Golay code) and  $\hbar = [\Omega]$ . The set  $\mathfrak{F}(\hbar) = \bar{\mathfrak{o}}_1$  is complete and admissible; listing iterated maximal subsets (see §4.5.4), after four steps of the algorithm we find out that there are no complete subsets  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$  of rank  $\text{rk } \mathfrak{L} \leq 21$  and size  $|\mathfrak{L}| \geq 385$  (cf. §8.1 below).

### 7.2 Configuration 2

We have  $|\text{stab } \hbar| = 11520$  and  $\hbar = \bar{1}_{\mathcal{R}}$ , where  $|\mathcal{R}| = 6$  and  $\mathcal{R}$  is a subset of an octad (cf. §7.11). We subdivide  $\bar{\mathfrak{o}}_3$  into 15 pairwise disjoint clusters

$$\mathfrak{c}_o := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_3 \mid \text{supp } \mathfrak{o} \supset o\}, \quad o \subset \mathcal{R}, \quad |o| = 4,$$

and use these clusters (see §4.5.2) to show that  $\mathcal{B}_{156}(\bar{\mathfrak{o}}_3) = \emptyset$ . (The set  $\mathfrak{b}(\mathfrak{o})$ ,  $\mathfrak{o} \subset \bar{\mathfrak{o}}_3$ , equals  $\{0, \dots, 4, 6, 8\}$ , which simplifies the computation.)

TABLE 4. The lattice  $N(24\mathbf{A}_1)$  (cf. Convention 5.2).

<b>1:</b>	-----	759	<b>759</b>
1:	-----	759	1 <u>1</u>
<b>2:</b>	○○○○○.○. . . . .	999	<b>507</b>
1:	○○. . . . .	15	1 <u>1</u>
2:	-----= . . . . .	6	4 <u>2</u>
3:	-----	60	16 <u>8</u>
<b>3:</b>	++----. . . . .	987	<b>507</b>
1:	○.○. . . . .	12	1 <u>1</u>
2:	-----= . . . . .	15	1 <u>1</u>
3:	-----	60	16 <u>8</u>
<b>4:</b>	+-----	807	<b>471</b>
1:	○○. . . . .	15	1 <u>1</u>
2:	-----= . . . . .	105	1 <u>1</u>
3:	-----	168	4 <u>2</u>
4:	.-----	15	1 <u>1</u>
<b>5:</b>	-----○-○. . . . .	875	<b>451</b>
1:	-----	1	1 <u>1</u>
2:	-----	72	4 <u>2</u>
3:	-----	64	8 <u>4</u>
4:	-----= . . . . .	8	1 <u>1</u>
5:	-----= . . . . .	12	1 <u>1</u>
6:	. . . . .○.○. . . . .	6	1 <u>1</u>
7:	. . . . .	3	16 <u>8</u>
<b>6:</b>	-----○-○. . . . .	751	<b>439</b>
1:	-----	30	1 <u>1</u>
2:	-----	192	2 <u>1</u>
3:	-----= . . . . .	96	1 <u>1</u>
4:	-----	60	4 <u>2</u>
5:	. . . . .○.○. . . . .	1	1 <u>1</u>
<b>7:</b>	+-----○- . . . . .	799	<b>407</b>
<b>8:</b>	+-----○- . . . . .	803	<b>407</b>
<b>9:</b>	-----○-○. . . . .	747	<b>393</b>
<b>10:</b>	●○○. . . . .	715	<b>379</b>
1:	○.○. . . . .	21	2 <u>2</u>
2:	-----	21	32 <u>16</u>
3:	.○○. . . . .	1	1 <u>1</u>
<b>11:</b>	○○○○○. . . . .	735	<b>375</b>
<b>12:</b>	-----○-○. . . . .	743	<b>375</b>

### 7.3 Configuration 3

We have  $|\text{stab } \bar{h}| = 11520$  and  $\bar{h} = [\mathcal{O}] + \bar{1}\mathcal{R}$ , where  $\mathcal{O}$  is an octad and  $\mathcal{R} \subset \mathcal{O}$  a 2-element set. Subdivide  $\bar{o}_3$  into 15 pairwise disjoint clusters

$$\mathbf{c}_o := \{\mathfrak{o} \subset \bar{o}_3 \mid \text{supp } \mathfrak{o} \supset o\}, \quad o \subset \mathcal{O} \setminus \mathcal{R}, \quad |o| = 2,$$

and use these clusters (see § 4.5.2) to show that  $\mathcal{B}_{156}(\bar{o}_3) = \emptyset$ . (The set  $\mathfrak{b}(\mathfrak{o})$ ,  $\mathfrak{o} \subset \bar{o}_3$ , equals  $\{0, \dots, 4, 6, 8\}$ , which simplifies the computation. In spite of the apparent similarity, this polarized lattice is not equivalent to #2; see § 7.2.)

### 7.4 Configuration 4

We have  $|\text{stab } \bar{h}| = 20160$  and  $\bar{h} = [\mathcal{O}] + r$ , where  $\mathcal{O} \ni r$  is a codeword of length 16. Let  $\mathcal{K} := \Omega \setminus \mathcal{O}$ .

We subdivide the orbit  $\bar{o}_3$  into 28 pairwise disjoint clusters

$$\mathbf{c}_k := \{\mathfrak{o} \subset \bar{o}_3 \mid \text{supp } \mathfrak{o} \supset k\}, \quad k \subset \mathcal{K}, \quad |k| = 2,$$

and, using these clusters and meta-patterns (see § 4.5.3), show that  $\mathcal{B}_{50}(\bar{o}_3) = \emptyset$ . (Note that  $|\mathcal{L} \cap \mathbf{c}_k| \in \{0, \dots, 8, 10, 12\}$  for each  $k$  and each geometric set  $\mathcal{L}$ .)

Now, consider the complete admissible set  $\mathcal{C} := \bar{o}_1 \cup \bar{o}_2 \cup \bar{o}_4$ . Listing iterated maximal subsets (see § 4.5.4), we find 34 sets  $\mathcal{L} \in \mathcal{B}_{70}(\mathcal{C})$ , of which all but two are ruled out by counting maximal orbits (see § 3.4.3), using  $\mathfrak{T} = \bar{o}_3$  for the test set.

One of the two exceptions is a set  $\mathcal{L}$  of size 87, for which  $|\mathfrak{T}_\mu| = 96$  (see § 3.4.3 for the notation). Extending the pattern  $\pi_\mathcal{L}$  (see § 3.4.2) via  $\mathfrak{o} \mapsto b(\mathfrak{o}) = 2$  for each orbit  $\mathfrak{o} \in \mathfrak{T}_\mu$  (and leaving the other values undefined) and applying the algorithm of § 4.5.1, we show that  $|\mathcal{L}'| \leq 350$  for each extension  $\mathcal{L}' \supset \mathcal{L}$  satisfying (3.10).

The other exception is  $\mathcal{L} = \bar{o}_2$ , with  $\mathfrak{T}_\mu = \mathfrak{T}$ . Arguing as in § 3.4.2 (extending  $\pi_\mathcal{L}$  by a single extra value  $\mathfrak{o} \mapsto 1$  or 2 for some orbit  $\mathfrak{o} \subset \bar{o}_3$  and analyzing the output), we observe that  $|\mathcal{L}' \cap \mathbf{c}_k| \in \{0, 12\}$  for each cluster  $\mathbf{c}_k$  and each extension  $\mathcal{L}' \supset \mathcal{L}$  satisfying (3.10). Using this observation and meta-patterns (see § 4.5.3) with the set of values restricted accordingly, we show that  $|\mathcal{L}'| \leq 350$  with the exception of a single, up to  $O_{\bar{h}}(N)$ , saturated set  $\mathfrak{G}_{357}^i$  of rank 21 and size 357. This set is characterised by any of the eight patterns

$$\pi_{\mathfrak{c}}(\mathfrak{o}) = \begin{cases} 1 & \text{if } \mathfrak{o} \subset \bar{o}_2, \\ 2 & \text{if } \mathfrak{o} \subset \mathfrak{c}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\mathfrak{c} := \{\mathfrak{o} \subset \bar{o}_3 \mid \text{supp } \mathfrak{o} \not\supset k\}$  for some fixed  $k \in \mathcal{K}$ . Alternatively,

$$\mathfrak{G}_{357}^i = \mathfrak{F}(\bar{h}) \cap \text{span}([\mathcal{K}], \bar{h} - 4r, k)^\perp. \quad (7.2)$$

### 7.5 Configuration 5

We have  $|\text{stab } \bar{h}| = 2304$  and  $\bar{h} = [\mathcal{O}] + \bar{1}\mathcal{R}$ , where  $\mathcal{O}$  is an octad and  $\mathcal{R} \subset \Omega \setminus \mathcal{O}$  a 4-element set, so that there is an octad  $o \supset \mathcal{R}$  such that  $|o \cap \mathcal{O}| = 4$  (cf. § 7.12). We subdivide  $\bar{o}_3$  into four pairwise disjoint clusters

$$\mathbf{c}_r := \{\mathfrak{o} \subset \bar{o}_3 \mid \text{supp } \mathfrak{o} \not\supset r\}, \quad r \in \mathcal{R},$$



and use these clusters (see § 4.5.2) to show that  $\mathcal{B}_{60}(\bar{\mathfrak{o}}_3) = \emptyset$ . Then, subdividing  $\bar{\mathfrak{o}}_2$  into six pairwise disjoint clusters

$$\mathfrak{c}'_s := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid \text{supp } \mathfrak{o} \supset s\}, \quad s \subset \mathcal{R}, \quad |s| = 2,$$

we show (see § 4.5.2) that  $\mathcal{B}_{39}(\bar{\mathfrak{o}}_2) = \emptyset$ .

### 7.6 Configuration 6

We have  $|\text{stab } \bar{h}| = 11520$  and  $\bar{h} = [\mathcal{O}] + \bar{1}_{\mathcal{R}}$ , where  $\mathcal{O}$  is a codeword of length 16 and  $\mathcal{R} \subset \Omega \setminus \mathcal{O}$  is a 2-element set. Let  $\mathcal{K} := \Omega \setminus (\mathcal{O} \cup \mathcal{R})$ . We subdivide  $\bar{\mathfrak{o}}_4$  into six clusters (not disjoint)

$$\mathfrak{c}_k := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_4 \mid \text{supp } \mathfrak{o} \ni k\}, \quad k \in \mathcal{K},$$

and use these clusters (see § 4.5.2) to show that  $\mathcal{B}_{35}(\bar{\mathfrak{o}}_4) = \emptyset$ . Next, we subdivide  $\bar{\mathfrak{o}}_2$  into twelve pairwise disjoint clusters

$$\mathfrak{c}'_s := \{\mathfrak{o} \subset \bar{\mathfrak{o}}_2 \mid \text{supp } \mathfrak{o} \supset s\}, \quad s \in \mathcal{R} \times \mathcal{K},$$

and show (using § 4.5.2) that  $\mathcal{B}_{22}(\bar{\mathfrak{o}}_2) = \emptyset$ . Finally, listing iterated maximal subsets (see § 4.5.4) in the complete admissible set  $\mathfrak{C} := \bar{\mathfrak{o}}_1 \cup \bar{\mathfrak{o}}_3 \cup \bar{\mathfrak{o}}_5$ , we obtain  $\mathcal{B}_{29}(\mathfrak{C}) = \emptyset$ .

### 7.7 Configuration 7

We have  $|\text{stab } \bar{h}| = 336$  and  $\bar{h} = [\mathcal{O}] + \bar{1}_{\mathcal{R}} + r$ , where  $\mathcal{O} \ni r$  is an octad and  $\mathcal{R} \subset \Omega \setminus \mathcal{O}$  a 2-element set. This configuration is equivalent to #1 in  $N(8\mathbf{A}_3)$ ; see § 5.2.1, where we treat pseudo-geometric sets as well.

### 7.8 Configuration 8

We have  $|\text{stab } \bar{h}| = 660$  and  $\bar{h} = [\mathcal{O}] + r + s$ , where  $\mathcal{O} \ni r$  is a dodecad and  $s \in \Omega \setminus \mathcal{O}$ . This configuration is equivalent to #1 in  $N(12\mathbf{A}_2)$ ; see § 6.1, where we treat pseudo-geometric sets as well.

### 7.9 Configuration 9

We have  $|\text{stab } \bar{h}| = 432$  and  $\bar{h} = [\mathcal{O}] + \bar{1}_{\mathcal{R}}$ , where  $\mathcal{O}$  is a dodecad and  $\mathcal{R} \subset \Omega \setminus \mathcal{O}$  a 3-element set. This configuration is equivalent to #3 in  $N(12\mathbf{A}_2)$ ; see § 6.2, where we mainly treat pseudo-geometric sets. The only exception, explained in Remark 6.2, applies to the present case as well, as the sublattice  $\text{span}_2(\bar{\mathfrak{o}}_3 \cup \bar{\mathfrak{o}}_6) \subset N$  has no proper mild extensions.

### 7.10 Configuration 10

We have  $|\text{stab } \bar{h}| = 40320$  and  $\bar{h} = \bar{1}_{\mathcal{R}} + 2r$ , where  $\mathcal{R}$  is a 2-element set and  $r \notin \mathcal{R}$ . Using patterns (see § 4.5.1), we show that  $\mathcal{B}_{28}(\bar{\mathfrak{o}}_2) = \emptyset$ . (Note that  $\mathfrak{b}(\mathfrak{o}) = \{0, \dots, 8, 10, 16\}$  for  $\mathfrak{o} \subset \bar{\mathfrak{o}}_2$ .)

### 7.11 Configuration 11

We have  $|\text{stab } \bar{h}| = 2160$  and  $\bar{h} = \bar{1}_{\mathcal{R}}$ , where  $|\mathcal{R}| = 6$  and  $\mathcal{R}$  is *not* a subset of an octad (cf. § 7.2). This configuration is equivalent to #1 in  $N(6\mathbf{D}_4)$ ; see § 5.1.1, where we treat pseudo-geometric sets as well.

### 7.12 Configuration 12

We have  $|\text{stab } \bar{h}| = 192$  and  $\bar{h} = [\mathcal{O}] + \bar{1}_{\mathcal{R}}$ , where  $\mathcal{O}$  is an octad and  $\mathcal{R} \subset \Omega \setminus \mathcal{O}$  a 4-element set, so that there is *no* octad  $o \supset \mathcal{R}$  such that  $|o \cap \mathcal{O}| = 4$  (cf. § 7.5). This configuration is equivalent to #2 in  $N(8\mathbf{A}_3)$ ; see § 5.2.2, where we treat pseudo-geometric sets as well.  $\square$

### 8. The Leech lattice

Let  $N := \Lambda$  be the Leech lattice. We prove the following theorem.

**THEOREM 8.1.** *Let  $\hbar \in \Lambda$ ,  $\hbar^2 = 12$ , and let  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$  be a geometric set  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ . Then, unless*

- $|\mathfrak{L}| = 405$  and  $\mathfrak{L} = \mathfrak{M}_{405}^i$ , see (8.4), or
- $|\mathfrak{L}| = 357$  and  $\mathfrak{L} = \mathfrak{S}_{357}^{ii}$ , see (8.5), or
- $|\mathfrak{L}| = 351$  and  $\mathfrak{L} = \mathfrak{L}_{351}^i$ , see (8.6),

one has  $|\mathfrak{L}| \leq 297$ .

*Proof.* It is known (see, for example, [CS88, Chapter 10, Theorem 28]) that any nonzero class  $[\hbar] \in \Lambda/2\Lambda$  is represented by a unique pair  $\pm a \in \Lambda$ , where  $a^2 = 4, 6$  or  $a^2 = 8$  and  $a$  is part of a fixed coordinate frame. Since, clearly,  $a^2 = \hbar^2 \pmod 4$ , and since  $\Lambda$  is positive definite and root free, for  $\hbar^2 = 12$  we have either

- (i)  $\hbar = a + 2b$ , where  $a^2 = b^2 = 4$  and  $a \cdot b = -2$  (type  $6_{22}$  in loc. cit.), or
- (ii)  $\hbar = a + 2b$ , where  $a^2 = 8, b^2 = 4$  and  $a \cdot b = -3$  (type  $6_{32}$  in loc. cit.)

In addition, a pair  $a, b \in \Lambda$  as in item (i) or (ii) is unique up to  $O(\Lambda)$ . Thus, there are two  $O(\Lambda)$ -orbits of square 12 vectors  $\hbar \in \Lambda$  (see Theorem 29 in loc. cit.)

The two cases are considered below. We use the shortcut  $F := \text{span}_2 \mathfrak{F}(\hbar)$ .

#### 8.1 Configuration 1: $\hbar = a \pmod{2\Lambda}$ , $a^2 = 4$

We have

$$|O_{\hbar}(F)| = 55180984320, \quad |\mathfrak{F}| = 891, \quad \text{rk } F = 23, \quad \text{discr } F = \mathcal{U} \oplus \langle \frac{7}{4} \rangle,$$

and all index 8 extensions of  $F \oplus \mathbb{Z}a$ ,  $a^2 = 4$ , are isomorphic and have vector  $\hbar$  of the same type. Listing iterated maximal subsets (see §4.5.4) in the full set  $\mathfrak{F}(\hbar)$ , which is obviously complete and admissible, after four steps we obtain 17 complete sets  $\mathfrak{L} \subset \mathfrak{F}(\hbar)$  of rank  $\text{rk } \mathfrak{L} \leq 21$  and size  $|\mathfrak{L}| \geq 285$ . Using Nikulin's theory [Nik79], one can easily obtain the following statement.

**LEMMA 8.2.** *None of the 17 sets above has a proper root free finite-index extension  $S \supset \text{span}_2 \mathfrak{L}$  (even abstract, not necessarily lying in  $\Lambda$ ) such that  $\hbar \in 4S^{\vee}$ .*

Only 8 of the 17 sets are geometric, namely, the sets  $\mathfrak{M}_{405}^i$ ,  $\mathfrak{S}_{357}^{ii}$  and  $\mathfrak{L}_{351}^i$  (the subscript indicating the number of planes) described below, two sets of size 297 (ranks 20 and 21) and three sets of size 285 (ranks 20, 21, and 21).

*Remark 8.3.* Technically, we work in  $F$  rather than in  $\Lambda$  itself, which gives us a slightly larger symmetry group: indeed,  $O_{\hbar}(\Lambda)$  induces on  $F$  an index 6 subgroup of  $O_{\hbar}(F) = \text{Aut } \mathfrak{F}(\hbar)$  (computed by the GRAPE package [McK90, MP14, Soi18] in GAP [GAP19]). That is why we have to consider *abstract* finite-index extensions.

The three large sets found can be described in terms of square 4 vectors in  $\Lambda$ , which are relatively easy to handle. Consider the lattice  $U := \mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c$  with Gram matrix

$$\begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix},$$

and let  $V := U + \mathbb{Z}v$ ,  $v^2 = 4$ , be its extension such that  $v \cdot a = 1$  and the other two products are as follows:

$$v \cdot b = 1, \quad v \cdot c = -2 \quad \text{for } \mathfrak{M}_{405}^i: \quad |\text{Aut } \mathfrak{L}| = 349920, \quad T \cong -[6, 3, 6], \quad (8.4)$$

$$v \cdot b = 1, \quad v \cdot c = 0 \quad \text{for } \mathfrak{S}_{357}^{ii}: \quad |\text{Aut } \mathfrak{L}| = 10080, \quad T \cong -[2, 1, 18], \quad (8.5)$$

$$v \cdot b = -2, \quad v \cdot c = 1 \quad \text{for } \mathfrak{L}_{351}^i: \quad |\text{Aut } \mathfrak{L}| = 31104, \quad T \cong -[6, 0, 6]. \quad (8.6)$$

(For the reader's convenience, we also list the size of the group  $\text{Aut } \mathfrak{L}$  of the graph automorphisms of  $\mathfrak{L}$ , computed via **GRAPE**, and the *transcendental lattice*  $T$ ; see § 9 below.) Up to  $O_{\hbar}(\Lambda)$ , there is a unique isometry  $V \hookrightarrow \Lambda$  such that  $a + 2b \mapsto \hbar$ . Then, the set in question is

$$\mathfrak{F}(\hbar) \cap ((\mathbb{Z}\hbar \oplus V^\perp) \otimes \mathbb{Q}).$$

## 8.2 Configuration 2: $\hbar = a \bmod 2\Lambda$ , $a^2 = 8$

We have

$$O_{\hbar}(\Lambda) = O_{\hbar}(F) = M_{24}, \quad |\mathfrak{F}| = 759, \quad \text{rk } F = 24, \quad \text{discr } F = \langle \frac{1}{4} \rangle \oplus \langle \frac{7}{4} \rangle.$$

This lattice has two index 4 extensions: one of them is  $\Lambda$ , and the other is  $N(24\mathbf{A}_1)$ . Therefore, this configuration is equivalent to #1 in  $N(24\mathbf{A}_1)$ ; see § 7.1, where we treat pseudo-geometric sets as well.  $\square$

## 9. Proofs of the main results

In this section, we complete the proof of the principal results of the paper, namely Theorems 1.1 and 1.2.

### 9.1 Proof of Theorem 1.1

Let  $X \subset \mathbb{P}^5$  be a smooth cubic 4-fold such that  $|\text{Fn } X| > 350$ . Applying the replanting procedure (see § 3.1) to the 3-polarized lattice  $M_X \ni h_X$ , we obtain a 12-polarized lattice  $S \ni \hbar$  embedded to a Niemeier lattice  $N$  (see Proposition 3.3) and such that

- one has  $\hbar \in 4S^\vee$ ,
- the torsion  $\text{Tors}(N/S)$  is a 2-group,
- $S$  satisfies all conditions of Proposition 3.4.

By Propositions 2.6 and 3.1, there is a bijection between the set of planes in  $X$  and the set

$$\mathfrak{L} = \mathfrak{F}(\hbar) \cap S \subset \mathfrak{F}(\hbar) = \{l \in N \mid l^2 = 4, l \cdot \hbar = 4\}$$

of  $\hbar$ -planes in  $S$ . Furthermore,  $\mathfrak{L}$  is geometric, essentially, by definition. Thus, Theorems 5.1, 6.1, 7.1 and 8.1 imply that we are in one of the following situations:

- $N \simeq N(24\mathbf{A}_1)$ , and  $\mathfrak{L}$  coincides with  $\mathfrak{S}_{357}^i$  up to  $O_{\hbar}(N)$ ;
- $N \simeq \Lambda$ , and  $\mathfrak{L}$  coincides with  $\mathfrak{M}_{405}^i$  up to  $O_{\hbar}(N)$ ;
- $N \simeq \Lambda$ , and  $\mathfrak{L}$  coincides with  $\mathfrak{S}_{357}^{ii}$  up to  $O_{\hbar}(N)$ ;
- $N \simeq \Lambda$ , and  $\mathfrak{L}$  coincides with  $\mathfrak{L}_{351}^i$  up to  $O_{\hbar}(N)$ .

The set  $\mathfrak{S}_{357}^{ii}$  is graph isomorphic to  $\mathfrak{S}_{357}^i$ ; this is established by the **GRAPE** package [McK90, MP14, Soi18] in **GAP** [GAP19].

PROPOSITION 9.1. *Let  $\mathfrak{L}$  be one of the sets  $\mathfrak{M}_{405}^i$ ,  $\mathfrak{G}_{357}^{ii}$  or  $\mathfrak{L}_{351}^i$ . Then, there exists a unique (up to  $PGL(\mathbb{C}, 6)$ ) smooth cubic  $X \subset \mathbb{P}^5$  such that  $\text{Fn } X$  is graph isomorphic to  $\mathfrak{L}$ .*

*Proof.* Let  $\mathfrak{L} = \mathfrak{M}_{405}^i$ . The lattice  $S = \text{span}_{\mathbb{Z}} \mathfrak{L} \subset \Lambda$  is of rank 21 and has no proper mild extension; see Lemma 8.2. According to Proposition 3.4, the 3-polarized lattice  $M \ni h$  obtained from  $S \ni \tilde{h}$  by the inverse replanting procedure (cf. §3.1) admits a primitive embedding to  $\mathbf{L}$  with even orthogonal complement. Thus, the existence of a smooth cubic  $X \subset \mathbb{P}^5$  such that  $\text{Fn } X$  is graph isomorphic to  $\mathfrak{L}$  follows from the surjectivity of the period map (see Theorem 2.3).

According to the global Torelli theorem, Theorem 2.1, the projective equivalence classes of smooth cubics  $X \subset \mathbb{P}^5$  such that  $\text{Fn } X$  is graph isomorphic to  $\mathfrak{L}$  (equivalently, the 3-polarized lattice  $M_X \ni h_X$  is isomorphic to  $M \ni h$ , cf. Lemma 8.2) are in a natural bijection with the  $O^+(\mathbf{L})$ -orbits of primitive embeddings  $M \hookrightarrow \mathbf{L}$  such that  $M^\perp$  is even, where  $O^+(\mathbf{L})$  is the group of autoisometries of  $\mathbf{L}$  preserving the positive sign structure.

The classification of embeddings can be obtained using Nikulin’s theory [Nik79]. For any embedding  $M \hookrightarrow \mathbf{L}$  such that  $M^\perp$  is even, the genus of the transcendental lattice  $T = M^\perp$  is determined by the discriminant  $\text{discr } M \cong -\text{discr } T$ . In our case, this implies that  $T \cong -[6, 3, 6]$ . The isomorphism classes of embeddings under consideration are in a bijection with

$$O(M) \backslash \text{Aut}(\text{discr } M) / O^+(T).$$

Using GRAPE, one can check that the natural homomorphism

$$\text{Aut } \mathfrak{L} = O_h(M) \rightarrow \text{Aut}(\text{discr } T)$$

is surjective, and to complete the proof of the uniqueness, there remains to notice that  $T$  admits an orientation-reversing autoisometry.

If  $\mathfrak{L} = \mathfrak{L}_{351}^i$ , the proof is literally the same except that now  $T \cong -[6, 0, 6]$ .

Likewise, in the case  $\mathfrak{L} = \mathfrak{G}_{357}^{ii}$ , the proof is literally the same except that now we have  $T \cong -[2, 1, 18]$ . In view of Lemma 8.2, we do not need to consider the graph-isomorphic set  $\mathfrak{G}_{357}^i \subset N(24\mathbf{A}_1)$ , as it results in the same lattice  $M$ .  $\square$

As an immediate consequence of the uniqueness given by Proposition 9.1, we conclude that  $\mathfrak{M}_{405}^i$  is graph isomorphic to the configuration of planes in the Fermat cubic; see §2.5.

LEMMA 9.2. *The configuration of planes in the Clebsch–Segre cubic  $Y$ , see §2.6, is graph isomorphic to  $\mathfrak{G}_{357}^i \simeq \mathfrak{G}_{357}^{ii}$ .*

*Proof.* The only other alternative would be  $\text{Fn } Y \simeq \mathfrak{M}_{405}^i$ . However,  $\mathfrak{M}_{405}^i$  does not admit a faithful action of  $\mathbb{S}_7$  (as  $7!$  does not divide  $|\text{Aut } \mathfrak{M}_{405}^i| = 349920$ ). Alternatively, by Corollary 2.2 and (8.4), the Fermat cubic does not admit a real structure with respect to which the classes of the real planes span a lattice of rank 21 (cf. a more detailed argument in §9.2).  $\square$

Theorem 1.1 is an immediate consequence of Proposition 9.1 and Lemma 9.2.  $\square$

## 9.2 Proof of Theorem 1.2

Let  $Z \subset \mathbb{P}^5$  be a smooth real cubic and  $c: Z \rightarrow Z$  the real structure. By Corollary 2.8, the classes of the real planes in  $Z$  span over  $\mathbb{Q}$  the sublattice  $M_Z^c := M_Z \cap \text{Ker}(1 - c^*)$ . Perturbing, if necessary, the period of  $Z$  (see Theorem 2.3), we can assume  $M_Z = M_Z^c$ ; that is, *all* planes contained in  $Z$  are real. Then, if  $Z$  contains at least 357 such planes, Theorem 1.1 implies that

$Z$  is projectively equivalent to either the Fermat cubic or the Clebsch–Segre cubic. In the latter case,  $T_Z \cong -[6, 3, 6]$ , and the assumption that all planes in  $Z$  are real contradicts Corollary 2.2.

### 9.3 Proof of Proposition 1.4

Given a cubic fourfold  $Y$  with a node  $p$ , the projection from  $p$  gives a birational map

$$\pi: Y \dashrightarrow \mathbb{P}^4,$$

which contracts the tangent cone of  $p$  to a  $K3$  surface  $S \subset \mathbb{P}^4$ . As one easily checks, if  $P \subset Y$  is a plane, then either  $\pi(P)$  is a line (if  $P$  contains  $p$ ), or  $\pi(P)$  is a plane in  $\mathbb{P}^4$  intersecting  $S$  in a (possibly reducible) conic. Now, given a plane  $V \subset \mathbb{P}^4$ , we claim that there can be at most one plane  $P \subset Y$  such that  $V = \pi(P)$ . Indeed, note that  $H = \pi^{-1}(V)$  is a  $\mathbb{P}^3$  passing through  $p$ . The cubic  $F$  defining  $Y$  splits when restricted to  $H$ , and only two cases can occur: either  $Y \cap H = P \cup P' \cup P''$  for two planes  $P'$  and  $P''$  (in which case both  $P'$  and  $P''$  must pass through  $p$ ), or  $Y \cap H = P \cup Q$ , where  $Q$  is an irreducible quadric surface. In either case, there is at most one plane in  $H$  that projects to  $P$ , so we get the claim.

In all, we see that the number of planes is bounded by the sum of the number of lines on  $S$  (which is at most 42, see [Deg19]) and the number of conics on  $S$  (which is at most 285, see [Deg21]). Thus, altogether, there can be at most  $42 + 285 = 327$  planes in a nodal cubic. However, a sextic  $K3$ -surface with 261 or more conics has no lines (see [Deg21]); hence the above bound is reduced down to  $42 + 260 = 302$ .  $\square$

*Example 9.3.* The 1-parameter family of sextic  $K3$ -surfaces with the maximal number 42 of lines has a member with 249 conics (counting both irreducible and reducible ones). Since the construction in §9.3 is obviously invertible, this gives us a nodal cubic fourfold with 291 planes. In fact, conjecturally, 249 is the maximal number of conics on a sextic  $K3$ -surface *containing at least one line*; this fact motivates our conjecture that 291 is the sharp upper bound on the number of planes in a nodal cubic fourfold.

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