NONLINEAR KINEMATICS IN IRREGULAR WAVES ON FINITE DEPTH

by

BJØRN HERVOLD RIISE

THESIS

for the degree of

MASTER OF SCIENCE

(Master i Anvendt matematikk og mekanikk)

Faculty of Mathematics and Natural Sciences
University of Oslo

May 2009

Det matematisk- naturvitenskapelige fakultet
Universitetet i Oslo

NONLINEAR KINEMATICS IN IRREGULAR WAVES ON FINITE DEPTH

by

BJØRN HERVOLD RIISE

Abstract

It has been developed a second order nonlinear Schrödinger method for prediction of kinematics in irregular waves on finite depth. The velocity potential and corresponding surface displacement has been developed under the assumption of slowly modulated amplitude. Shallow water waves have been identified giving contribution to the surface displacement and the horizontal current.

In the absence of measurements on finite depth to verify the accuracy of the method, deep water measurements have been compared with theory. Schrödinger theory has shown good results when compared with measurements of the horizontal velocity in the crest. Determination of the induced mean flow was indicated a return flow confined to the region around the still water level, adding more curvature to the horizontal velocity profile.

Contents

Preface

After finishing my bachelor's degree in Marine Engineering at Bergen University College in the spring of 2007 a desire to know more about hydrodynamics and the need of increasing my mathematical skills made me continue my education at the University of Oslo. At the University of Oslo I have had the privilege of studying hydrodynamics at the Mechanics Division. My major area of studying has been kinematics in water waves and this thesis is dealing with this subject. My achievements would not have been possible without the help from Professor Karsten Trulsen and Professor Atle Jensen. I would like to thank them for plenty of good advice and guidance.

Bjørn Hervold Riise
Oslo, May 2009

Chapter 1

Introduction

In the field of marine hydrodynamics a subject of great importance is the prediction of wave loads on marine structures. One of the most common approximations of wave loads is the widely used *Morison's formula*. By using the formula with an undisturbed wave field and some proper coefficients Morison et al. [7] were able to show good agreement with measurements. In this thesis it is the undisturbed wave field and the corresponding kinematics which are of interest. It should be emphasized that the coefficients in Morison's formula depend on the corresponding wave theory and that a change of wave theory must be followed by a change of coefficients.

For regular waves the most known theories are the linear *Airy theory* [1] and the nonlinear *Stokes theory* [1] developed in the 19th century. Airy theory solves the linear boundary value problem defined for waves propagating in a fluid by using potential theory. To justify the linearization it is assumed that the waves in question have small amplitudes. To solve the nonlinear problem for steeper waves, Stokes introduced power series in terms of a parameter including the amplitude. By using Stokes theory Skjelbreia et al. [9] and Fenton [4] developed fifth order theories for calculating the kinematics in regular waves.

Ocean waves are not regular, but random in form and propagation and these waves, also known as irregular waves, are topics of research in recent time. Irregular waves may be composed by superposing plane waves with different amplitudes and phases. In agreement with the principle of linear superposition, one of the simplest methods to predict kinematics is by using linear Airy theory where components from individual wave components are superposed. As for regular waves the linear theory is limited by the steepness of the waves. The Airy theory is known to over predict in the region near and above mean water level and empirical models have been developed to improve the accuracy. Wheeler developed the widely used *Wheeler stretching method* [13] by stretching the profiles of the kinematics. Wheeler stretching and other similar methods are based on empirical observations and these methods are not well founded in hydrodynamic theory. The coefficients used in Morison's formula are found empirically from force measurements and when the wave theory also is quite influenced by empirical adjustments the traceability from force prediction to rigid mathematical theory decreases.

Grue's method is a simple way to predict kinematics in deep water. Grue et al. [6] used third-order Stokes theory and a proper normalization to get the horizontal velocity to coincide with the exponential profile. They used the local wave to find the governing parameter and the method has shown good agreement with measurements.

Another way of representing irregular waves is by assuming slowly modulated amplitude and phase. The method is known as the Schrödinger method [3] where the Schrödinger equation is governing the propagation of the amplitude and phase. The Schrödinger method is using the entire wave field to find the governing parameters as distinct from those methods using the local wave. Parameters governing the development are steepness, bandwidth and depth, where modulation of the amplitude is related to bandwidth. The use of nonlinear Schrödinger theory to calculate kinematics has been utilized by Trulsen [10] for deep water and by Trulsen et al. [11] for finite depth. With third order nonlinear contribution, Schrödinger kinematics has shown excellent comparisons with measurements [12].

The purpose of this thesis is to develop a method for use in irregular waves at finite depth. For use in conventional engineering the method has to be both accurate and simple to implement. Schrödinger kinematics have shown great accuracy, but the methods already developed are complex, especially at finite depth. Ocean waves are normally specified by their deterministic or statistical characteristics through a measured time series or a wave spectrum. For conventional use the method has to be able to generate kinematics from both a time series and a spectrum. To be used in force prediction the method has to display the velocities and the accelerations throughout the water column at arbitrary time.

The first part of this thesis will be a review of the established methods previously mentioned and in use today. Then there will be developed a simplified nonlinear Schödinger method with details of implementation. Finally the Schrödinger kinematics will be implemented and compared with the established methods.

Chapter 2

Established engineering methods

2.1 Boundary value problem

Like in Newman [8] the fluid is assumed to be incompressible, inviscid and irrotational. Then the velocity vector may be represented as the gradient of the velocity potential $\vec{v} = \nabla\phi$. The velocity potential ϕ and the surface displacement η is found from solving Laplace's equation (2.1) for the given boundary condition. The kinematic condition (2.2) is found with the requirement that a particle on the free surface will stay on the free surface. This requirement is taken care of by the substantial derivative of the surface displacement being equal to the vertical component of the velocity on the free surface. The pressure on the free surface is assumed constant and equal to the atmospheric pressure and the dynamic condition (2.3) is found from Bernoulli's equation. The bottom condition (2.4) is found from the requirement that there is no flow through the sea floor. For uniform depth the boundary value problem to be solved is

$$\nabla^2\phi = 0 \quad \text{for } -h < z < \eta, \tag{2.1}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial z} = 0 \quad \text{at } z = \eta, \tag{2.2}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz = 0 \quad \text{at } z = \eta \quad \text{and} \tag{2.3}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at } z = -h. \tag{2.4}$$

Here a two dimensional Cartesian coordinate system is adopted where x is horizontal axis and z is vertical axis where the plane $z = 0$ coincides with the undisturbed free surface. ∇ is $(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial z}\vec{k})$ where $\vec{i}$ and $\vec{k}$ respectively are unit vectors in horizontal and vertical direction. g is the acceleration due to gravity.

2.2 Airy theory

The simplest way to predict wave kinematics is by using linear *Airy theory.* The surface conditions are approximated with Taylor series around the plane $z = 0$ and subsequently made linear under the assumption that the surface displacement η is sufficiently small compared to the wavelength λ. This is taken care of by the requirement $ak << 1$ where a is the wave amplitude and $k = 2\pi/\lambda$ is the wave number. For a regular wave a possible solution to the linear problem is

$$\eta = a\cos(kx - \omega t + \varphi) \quad \text{and} \tag{2.5}$$

$$\phi = a\omega \frac{\cosh k(z+h)}{k \sinh kh} \sin(kx - \omega t + \varphi), \tag{2.6}$$

where $\omega = 2\pi/T$ is the angular frequency, T is the wave period, k and ω are related through the linear dispersion relation

$$\omega^2 = gk \tanh kh, \tag{2.7}$$

and φ is the phase displacement, which may be set equal to zero with a suitable choice of origin.

Irregular waves may be composed by superposing plane wave components with different amplitudes and phases. With the use of linear Airy theory the surface displacemnt and the velocity potential may be presented as

$$\eta = \frac{1}{2} \sum_j \left(a_j \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} + \mathrm{c.c} \right) \quad \text{and} \tag{2.8}$$

$$\phi = -\frac{1}{2} \sum_j \left(\mathrm{i} a_j \omega_j \frac{\cosh k_j(z+h)}{k_j \sinh k_j h} \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} + \mathrm{c.c} \right). \tag{2.9}$$

and the velocities and accelerations as

$$u = \frac{\partial \phi}{\partial x} = \frac{1}{2} \sum_j \left(a_j \omega_j \frac{\cosh k_j(z+h)}{\sinh k_j h} \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} + \mathrm{c.c} \right), \tag{2.10}$$

$$w = \frac{\partial \phi}{\partial z} = -\frac{1}{2} \sum_j \left(\mathrm{i} a_j \omega_j \frac{\sinh k_j(z+h)}{\sinh k_j h} \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} + \mathrm{c.c} \right), \tag{2.11}$$

$$a_x = \frac{\partial u}{\partial t} = -\frac{1}{2} \sum_j \left(\mathrm{i} a_j \omega_j^2 \frac{\cosh k_j(z+h)}{\sinh k_j h} \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} + \mathrm{c.c} \right) \quad \text{and} \tag{2.12}$$

$$a_z = \frac{\partial w}{\partial t} = -\frac{1}{2} \sum_j \left(a_j \omega_j^2 \frac{\sinh k_j(z+h)}{\sinh k_j h} \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} + \mathrm{c.c} \right), \tag{2.13}$$

where a_j is complex amplitude containing both length of the amplitude $|a_j|$ and phase displacement $\varphi_j = \arg a_j$.

2.3 Wave spectrum

Ocean waves are normally specified by their deterministic characteristics with a measured time series or by their statistical characteristics with a wave spectrum. Like in Gran [5] the wave spectrum $S(\omega)$ is defined analogous to the power spectral density function $\Delta P/\Delta\omega$. Looking at the frequency domain, ΔP is the power contribution from waves within the range $\Delta\omega$. Within the range $\Delta\omega$, waves are fairly monochromatic and may be presented by a pure sinusoidal function $a \cos \omega t$. The power contribution is proportional to the mean square of the wave and with

$$\Delta P = \frac{1}{T} \int_0^T (a \cos \omega t)^2 \, \mathrm{d}t = a^2 \frac{1}{T} \int_0^T \frac{1}{2}(1 + \cos 2\omega t) \, \mathrm{d}t = \frac{1}{2} a^2. \tag{2.14}$$

the wave spectrum is defined as

$$S(\omega) = \frac{\Delta P}{\Delta \omega} = \frac{a^2}{2\Delta\omega} \quad \text{which gives} \quad a(\omega) = \sqrt{2S(\omega)\Delta\omega}, \tag{2.15}$$

and the total energy flux as

$$P = \int_P \mathrm{d}P = \lim_{\Delta P \to 0} \sum_i \Delta P_i = \lim_{\Delta\omega \to 0} \sum_i S(\omega_i)\Delta\omega_i = \int_\omega S(\omega)\,\mathrm{d}\omega = \sum_i \frac{1}{2} a_i^2. \tag{2.16}$$

Then irregular waves may be presented as

$$\eta = \sum_j a(\omega_j)\cos(k_j x - \omega_j t + \varphi_j), \tag{2.17}$$

where k_j and ω_j are related through the linear dispersion relation (2.7) and φ_j is randomly distributed between 0 and 2π. The wave spectrum may also be derived from the auto-correlation function for the surface displacement which forms a Fourier transform pair with the power spectral density function.

The wave spectrum may be estimated by using the Fourier transform $\hat{\eta}$ of a measured time series η which gives

$$S(\omega) = \frac{2}{\Delta\omega}|\hat{\eta}(\omega)|^2 \quad \text{where} \quad \Delta\omega = 2\pi/\Delta T. \tag{2.18}$$

Another way to estimate the wave spectrum is by using analytical functions. The *Pierson-Moskowitz* spectrum and the *JONSWAP* spectrum are frequently applied for wind generated waves. In accordance with DNV [2] the Pierson-Moskowitz spectrum S_{PM} is given by

$$S_{PM}(\omega) = \frac{5}{16}(H_{1/3})^2(\omega_P)^4(\omega)^{-5}\exp\Big(-\frac{5}{4}\Big(\frac{\omega}{\omega_P}\Big)^{-4}\Big) \tag{2.19}$$

where $H_{1/3}$ is average of the highest one-third waves, $\omega_P = 2\pi/T_P$, $T_P = 1/f_P$ and f_P is the frequency where the wave spectrum has its maximum value. The JONSWAP spectrum $S_J(\omega)$ is given as a modification of the Pierson-Moskowitz spectrum and in accordance with DNV

$$S_J(\omega) = A_\gamma S_{PM}(\omega)\gamma^{\exp\big(-0.5\big(\frac{\omega-\omega_P}{\Omega\omega_P}\big)^2\big)} \quad \text{where} \quad A_\gamma = 1 - 0.287\ln(\gamma). \tag{2.20}$$

γ is the peak shape parameter where

$$\gamma = \begin{cases} 5 & \text{for} \quad T_P/\sqrt{H_{1/3}} \leq 3.6 \\ \exp(5.75 - 1.15\frac{T_P}{\sqrt{H_{1/3}}}) & \text{for} \quad 3.6 < T_P/\sqrt{H_{1/3}} < 5 \\ 1 & \text{for} \quad 5 \leq T_P/\sqrt{H_{1/3}} \end{cases} \tag{2.21}$$

may be applied. Ω is the spectral width parameter where

$$\Omega = \begin{cases} \Omega_a & \text{for} \quad \omega \leq \omega_P \\ \Omega_b & \text{for} \quad \omega > \omega_P. \end{cases} \tag{2.22}$$

For $3.6 < T_P/\sqrt{H_{1/3}} < 5$ the JONSWAP spectrum is expected to be a reasonable model. Average values for the spectral parameter are $\gamma = 3.3$, $\Omega_a = 0.07$ and $\Omega_b = 0.09$.

2.4 Why Airy thoery breaks down

The mathematical foundation is the linearization of the surface conditions. The surface displacement is assumed to be sufficiently small to make nonlinear terms minimal compared to linear terms. Subsequently the nonlinear terms are neglected and the linear problem arises. When the nonlinear terms are neglected from the Taylor series the solution for $z = \eta$ equals the solution for $z = 0$. Rigorously this means that the kinematics should be constant throughout the water column from trough to crest. But in this region the kinematics is not constant but drastically changing and the kinematics is extrapolated by using exact vertical position.

For maximum horizontal velocity (2.8) is substituted into (2.10) which gives

$$\big[u\big]_{z=\eta} = \frac{1}{2}\sum_i \sum_j \Big(\frac{a_j\omega_j}{\sinh k_j h}\cosh\Big(\frac{1}{2}(a_i k_j \mathrm{e}^{\mathrm{i}(k_i x-\omega_i t)} + \mathrm{c.c}) + k_j h\Big)\mathrm{e}^{\mathrm{i}(k_j x-\omega_j t)} + \mathrm{c.c}\Big). \tag{2.23}$$

To justify the linearization $a_j k_j << 1$ which leads to a maximum value of the wave spectrum for a given low pass frequency ω_{lp}.

The low pass frequency ω_{lp} is the frequency immediately above the highest frequency considered in the wave spectrum and may also be known as low pass filter. Likewise, the high pass frequency ω_{hp} is the frequency immediately beneath the lowest frequency considered and may also be known as high pass filter. Both frequencies are frequently denoted cut off frequencies.

The shortest waves in the spectrum are assumed to be deep water waves where $kh >> 1$ and for ω_{lp} the linear dispersion relation valid for deep water may be applied and

$$\omega_{\mathrm{lp}} = \sqrt{g k_{\mathrm{max}}} \quad \text{which gives} \quad k_{\mathrm{max}} = \frac{\omega_{\mathrm{lp}}^2}{g}. \tag{2.24}$$

When $a_i k_j << 1$ and $a_i = \sqrt{2S(\omega_i)\Delta\omega}$

$$S(\omega_i) << \frac{g^2}{2\Delta\omega}\omega_{\mathrm{lp}}^{-4}. \tag{2.25}$$

This and $k_j\eta << 1$ are both necessary requirements to justify the linearization and confines Airy theory to very small amplitudes. Violation of these requirements is the reason why Airy theory breaks down.

2.5 Wheeler streching

To improve the Airy theory many empirical methods have been developed. One of the best known methods is the widely used *Wheeler stretching* [13] which adjust the kinematics with a modification of the effective height. The modification of the height was suggested by calculations made in connection with wave tank studies. The accuracy of the velocities improved when the mean height of the water particle orbit was used instead of the surface displacement. The vertical axis is stretched from the old z to the new z_s with

$$(z+h) = h(z_s+h)/(\eta+h) \quad \text{for} \quad -h < z_s < \eta. \tag{2.26}$$

The Wheeler stretching is based on Airy theory using a lowpass filter. DNV [2] recommends a low pass frequency equal to four times the peak frequency, $\omega_{\mathrm{lp}} = 4\omega_P$. If the low pass frequency is set too high and too much of the energy is extracted from the wave spectrum this may be the source of error that Wheeler stretching is compensating for. Wheeler stretching and other similar methods which are based on empirical observations are not well founded in mathematical theory.

2.6 Stokes theory

To solve the nonlinear boundary value problem (2.1)-(2.4) Stokes introduced harmonic power series in terms of a parameter containing the amplitude. In Stokes' original theory the parameter $\epsilon = ak$ was used to expand the series. Looking at the surface displacement the most noticeable higher order effect is the second order contribution which makes both the crest and trough higher. Skjelbreia et al. [9] developed a fifth order method for calculating both the surface displacement and the kinematics with $\epsilon = ak$ as the expansion parameter. Later Fenton [4] showed this method to be wrong at fifth order and introduced the wave steepness $\epsilon = kH/2$, where H is the wave height from crest to trough, as the expansion parameter. Fenton's method for steady propagating waves at finite depth is

$$\phi = (c - \bar{u})x + C_0\sqrt{g/k^3}\sum_{i=1}^{5}\epsilon^i\sum_{j=1}^{i}A_{ij}\cosh jk(z+h)\sin jk(x-ct) + O(\epsilon^6), \tag{2.27}$$

$$\bar{u}\sqrt{k/g} = C_0 + \epsilon^2 C_2 + \epsilon^4 C_4 + O(\epsilon^6) \quad \text{and} \tag{2.28}$$

$$\begin{aligned} k\eta = {} & kh + \epsilon\cos k(x-ct) + \epsilon^2 B_{22}\cos 2k(x-ct) + \epsilon^3 B_{31}\big(\cos k(x-ct) - \cos 3k(x-ct)\big) \\ & + \epsilon^4\big(B_{42}\cos 2k(x-ct) + B_{44}\cos 4k(x-ct)\big) \\ & + \epsilon^5\big(-\big(B_{53}+B_{55}\big)\cos k(x-ct) + B_{53}\cos 3k(x-ct) + B_{55}\cos 5k(x-ct)\big) + O(\epsilon^6), \end{aligned} \tag{2.29}$$

where $c = \lambda/T$ and H is known. For cases where λ and T are known the theory can be directly applied. Otherwise it is necessary to specify the current or the mass flux. The coefficients A, B and C and additional ways to find the wave length and the wave period can be found in Fenton's work [4].

Grue's method is a simple way to predict kinematics in deep water. For deep water a possible solution to the boundary value problem (2.1)-(2.4) using Stokes original parameter $\epsilon = ak$ for the expansion is

$$\frac{k\phi}{\sqrt{g/k}} = \epsilon e^{ky}\sin(kx - \omega t) + O(\epsilon^4), \tag{2.30}$$

$$k\eta = \big(1 + \frac{1}{8}\epsilon^2\big)\epsilon\cos(kx-\omega t) + \frac{1}{2}\epsilon^2\cos 2(kx-\omega t) + \frac{3}{8}\epsilon^3\cos 3(kx-\omega t) + O(\epsilon^4), \tag{2.31}$$

$$\frac{\omega^2}{gk} = 1 + \epsilon^2 + O(\epsilon^3), \tag{2.32}$$

and for a wave event the maximum surface displacement is denoted η_m and

$$k\eta_{\mathrm{m}} = \epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon^3 + O(\epsilon^4). \tag{2.33}$$

The parameter ϵ is found through solving the two equations concerning dispersion (2.32) and wave steepness (2.33). $\omega = 2\pi/T$, where T is the local wave period from trough-to-trough for the wave event. The reconstruction formula for the horizontal velocity is

$$u = \frac{\partial\phi}{\partial x} = \epsilon\sqrt{g/k}e^{ky}\cos(kx-\omega t). \tag{2.34}$$

For the crest the horizontal velocity reaches the maximum $u_{\mathrm{crest}} = \epsilon\sqrt{g/k}e^{ky}$. When the velocity is made dimensionless with the reference velocity $u_{\mathrm{ref}} = \epsilon\sqrt{g/k}$ it becomes the exponential profile under the wave crest $u_{\mathrm{crest}}/u_{\mathrm{ref}} = e^{ky}$.

The foundation of the simplicity in Grue's method is the deep water condition, where $kh >> 1$. When $kh >> 1$ the coefficients from the velocity potential in connection with the second and third order terms converges towards zero. Looking at the velocity potential the only correction from first to third order is in the dispersion relation where the amplitude is included. For finite depth the solution satisfying the third order boundary conditions becomes considerably more complex.

Chapter 3

Nonlinear Schrödinger theory

3.1 Background

Airy theory is developed by using the assumption that the amplitude is small and the theory is limited by the steepness. When Airy theory is used on irregular waves almost the entire wave field is taken into account and the theory is not restricted by the bandwidth. The Airy theory is known to over predict in the region near and above mean water level. The Wheeler stretching method compensates for the over prediction by using an empirical modification, but this is not well founded in mathematical theory. Distinguished from linear theory Grue's method is not limited by the steepness. The method is based on the local wave and is restricted to small bandwidth.

Based on the above considerations a better approach would be to take in consideration both the steepness and the bandwidth. A method that is considering both the steepness and the bandwidth is the Schrödinger theory where the velocity potential and surface displacement are presented by a perturbation expansion in terms of the steepness. The use of nonlinear Schrödinger theory to calculate the kinematics has been utilized by Trulsen [10] for deep water and Trulsen et al. [11] for finite depth. In Trulsen [10] and Trulsen et al. [11] bandwidth and modulation of the amplitude are assumed to be of the same magnitude. To decrease the complexity in this thesis, modulation of the amplitude, according to a characteristic frequency, is assumed to be of same magnitude as the steepness. Distinct from the work on finite depth by Trulsen et al. [11] this thesis is taking into account the contribution from shallow water waves.

3.2 The boundary value problem

The boundary value problem (2.1)-(2.4) is to be considered. The waves are assumed to be weakly nonlinear where $a_c k_c = \epsilon$ and ϵ is a small parameter $<< 1$. a_c is the characteristic amplitude and k_c is the characteristic wave number which is related to the characteristic frequency ω_c though the linear dispersion relation (2.7). The modulation of the amplitude is related to the characteristic parameter whith

$$\frac{\Delta k}{k_c} = \frac{\Delta \omega}{\omega_c} = O(\epsilon) \tag{3.1}$$

and the depth is assumed to be finite where $k_c h = O(1)$. The characteristic scale a_c, k_c and ω_c is used to make the variables (x, z, h), the velocity potential ϕ and the surface displacement η dimensionless and normalize where

$$(x^\star, z^\star, h^\star) = k_c(x, z, h), \quad t^\star = \omega_c t, \quad \phi^\star = \frac{k_c}{a_c\omega_c}\phi, \quad \eta^\star = \frac{1}{a_c}\eta \tag{3.2}$$

and $\star$ denotes the dimensionless parameters. To get the boundary value problem on dimensionless form (3.2) is substituted into (2.1) - (2.4) and

$$\frac{a_c\omega_c k_c^2}{k_c}\nabla^2\phi^\star = 0 \quad \text{for } -h^\star < z^\star < a_c k_c \eta^\star, \tag{3.3}$$

$$a_c\omega_c\frac{\partial\eta^\star}{\partial t^\star} + \frac{a_c^2\omega_c k_c^2}{k_c}\frac{\partial\phi^\star}{\partial x^\star}\frac{\partial\eta^\star}{\partial x^\star} - \frac{a_c\omega_c k_c}{k_c}\frac{\partial\phi^\star}{\partial z^\star} = 0 \quad \text{at } z^\star = a_c k_c\eta^\star, \tag{3.4}$$

$$\frac{a_c\omega_c\omega_c}{k_c}\frac{\partial\phi^\star}{\partial t^\star} + \frac{a_c^2\omega_c^2 k_c^2}{k_c^2}\frac{1}{2}(\nabla\phi)^2 + \frac{g}{k_c}z^\star = 0 \quad \text{at } z^\star = a_c k_c\eta^\star \quad \text{and} \tag{3.5}$$

$$\frac{a_c\omega_c k_c}{k_c}\frac{\partial\phi^\star}{\partial z^\star} = 0 \quad \text{at } z^\star = -h^\star. \tag{3.6}$$

After (3.3) is multiplied with $(1/\omega_c)$, (3.5) is multiplied with (k_c^2/ω_c^2), (3.4) and (3.6) are multiplied with (k_c/ω_c), the linear dispersion relation (2.7) is utilized, $(\tanh k_c h)$ is replaced with σ and $(a_c k_c)$ is replaced with ϵ the dimensionless boundary value problem becomes

$$\epsilon\nabla^2\phi^\star = 0 \quad \text{for } -h^\star < z^\star < \epsilon\eta^\star, \tag{3.7}$$

$$\epsilon\frac{\partial\eta^\star}{\partial t^\star} + \epsilon^2\frac{\partial\phi^\star}{\partial x^\star}\frac{\partial\eta^\star}{\partial x^\star} - \epsilon\frac{\partial\phi^\star}{\partial z^\star} = 0 \quad \text{at } z^\star = \epsilon\eta^\star, \tag{3.8}$$

$$\epsilon\frac{\partial\phi^\star}{\partial t^\star} + \epsilon^2\frac{1}{2}(\nabla\phi)^2 + \frac{1}{\sigma}z^\star = 0 \quad \text{at } z^\star = \epsilon\eta^\star \quad \text{and} \tag{3.9}$$

$$\epsilon\frac{\partial\phi^\star}{\partial z^\star} = 0 \quad \text{at } z^\star = -h^\star. \tag{3.10}$$

For the following calculations the $\star$ is removed and x, z, h, t, ϕ and η are dimensionless. According to the assumption about modulation of the amplitude (3.1) a slow variation in both space x_1 and time t_1 is introduced where $(x_1, t_1) = \epsilon(x, t)$ which gives additional derivatives in space and time where

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x} = \frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{\partial}{\partial t_1}\frac{\partial t_1}{\partial t} = \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial t_1}, \tag{3.11}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial x}\Big) \to \Big(\frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial x_1}\Big)\Big(\frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial x_1}\Big) = \frac{\partial^2}{\partial x^2} + 2\epsilon\frac{\partial^2}{\partial x\partial x_1} + \epsilon^2\frac{\partial^2}{\partial x_1^2} \quad \text{and} \tag{3.12}$$

$$(\nabla\phi)^2 \to (\nabla\phi)^2 + 2\epsilon\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x_1} + \epsilon^2(\frac{\partial\phi}{\partial x_1})^2. \tag{3.13}$$

The slow variations (3.11)-(3.13) are included in (3.7)-(3.10) and the boundary value problem becomes

$$\epsilon\frac{\partial^2\phi}{\partial x^2} + 2\epsilon^2\frac{\partial^2\phi}{\partial x\partial x_1} + \epsilon^3\frac{\partial^2\phi}{\partial x_1^2} + \epsilon\frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{for } -h < z < \epsilon\eta, \tag{3.14}$$

$$\epsilon\frac{\partial\eta}{\partial t} + \epsilon^2\frac{\partial\eta}{\partial t_1} + \epsilon^2\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \epsilon^3\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x_1} + \epsilon^3\frac{\partial\phi}{\partial x_1}\frac{\partial\eta}{\partial x} - \epsilon\frac{\partial\phi}{\partial z} + O(\epsilon^4) = 0 \quad \text{at } z = \epsilon\eta, \tag{3.15}$$

$$\epsilon\frac{\partial\phi}{\partial t}+\epsilon^2\frac{\partial\phi}{\partial t_1}+\epsilon^2\frac{1}{2}(\nabla\phi)^2+\epsilon^3\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x_1}+\frac{1}{\sigma}z+O(\epsilon^4)=0 \quad \text{at } z=\epsilon\eta \quad \text{and} \tag{3.16}$$

$$\epsilon\frac{\partial\phi}{\partial z}=0 \quad \text{at } z=-h. \tag{3.17}$$

The kinematic and dynamic boundary conditions are approximated with Taylor series expanded around $z = 0$. The kinematic condition becomes

$$\begin{aligned}&\left[\epsilon\frac{\partial\eta}{\partial t}+\epsilon^2\frac{\partial\eta}{\partial t_1}+\epsilon^2\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}+\epsilon^3\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x_1}+\epsilon^3\frac{\partial\phi}{\partial x_1}\frac{\partial\eta}{\partial x}-\epsilon\frac{\partial\phi}{\partial z}+O(\epsilon^4)\right]_{z=\epsilon\eta}\\ &=\left[\epsilon\frac{\partial\eta}{\partial t}+\epsilon^2\frac{\partial\eta}{\partial t_1}+\epsilon^2\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}+\epsilon^3\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x_1}+\epsilon^3\frac{\partial\phi}{\partial x_1}\frac{\partial\eta}{\partial x}\right.\\ &\left.-\epsilon\frac{\partial\phi}{\partial z}+\epsilon^3\eta\frac{\partial^2\phi}{\partial z\partial x}\frac{\partial\eta}{\partial x}+\epsilon^3\eta\frac{\partial^2\phi}{\partial z\partial x}\frac{\partial\eta}{\partial x}-\epsilon^2\eta\frac{\partial^2\phi}{\partial z^2}-\epsilon^3\eta^2\frac{1}{2}\frac{\partial^3\phi}{\partial z^3}+O(\epsilon^4)\right]_{z=0}=0\end{aligned} \tag{3.18}$$

and the dynamic condition becomes

$$\begin{aligned}&\left[\epsilon\frac{\partial\phi}{\partial t}+\epsilon^2\frac{\partial\phi}{\partial t_1}+\epsilon^2\frac{1}{2}(\nabla\phi)^2+\epsilon^3\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x_1}+\frac{1}{\sigma}z+O(\epsilon^4)\right]_{z=\epsilon\eta}\\ &=\left[\epsilon\frac{\partial\phi}{\partial t}+\epsilon^2\frac{\partial\phi}{\partial t_1}+\epsilon^2\frac{1}{2}(\nabla\phi)^2+\epsilon^3\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x_1}+\epsilon^2\eta\frac{\partial^2\phi}{\partial z\partial t}+\epsilon^3\eta\frac{\partial^2\phi}{\partial z\partial t_1}\right.\\ &\left.+\epsilon^3\eta\nabla\phi\frac{\partial}{\partial z}(\nabla\phi)+\epsilon^3\eta^2\frac{1}{2}\frac{\partial^3\phi}{\partial z^2\partial t}+O(\epsilon^4)\right]_{z=0}+\epsilon\eta\frac{1}{\sigma}=0.\end{aligned} \tag{3.19}$$

A harmonic expansion of the velocity potential and the surface displacement is proposed as a possible solution where

$$\phi=\phi_1+\epsilon\phi_2+\epsilon^2\phi_3+\cdots+ \quad \text{and} \quad \eta=\eta_1+\epsilon\eta_2+\epsilon^2\eta_3+\cdots+ \tag{3.20}$$

with

$$\phi_1=\acute{A}_{10}+\frac{1}{2}(\acute{A}_{11}\mathrm{e}^{\mathrm{i}\chi}+\mathrm{c.c}), \tag{3.21}$$

$$\phi_2=\acute{A}_{20}+\frac{1}{2}(\acute{A}_{21}\mathrm{e}^{\mathrm{i}\chi}+\acute{A}_{22}\mathrm{e}^{2\mathrm{i}\chi}+\mathrm{c.c}), \tag{3.22}$$

$$\phi_3=\acute{A}_{30}+\frac{1}{2}(\acute{A}_{31}\mathrm{e}^{i\chi}+\acute{A}_{32}\mathrm{e}^{2i\chi}+\acute{A}_{33}\mathrm{e}^{3i\chi}+\mathrm{c.c}), \tag{3.23}$$

$$\eta_1=B_{10}+\frac{1}{2}(B_{11}\mathrm{e}^{\mathrm{i}\chi}+\mathrm{c.c}), \tag{3.24}$$

$$\eta_2=B_{20}+\frac{1}{2}(B_{21}\mathrm{e}^{\mathrm{i}\chi}+B_{22}\mathrm{e}^{2\mathrm{i}\chi}+\mathrm{c.c}), \quad \text{and} \tag{3.25}$$

$$\eta_3=B_{30}+\frac{1}{2}(B_{31}\mathrm{e}^{i\chi}+B_{32}\mathrm{e}^{2i\chi}+B_{33}\mathrm{e}^{3i\chi}+\mathrm{c.c}). \tag{3.26}$$

Here $\acute{A}_{nm} = \acute{A}_{nm}(x_1, t_1, z)$, depending on the slow variation and the vertical position and $B_{nm} = B_{nm}(x_1, t_1)$, depending only on the slow variation. $\chi = (x - t)$ is the phase function and c.c denotes the complex conjugate. Finally the velocity potential and the surface displacement (3.20) are substituted into Laplace (3.14), the bottom condition (3.17), the dynamic condition (3.19) and the kinematic condition (3.18). The equations are reorganized according to the magnitude of ϵ and the boundary value problem

to be solved is
Laplace for $-h < z < \epsilon\eta$

$$\epsilon\frac{\partial^2\phi}{\partial x^2} + 2\epsilon^2\frac{\partial^2\phi}{\partial x\partial x_1} + \epsilon\frac{\partial^2\phi}{\partial z^2}$$
$$= \epsilon\Big(\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial z^2}\Big) + \epsilon^2\Big(\frac{\partial^2\phi_2}{\partial x^2} + 2\frac{\partial^2\phi_1}{\partial x\partial x_1} + \frac{\partial^2\phi_2}{\partial z^2}\Big)$$
$$+\epsilon^3\Big(\frac{\partial^2\phi_3}{\partial x^2} + 2\frac{\partial^2\phi_2}{\partial x\partial x_1} + \frac{\partial^2\phi_1}{\partial x_1^2} + \frac{\partial^2\phi_3}{\partial z^2}\Big) + O(\epsilon^4) = 0, \tag{3.27}$$

the kinematic condition at $z = 0$

$$\epsilon\frac{\partial\eta}{\partial t} + \epsilon^2\frac{\partial\eta}{\partial t_1} + \epsilon^2\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \epsilon^3\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x_1} + \epsilon^3\frac{\partial\phi}{\partial x_1}\frac{\partial\eta}{\partial x}$$
$$-\epsilon\frac{\partial\phi}{\partial z} + \epsilon^3\eta\frac{\partial^2\phi}{\partial z\partial x}\frac{\partial\eta}{\partial x} - \epsilon^2\eta\frac{\partial^2\phi}{\partial z^2} - \epsilon^3\eta^2\frac{1}{2}\frac{\partial^3\phi}{\partial z^3} + O(\epsilon^4)$$
$$= \epsilon\Big(\frac{\partial\eta_1}{\partial t} - \frac{\partial\phi_1}{\partial z}\Big) + \epsilon^2\Big(\frac{\partial\eta_2}{\partial t} + \frac{\partial\eta_1}{\partial t_1} + \frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x} - \frac{\partial\phi_2}{\partial z} - \eta_1\frac{\partial^2\phi_1}{\partial z^2}\Big)$$
$$+\epsilon^3\Big(\frac{\partial\eta_3}{\partial t} + \frac{\partial\eta_2}{\partial t_1} + \frac{\partial\phi_1}{\partial x}\frac{\partial\eta_2}{\partial x} + \frac{\partial\phi_2}{\partial x}\frac{\partial\eta_1}{\partial x} + \frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x_1} + \frac{\partial\phi_1}{\partial x_1}\frac{\partial\eta_1}{\partial x}$$
$$-\frac{\partial\phi_3}{\partial z} + \eta_1\frac{\partial^2\phi_1}{\partial z\partial x}\frac{\partial\eta_1}{\partial x} - \eta_1\frac{\partial^2\phi_2}{\partial z^2} - \eta_2\frac{\partial^2\phi_1}{\partial z^2} - \eta_1^2\frac{1}{2}\frac{\partial^3\phi_1}{\partial z^3}\Big) + O(\epsilon^4) = 0, \tag{3.28}$$

the dynamic condition at $z = 0$

$$\epsilon\frac{\partial\phi}{\partial t} + \epsilon^2\frac{\partial\phi}{\partial t_1} + \epsilon^2\frac{1}{2}(\nabla\phi)^2 + \epsilon^3\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x_1} + \epsilon^2\eta\frac{\partial^2\phi}{\partial z\partial t} + \epsilon^3\eta\frac{\partial^2\phi}{\partial z\partial t_1}$$
$$+\epsilon^3\eta\nabla\phi\frac{\partial}{\partial z}(\nabla\phi) + \epsilon^3\eta^2\frac{1}{2}\frac{\partial^3\phi}{\partial z^2\partial t} + \epsilon\eta\frac{1}{\sigma} + O(\epsilon^4)$$
$$= \epsilon\Big(\frac{\partial\phi_1}{\partial t} + \eta_1\frac{1}{\sigma}\Big) + \epsilon^2\Big(\frac{\partial\phi_2}{\partial t} + \frac{\partial\phi_1}{\partial t_1} + \frac{1}{2}(\nabla\phi_1)^2 + \eta_1\frac{\partial^2\phi_1}{\partial z\partial t} + \eta_2\frac{1}{\sigma}\Big)$$
$$+\epsilon^3\Big(\frac{\partial\phi_3}{\partial t} + \frac{\partial\phi_2}{\partial t_1} + \frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial x} + \frac{\partial\phi_1}{\partial z}\frac{\partial\phi_2}{\partial z} + \frac{\partial\phi_1}{\partial x}\frac{\partial\phi_1}{\partial x_1} + \eta_1\frac{\partial^2\phi_2}{\partial z\partial t}$$
$$+\eta_2\frac{\partial^2\phi_1}{\partial z\partial t} + \eta_1\frac{\partial^2\phi_1}{\partial z\partial t_1} + \eta_1\nabla\phi_1\frac{\partial}{\partial z}(\nabla\phi_1) + \eta^2\frac{1}{2}\frac{\partial^3\phi_1}{\partial z^2\partial t} + \eta_3\frac{1}{\sigma}\Big) + O(\epsilon^4) = 0 \quad \text{and} \tag{3.29}$$

the bottom condition at $z = -h$

$$\epsilon\frac{\partial\phi}{\partial z} = \epsilon\frac{\partial\phi_1}{\partial z} + \epsilon^2\frac{\partial\phi_2}{\partial z} + \epsilon^3\frac{\partial\phi_3}{\partial z} + O(\epsilon^4) = 0. \tag{3.30}$$

3.3 1st order problem

From the boundary value problem (3.27) - (3.30) the parts of $O(\epsilon)$ is extracted and the 1st order problem is established. The 1st order problem is linear and equivalent to the problem solved by the Airy theory and is

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial z^2} = 0 \quad \text{for } -h < z < \eta, \tag{3.31}$$

$$\frac{\partial\eta_1}{\partial t} - \frac{\partial\phi_1}{\partial z} = 0 \quad \text{at } z = 0, \tag{3.32}$$

$$\frac{\partial \phi_1}{\partial t} + \frac{1}{\sigma}\eta_1 = 0 \quad \text{at } z = 0 \quad \text{and} \tag{3.33}$$

$$\frac{\partial \phi_1}{\partial z} = 0 \quad \text{at } z = -h. \tag{3.34}$$

To find the 1st order potential (3.21) and the corresponding surface displacement (3.24) Laplace (3.31) will be solved for the given boundary condition (3.32)-(3.34). At first the 1st order potential is substituted into the Laplace and

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = -\frac{1}{2}\acute{A}_{11}\mathrm{e}^{\mathrm{i}\chi} + \frac{\partial^2 \acute{A}_{10}}{\partial z^2} + \frac{1}{2}\frac{\partial^2 \acute{A}_{11}}{\partial z^2}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c} = 0. \tag{3.35}$$

The equation is valid for arbitrary time and may therefore be divided into different harmonies which gives two separate equations where $A_{1n} = A_{1n}(x_1, t_1)$ and $C_{1n} = C_{1n}(x_1, t_1)$ for $n = (0, 1)$. From the 0th harmonic

$$\frac{\partial^2 \acute{A}_{10}}{\partial z^2} = 0 \quad \text{with possible solution} \quad \acute{A}_{10} = A_{10}z + C_{10}. \tag{3.36}$$

From the 1st harmonic

$$\frac{\partial^2 \acute{A}_{11}}{\partial z^2} - \acute{A}_{11} = 0 \quad \text{with possible solution} \quad \acute{A}_{11} = A_{11}\cosh(z+h) + C_{11}\sinh(z+h). \tag{3.37}$$

By substituting (3.36) and (3.37) into (3.21) the 1st order potential becomes

$$\phi_1 = (A_{10}z + C_{10}) + \frac{1}{2}\Big(\big(A_{11}\cosh(z+h) + C_{11}\sinh(z+h)\big)\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big). \tag{3.38}$$

Next the 1st order potential (3.38) is substituted into the bottom condition (3.34) and

$$\left[\frac{\partial \phi_1}{\partial z}\right]_{z=-h} = A_{10} + \Big(\frac{1}{2}\big(A_{11}\sinh(-h+h) + C_{11}\cosh(-h+h)\big)\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) = 0 \tag{3.39}$$

which gives $A_{10} = C_{11} = 0$. When $C_{10}(x_1, t_1)$ is replaced with $\beta(x_1, t_1)$ and utilize that any linear combination is also a solution of the differential equation the 1st order potential may be

$$\phi_1 = \beta + \frac{1}{2}\Big(A_{11}\frac{\cosh(z+h)}{\cosh h}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big). \tag{3.40}$$

Finally the 1st order potential (3.40) and the 1st order surface displacement (3.24) is substituted into the kinematic condition (3.32) and the dynamic condition (3.33) and

$$\frac{\partial \eta_1}{\partial t} - \frac{\partial \phi_1}{\partial z} = -\frac{1}{2}\mathrm{i}B_{11}\mathrm{e}^{\mathrm{i}\chi} - \frac{1}{2}A_{11}\sigma\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c} = 0 \quad \text{and} \tag{3.41}$$

$$\frac{\partial \phi_1}{\partial t} + \frac{1}{\sigma}\eta_1 = \frac{1}{\sigma}B_{10} + \Big(-\mathrm{i}\frac{1}{2}A_{11}\mathrm{e}^{\mathrm{i}\chi} + \frac{1}{2\sigma}B_{11}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) = 0. \tag{3.42}$$

From the kinematic condition (3.41)

$$\mathrm{i}B_{11} + A_{11}\sigma = 0. \tag{3.43}$$

From the dynamic condition (3.42) the 0th harmonic gives $B_{10} = 0$ and the 1st harmonic gives

$$-\mathrm{i}A_{11} + \frac{1}{\sigma}B_{11} = 0. \tag{3.44}$$

The equations (3.43) and (3.44) forms a homogeneous system of linear equations. With the use of methods from linear algebra the system may be presented as

$$\begin{bmatrix} -\mathrm{i} & 1/\sigma \\ \sigma & \mathrm{i} \end{bmatrix} \begin{Bmatrix} A_{11} \\ B_{11} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3.45}$$

To find the non trivial solution the determinant is required to equal zero which gives one free variable. B_{11} is representing the linear amplitude of the surface displacement and is chosen to be the free variable. After replacing $A_{11}(x_1, t_1)$ with $A(x_1, t_1)$ and $B_{11}(x_1, t_1)$ with $B(x_1, t_1)$ the solution to the 1st order problem is

$$\phi_1 = \beta + \frac{1}{2}\Big(A\frac{\cosh(z+h)}{\cosh h}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) \tag{3.46}$$

$$\text{where} \quad A = -\mathrm{i}\frac{B}{\sigma} \quad \text{and} \tag{3.47}$$

$$\eta_1 = \frac{1}{2}\big(B\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\big). \tag{3.48}$$

3.4 2nd order problem

From the boundary value problem (3.27) - (3.30) the parts of $O(\epsilon^2)$ is extracted and the 2nd order problem is established and is

$$\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial z^2} + 2\frac{\partial^2\phi_1}{\partial x\partial x_1} = 0 \qquad \text{for } -h < z < \eta, \tag{3.49}$$

$$\frac{\partial\eta_2}{\partial t} + \frac{\partial\eta_1}{\partial t_1} + \frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x} - \frac{\partial\phi_2}{\partial z} - \eta_1\frac{\partial^2\phi_1}{\partial z^2} = 0 \qquad \text{at } z = 0, \tag{3.50}$$

$$\frac{\partial\phi_2}{\partial t} + \frac{\partial\phi_1}{\partial t_1} + \frac{1}{2}\Big((\frac{\partial\phi_1}{\partial x})^2 + (\frac{\partial\phi_1}{\partial z})^2\Big) + \eta_1\frac{\partial^2\phi_1}{\partial z\partial t} + \frac{1}{\sigma}\eta_2 = 0 \qquad \text{at } z = 0 \quad \text{and} \tag{3.51}$$

$$\frac{\partial\phi_2}{\partial z} = 0 \qquad \text{at } z = -h. \tag{3.52}$$

To find the 2nd order potential (3.22) and the corresponding surface displacement (3.25) Laplace (3.49) will be solved for the given boundary condition (3.50)-(3.52). First the 2nd order potential and the 1st order potential (3.46) is substituted into Laplace which is

$$\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial z^2} = -2\frac{\partial^2\phi_1}{\partial x\partial x_1}. \tag{3.53}$$

The left hand side of the equation becomes

$$\begin{aligned}&\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial z^2} \\ &\quad = \frac{\partial^2\acute{A}_{20}}{\partial z^2} + \bigg(-\Big(\frac{1}{2}\acute{A}_{21}\mathrm{e}^{\mathrm{i}\chi} + 2\acute{A}_{22}\mathrm{e}^{2\mathrm{i}\chi}\Big) + \frac{1}{2}\Big(\frac{\partial^2\acute{A}_{21}}{\partial z^2}\mathrm{e}^{\mathrm{i}\chi} + \frac{\partial^2\acute{A}_{22}}{\partial z^2}\mathrm{e}^{2\mathrm{i}\chi}\Big) + \mathrm{c.c}\bigg)\end{aligned} \tag{3.54}$$

and the right hand side of the equation becomes

$$-2\frac{\partial^2\phi_1}{\partial x\partial x_1} = -\frac{1}{\sigma}\frac{\partial B}{\partial x_1}\frac{\cosh(z+h)}{\cosh h}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c.} \tag{3.55}$$

The equation is valid for arbitrary time and may therefore be divided into different harmonies which gives three separate equations where $A_{2n} = A_{2n}(x_1, t_1)$ and $C_{2n} = C_{2n}(x_1, t_1)$ for $n = (0, 1, 2)$. From the 0th harmonic

$$\frac{\partial^2 \acute{A}_{20}}{\partial z^2} = 0 \quad \text{with possible solution} \quad \acute{A}_{20} = A_{20} z + C_{20}. \tag{3.56}$$

From the 1st harmonic

$$\frac{\partial^2 \acute{A}_{21}}{\partial z^2} - \acute{A}_{21} = -\frac{2}{\sigma}\frac{\partial B}{\partial x_1}\frac{\cosh(z+h)}{\cosh h}. \tag{3.57}$$

To find a solution to (3.57) $\acute{A}_{21}$ is divided into a solution to the homogeneous problem $\acute{A}_{21H}$ and a solution to the particular problem $\acute{A}_{21P}$ where $\acute{A}_{21} = \acute{A}_{21H} + \acute{A}_{21P}$. A possible solution to the homogeneous problem is

$$\acute{A}_{21H} = A_{21}\cosh(z+h) + C_{21}\sinh(z+h) \tag{3.58}$$

and a possible solution to the particular problem is

$$\acute{A}_{21P} = D_{21} z\cosh(z+h) + E_{21} z\sinh(z+h). \tag{3.59}$$

Equation (3.59) is subtituted into (3.57)

$$\frac{\partial^2 \acute{A}_{21P}}{\partial z^2} - \acute{A}_{21P} = 2D_{21}\sinh(z+h) + 2E_{21}\cosh(z+h) = -\frac{2}{\sigma}\frac{\partial B}{\partial x_1}\frac{\cosh(z+h)}{\cosh h} \tag{3.60}$$

which gives

$$D_{21} = 0, \quad E_{21} = -\frac{1}{\sigma\cosh h}\frac{\partial B}{\partial x_1} \quad \text{and} \tag{3.61}$$

(3.61) is substituted into (3.59) which gives

$$\begin{aligned}\acute{A}_{21} &= A_{21}\cosh(z+h) + C_{21}\sinh(z+h) - \frac{z}{\sigma\cosh h}\frac{\partial B}{\partial x_1}\sinh(z+h)\\ &= A_{21}\cosh(z+h) + \left(C_{21} - \frac{z}{\sinh h}\frac{\partial B}{\partial x_1}\right)\sinh(z+h).\end{aligned} \tag{3.62}$$

From the 2nd harmonic

$$\frac{\partial^2 \acute{A}_{22}}{\partial z^2} - 4\acute{A}_{22} = 0 \quad \text{with possible solution} \quad \acute{A}_{22} = A_{22}\cosh 2(z+h) + C_{22}\sinh 2(z+h). \tag{3.63}$$

By using (3.56), (3.62) and (3.63) the 2nd order potential (3.22) becomes

$$\begin{aligned}\phi_2 = \Big(A_{20}z + C_{20}\Big) + \frac{1}{2}\Bigg(&\Big(A_{21}\cosh(z+h) + \Big(C_{21} - \frac{z}{\sinh h}\frac{\partial B}{\partial x_1}\Big)\sinh(z+h)\Big)\mathrm{e}^{\mathrm{i}\chi}\\ &+\Big(A_{22}\cosh 2(z+h) + C_{22}\sinh 2(z+h)\Big)\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Bigg).\end{aligned} \tag{3.64}$$

Next the 2nd order potential (3.64) and the 1st order potential (3.46) is substituted into the bottom condition (3.52) and

$$\left[\frac{\partial\phi_2}{\partial z}\right]_{z=-h} = A_{20} + \left(\frac{1}{2}\Big(C_{21} + \frac{h}{\sinh h}\frac{\partial B}{\partial x_1}\Big)\mathrm{e}^{\mathrm{i}\chi} + C_{22}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\right) = 0. \tag{3.65}$$

The equation is valid for arbitrary time and may therefore be divided into different harmonies which gives

$$A_{20} = 0, \quad C_{21} = -\frac{h}{\sinh h}\frac{\partial B}{\partial x_1} \quad \text{and} \quad C_{22} = 0. \tag{3.66}$$

When (3.66) is substituted into (3.64), $C_{20}(x_1, t_1)$ is replaced with $\beta_2(x_1, t_1)$ and utilize that any linear combination is also a solution of the differential equation the 2nd order potential may be

$$\begin{aligned}
\phi_2 &= C_{20} + \frac{1}{2}\bigg(\Big(A_{21}\cosh(z+h) - \Big(\frac{h}{\sinh h}\frac{\partial B}{\partial x_1} + \frac{z}{\sinh h}\frac{\partial B}{\partial x_1}\Big)\sinh(z+h)\Big)\mathrm{e}^{\mathrm{i}\chi} \\
&+A_{22}\cosh 2(z+h)\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\bigg) \\
&= \beta_2 + \frac{1}{2}\bigg(\Big(A_{21}\frac{\cosh(z+h)}{\cosh h} - (h+z)\frac{\sinh(z+h)}{\sinh h}\frac{\partial B}{\partial x_1}\Big)\mathrm{e}^{\mathrm{i}\chi} \\
&+A_{22}\frac{\cosh 2(z+h)}{\cosh 2h}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\bigg).
\end{aligned} \tag{3.67}$$

Next the 2nd order potential (3.67) and the 2nd order surface displacement (3.25) together with the solution to the 1st order problem (3.46) and (3.48) is substituted into the kinematic condition (3.50) which is

$$\frac{\partial \phi_2}{\partial z} - \frac{\partial \eta_2}{\partial t} = \frac{\partial \eta_1}{\partial t_1} + \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x} - \eta_1\frac{\partial^2 \phi_1}{\partial z^2}. \tag{3.68}$$

The left hand side of the equation becomes

$$\begin{aligned}
&\frac{\partial \phi_2}{\partial z} - \frac{\partial \eta_2}{\partial t} \\
&= \frac{1}{2}\Big(A_{21}\sigma - (1+\frac{h}{\sigma})\frac{\partial B}{\partial x_1}\Big)\mathrm{e}^{\mathrm{i}\chi} + \frac{1}{2}\Big(A_{21}^*\sigma - (1+\frac{h}{\sigma})\frac{\partial B^*}{\partial x_1}\Big)\mathrm{e}^{-\mathrm{i}\chi} \\
&+A_{22}\sigma_2\mathrm{e}^{2\mathrm{i}\chi} + A_{22}^*\sigma_2\mathrm{e}^{-2\mathrm{i}\chi} + \frac{1}{2}\mathrm{i}B_{21}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{i}B_{22}\mathrm{e}^{2\mathrm{i}\chi} - \frac{1}{2}\mathrm{i}B_{21}^*\mathrm{e}^{-\mathrm{i}\chi} - \mathrm{i}B_{22}^*\mathrm{e}^{-2\mathrm{i}\chi}
\end{aligned} \tag{3.69}$$

where $*$ denotes the complex conjugate and

$$\sigma_2 = \tanh 2h = \frac{2\sigma}{1+\sigma^2} \tag{3.70}$$

and the right hand side of the equation becomes

$$\begin{aligned}
&\frac{\partial \eta_1}{\partial t_1} + \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x} - \eta_1\frac{\partial^2 \phi_1}{\partial z^2} \\
&= \frac{1}{2}\Big(\frac{\partial B}{\partial t_1}\mathrm{e}^{\mathrm{i}\chi} + \frac{\partial B^*}{\partial t_1}\mathrm{e}^{-\mathrm{i}\chi}\Big) + \frac{1}{2}\Big(\mathrm{i}A\mathrm{e}^{\mathrm{i}\chi} - \mathrm{i}A^*\mathrm{e}^{-\mathrm{i}\chi}\Big)\frac{1}{2}\Big(\mathrm{i}B\mathrm{e}^{\mathrm{i}\chi} - \mathrm{i}B^*\mathrm{e}^{-\mathrm{i}\chi}\Big) \\
&-\frac{1}{2}\Big(B\mathrm{e}^{\mathrm{i}\chi} + B^*\mathrm{e}^{-\mathrm{i}\chi}\Big)\frac{1}{2}\Big(A\mathrm{e}^{\mathrm{i}\chi} + A^*\mathrm{e}^{-\mathrm{i}\chi}\Big) \\
&= \frac{1}{2}\Big(\frac{\partial B}{\partial t_1}\mathrm{e}^{\mathrm{i}\chi} + \frac{\partial B^*}{\partial t_1}\mathrm{e}^{-\mathrm{i}\chi}\Big) - \frac{1}{4}\Big(AB\mathrm{e}^{2\mathrm{i}\chi} + A^*B^*\mathrm{e}^{-2\mathrm{i}\chi}\Big) - \frac{1}{4}\Big(BA\mathrm{e}^{2\mathrm{i}\chi} + B^*A^*\mathrm{e}^{-2\mathrm{i}\chi}\Big) \\
&= \frac{1}{2}\Big(\frac{\partial B}{\partial t_1}\mathrm{e}^{\mathrm{i}\chi} + \frac{\partial B^*}{\partial t_1}\mathrm{e}^{-\mathrm{i}\chi}\Big) - \frac{1}{2}\Big(AB\mathrm{e}^{2\mathrm{i}\chi} + A^*B^*\mathrm{e}^{-2\mathrm{i}\chi}\Big).
\end{aligned} \tag{3.71}$$

The equation is valid for arbitrary time and may therefore be divided into different harmonies which gives two separate equations. From the 1st harmonic

$$A_{21}\sigma - (1+\frac{h}{\sigma})\frac{\partial B}{\partial x_1} + \mathrm{i}B_{21} = \frac{\partial B}{\partial t_1} \quad \Rightarrow \quad A_{21}\sigma + \mathrm{i}B_{21} = \frac{\partial B}{\partial t_1} + (1+\frac{h}{\sigma})\frac{\partial B}{\partial x_1} \tag{3.72}$$

From the 2nd harmonic and with the use of (3.47)

$$A_{22}\sigma_2 + \mathrm{i}B_{22} = -\frac{1}{2}AB \quad \Rightarrow \quad A_{22}\sigma_2 + \mathrm{i}B_{22} = \mathrm{i}\frac{B^2}{2\sigma} \tag{3.73}$$

Then the 2nd order potential (3.67) and the 2nd order surface displacement (3.25) together with solutions to the 1st order problem (3.46) and (3.48) is substituted into the dynamic condition (3.51) which is

$$\frac{\partial \phi_2}{\partial t} + \frac{1}{\sigma}\eta_2 = -\Big(\frac{\partial \phi_1}{\partial t_1} + \frac{1}{2}(\frac{\partial \phi_1}{\partial x})^2 + \frac{1}{2}(\frac{\partial \phi_1}{\partial z})^2 + \eta_1\frac{\partial^2 \phi_1}{\partial z \partial t}\Big). \tag{3.74}$$

The left hand side of the equations becomes

$$\begin{aligned}\frac{\partial \phi_2}{\partial t} + \frac{1}{\sigma}\eta_2 = &-\frac{1}{2}\mathrm{i}\Big((A_{21} - h\frac{\partial B}{\partial x_1})\mathrm{e}^{\mathrm{i}\chi} - (A_{21}^* - h\frac{\partial B^*}{\partial x_1})\mathrm{e}^{-\mathrm{i}\chi}\Big)\\ &-\mathrm{i}\Big(A_{22}\mathrm{e}^{2\mathrm{i}\chi} - A_{22}^*\mathrm{e}^{-2\mathrm{i}\chi}\Big) + \frac{1}{\sigma}\Big(B_{20} + \frac{1}{2}(B_{21}\mathrm{e}^{\mathrm{i}\chi} + B_{22}\mathrm{e}^{2\mathrm{i}\chi} + B_{21}^*\mathrm{e}^{-\mathrm{i}\chi} + B_{22}^*\mathrm{e}^{-2\mathrm{i}\chi})\Big)\end{aligned} \tag{3.75}$$

and the right hand side of the equation becomes

$$\begin{aligned}&-\Big(\frac{\partial \phi_1}{\partial t_1} + \frac{1}{2}(\frac{\partial \phi_1}{\partial x})^2 + \frac{1}{2}(\frac{\partial \phi_1}{\partial z})^2 + \eta_1\frac{\partial^2 \phi_1}{\partial z \partial t}\Big)\\ &= -\Big(\frac{\beta}{\partial t_1} + \frac{1}{2}\frac{\partial A}{\partial t_1}\mathrm{e}^{\mathrm{i}\chi} + \frac{1}{2}\frac{\partial A^*}{\partial t_1}\mathrm{e}^{-\mathrm{i}\chi} + \frac{1}{2}(\frac{1}{2}\mathrm{i}A\mathrm{e}^{\mathrm{i}\chi} - \frac{1}{2}\mathrm{i}A^*\mathrm{e}^{-\mathrm{i}\chi})^2\\ &+\frac{1}{2}(\frac{1}{2}A\sigma^{\mathrm{i}\chi} + \frac{1}{2}A^*\sigma^{-\mathrm{i}\chi})^2 + (\frac{1}{2}a\mathrm{e}^{\mathrm{i}\chi} + \frac{1}{2}B^*\mathrm{e}^{-\mathrm{i}\chi})(-\frac{1}{2}\mathrm{i}A\sigma^{\mathrm{i}\chi} + \frac{1}{2}\mathrm{i}A^*\sigma^{-\mathrm{i}\chi})\Big)\\ &= -\Big(\frac{\partial\beta}{\partial t_1} + \frac{1}{2}\frac{\partial A}{\partial t_1}\mathrm{e}^{\mathrm{i}\chi} + \frac{1}{2}\frac{\partial A^*}{\partial t_1}\mathrm{e}^{-\mathrm{i}\chi}\Big) + \frac{1}{8}\Big(A^2\mathrm{e}^{2\mathrm{i}\chi} - 2AA^* + (A^*)^2\mathrm{e}^{-2\mathrm{i}\chi}\Big)\\ &-\frac{1}{8}\sigma^2\Big(A^2\mathrm{e}^{2\mathrm{i}\chi} + 2AA^* + (A^*)^2\mathrm{e}^{-2\mathrm{i}\chi}\Big) + \frac{1}{4}\mathrm{i}\sigma\Big(aA\mathrm{e}^{2\mathrm{i}\chi} - 2aA^* - B^*A^*\mathrm{e}^{-2\mathrm{i}\chi}\Big).\end{aligned} \tag{3.76}$$

The equation is valid for arbitrary time and may therefore be divided into different harmonies which gives three separate equations. From the 0th harmonic

$$\begin{aligned}\frac{1}{\sigma}B_{20} &= -\frac{\partial\beta}{\partial t_1} - \frac{1}{4}AA^* - \frac{1}{4}\sigma^2AA^* - \frac{1}{2}\mathrm{i}\sigma BA^*\\ &= -\frac{\partial\beta}{\partial t_1} - \frac{1}{4}(-\mathrm{i}\frac{B}{\sigma})(\mathrm{i}\frac{B^*}{\sigma})(1+\sigma^2) + \frac{1}{2}\mathrm{i}\sigma B(\mathrm{i}\frac{B^*}{\sigma})\\ &= -\frac{\partial\beta}{\partial t_1} + \Big(\frac{\sigma^2-1}{4\sigma^2}\Big)|B|^2\end{aligned} \tag{3.77}$$

which gives

$$B_{20} = -\sigma\frac{\partial\beta}{\partial t_1} + \Big(\frac{\sigma^2-1}{4\sigma}\Big)|B|^2 = -\sigma\frac{\partial\beta}{\partial t_1} + \gamma_1|B|^2 \tag{3.78}$$

where

$$\gamma_1 = \Big(\frac{\sigma^2-1}{4\sigma}\Big). \tag{3.79}$$

From the 1st harmonic with the use of (3.47), multiplied with $(i\sigma)$ and rearranged

$$-\mathrm{i}\Big(A_{21} - h\frac{\partial B}{\partial x_1}\Big)\mathrm{e}^{\mathrm{i}\chi} + \frac{1}{\sigma}B_{21}\mathrm{e}^{\mathrm{i}\chi} = -\frac{\partial A}{\partial t_1}\mathrm{e}^{\mathrm{i}\chi} \quad \Rightarrow \quad A_{21}\sigma + \mathrm{i}B_{21} = -\frac{\partial B}{\partial t_1} + h\sigma\frac{\partial B}{\partial x_1}. \tag{3.80}$$

From the 2nd harmonic

$$\begin{aligned} -\mathrm{i}A_{22} + \frac{1}{2\sigma}B_{22} &= \frac{1}{8}A^2 - \frac{1}{8}\sigma^2A^2 + \frac{1}{4}\mathrm{i}\sigma BA = \frac{1}{8}A^2(1-\sigma^2) + \frac{1}{4}\mathrm{i}\sigma BA \\ &= \frac{1}{8}\Big(-\mathrm{i}\frac{B}{\sigma}\Big)^2\Big(1-\sigma^2\Big) + \frac{1}{4}\mathrm{i}\sigma B\Big(-\mathrm{i}\frac{B}{\sigma}\Big) = \Big(\frac{3\sigma^2-1}{8\sigma^2}\Big)B^2 \end{aligned} \tag{3.81}$$

when multiplied with σ gives

$$-\mathrm{i}\sigma A_{22} + \frac{1}{2}B_{22} = \Big(\frac{3\sigma^2-1}{8\sigma}\Big)B^2 = \gamma_2 B^2 \tag{3.82}$$

where

$$\gamma_2 = \Big(\frac{3\sigma^2-1}{8\sigma}\Big) \tag{3.83}$$

The equations from the kinematic and dynamic conditions are forming two systems of linear equations. From the 1st harmonic (3.72) and (3.80) may be presented as

$$\begin{bmatrix} \sigma & \mathrm{i} \\ \sigma & \mathrm{i} \end{bmatrix} \begin{Bmatrix} A_{21} \\ B_{21} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial B}{\partial t_1} + h\sigma\frac{\partial B}{\partial x_1} \\ \frac{\partial B}{\partial t_1} + (1+\frac{h}{\sigma})\frac{\partial B}{\partial x_1} \end{Bmatrix} \tag{3.84}$$

The determinant equals zero which gives one free variable and the sum of the right side equal to zero. The sum of the right side is

$$\begin{aligned} -\frac{\partial B}{\partial t_1} + h\sigma\frac{\partial B}{\partial x_1} - \Big(\frac{\partial B}{\partial t_1} + (1+\frac{h}{\sigma})\frac{\partial B}{\partial x_1}\Big) &= -2\frac{\partial B}{\partial t_1} + (h\sigma - 1 - \frac{h}{\sigma})\frac{\partial B}{\partial x_1} \\ = -2\frac{\partial B}{\partial t_1} + \Big(\frac{h\sigma^2-\sigma-h}{\sigma}\Big)\frac{\partial B}{\partial x_1} &= 0 \end{aligned} \tag{3.85}$$

which gives

$$\frac{\partial B}{\partial t_1} = \Big(\frac{h\sigma^2-\sigma-h}{2\sigma}\Big)\frac{\partial B}{\partial x_1} = -\gamma_3\frac{\partial B}{\partial x_1}. \tag{3.86}$$

where

$$\gamma_3 = \Big(\frac{\sigma+h-h\sigma^2}{2\sigma}\Big) = \frac{1}{2}\Big(1 - \frac{(\sigma^2-1)h}{\sigma}\Big) = c_g \tag{3.87}$$

c_g denotes the dimensionless group velocity which is derived in appendix A.1. After (3.87) is substituted into (3.86), utilizing $(x_1, t_1) = \epsilon(x, t)$ and rearranging

$$\frac{\partial B}{\partial t_1} + c_g\frac{\partial B}{\partial x_1} = \frac{1}{\epsilon}\frac{\partial B}{\partial t_1} + c_g\frac{1}{\epsilon}\frac{\partial B}{\partial x_1} = \frac{\partial B}{\partial t} + c_g\frac{\partial B}{\partial x} = 0 \tag{3.88}$$

which is the linear Schrödinger equation. The equation is a 1st order partial differential equation, also known as a wave equation, and has a general solution $B = f(x - c_g t)$ where f is an arbitrary complex function for a wave propagating with the group velocity. This shows the evolution of the linear amplitude and the equation will be denoted the evolution equation. B_{21} is chosen to be free and equal to zero. When $B_{21} = 0$ is substituted into (3.80), multiplied with $(1/\sigma)$ and utilizing (3.88)

$$A_{21} = -\frac{1}{\sigma}\frac{\partial B}{\partial t_1} + h\frac{\partial B}{\partial x_1} = -\frac{c_g}{\sigma}\frac{\partial B}{\partial x_1} + h\frac{\partial B}{\partial x_1} = (h - \frac{c_g}{\sigma})\frac{\partial B}{\partial x_1} \tag{3.89}$$

For the 2nd harmonic (3.73) and (3.82) may be presented as

$$\begin{bmatrix} -\mathrm{i}\sigma & 1/2 \\ \sigma_2 & \mathrm{i} \end{bmatrix} \begin{Bmatrix} A_{22} \\ B_{22} \end{Bmatrix} = \begin{Bmatrix} \gamma_2 \\ \mathrm{i}/(2\sigma) \end{Bmatrix} B^2 \tag{3.90}$$

which gives

$$\begin{Bmatrix} A_{22} \\ B_{22} \end{Bmatrix} = \frac{2}{2\sigma - \sigma_2} \begin{bmatrix} \mathrm{i} & -1/2 \\ -\sigma_2 & -\mathrm{i}\sigma \end{bmatrix} \begin{Bmatrix} \gamma_2 \\ \mathrm{i}/(2\sigma) \end{Bmatrix} B^2. \tag{3.91}$$

The explicit solutions of A_{22} is

$$\begin{aligned} A_{22} &= 2\Big(2\sigma - \sigma_2\Big)^{-1}\Big(\mathrm{i}\gamma_2 - \mathrm{i}\frac{1}{4\sigma}\Big)B^2 = 2\Big(\frac{1+\sigma^2}{2\sigma^3}\Big)\Big((\frac{3\sigma^2 - 1}{8\sigma}) - \frac{1}{4\sigma}\Big)\mathrm{i}B^2 \\ &= \Big(\frac{1+\sigma^2}{\sigma^3}\Big)\Big(\frac{3(\sigma^2-1)}{8\sigma}\Big)\mathrm{i}B^2 = \mathrm{i}\Big(\frac{3(\sigma^2+1)(\sigma^2-1)}{8\sigma^4}\Big)B^2 = \mathrm{i}\gamma_4 B^2 \end{aligned} \tag{3.92}$$

where

$$\gamma_4 = \Big(\frac{3(\sigma^2+1)(\sigma^2-1)}{8\sigma^4}\Big). \tag{3.93}$$

The explicit solution of B_{22} is

$$\begin{aligned} B_{22} &= 2\Big(2\sigma - \sigma_2\Big)^{-1}\Big(-\sigma_2\gamma_2 + \frac{1}{2}\Big)B^2 = 2\Big(\frac{1+\sigma_2}{2\sigma^3}\Big)\Big(-(\frac{2\sigma}{1+\sigma^2})(\frac{3\sigma^2-1}{8\sigma}) + \frac{1}{2}\Big)B^2 \\ &= \Big(\frac{1+\sigma^2}{\sigma^3}\Big)\Big(\frac{3-\sigma^2}{4(1+\sigma^2)}\Big)B^2 = \Big(\frac{3-\sigma^2}{4\sigma^3}\Big)B^2 = \gamma_5 B^2 \end{aligned} \tag{3.94}$$

where

$$\gamma_5 = \Big(\frac{3-\sigma^2}{4\sigma^3}\Big). \tag{3.95}$$

After (3.78), (3.92) and (3.94) is substituted into (3.67) and (3.25) the solution to the 2nd order problem is

$$\begin{aligned} \phi_2 = \beta_2 + \frac{1}{2}\bigg(&\Big((h - \frac{c_g}{\sigma})\frac{\cosh(z+h)}{\cosh h} - (h+z)\frac{\sinh(z+h)}{\sinh h}\Big)\frac{\partial B}{\partial x_1}\mathrm{e}^{\mathrm{i}\chi} \\ &+\mathrm{i}\gamma_4 B^2 \frac{\cosh 2(z+h)}{\cosh 2h}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\bigg) \quad \text{and} \end{aligned} \tag{3.96}$$

$$\eta_2 = -\sigma\frac{\partial \beta}{\partial t_1} + \gamma_1|B|^2 + \frac{1}{2}\Big(\gamma_5 B^2 \mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big). \tag{3.97}$$

3.5 3rd order problem

To find a solution to β the 0th harmonic from the 3rd order kinematic boundary condition will be solved. From the boundary value problem (3.27)-(3.30) the parts of $O(\epsilon^3)$ is extracted and the 3rd order problem is established and is

$$\frac{\partial^2\phi_3}{\partial x^2}+2\frac{\partial^2\phi_2}{\partial x\partial x_1}+\frac{\partial^2\phi_1}{\partial x_1^2}+\frac{\partial^2\phi_3}{\partial z^2}=0 \quad \text{for} \quad -h<z<\eta, \tag{3.98}$$

$$\begin{aligned}&\frac{\partial\eta_3}{\partial t}+\frac{\partial\eta_2}{\partial t_1}+\frac{\partial\phi_1}{\partial x}\frac{\partial\eta_2}{\partial x}+\frac{\partial\phi_2}{\partial x}\frac{\partial\eta_1}{\partial x}+\frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x_1}+\frac{\partial\phi_1}{\partial x_1}\frac{\partial\eta_1}{\partial x}\\ &\quad-\frac{\partial\phi_3}{\partial z}+\eta_1\frac{\partial^2\phi_1}{\partial z\partial x}\frac{\partial\eta_1}{\partial x}-\eta_1\frac{\partial^2\phi_2}{\partial z^2}-\eta_2\frac{\partial^2\phi_1}{\partial z^2}-\eta_1^2\frac{1}{2}\frac{\partial^3\phi_1}{\partial z^3}=0 \quad \text{at} \quad z=0,\end{aligned} \tag{3.99}$$

$$\begin{aligned}&\frac{\partial\phi_3}{\partial t}+\frac{\partial\phi_2}{\partial t_1}+\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial x}+\frac{\partial\phi_1}{\partial z}\frac{\partial\phi_2}{\partial z}+\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_1}{\partial x_1}+\eta_1\frac{\partial^2\phi_2}{\partial z\partial t}\\ &\quad+\eta_2\frac{\partial^2\phi_1}{\partial z\partial t}+\eta_1\frac{\partial^2\phi_1}{\partial z\partial t_1}+\eta_1\nabla\phi_1\frac{\partial}{\partial z}(\nabla\phi_1)+\eta^2\frac{1}{2}\frac{\partial^3\phi_1}{\partial z^2\partial t}+\eta_3\frac{1}{\sigma}=0 \quad \text{at} \quad z=0 \quad \text{and}\end{aligned} \tag{3.100}$$

$$\frac{\partial\phi_3}{\partial z}=0 \quad \text{at} \quad z=-h. \tag{3.101}$$

The 3rd order potential (3.23), the 2nd order potential (3.96) and the 1st order potential (3.46) is substituted into Laplace (3.98) and from the 0th harmonic

$$\frac{\partial^2\acute{A}_{30}}{\partial z^2}=-\frac{\partial^2\beta}{\partial x_1^2} \quad \text{with possible solution} \quad \acute{A}_{30}=-\frac{1}{2}\frac{\partial^2\beta}{\partial x_1^2}z^2+A_{30}z+C_{30} \tag{3.102}$$

where $A_{30}=A_{30}(x_1,t_1)$ and $C_{30}=C_{30}(x_1,t_1)$. Next (3.102) is substituted into the bottom condition (3.101) and

$$\frac{\partial\acute{A}_{30}}{\partial z}=\frac{\partial^2\beta}{\partial x_1^2}h+A_{30}=0 \quad \text{which gives} \quad A_{30}=-\frac{\partial^2\beta}{\partial x_1^2}h. \tag{3.103}$$

(3.103) is substituted into (3.102) and $C_{30}(x_1,t_1)$ is replaced with $\beta_3(x_1,t_1)$ and

$$\acute{A}_{30}=\beta_3-\frac{\partial^2\beta}{\partial x_1^2}(\frac{1}{2}z^2+hz) \tag{3.104}$$

The three potentials and corresponding surface displacements (3.21)-(3.26) are substituted into the 3rd order kinematic boundary condition and with $B_{10}=B_{21}=0$ the 0th harmonic becomes

$$\begin{aligned}&\Big[\frac{\partial\acute{A}_{30}}{\partial z}=\frac{\partial B_{20}}{\partial t_1}+\frac{1}{4}(\acute{A}_{21}B^*+\acute{A}_{21}^*B)+\mathrm{i}\frac{1}{4}(\acute{A}_{11}\frac{\partial B^*}{\partial x_1}-\acute{A}_{11}^*\frac{\partial B}{\partial x_1})\\ &\quad-\mathrm{i}\frac{1}{4}(\frac{\partial\acute{A}_{11}}{\partial x_1}B^*-\frac{\partial\acute{A}_{11}^*}{\partial x_1}B)+\frac{1}{4}(B\frac{\partial^2\acute{A}_{21}^*}{\partial z^2}+B^*\frac{\partial^2\acute{A}_{21}}{\partial z^2})\Big]_{z=0}\\ &\quad=\frac{\partial B_{20}}{\partial t_1}+\frac{1}{4}\Big(-\frac{c_g}{\sigma}(B\frac{\partial B^*}{\partial x_1}+\text{c.c})+\frac{1}{\sigma}(B\frac{\partial B^*}{\partial x_1}+\text{c.c})\\ &\quad-\frac{1}{\sigma}(B\frac{\partial B^*}{\partial x_1}+\text{c.c})-\frac{1}{\sigma}(c_g+2)(B\frac{\partial B^*}{\partial x_1}+\text{c.c})\Big)\\ &\quad=\frac{\partial B_{20}}{\partial t_1}-\frac{1}{2\sigma}(c_g+1)(B\frac{\partial B^*}{\partial x_1}+\text{c.c})=\frac{\partial B_{20}}{\partial t_1}+\frac{1}{2\sigma}(1+1/c_g)\frac{\partial}{\partial t_1}|B|^2\end{aligned} \tag{3.105}$$

where explicit solutions of A_{nm}, B_{nm} and their derivatives at $z = 0$ may be seen in appendix B.1. (3.104) and (3.78) are substituted into (3.105) and

$$-h\frac{\partial^2\beta}{\partial x_1^2} = -\sigma\frac{\partial^2\beta}{\partial t_1^2} + \gamma_1\frac{\partial}{\partial t_1}|B|^2 + \frac{1}{2\sigma}(1+1/c_g)\frac{\partial}{\partial t_1}|B|^2. \tag{3.106}$$

Rearranging (3.106) gives

$$\frac{\partial^2\beta}{\partial t_1^2} - \frac{h}{\sigma}\frac{\partial^2\beta}{\partial x_1^2} = \gamma_6\frac{\partial}{\partial t_1}|B|^2 \tag{3.107}$$

where

$$\gamma_6 = \gamma_1 + 1/2\sigma + 1/2\sigma c_g. \tag{3.108}$$

The partial differential equation (3.107) is classified as 2nd order, inhomogeneous, linear and hyperbolic. It is also known as a wave equation with a source. After $(x_1, t_1) = \epsilon(x, t)$ is utilized the equation is

$$\frac{\partial^2\beta}{\partial t^2} - c_0^2\frac{\partial^2\beta}{\partial 1^2} = \epsilon\gamma_6\frac{\partial}{\partial t}|B|^2 \quad \text{where} \quad c_0 = \sqrt{\frac{h}{\sigma}}. \tag{3.109}$$

c_0 is the shallow water velocity which is derived in appendix A.2. To solve equation 3.109 the solution is divided into a homogeneous part and a particular part. The right hand side of the equation is depending on B where B has the leading order behavior that modulations are advected with the group velocity c_g, $B = B(x - c_g t)$. Thus we search for a leading order particular solution for β that has the same propagation speed and it seems possible to find a leading order solution of the form $\beta = \beta_{\text{free}}(x - c_0 t) + \beta_{\text{bound}}(x - c_g t)$ as long as $c_0 \neq c_g$. Long waves propagates faster than short waves which gives $c_0 > c_g$. $\beta_{\text{free}}(x - c_0 t)$ is the solution to the homogeneous problem and $\beta_{\text{bound}}(x - c_g t)$ is the solution to the particular problem. The homogeneous problem

$$\frac{\partial^2\beta_{\text{f}}}{\partial t^2} - c_0^2\frac{\partial^2\beta_{\text{f}}}{\partial x^2} = 0 \tag{3.110}$$

has possible solution

$$\beta_{\text{f}} = \frac{1}{2}\sum_j\left(D_j^+ \text{e}^{\text{i}(k_j x - \omega_j t)} + \text{c.c}\right) + \frac{1}{2}\sum_j\left(D_j^- \text{e}^{-\text{i}(k_j x + \omega_j t)} + \text{c.c}\right) + U_{\text{f}}x + E_{\text{f}}t + F_{\text{f}} \tag{3.111}$$

where $\omega_j/k_j = c_0$ and $k_j h << 1$. The particular problem

$$\frac{\partial^2\beta_{\text{b}}}{\partial t^2} - c_0^2\frac{\partial^2\beta_{\text{b}}}{\partial x^2} = \epsilon\gamma_6\frac{\partial}{\partial t}|B|^2. \tag{3.112}$$

is solved in agreement of the assumption of β_{b} having the same form as B and that the linear Schrödinger equation also is valid for β_{b} . Then

$$\frac{\partial^2\beta_{\text{b}}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\beta_{\text{b}}}{\partial t}\right) = -c_g\frac{\partial}{\partial x}\left(-c_g\frac{\partial\beta_{\text{b}}}{\partial x}\right) = c_g^2\frac{\partial^2\beta_{\text{b}}}{\partial x^2} \tag{3.113}$$

which gives two equations

$$c_g^2\frac{\partial^2\beta_{\text{b}}}{\partial x^2} - c_0^2\frac{\partial^2\beta_{\text{b}}}{\partial x^2} = -\epsilon c_g\gamma_6\frac{\partial}{\partial x}|B|^2 \quad \Rightarrow \quad \frac{\partial^2\beta_{\text{b}}}{\partial x^2} = -\frac{\epsilon c_g\gamma_6}{(c_g^2 - c_0^2)}\frac{\partial}{\partial x}|B|^2 \quad \text{and} \tag{3.114}$$

$$\frac{\partial^2 \beta_\mathrm{b}}{\partial t^2} - \frac{c_0^2}{c_g^2}\frac{\partial^2 \beta_\mathrm{b}}{\partial t^2} = \epsilon\gamma_6 \frac{\partial}{\partial t}|B|^2 \quad \Rightarrow \quad \frac{\partial^2 \beta_\mathrm{b}}{\partial t^2} = \frac{\epsilon c_g^2 \gamma_6}{(c_g^2 - c_0^2)}\frac{\partial}{\partial t}|B|^2. \tag{3.115}$$

The reconstruction fomulas require only the derivatives of β and when (3.114) and (3.115) are integrated once

$$\frac{\partial \beta_\mathrm{b}}{\partial x} = -\frac{\epsilon c_g \gamma_6}{(c_g^2 - c_0^2)}|B|^2 \quad \text{and} \tag{3.116}$$

$$\frac{\partial \beta_\mathrm{b}}{\partial t} = \frac{\epsilon c_g^2 \gamma_6}{(c_g^2 - c_0^2)}|B|^2 \tag{3.117}$$

the derivatives of β becomes

$$\begin{aligned}\frac{\partial \beta}{\partial x} &= \frac{\partial \beta_\mathrm{f}}{\partial x} + \frac{\partial \beta_\mathrm{b}}{\partial x}\\ &= \frac{1}{2}\sum_j \left(\mathrm{i}k_j D_j^+ \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} + \mathrm{c.c}\right) - \frac{1}{2}\sum_j \left(\mathrm{i}k_j D_j^- \mathrm{e}^{-\mathrm{i}(k_j x + \omega_j t)} + \mathrm{c.c}\right) + U_\mathrm{f}\\ &\quad - \frac{\epsilon c_g \gamma_6}{(c_g^2 - c_0^2)}|B|^2 \quad \text{and}\end{aligned} \tag{3.118}$$

$$\begin{aligned}\frac{\partial \beta}{\partial t} &= \frac{\partial \beta_\mathrm{f}}{\partial t} + \frac{\partial \beta_\mathrm{b}}{\partial t}\\ &= -\Big(\frac{1}{2}\sum_j \left(\mathrm{i}\omega_j D_j^+ \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} + \mathrm{c.c}\right) + \frac{1}{2}\sum_j \left(\mathrm{i}\omega_j D_j^- \mathrm{e}^{-\mathrm{i}(k_j x + \omega_j t)} + \mathrm{c.c}\right)\Big) + E_\mathrm{f}\\ &\quad + \frac{\epsilon c_g^2 \gamma_6}{(c_g^2 - c_0^2)}|B|^2.\end{aligned} \tag{3.119}$$

In Trulsen et al. [11] the contribution from free shallow water waves was not considered. Instead of the internal independent constant U_f and E_f in (3.111) they used a constant of integration U added to the particular solution. U was found from the requirement that the surface displacement averaged over the entire period is zero. Here the constants are assumed to be independent and have to be found from two different requirements.

Chapter 4

Implementation

4.1 Reconstruction formulas

The reconstruction formulas are formulated with a dimensional output. The input are made dimensionless according to (3.2). To denote the dimensionless parameters the $\star$ is invoked and for the rest of the calculations the $\star$ denotes the dimensionless parameters. The formulas are developed for the condition that the surface displacement is given as a measured time series. $(x_1, t_1) = \epsilon(x, t)$ and the evolution equation (3.88) is utilized to get the formulas depending on the time t. To get the formulas to work with a space series the only change required is changing the derivative of B with the use of the evolution equation.

The velocity potential is

$$\phi = \frac{a_c \omega_c}{k_c}\phi^\star = \frac{a_c \omega_c}{k_c}(\phi_1^\star + \epsilon\phi_2^\star) + O(\epsilon^2) \tag{4.1}$$

with the dimensionless

$$\phi_1^\star = \beta - \frac{1}{2}(\mathrm{i}B\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}) \quad \text{and} \tag{4.2}$$

$$\begin{aligned}\phi_2^\star = \beta_2 + \frac{1}{2}\Big(\frac{1}{c_g^\star}\Big((c_g^\star/\sigma - h^\star)\frac{\cosh(z^\star + h^\star)}{\cosh h^\star} + (h^\star + z^\star)\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\Big)\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} \\ + \mathrm{i}\gamma_4 B^2\frac{\cosh 2(z^\star + h^\star)}{\cosh 2h^\star}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big).\end{aligned} \tag{4.3}$$

The velocities are given by the gradient of the velocity potential

$$\nabla\phi = u\vec{i} + w\vec{k} = a_c\omega_c u^\star\vec{i} + a_c\omega_c w^\star\vec{k}. \tag{4.4}$$

Explicite solution for the horizontal velocity is

$$u = \frac{a_c\omega_c k_c}{k_c}u^\star = \epsilon\frac{\omega_c}{k_c}(\frac{\partial\phi^\star}{\partial x^\star} + \epsilon\frac{\partial\phi^\star}{\partial x_1^\star}) = a_c\omega_c(\frac{\partial\phi_1^\star}{\partial x^\star} + \epsilon\frac{\partial\phi_2^\star}{\partial x^\star} + \epsilon\frac{\partial\phi_1^\star}{\partial x_1^\star}) + O(\epsilon^3) \tag{4.5}$$

where

$$\frac{\partial\phi_1^\star}{\partial x^\star} = \frac{1}{2}\Big(B\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big), \tag{4.6}$$

$$\frac{\partial \phi_2^\star}{\partial x^\star} = \frac{1}{2}\Big(\mathrm{i}\frac{1}{c_g^\star}\Big((c_g^\star/\sigma - h^\star)\frac{\cosh(z^\star + h^\star)}{\cosh h^\star} + (h^\star + z^\star)\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\Big)\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi}$$
$$-2\gamma_4 B^2 \frac{\cosh 2(z^\star + h^\star)}{\cosh 2h^\star}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big) \quad \text{and} \tag{4.7}$$

$$\frac{\partial \phi_1^\star}{\partial x_1^\star} = \frac{1}{\epsilon}\frac{\partial \beta}{\partial x^\star} + \frac{1}{2}\Big(\mathrm{i}\frac{1}{c_g^\star}\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big). \tag{4.8}$$

Explicit solutions for the rest of the kinematics including accelerations are given in appendix C.1.

The surface displacement is

$$\eta = a_c \eta^\star = a_c(\eta_1^\star + \epsilon\eta_2^\star) + O(\epsilon^2) \tag{4.9}$$

with the nondimensional

$$\eta_1^\star = \frac{1}{2}(B\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}) \quad \text{and} \tag{4.10}$$

$$\eta_2^\star = -\sigma\frac{\partial \beta}{\partial t_1^\star} + \gamma_1|B|^2 + \frac{1}{2}\Big(\gamma_5 B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big) \tag{4.11}$$

which gives

$$\eta = a_c\Big(-\sigma\frac{\partial \beta}{\partial t^\star} + \epsilon\gamma_1|B|^2 + \frac{1}{2}\big(B\mathrm{e}^{\mathrm{i}\chi} + \epsilon\gamma_5 B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\big)\Big) + O(\epsilon^2). \tag{4.12}$$

4.2 A measured time series

Ocean waves may be specified by their deterministic or statistical characteristics, normally through a measured time series or a wave spectrum, and the nonlinear Schrödinger method may be applied to either description. For a measured space series the approach would be close to the approach with the time series, but will not be further commented. For a measured time series N is the number of elements of the sample, $f_s = 1/\Delta T$ where ΔT is time between two subsequent samples and $T = N\Delta T$ is defined as the period. The significant wave height H_s and the characteristic amplitude a_c are defined as

$$H_s = 4\sigma_\eta \quad \text{and} \quad a_c = \sqrt{2}\sigma_\eta \tag{4.13}$$

where σ_η is the stantdard deviation

$$\sigma_\eta = \sqrt{\frac{1}{N}\sum_n(\eta_n - \bar{\eta})^2}. \tag{4.14}$$

The discrete Fourier transform pair is used on the time series where

$$\eta_n = \sum_{-N/2<j\leq N/2} \hat{\eta}_j \mathrm{e}^{-\mathrm{i}\omega_j t_n} \quad \text{and} \quad \hat{\eta}_j = \frac{1}{N}\sum_{n=0}^{N-1}\eta_n \mathrm{e}^{\mathrm{i}\omega_j t_n} \tag{4.15}$$

with $\omega_j = j2\pi/T$ and $t_n = n\Delta T$. If $\eta(t)$ is real, which it would be for a measured surface displacement,

$$\hat{\eta}(-\omega) = \frac{1}{N}\sum_{n=0}^{N-1}\eta_n \mathrm{e}^{-\mathrm{i}\omega t_n} = \hat{\eta}^*(\omega) \tag{4.16}$$

where $*$ denotes the complex conjugate. Then it is sufficient to look at non-negative frequencies below the Nyquist frequency $f_n = f_s/2$. From the Fourier transform of η the energy density spectrum S may be estimated by

$$S_j = S(\omega_j) = \frac{2}{\Delta\omega}|\hat{\eta}(\omega_j)|^2 = \frac{2}{\Delta\omega}|\hat{\eta}_j|^2. \tag{4.17}$$

where $\Delta\omega = 2\pi/T$.

To define the characteristic frequency in accordance with (3.2) there are several alternatives. The peak frequency ω_P is widely used to define wave spectrum. The peak frequency is the frequency corresponding the highest energy in the wave spectrum and $\omega_P = \omega_P(S_{\max})$. The mean frequency is the frequency corresponding to the mean of the energy where

$$\omega_m = \frac{\sum_j \omega_j S_j}{\sum_j S_j} \quad \text{for} \quad j = (0, 1, ...N/2). \tag{4.18}$$

The local frequency ω_{TT}, used in Grue's method, is the frequency corresponding to the local wave period T_{TT} defined from trough to trough.

4.3 The linear amplitude B

As in Trulsen [11] the linear amplitude B is constructed by extracting the assumed linear contribution from the surface displacement. The Fourier transform pair is used on the amplitude and

$$B_m = \sum_{-M/2<j\leq M/2} \hat{B}_j \mathrm{e}^{-\mathrm{i}\omega_j t_m} \quad \text{and} \quad \hat{B}_j = \frac{1}{M}\sum_{m=0}^{M-1} B_m \mathrm{e}^{\mathrm{i}\omega_j t_m} \tag{4.19}$$

where $t_m = mT/M$. $\hat{B}$ is constructed by band passing a suitable number of modes from $\hat{\eta}$ with

$$\hat{B}_j = 2\hat{\eta}_{j+M_0} \quad \text{for} \quad -M/2 < j \leq M/2 \quad \text{where} \quad M/2 \leq M_0. \tag{4.20}$$

M_0 is found from the characteristic frequency and rounded to the nearest integer which gives $M_0 \cong \omega_c T/2\pi$. The upper cut-off frequency should be below the Nyquist frequency and the bandwidth should be wide enough to incorporate the assumed modulation in accordance with(3.1) which gives

$$M_0 + \frac{1}{2}M < \frac{1}{2}N \quad \text{and} \quad \frac{1}{2}M/M_0 > \epsilon. \tag{4.21}$$

For implementation the amplitude is interpolated and

$$B_n = \sum_{-N/2<j\leq N/2} \hat{B}_j \mathrm{e}^{-\mathrm{i}\omega_j t_n} \quad \text{where} \quad t_n = nT/N, \tag{4.22}$$

$$\left(\frac{\partial B}{\partial t^\star}\right)_n = \frac{1}{\omega_c}\left(\frac{\partial B}{\partial t}\right)_n = \sum_{-N/2<j\leq N/2} -\mathrm{i}\frac{\omega_j}{\omega_c}\hat{B}_j \mathrm{e}^{-\mathrm{i}\omega_j t_n}, \tag{4.23}$$

$$(B^2)_n = (B_n)^2 \quad \text{and} \quad (|B|^2)_n = |B_n|^2. \tag{4.24}$$

Contributions originating from β are implemented as

$$\left(\frac{\partial \beta}{\partial x^\star}\right)_n = \frac{1}{2}\sum_j \mathrm{i}k_j^\star\Big(\big(D_j^+ - D_j^-\big)\mathrm{e}^{-\mathrm{i}\omega_j^\star t_n^\star} + \mathrm{c.c}\Big) + U_\mathrm{f} - \frac{c_g\epsilon\gamma_6}{(c_g^2 - c_0^2)}\big(|B|^2\big)_n \quad \text{and} \tag{4.25}$$

$$\left(\frac{\partial\beta}{\partial t^\star}\right)_n = -\frac{1}{2}\sum_j \mathrm{i}\omega_j^\star\Big(\big(D_j^+ + D_j^-\big)\mathrm{e}^{-\mathrm{i}\omega_j^\star t_n} + \mathrm{c.c}\Big) + E_\mathrm{f} + \frac{c_g^2\epsilon\gamma_6}{(c_g^2 - c_0^2)}\big(|B|^2\big)_n \tag{4.26}$$

for $k_j^\star h^\star << 1$ where $\omega_j^\star = j2\pi/\omega_c T$ and $\omega_j^\star/k_j^\star = c_0 = \sqrt{h^\star/\sigma}$ which gives $k_j^\star = \omega_j^\star\sqrt{h^\star/\sigma}$.

4.4 Horizontal current

To find the horizontal current U_f two methods are suggested. The first method is assuming volume flux integrated over one period to be zero. This may be true for an enclosed wave tank where the amount of fluid must be conserved. $U_\mathrm{f} = U_{\mathrm{f}0}$ is found when the horizontal volume flux integrated over one period is

$$\int_{t_0}^{t_0+T}\int_{-h}^{\eta} u \,\mathrm{d}z\,\mathrm{d}t = a_c\omega_c\int_T\int_{-h}^{\eta}\big(\frac{\partial\phi_1^\star}{\partial x^\star} + \epsilon\frac{\partial\phi_2^\star}{\partial x^\star} + \epsilon\frac{\partial\phi^\star}{\partial x_1^\star}\big)\,\mathrm{d}z\,\mathrm{d}t = 0 \tag{4.27}$$

By using

$$\int_{z=-h}^{z=\eta} f(z^\star)\,\mathrm{d}z = \frac{1}{k_c}\int_{z^\star=-h^\star}^{z^\star=k_c\eta} f(z^\star)\,\mathrm{d}z^\star, \quad k_c\eta = k_c a_c\eta^\star = \epsilon\eta^\star, \tag{4.28}$$

$$\begin{aligned}\int_{-h^\star}^{\epsilon\eta^\star} f(z^\star)\,\mathrm{d}z^\star &= \int_{-h^\star}^{0} f(z^\star)\,\mathrm{d}z^\star + \int_0^{\epsilon\eta^\star} f(z^\star)\,\mathrm{d}z^\star\\ &= \int_{-h^\star}^{0} f(z^\star)\,\mathrm{d}z^\star + \int_0^{\epsilon\eta^\star}\Big(\big[f\big]_{z=0} + z\big[\frac{\partial f}{\partial z}\big]_{z=0} + \cdots + \Big)\,\mathrm{d}z^\star\\ &= \int_{-h^\star}^{0} f(z^\star)\,\mathrm{d}z^\star + \epsilon\eta^\star\big[f\big]_{z=0} + \frac{1}{2}(\epsilon\eta^\star)^2\big[\frac{\partial f}{\partial z}\big]_{z=0} + \cdots +\end{aligned} \tag{4.29}$$

and $\eta^\star = \eta_1^\star + \epsilon\eta_2^\star + \cdots +$ the horizontal volume flux integrated over one period may be presented as

$$\begin{aligned}&\int_T\int_{-h^\star}^{\epsilon\eta^\star}\big(\frac{\partial\phi_1^\star}{\partial x^\star} + \epsilon\frac{\partial\phi_2^\star}{\partial x^\star} + \epsilon\frac{\partial\phi^\star}{\partial x_1^\star}\big)\,\mathrm{d}z^\star\,\mathrm{d}t\\ &= \int_T\int_{-h^\star}^{0}\big(\frac{\partial\phi_1^\star}{\partial x^\star} + \epsilon\frac{\partial\phi_2^\star}{\partial x^\star} + \epsilon\frac{\partial\phi^\star}{\partial x_1^\star}\big)\,\mathrm{d}z^\star\,\mathrm{d}t + \epsilon\eta_1^\star\big[\frac{\partial\phi_1^\star}{\partial x^\star}\big]_{z=0} + O(\epsilon^2) = 0.\end{aligned} \tag{4.30}$$

The 0th order problem is

$$\begin{aligned}\int_T\int_{-h^\star}^{0}\frac{\partial\phi_1^\star}{\partial x^\star}\,\mathrm{d}z^\star\,\mathrm{d}t &= \int_T\int_{-h^\star}^{0}\frac{1}{2}\Big(B\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big)\,\mathrm{d}z^\star\,\mathrm{d}t\\ &= \int_T\frac{1}{2}(B\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c})\,\mathrm{d}t = 0.\end{aligned} \tag{4.31}$$

The 1st order problem is

$$\int_T\int_{-h^\star}^{0}\frac{\partial\phi_2^\star}{\partial x^\star}\,\mathrm{d}z^\star\,\mathrm{d}t + \int_T\int_{-h^\star}^{0}\frac{\partial\phi_1^\star}{\partial x_1^\star}\,\mathrm{d}z^\star\,\mathrm{d}t + \int_T\eta_1^\star\big[\frac{\partial\phi_1^\star}{\partial x^\star}\big]_{z=0}\,\mathrm{d}t = 0 \tag{4.32}$$

where

$$\begin{aligned}&\int_{-h^\star}^{0}\frac{\partial\phi_2^\star}{\partial x^\star}\,\mathrm{d}z^\star\\ &= \int_{-h^\star}^{0}\frac{1}{2\epsilon c_g}\Big(\mathrm{i}\Big((c_g/\sigma - h^\star)\frac{\cosh(z^\star + h^\star)}{\cosh h^\star} + (h^\star + z^\star)\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\Big)\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big)\,\mathrm{d}z^\star\\ &- \int_{-h^\star}^{0}\Big(\gamma_4\frac{\cosh 2(z^\star + h^\star)}{\cosh 2h^\star}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big)\,\mathrm{d}z^\star\\ &= \frac{1}{2}\Big(\mathrm{i}\frac{\gamma_7}{\epsilon c_g}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} - \gamma_4 B^2\sigma_2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big),\end{aligned} \tag{4.33}$$

$$\gamma_7 = \Big(c_g + \frac{h^\star(1-\sigma^2)}{\sigma} - 1\Big), \quad \sigma_2 = \tanh(2h^\star), \tag{4.34}$$

$$\begin{aligned}\int_{-h^\star}^{0} \frac{\partial \phi_1^\star}{\partial x_1^\star}\,\mathrm{d}z^\star \\ &= \int_{-h^\star}^{0} \frac{1}{\epsilon}\frac{\partial \beta}{\partial x^\star}\,\mathrm{d}z^\star + \int_{-h^\star}^{0} \frac{1}{2\epsilon c_g}\big(\mathrm{i}\frac{\cosh(z^\star+h^\star)}{\sinh h^\star}\frac{\partial \beta}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\big)\,\mathrm{d}z^\star \\ &= \frac{h}{\epsilon}\frac{\partial \beta}{\partial x} + \frac{1}{2\epsilon c_g}\big(\mathrm{i}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\big) \quad \text{and}\end{aligned} \tag{4.35}$$

$$\begin{aligned}\eta_1^\star\big[\frac{\partial \phi_1^\star}{\partial x^\star}\big]_{z=0} &= \frac{1}{2}(B\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c})\big[\frac{1}{2}\big(B\frac{\cosh(z^\star+h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\big)\big]_{z=0} \\ &= \frac{1}{4\sigma}|B|^2 + \frac{1}{4\sigma}(B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}).\end{aligned} \tag{4.36}$$

Then the 1st order problem becomes

$$\begin{aligned}&\int_T \frac{1}{2}\Big(\frac{\mathrm{i}}{\epsilon c_g}(\gamma_8+1)\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \big(\frac{1}{2\sigma} - \gamma_4\sigma_2\big)B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big)\,\mathrm{d}t \\ &\quad + \int_T \frac{h^\star}{\epsilon}\frac{\partial \beta}{\partial x^\star}\,\mathrm{d}t + \int_T \frac{1}{4\sigma}|B|^2\,\mathrm{d}t = 0\end{aligned} \tag{4.37}$$

where

$$\int_T \frac{h^\star}{\epsilon}\frac{\partial \beta}{\partial x^\star}\,\mathrm{d}t = \frac{h^\star}{\epsilon}U_{\mathrm{f0}}T - \int_T \frac{h^\star c_g\gamma_6}{(c_g^2 - c_0^2)}|B|^2\,\mathrm{d}t \tag{4.38}$$

with

$$\int_T \frac{h^\star}{\epsilon k_c}\frac{1}{2}\sum_j \mathrm{i}\kappa_j\big(D_j^+\mathrm{e}^{\mathrm{i}\epsilon(\kappa_j x - \omega_j t)} + D_j^-\mathrm{e}^{-\mathrm{i}\epsilon(\kappa_j x + \omega_j t)} + \mathrm{c.c}\big)\,\mathrm{d}t = 0. \tag{4.39}$$

This gives

$$U_{\mathrm{f0}} = -\gamma_8 \tag{4.40}$$

where

$$\begin{aligned}\gamma_8 = &\frac{1}{2h^\star T}\int_T \Big(\frac{\mathrm{i}}{c_g}(\gamma_7+1)\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \epsilon\big(\frac{1}{2\sigma} - \gamma_4\sigma_2\big)B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big)\,\mathrm{d}t \\ &- \frac{\epsilon c_g\gamma_6}{(c_g^2 - c_0^2)T}\int_T |B|^2\,\mathrm{d}t + \frac{\epsilon}{4h^\star\sigma T}\int_T |B|^2\,\mathrm{d}t.\end{aligned} \tag{4.41}$$

The second method is depending on measurements. By stating that horizontal current U_f is incorporated in the measurements U_f is found when adjusting theory to measurements. $U_\mathrm{f} = U_{\mathrm{fm}}$ is found by

$$U_{\mathrm{fm}} = \frac{1}{M}\sum_{m=1}^{M}\big(u_{\mathrm{meas}}(z_m) - u_{\mathrm{theory}}(z_m)\big) \tag{4.42}$$

for all z_m with measurement.

4.5 Set down

The set down E_{f} is found from the time average of η which is

$$\begin{aligned}\bar{\eta} &= \frac{1}{T}\int_{t_0}^{t_0+T} \eta \,\mathrm{d}t \\ &= \frac{1}{T}\int_T a_c\big(-\sigma\frac{\partial\beta}{\partial t^\star} + \epsilon\gamma_1|B|^2 + \frac{1}{2}(B\mathrm{e}^{\mathrm{i}\chi} + \epsilon\gamma_5 B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c})\big)\,\mathrm{d}t \\ &= \frac{1}{T}\int_T a_c\big(-\sigma E_{\mathrm{f}} - \sigma\frac{c_g^2}{(c_g^2 - c_0^2)}\epsilon\gamma_6|B|^2 \\ &\quad + \epsilon\gamma_1|B|^2 + \frac{1}{2}(B\mathrm{e}^{\mathrm{i}\chi} + \epsilon\gamma_5 B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c})\big)\,\mathrm{d}t\end{aligned} \tag{4.43}$$

with

$$\frac{1}{T}\int_T \frac{a_c\sigma}{\omega_c}\frac{1}{2}\sum_j \mathrm{i}\omega_j\big(D_j^+\mathrm{e}^{\mathrm{i}\epsilon(\kappa_j x-\omega_j t)} + D_j^-\mathrm{e}^{-\mathrm{i}\epsilon(\kappa_j x+\omega_j t)} + \mathrm{c.c}\big)\,\mathrm{d}t = 0 \tag{4.44}$$

and from (4.31)

$$\frac{1}{T}\int_T (B\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c})\,\mathrm{d}t = 0. \tag{4.45}$$

This gives

$$E_{\mathrm{f}} = \gamma_7 \tag{4.46}$$

where

$$\gamma_7 = \epsilon\big(\frac{\gamma_1}{\sigma} - \frac{c_g^2\gamma_6}{(c_g^2 - c_0^2)}\big)\frac{1}{T}\int_T |B|^2\,\mathrm{d}t + \frac{\epsilon\gamma_5}{2\sigma T}\int_T (B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c})\,\mathrm{d}t - \frac{1}{a_c\sigma}\bar{\eta}. \tag{4.47}$$

For numerical implementation

$$\int_T f(t)\,\mathrm{d}t = \sum_{n=0}^{N-1} f_n\Delta T, \quad \frac{1}{T}\int_T f(t)\,\mathrm{d}t = \frac{1}{N}\sum_{n=0}^{N-1} f_n \quad \text{and} \quad \bar{\eta} = \frac{1}{N}\sum_{n=0}^{N-1}\eta_n = \hat{\eta}(0). \tag{4.48}$$

4.6 Shallow water waves

To find the complex amplitudes D_j^+ and D_j^- a method is suggested which require the surface displacement to be measured by two wave gauges. It is assumed that η can be decomposed into $\eta = \eta_S + \eta_B$, where η_S is the shallow water waves, independent of the amplitude B then

$$\eta_S = a_c\big(-\sigma\frac{\partial\beta_{\mathrm{f}}}{\partial t^\star}\big) = \frac{a_c\sigma}{2}\Big(\sum_j \mathrm{i}\omega_j^\star\big(D_j^+\mathrm{e}^{\mathrm{i}(k_j^\star x^\star - \omega_j^\star t^\star)} + D_j^-\mathrm{e}^{-\mathrm{i}(k_j^\star x^\star + \omega_j^\star t^\star)}\big) + \mathrm{c.c}\Big). \tag{4.49}$$

The phase functions are extracted from the exponential function and with $\omega_j^\star = \omega_j/\omega_c$, $k_j^\star = \omega_j/\omega_c c_0$, $t^\star = \omega_c t$ and $x^\star = k_x x$ they becomes

$$(k_j^\star x^\star - \omega_j^\star t^\star) = (\frac{\omega_j^\star}{c_0}k_c x - \frac{\omega_j}{\omega_c}\omega_c t) = (\kappa_j x - \omega_j t) \quad \text{and} \tag{4.50}$$

$$-(k_j^\star x^\star + \omega_j^\star t^\star) = -(\frac{\omega_j^\star}{c_0} k_c x + \frac{\omega_j}{\omega_c}\omega_c t) = -(\kappa_j x + \omega_j t), \tag{4.51}$$

where $\kappa_j = \omega_j k_c/\omega_c c_0$ and $\omega_j = j2\pi/T$. By using (4.50) and (4.51) the shallow water waves (4.49) becomes

$$\eta_S = \frac{a_c\sigma}{2\omega_c}\big(\sum_j \mathrm{i}\omega_j\big(D_j^+ \mathrm{e}^{i(\kappa_j x-\omega_j t)} + D_j^- \mathrm{e}^{-\mathrm{i}(\kappa_j x+\omega_j t)}\big)\big) + \text{c.c.} \tag{4.52}$$

In postition $x = 0$

$$\begin{aligned}\big[\eta_S\big]_{x=0} &= \frac{a_c\sigma}{2}\Big(\sum_j \mathrm{i}\omega_j^\star\big(D_j^+\mathrm{e}^{-\mathrm{i}\omega_j t} + D_j^-\mathrm{e}^{-\mathrm{i}\omega_j t}\big) + \text{c.c}\Big)\\ &= \frac{a_c\sigma}{2\omega_c}\Big(\sum_j \mathrm{i}\omega_j\big(D_j^+ + D_j^-\big)\mathrm{e}^{-\mathrm{i}\omega_j t} + \text{c.c}\Big)\end{aligned} \tag{4.53}$$

and in position $x = \Delta x$

$$\begin{aligned}\big[\eta_s\big]_{x=\Delta x} &= \frac{a_c\sigma}{2\omega_c}\Big(\sum_j \mathrm{i}\omega_j\big(D_j^+\mathrm{e}^{\mathrm{i}(\kappa_j\Delta x-\omega_j t)} + D_j^-\mathrm{e}^{-\mathrm{i}(\kappa_j\Delta x+\omega_j t)}\big) + \text{c.c}\Big)\\ &= \frac{a_c\sigma}{2\omega_c}\Big(\sum_j \mathrm{i}\omega_j\big(D_j^+\mathrm{e}^{\mathrm{i}\kappa_j\Delta x} + D_j^-\mathrm{e}^{-\mathrm{i}\kappa_j\Delta x}\big)\mathrm{e}^{-\mathrm{i}\omega_j t} + \text{c.c}\Big).\end{aligned} \tag{4.54}$$

The shallow water waves may be presented with the Fourier transform of the surface displacement in two positions with distance Δx which gives

$$\big[\eta_s\big]_{x=0} = \sum_j \big[\hat\eta_j\big]_{x=0}\mathrm{e}^{-\mathrm{i}\omega_j t} + \text{c.c} \quad \text{and} \quad \big[\eta_s\big]_{x=\Delta x} = \sum_j \big[\hat\eta_j\big]_{x=\Delta x}\mathrm{e}^{-\mathrm{i}\omega_j t} + \text{c.c} \tag{4.55}$$

where $\omega_j = j2\pi/T$ for $j > 0$. For shallow water waves $\omega_j = k_j\sqrt{gh}$ and $k_j h << 1$ and

$$\omega_j = j\frac{2\pi}{T} = k_j\sqrt{gh} << \frac{1}{h}\sqrt{gh} \quad \text{which gives} \quad j << \frac{T}{2\pi}\sqrt{\frac{g}{h}}. \tag{4.56}$$

According to the Nyquist theorem the wave numbers to be reconstructed cannot exceed $\frac{1}{2}(2\pi/|\Delta x|)$ and

$$k_j = \frac{\omega_j}{\sqrt{gh}} = j\frac{2\pi}{T\sqrt{gh}} < \frac{1}{2}(\frac{2\pi}{|\Delta x|}) \quad \text{which gives} \quad j < \frac{T\sqrt{gh}}{2|\Delta x|}. \tag{4.57}$$

For the possition $x = 0$

$$\begin{aligned}\big[\eta_s\big]_{x=0} &= \sum_j \big[\hat\eta_j\big]_{x=0}\mathrm{e}^{-\mathrm{i}\omega_j t} + \text{c.c}\\ &= \frac{a_c\sigma}{2\omega_c}\Big(\sum_j \mathrm{i}\omega_j\big(D_j^+ + D_j^-\big)\mathrm{e}^{-\mathrm{i}\omega_j t} + \text{c.c}\Big),\end{aligned} \tag{4.58}$$

which gives

$$\sum_j \Big(\big[\hat\eta_j\big]_{x=0}\mathrm{e}^{-\mathrm{i}\omega_j t} - \frac{a_c\sigma}{2\omega_c}\mathrm{i}\omega_j\big(D_j^+ + D_j^-\big)\mathrm{e}^{-\mathrm{i}\omega_j t}\Big) = 0. \tag{4.59}$$

For the possition $x = \Delta x$

$$\begin{aligned}\big[\eta_s\big]_{x=\Delta x} &= \sum_j \big[\hat\eta_j\big]_{x=\Delta x}\mathrm{e}^{-\mathrm{i}\omega_j t} + \text{c.c}\\ &= \frac{a_c\sigma}{2\omega_c}\Big(\sum_j \mathrm{i}\omega_j\big(D_j^+\mathrm{e}^{\mathrm{i}\kappa_j\Delta x} + D_j^-\mathrm{e}^{-\mathrm{i}\kappa_j\Delta x}\big)\mathrm{e}^{-\mathrm{i}\omega_j t} + \text{c.c}\Big),\end{aligned} \tag{4.60}$$

which gives

$$\sum_j \Big(\big[\hat{\eta}_j\big]_{x=\Delta x} \mathrm{e}^{-\mathrm{i}\omega_j t} - \frac{a_c \sigma}{2\omega_c} \mathrm{i}\omega_j \big(D_j^+ \mathrm{e}^{\mathrm{i}\kappa_j \Delta x} + D_j^- \mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} \big) \mathrm{e}^{-\mathrm{i}\omega_j t} \Big) = 0. \tag{4.61}$$

One possible solution which gives two equations and two unknown for all j is

$$\big[\hat{\eta}_j\big]_{x=0} \mathrm{e}^{-\mathrm{i}\omega_j t} - \frac{a_c \sigma}{2\omega_c} \mathrm{i}\omega_j \big(D_j^+ + D_j^- \big) \mathrm{e}^{-\mathrm{i}\omega_j t} = 0 \quad \text{and} \tag{4.62}$$

$$\big[\hat{\eta}_j\big]_{x=\Delta x} \mathrm{e}^{-\mathrm{i}\omega_j t} - \frac{a_c \sigma}{2\omega_c} \mathrm{i}\omega_j \big(D_j^+ \mathrm{e}^{\mathrm{i}\kappa_j \Delta x} + D_j^- \mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} \big) \mathrm{e}^{-\mathrm{i}\omega_j t} = 0. \tag{4.63}$$

These two equation are rearranged and with the use of methods from linear algebra

$$D_j^+ + D_j^- = -\mathrm{i} \frac{2\omega_c}{a_c \sigma \omega_j} \big[\hat{\eta}_j\big]_{x=0}, \tag{4.64}$$

$$D_j^+ \mathrm{e}^{\mathrm{i}\kappa_j \Delta x} + D_j^- \mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} = -\mathrm{i} \frac{2\omega_c}{a_c \sigma \omega_j} \big[\hat{\eta}_j\big]_{x=\Delta x}, \tag{4.65}$$

$$\begin{bmatrix} 1 & 1 \\ \mathrm{e}^{\mathrm{i}\kappa_j \Delta x} & \mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} \end{bmatrix} \begin{Bmatrix} D_j^+ \\ D_j^- \end{Bmatrix} = \frac{-\mathrm{i}2\omega_c}{a_c \sigma \omega_j} \begin{Bmatrix} \big[\hat{\eta}_j\big]_{x=0} \\ \big[\hat{\eta}_j\big]_{x=\Delta x} \end{Bmatrix} \quad \text{and} \tag{4.66}$$

$$\begin{Bmatrix} D_j^+ \\ D_j^- \end{Bmatrix} = \frac{-\mathrm{i}2\omega_c}{a_c \sigma \omega_j (\mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} - \mathrm{e}^{\mathrm{i}\kappa_j \Delta x})} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} & -1 \\ -\mathrm{e}^{\mathrm{i}\kappa_j \Delta x} & 1 \end{bmatrix} \begin{Bmatrix} \big[\hat{\eta}_j\big]_{x=0} \\ \big[\hat{\eta}_j\big]_{x=\Delta x} \end{Bmatrix}, \tag{4.67}$$

which gives the explicit solution to the shallow water wave amplitudes

$$D_j^+ = \frac{-\mathrm{i}2\omega_c}{a_c \sigma \omega_j (\mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} - \mathrm{e}^{\mathrm{i}\kappa_j \Delta x})} \big(\big[\hat{\eta}_j\big]_{x=0} \mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} - \big[\hat{\eta}_j\big]_{x=\Delta x} \big) \quad \text{and} \tag{4.68}$$

$$D_j^- = \frac{\mathrm{i}2\omega_c}{a_c \sigma \omega_j (\mathrm{e}^{-\mathrm{i}\kappa_j \Delta x} - \mathrm{e}^{\mathrm{i}\kappa_j \Delta x})} \big(\big[\hat{\eta}_j\big]_{x=0} \mathrm{e}^{\mathrm{i}\kappa_j \Delta x} - \big[\hat{\eta}_j\big]_{x=\Delta x} \big). \tag{4.69}$$

Chapter 5

Comparison

5.1 Measurements

In the absence of measurements on finite depth to verify the accuracy of the theory, deep water measurements presented by Grue et al. [6] have been compared with reconstruction of the horizontal velocities in the crest. Six time series of irregular waves were generated in a wave tank by using the JONSWAP spectrum. Within each time series five wave events were identified and the horizontal velocity profile below the crest was measured. The first of these six series have been analyzed in an attempt to indicate the accuracy of the Schrödinger method. It must be emphasized that deep water measurements only gives an indication and the accuracy is expected to increase on finite depth.

The JONSWAP spectrum was estimated by using $\gamma = 3.3$, $H_s = 6.55$m and $T_P = 0.939$s. $h = 72$cm is the depth of the wave tank and $g = 9.81\text{m/s}^2$ is acceleration due to gravity. For the measured time series $N = 16384$ is the number of elements in the sample, $\Delta T = 0.02$s is the time between two subsequent samples and the period of the sample is defined as $T = N\Delta T = 327.68$s.

	Peak	Mean	TT	
T	0.97	0.85	0.76	[s]
ω	6.46	7.38	8.27	[1/s]
k	4.28	5.56	6.97	[1/m]
ϵ	0.10	0.12	0.16	[1]
kh	3.08	4.00	5.02	[1]

Table 5.1: Characteristic of the time series

Wave	time	η_{m}	T_{TT}	k_{TT}	ϵ_{TT}	u_{ref}
1	124.45	5.03	0.76	6.47	0.28	0.34
2	133.24	5.24	0.80	5.88	0.26	0.34
3	165.27	7.55	0.76	6.13	0.40	0.47
4	214.01	3.75	0.82	5.77	0.19	0.25
5	283.43	7.39	0.80	5.63	0.34	0.45
	[s]	[m]	[s]	[1/m]	[1]	[m/s]

Table 5.2: Characteristic of the waves

Two measurements of the surface displacement were available giving the opportunity to find the contribution from shallow water waves. One wave gauge was placed in x=0 and the other was placed in positive direction with a distance $\Delta x = 1.17$. Possible characteristic parameters were taken from the surface displacement at $x = 0$ and is shown in Tab. 5.1. The characteristic frequency is found from the mean value and is given by $\omega_c = M_0 2\pi/T$ where M_0 is nearest integer of T/T_{Mean}. Then $\omega_c = 7.38\text{s}^{-1}$, $T_c = 0.85$s, $k_c = 5.56\text{m}^{-1}$, $a_c = 2.2$cm, $\epsilon = 0.12$ and $k_c h = 4.00$. The wave spectrum is shown in Fig. 5.1 with peak frequency ω_P, mean frequency ω_M and trough to trough frequency ω_{TT} for the highest crest.

Parameters of the five wave events are shown in Tab. 5.2. The local wave number k_{TT} and the local steepness ϵ_{TT} are found from (2.32) and (2.33). Then the reference velocity is given by $u_{\mathrm{ref}} = \epsilon_{\mathrm{TT}}\sqrt{g/k_{\mathrm{TT}}}$. The five wave events and corresponding wave groups may be seen in appendix D.1.

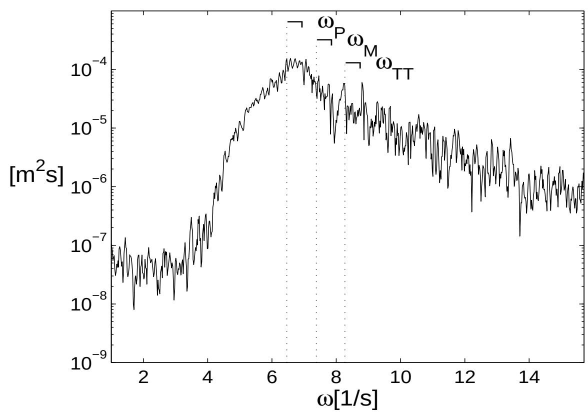


Figure 5.1: Wave spectrum

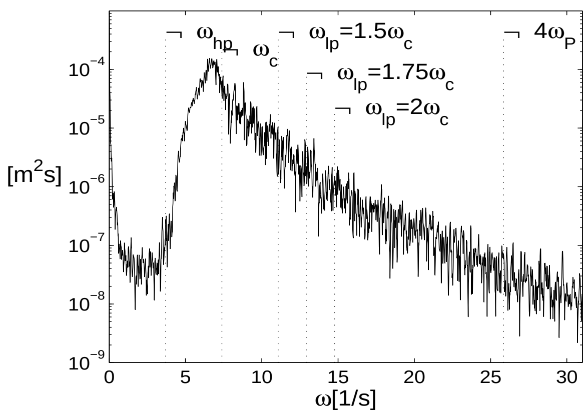


Figure 5.2: Wave spectrum and bandwidth

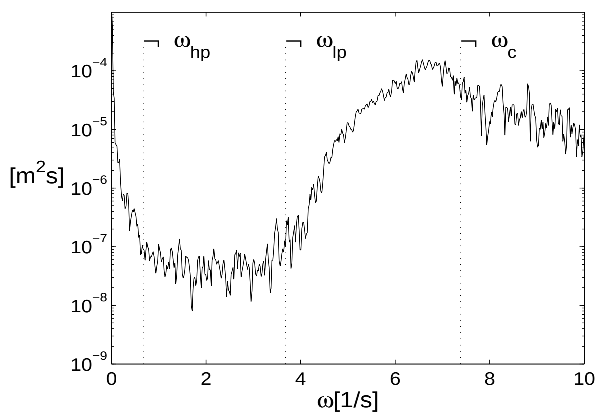


Figure 5.3: Wave spectrum and bandwidth for shallow water waves

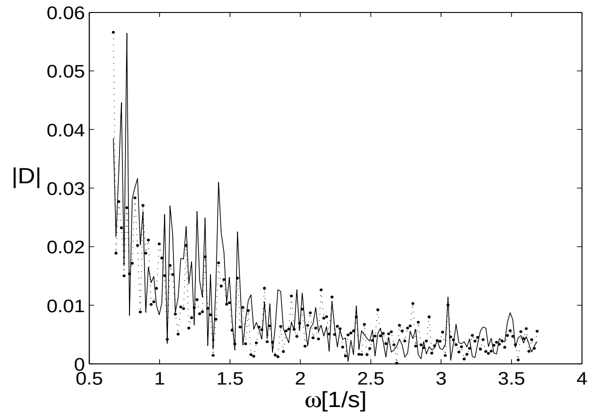


Figure 5.4: Shallow water wave amplitudes. Solid line (–): D_j^+. Dotted line ($\cdot \bullet \cdot$): D_j^-.

The longest wave is assumed to be a standing wave with wave length equal to the length of the wave tank. The length of the wave tank is 24.6m which gives maximum wave length $\lambda_{\max} = 24.6$m and minimum wave number $k_{\min} = 2\pi/\lambda_{\max}$. For shallow water waves $\omega = k\sqrt{gh}$ which gives

$$\omega_{\mathrm{hp}} = j_{\mathrm{hp}}\frac{2\pi}{T} = k_{\min}\sqrt{gh} = \frac{2\pi}{\lambda_{\max}}\sqrt{gh} \quad \text{and} \quad j_{\mathrm{hp}} = \frac{T}{\lambda_{\max}}\sqrt{gh}. \tag{5.1}$$

This becomes the cut off frequency used as high pass filter when extracting the shallow water waves. Contributions below this frequency are assumed to be noise. Fig. 5.3 is showing ω_{hp} together with the low pass frequency ω_{lp}. ω_{lp} is defined by (4.56) to ensure that the extracted waves are shallow water waves. In Fig. 5.4 shallow water wave amplitudes found by using (4.68) and (4.69) are shown, where D_j^+

are amplitudes representing waves propagating in positive direction and D_j^- are amplitudes representing waves propagating in negative direction.

The linear amplitude B is constructed by using the shallow water wave low pass frequency (4.56) as high pass frequency ω_{hp} making sure that all the energy in this region is considered. ω_{hp} is shown together with possible low pass frequencies $\omega_{\text{lp}} = (1.5\omega_c, 1.75\omega_c, 2\omega_c)$ in Fig. 5.2. In addition, the cut off frequency $4\omega_P$ recommended by DNV [2] as low pass frequency used when reconstructing Airy and Wheeler kinematics is shown. The bandwidth is $\omega_{\text{bw}} = \omega_{\text{lp}} - \omega_{\text{hp}}$

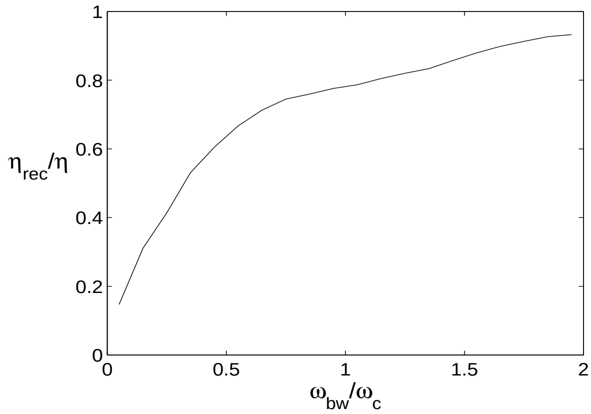
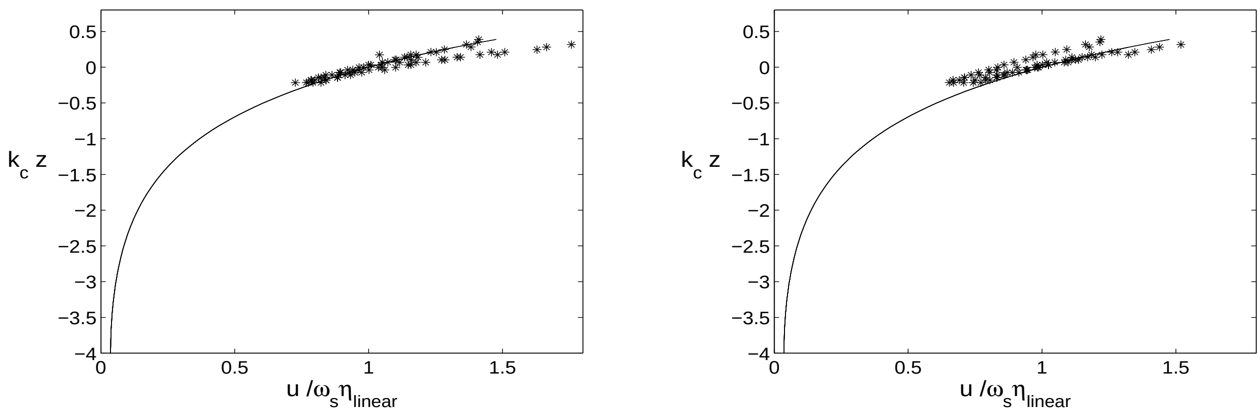


Figure 5.5: Ratio of reconstructed η

Both the reconstruction of surface displacement and the reconstruction of kinematics are depending on the bandwidth. Fig 5.5 is showing the ratio between reconstructed and measured η of the highest crest as a function of ratio between bandwidth ω_{bw} and characteristic frequency ω_c. The accuracy of reconstructed surface is increasing rapidly until $\omega_{\text{bw}} \approx 0.75\omega_c$, while for wider bandwidth there is considerable less change in accuracy.



(a) $\omega_{\text{lp}} = 1.5\omega_c$ (b) $\omega_{\text{lp}} = 2\omega_c$

Figure 5.6: Solid line (—): $\cosh k_c(z+h)/\sinh k_c h$. Asterisk ($*$): measurements

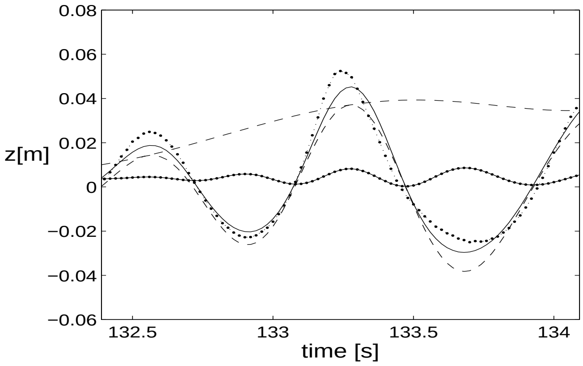
In Fig. 5.6 the dimensionless decay function $\cosh k_c(z+h)/\sinh k_c h$ is plotted together with measurements of the horizontal velocity from the five wave events. The measurements are from the crest of the five wave events and are made dimensionless by

$$\omega_c \eta_{\text{linear}} = \omega_c a_c \frac{1}{2}(B\mathrm{e}^{-\mathrm{i}\chi} + \text{c.c}). \tag{5.2}$$

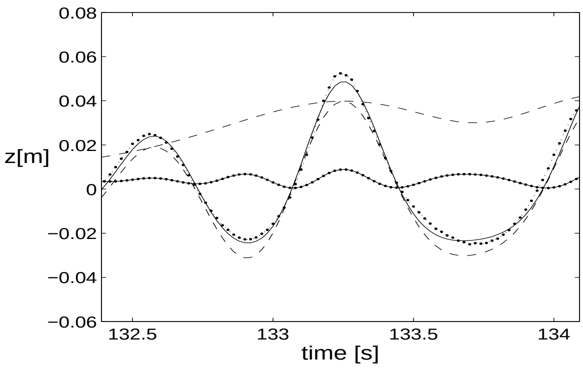
The difference between Fig. 5.6 (a) and Fig. 5.6 (b) is the bandwidth. To increase the accuracy the profile need to be more curved.

5.2 Reconstructed surface

As an example of reconstruction of surface the crest at $t = 133.24$ is shown in Fig. 5.2 and the 2nd order contribution is shown in Fig. 5.2. Two different bandwidths are shown and the reconstruction is approaching the measurements by increasing the bandwidth. By increasing the bandwidth the reconstructed surface is approaching the measured.

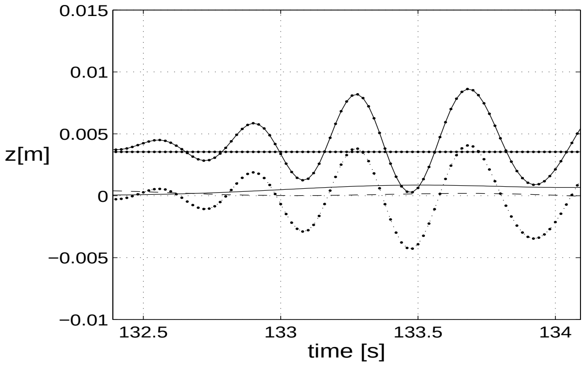


(a) $\omega_{\text{lp}} = 1.5\omega_c$

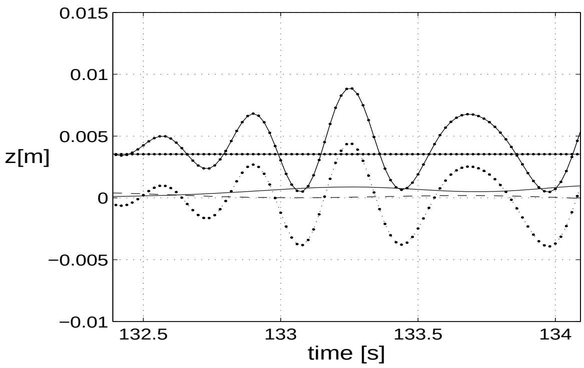


(b) $\omega_{\text{lp}} = 2\omega_c$

Figure 5.7: Reconstructed surface at $t = 133.24$. Dotted line (· • ·): η. Solid line (—): η_{rec}. Broken line (- -): 1st order. Dotted solid line (-•-): 2nd order contribution. Broken line (- -): amplitude B.



(a) $\omega_{\text{lp}} = 1.5\omega_c$

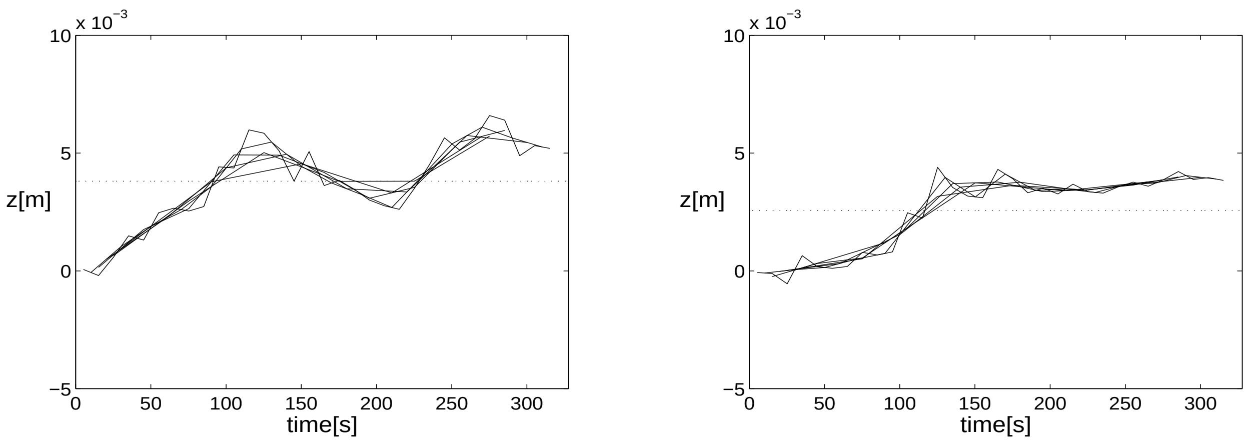


(b) $\omega_{\text{lp}} = 2\omega_c$

Figure 5.8: 2nd order contribution at $t = 133.24$. Dotted solid line (-•-): total 2nd order contribution. Broken line (- -): shallow water waves. Solid line (—): 0th harmonic $\propto |B|^2$. Dotted solid line (-•-): set down $\propto E_{\text{f}}$. Dotted line (· • ·): 2nd harmonic.

5.3 Reconstructed horizontal velocity

The mean surface was calculated in equal intervals throughout the time series. Calculations were done for intervals of 10, 20, 30, 40, 50 and 60 seconds and are shown in Fig. 5.9 together with the total mean surface. Both gauges indicate an accumulation of water at the end of the tank. A return flow as a consequence of this has to be considered.



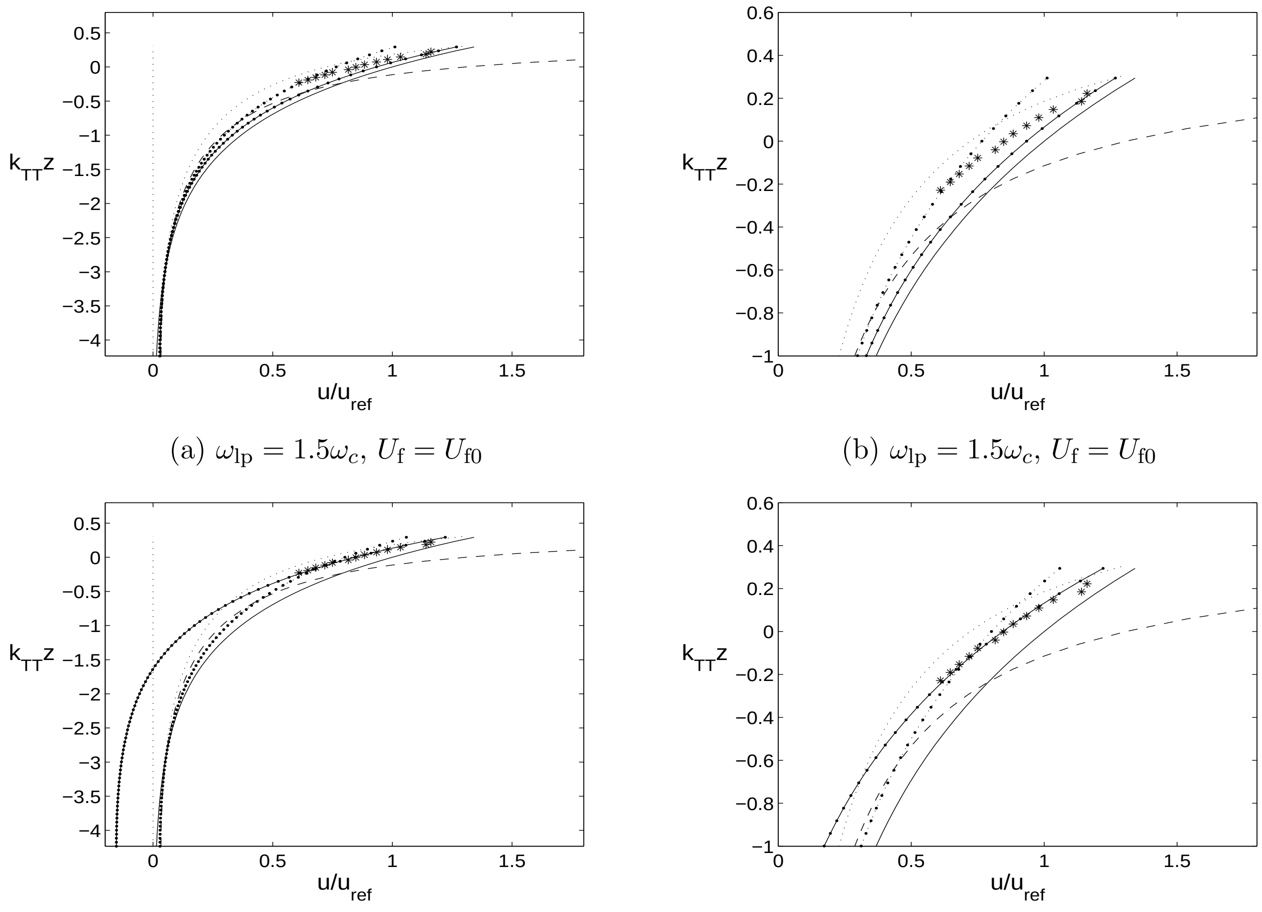
(a) $x = 0$ (b) $x = \Delta x$

Figure 5.9: Mean. Solid lines (—): change in mean of η. Dotted line ($\cdots$): total mean of η

To determine the free stream U_{f} two methods have been suggested, U_{f0} (4.40) and U_{fm} (4.42). When using U_{f0} it is assumed that the total flux integrated over one period is zero. When the behavior of the return flow is not known the magnitude of U_{f}, determined by using U_{f0}, is highly uncertain.

In Fig. 5.10 Airy theory, Wheeler stretching, 1st and 2nd order Schr̈dinger and the exponential profile are compared with measurements of the horizontal velocity in the crest at $t = 133.24$. All the measured velocities analyzed may be seen in appendix D.1. The velocities are made dimensionless with the reference velocity u_{ref}. In Figs. 5.10 (a) and (b) the bandwidth $\omega_{\text{bw}} = \omega_c$ and the horizontal current $U_{\text{f}} = U_{\text{f0}}$ are used. Comparing 2nd order Schrödinger with the measurements indicates a need for adding more curvature to the profile and the return flow to be stronger. Increasing the bandwidth adds more curvature to the profile and by using $U_{\text{f}} = U_{\text{fm}}$ the return flow increases. With these modifications 2nd order Schrödinger shows good agreement with measurements in the crest as shown in Fig. 5.10 (d). When looking at the entire water column in Fig. 5.10 the increase in bandwidth and return flow has resulted in negative velocities in about half of the water column. If the velocities in the lower part of the water column are less as negative than the theory estimates, it would indicat a return flow confined to the region around still water level causing more curvature of the profile.

In Fig. 5.11 (a) 2nd order contribution incorporated in Figs. 5.10 (a) and (b) is shown and in Fig. 5.11 (b) 2nd order contribution incorporated in Figs. 5.10 (c) and (d) is shown. The contribution from the shallow water waves are close to zero. Justified by the uncertainty about the magnitude of the induced mean flow and for computational convenience, the shallow water waves are neglected for the rest of the calculations.



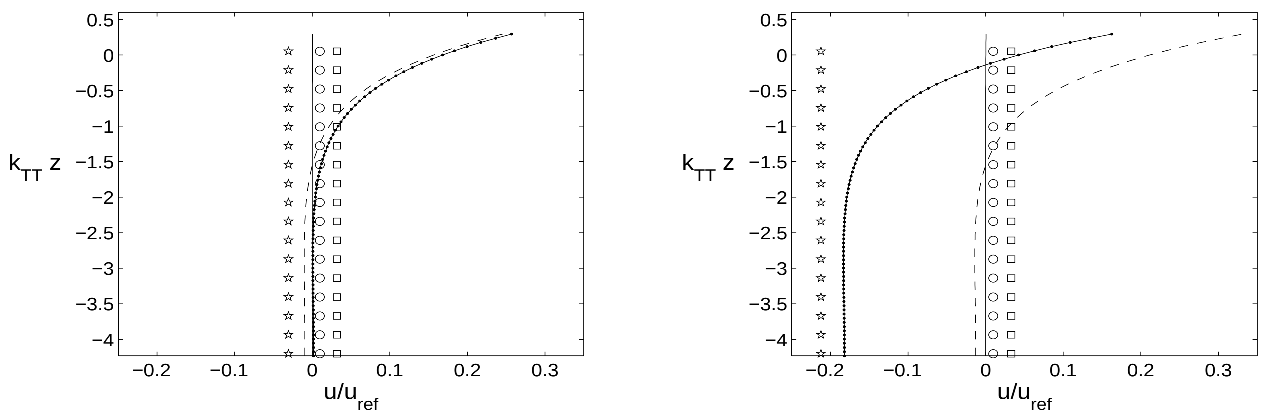
(a) $\omega_{\mathrm{lp}} = 1.5\omega_c$, $U_{\mathrm{f}} = U_{\mathrm{f0}}$

(b) $\omega_{\mathrm{lp}} = 1.5\omega_c$, $U_{\mathrm{f}} = U_{\mathrm{f0}}$

(c) $\omega_{\mathrm{lp}} = 1.75\omega_c$, $U_{\mathrm{f}} = U_{\mathrm{fm}}$

(d) $\omega_{\mathrm{lp}} = 1.75\omega_c$, $U_{\mathrm{f}} = U_{\mathrm{fm}}$

Figure 5.10: Horizontal velocity at $t = 133.24$. Broken line (- -): Airy theory. Dotted line ($\cdots$): Wheeler stretching. Extra dotted line ($\cdot\bullet\cdot$): 1st order Schrödinger. Dotted solid line (-$\bullet$-): 2nd order Schrödinger. Asterisk ($*$): measurements.



(a) $\omega_{\mathrm{lp}} = 1.5\omega_c$, $U_{\mathrm{f}} = U_{\mathrm{f0}}$

(b) $\omega_{\mathrm{lp}} = 1.75\omega_c$, $U_{\mathrm{f}} = U_{\mathrm{fm}}$

Figure 5.11: 2nd order contribution at $t = 133.24$. Dotted solid line (-$\bullet$-):total 2nd order contribution. Broken line (- -): 1st harmonic. Solid line (—): 2nd harmonic. Circles ($\circ$): contribution from shallow water waves is. Squares ($\square$): 0th harmonic $\propto |B|^2$. Stars ($\star$): Set down $\propto E_{\mathrm{f}}$.

5.4 Deviation

To give an indication about the accuracy the standard deviation according to the measured velocities are calculated for all the five wave events. Standard deviation, also known as the root mean square is given by

$$RMS = \sqrt{\frac{1}{M}\sum_{m=1}^{M}\left(u_{meas}(z_m) - u_{theory}(z_m)\right)^2} \tag{5.3}$$

for all z_m with measurements.

At $t = 165.27$, Fig. D.6, the two measurements highest in the crest are assumed to be diverging from the measured velocity profile and are therefore not considered when calculating RMS. For the same reason the measurements highest in the crest at $t = 214.01$, Fig. D.8, is not considered.

Fig. 5.12 is showing the root mean square for all the five wave events. 1st order Schrödinger is showing better approximation than both Wheeler stretching and the exponential profile. Apart from wave 1 and 5, 2nd order Schrödinger is also showing better approximation than both Wheeler stretching and the exponential profile. Figs. D.2 and D.10 respectively are showing considerably over prediction by 2nd order for wave 1 and 5. 2nd order Schrödinger fitted for best approximation by using $U_{\mathrm{f}} = U_{\mathrm{fm}}$ results in the best result which shows that increasing the bandwidth gets the right curvature of the horizontal profile under the crest. When Wheeler stretching is showing better approximation than the exponential profile, it must be emphasized that Wheeler stretching is constantly under predicting in the crest.

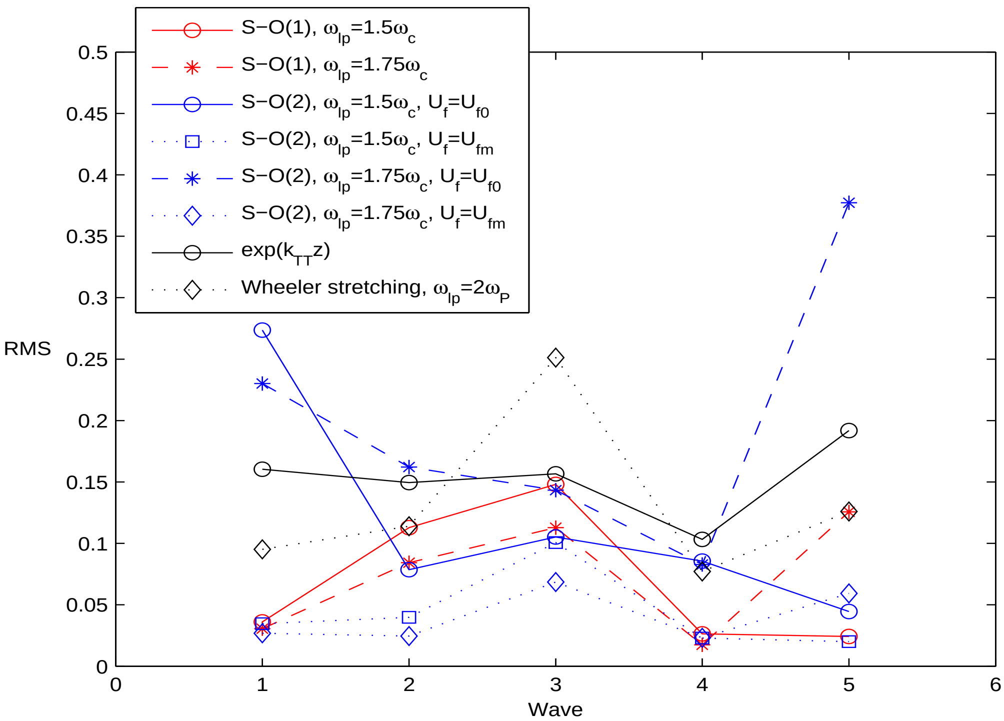


Figure 5.12: Root mean square

Chapter 6

Conclusion

A second order nonlinear Schrödinger method for prediction of kinematics in irregular waves on finite depth has been developed. The main difference between the method presented by Trulsen et al. [11] and the method presented in this thesis is the relationship between modulation of amplitude $\Delta\omega$ and the characteristic frequency ω_c. Trulsen et al. assumed $\Delta\omega/\omega_c = O(\epsilon^{1/2})$ while this thesis is assuming $\Delta\omega/\omega_c = O(\epsilon)$. This change in assumption gives a noticeable decrease in complexity. In addition, the contribution from the shallow water waves has been considered.

The boundary value problem defined for the fluid was made dimensionless with characteristic parameters and a slow variation was introduced according to the assumption about modulation of amplitude. The boundary condition at the surface was approximated with Taylor series around the still water level and the problem was solved with a harmonic expansion of the velocity potential and the surface displacement.

One of the best known methods in use today is the Wheeler stretching. Wheeler stretching is based on linear Airy theory which is over predicting under the crest. Wheeler stretching is improving Airy theory with an empirical adjustment of the vertical coordinate. Like Airy the Schrödinger method is using the exact vertical position in crest, but compared to Airy and Wheeler the low pass frequency is considerably lower and the amount of energy extracted from the wave spectrum is therefore considerably less and the theory does not break down.

In the absence of measurements on finite depth to verify the accuracy of the method, deep water measurements have been compared with the theory. Schrödinger theory has been compared with measurements of the horizontal velocity in the crest, Airy theory, Wheeler stretching and the exponential profile.

Different bandwidths have been considered when comparing theory with measurements. For the reconstructed surface the accuracy increased when the low pass frequency was increased from $1.5\omega_c$ until $2\omega_c$. For the reconstructed horizontal velocity an over prediction was indicated when the low pass frequency reached $1.75\omega_c$.

To find the free stream two methods where suggested. The first method suggested is requiring the horizontal volume flux to be equal zero while the other is requiring measurements. Neither did show satisfying results throughout the entire water column, and the determination of the mean flow is highly uncertain and is indicating a return flow confined to the region around still water level adding more curvature to the velocity profile.

Extracting the shallow water waves require measurements of the surface made by two wave gauges. The contribution from the shallow water waves turned out to be close to zero. Justified by the uncertainty about the magnitude of the induced mean flow and for computational convenience the shallow water waves has been neglected for the rest of the calculations.

To give an indication about the accuracy according to measurements of horizontal velocity in the crest the standard deviation, also known as the root mean square was calculated. The root mean square did show 1st order Schrödinger to be more accurate than both Wheeler stretching and the exponential profile. 2nd order was showing good but more changing results.

This may be due to both the uncertainty about the bandwidth and both magnitude and behavior of the return flow. The best results was given by the 2nd order with the widest bandwidth and then adjusted with the return flow to fit the measurements. This shows that increasing the bandwidth adds more curvature to the profile and the Schrödinger method converges towards the profile of the measurements.

The unknown behavior of the return flow may be discovered by higher order theory and to verify the behavior and magnitude of the return flow there is need for measured velocity profiles throughout the entire water column. There is also need for measurements of velocities and accelerations throughout the wave group to fully verify the theory.

Appendix A

A.1 Group velocity

For the following calculations $\star$ denotes the dimensionless. The group velocity c_g is given by

$$c_g = \frac{\partial \omega_c}{\partial k_c} = \frac{1}{2}\Big(\frac{g \tanh k_c h + g k_c (1 - \tanh^2 k_c h) h}{\omega_c}\Big) \tag{A.1}$$

And made dimensionless with

$$\begin{aligned} c_g^\star &= \frac{k_c}{\omega_c} c_g = \frac{k_c}{\omega_c}\frac{1}{2}\Big(\frac{g \tanh h^\star + g(1 - \tanh^2) h^\star)}{\omega_c}\Big) = \frac{1}{2\sigma}\Big(\sigma + (1-\sigma) h^\star\Big) \\ &= \frac{1}{2}\Big(1 - \frac{(\sigma^2 - 1) h^\star}{\sigma}\Big) = \gamma_3 \end{aligned} \tag{A.2}$$

where $h^\star = k_c h$ and the substitution

$$\frac{1}{\cosh^2 k_c h} = \frac{1 + \sinh^2 k_c h - \sinh^2 k_c h}{\cosh^2 k_c h} = \frac{1 + (\cosh^2 k_c h - 1) - \sinh^2 k_c h}{\cosh^2 k_c h} = 1 - \tanh^2 k_c h \tag{A.3}$$

has been used.

A.2 Shallow water velocity

For the following calculations $\star$ denotes the dimensionless. For non dispersive shallow water waves the dispersion relation is

$$\omega_s = k\sqrt{gh} \quad \text{which gives} \quad c_s = \omega_s/k = \sqrt{gh} \tag{A.4}$$

where c_s denotes the shallow water velocity which is made dimensioless with

$$c_s^\star = \frac{k_c}{\omega_c} c_s = \frac{k_c\sqrt{gh}}{\sqrt{g k_c \tanh(k_c h)}} = \frac{\sqrt{k_c h}}{\sqrt{\tanh(k_c h)}} = \sqrt{\frac{h^\star}{\sigma}} = c_0. \tag{A.5}$$

For shallow water waves the group velocity equals the phase velocity. Longer waves travel faster than shorter waves and the maximum traveling velocity for waves is the shallow water velocity. This gives

$$c_0 \geq c_g^\star. \tag{A.6}$$

Appendix B

B.1 A_{nm}, B_{nm} and their derivatives $z = 0$

Explicitly solutions of A_{nm}, B_{nm} and their derivatives at $z = 0$ from (3.105). For the following calculations x, z, t are made dimensionless according to (3.2), c_g is made dimensionless according to A.2 and the $\star$ is removed.

$$\Big[\acute{A}_{11}\Big]_{z=0} = \Big[-\mathrm{i}\frac{B}{\sigma}\frac{\cosh(z+h)}{\cosh h}\Big]_{z=0} = -\mathrm{i}\frac{B}{\sigma}, \tag{B.1}$$

$$\Big[\frac{\partial \acute{A}_{11}}{\partial x_1}\Big]_{z=0} = \Big[-\mathrm{i}\frac{1}{\sigma}\frac{\partial B}{\partial x_1}\frac{\cosh(z+h)}{\cosh h}\Big]_{z=0} = -\mathrm{i}\frac{1}{\sigma}\frac{\partial B}{\partial x_1}, \tag{B.2}$$

$$\Big[\acute{A}_{21}\Big]_{z=0} = \Big[\Big((h - c_g/\sigma)\frac{\cosh(z+h)}{\cosh h} - (h+z)\frac{\sinh(z+h)}{\sinh h}\Big)\frac{\partial B}{\partial x_1}\Big]_{z=0} = -\frac{c_g}{\sigma}\frac{\partial B}{\partial x_1}, \tag{B.3}$$

$$\begin{aligned}\Big[\frac{\partial^2 \acute{A}_{21}}{\partial z^2}\Big]_{z=0} &= \Big[\Big((h - c_g/\sigma)\frac{\cosh(z+h)}{\cosh h} - (h+z)\frac{\sinh(z+h)}{\sinh h} - 2\frac{\cosh(z+h)}{\sinh h}\Big)\frac{\partial B}{\partial x_1}\Big]_{z=0} \\ &= -\frac{1}{\sigma}(c_g + 2)\frac{\partial B}{\partial x_1},\end{aligned} \tag{B.4}$$

$$\Big[\frac{\partial B_{20}}{\partial t_1}\Big]_{z=0} = -\sigma\frac{\partial^2 \beta}{\partial t_1^2} + \gamma_1\frac{\partial}{\partial t_1}|B|^2, \tag{B.5}$$

$$\Big[\frac{\partial \acute{A}_{30}}{\partial z}\Big]_{z=0} = \Big[-\frac{\partial^2 \beta}{\partial x_1^2}(z+h)\Big]_{z=0} = -\frac{\partial^2 \beta}{\partial x_1^2}h, \tag{B.6}$$

$$B\frac{\partial B^*}{\partial x_1} + \mathrm{c.c} = B\frac{\partial B^*}{\partial x_1} + B^*\frac{\partial B}{\partial x_1} = \frac{\partial}{\partial x_1}(BB^*) = \frac{\partial}{\partial x_1}|B|^2 = 2|B|\frac{\partial}{\partial x_1}|B| \tag{B.7}$$

$$\frac{\partial B}{\partial x_1} = -\frac{1}{c_g}\frac{\partial B}{\partial t_1} \quad \text{and} \tag{B.8}$$

$$B\frac{\partial B^*}{\partial x_1} + \mathrm{c.c} = -\frac{1}{c_g}\frac{\partial}{\partial t_1}|B|^2. \tag{B.9}$$

Appendix C

C.1 Reconstruction formulas

For the following calculations $\star$ denotes the dimensionless.

The velocity potential is

$$\phi = \frac{a_c \omega_c}{k_c}\phi^\star = \frac{a_c \omega_c}{k_c}(\phi_1^\star + \epsilon \phi_2^\star) + O(\epsilon^2) \tag{C.1}$$

with

$$\phi_1^\star = \beta - \frac{1}{2}(\mathrm{i}B\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}) \quad \text{and} \tag{C.2}$$

$$\phi_2^\star = \beta_2 + \frac{1}{2}\Bigg(\frac{1}{c_g^\star}\Big((c_g^\star/\sigma - h^\star)\frac{\cosh(z^\star + h^\star)}{\cosh h^\star} + (h^\star + z^\star)\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\Big)\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi}$$
$$+\mathrm{i}\gamma_4 B^2 \frac{\cosh 2(z^\star + h^\star)}{\cosh 2h^\star}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Bigg). \tag{C.3}$$

The surface displacement is

$$\eta = a_c \eta^\star = a_c(\eta_1^\star + \epsilon \eta_2^\star) + O(\epsilon^2) \tag{C.4}$$

with

$$\eta_1^\star = \frac{1}{2}(B\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}) \quad \text{and} \tag{C.5}$$

$$\eta_2^\star = -\sigma\frac{\partial \beta}{\partial t_1^\star} + \gamma_1 |B|^2 + \frac{1}{2}\Big(\gamma_5 B^2 \mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Big) \tag{C.6}$$

which gives

$$\eta = a_c\Big(-\sigma\frac{\partial \beta}{\partial t^\star} + \epsilon\gamma_1|B|^2 + \frac{1}{2}\big(B\mathrm{e}^{\mathrm{i}\chi} + \epsilon\gamma_5 B^2\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\big)\Big) + O(\epsilon^2). \tag{C.7}$$

The velocity is given with $\nabla\phi = \{u, w\}$ and for 1st order

$$u = \frac{a_c\omega_c k_c}{k_c}u^\star = \epsilon\frac{\omega_c}{k_c}\frac{\partial \phi_1^\star}{\partial x^\star} + O(\epsilon^2) = a_c\omega_c\frac{1}{2}\Big(B\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) + O(\epsilon^2) \tag{C.8}$$

and

$$w = \frac{a_c\omega_c k_c}{k_c}w^\star = \epsilon\frac{\omega_c}{k_c}\frac{\partial \phi_1^\star}{\partial z^\star} + O(\epsilon^2) = -a_c\omega_c\frac{1}{2}\Big(\mathrm{i}B\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) + O(\epsilon^2). \tag{C.9}$$

The only constribution to the coresponding accelerations is the local accelearation and for 1st order

$$a_x = \frac{\partial u}{\partial t} = \epsilon\frac{\omega_c^2}{k_c}\frac{\partial u^\star}{\partial t^\star} = \epsilon\frac{\omega_c^2}{k_c}\frac{\partial \phi_1^\star}{\partial x^\star \partial t^\star} + O(\epsilon^2) = -a_c\omega_c^2\frac{1}{2}\Big(\mathrm{i}B\frac{\cosh(z^\star+h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) + O(\epsilon^2) \quad \text{(C.10)}$$

and

$$a_z = \frac{\partial w}{\partial t} = \epsilon\frac{\omega_c^2}{k_c}\frac{\partial w^\star}{\partial t^\star} = \epsilon\frac{\omega_c^2}{k_c}\frac{\partial \phi_1^\star}{\partial z^\star \partial t^\star} + O(\epsilon^2) = -a_c\omega_c^2\frac{1}{2}\Big(B\frac{\sinh(z^\star+h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) + O(\epsilon^2). \quad \text{(C.11)}$$

For 2nd order the horizontal velocity is

$$u = \frac{a_c\omega_c k_c}{k_c}u^\star = \epsilon\frac{\omega_c}{k_c}\Big(\frac{\partial\phi^\star}{\partial x^\star} + \epsilon\frac{\partial\phi^\star}{\partial x_1^\star}\Big) = a_c\omega_c\Big(\frac{\partial\phi_1^\star}{\partial x^\star} + \epsilon\frac{\partial\phi_2^\star}{\partial x^\star} + \epsilon\frac{\partial\phi_1^\star}{\partial x_1^\star}\Big) + O(\epsilon^3) \quad \text{(C.12)}$$

where

$$\frac{\partial\phi_1^\star}{\partial x^\star} = \frac{1}{2}\Big(B\frac{\cosh(z^\star+h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big), \quad \text{(C.13)}$$

$$\frac{\partial\phi_2^\star}{\partial x^\star} = \frac{1}{2}\Bigg(\mathrm{i}\frac{1}{c_g^\star}\Big((c_g^\star/\sigma - h^\star)\frac{\cosh(z^\star+h^\star)}{\cosh h^\star} + (h^\star+z^\star)\frac{\sinh(z^\star+h^\star)}{\sinh h^\star}\Big)\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi}$$
$$-2\gamma_4 B^2\frac{\cosh 2(z^\star+h^\star)}{\cosh 2h^\star}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Bigg) \quad \text{and} \quad \text{(C.14)}$$

$$\frac{\partial\phi_1^\star}{\partial x_1^\star} = \frac{1}{\epsilon}\frac{\partial\beta}{\partial x^\star} + \frac{1}{2}\Big(\mathrm{i}\frac{1}{c_g^\star}\frac{\cosh(z^\star+h^\star)}{\sinh h^\star}\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big). \quad \text{(C.15)}$$

For 2nd order the vertical velocity is

$$w = \frac{a_c\omega_c k_c}{k_c}w^\star = \epsilon\frac{\omega_c}{k_c}\Big(\frac{\partial\phi^\star}{\partial z^\star}\Big) = a_c\omega_c\Big(\frac{\partial\phi_1^\star}{\partial z^\star} + \epsilon\frac{\partial\phi_2^\star}{\partial z^\star}\Big) + O(\epsilon^3) \quad \text{(C.16)}$$

where

$$\frac{\partial\phi_1^\star}{\partial z^\star} = -\frac{1}{2}\Big(\mathrm{i}B\frac{\sinh(z^\star+h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) \quad \text{and} \quad \text{(C.17)}$$

$$\frac{\partial\phi_2^\star}{\partial z^\star} = \frac{1}{2}\Bigg(\frac{1}{c_g^\star}\Big((c_g^\star/\sigma - h^\star)\frac{\sinh(z^\star+h^\star)}{\cosh h^\star} + (h^\star+z^\star)\frac{\cosh(z^\star+h^\star)}{\sinh h^\star}$$
$$+\frac{\sinh(z^\star+h^\star)}{\sinh h^\star}\Big)\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + 2\mathrm{i}\gamma_4 B^2\frac{\sinh 2(z^\star+h^\star)}{\cosh 2h^\star}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Bigg). \quad \text{(C.18)}$$

The accelearation is given by

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} \quad \text{(C.19)}$$

and for the horizontal and vertical direction respectively the accelerations are

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \quad \text{and} \quad a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}. \quad \text{(C.20)}$$

For 2nd order the horizonatal local acceleration is

$$\frac{\partial u}{\partial t} = \epsilon\frac{\omega_c^2}{k_c}\Big(\frac{\partial u^\star}{\partial t^\star} + \epsilon\frac{\partial u^\star}{\partial t_1^\star}\Big) = a_c\omega_c^2\Big(\frac{\partial^2\phi_1^\star}{\partial x^\star\partial t^\star} + \epsilon\frac{\partial^2\phi_2^\star}{\partial x^\star\partial t^\star} + \epsilon\frac{\partial^2\phi_1^\star}{\partial x_1^\star\partial t^\star} + \epsilon\frac{\partial^2\phi_1^\star}{\partial x^\star\partial t_1^\star}\Big) + O(\epsilon^3) \quad \text{(C.21)}$$

where

$$\frac{\partial^2 \phi_1^\star}{\partial x^\star \partial t^\star} = -\frac{1}{2}\big(\mathrm{i}B\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\big), \tag{C.22}$$

$$\frac{\partial^2 \phi_2^\star}{\partial x^\star \partial t^\star} = \frac{1}{2}\Bigg(\frac{1}{c_g^\star}\Big((c_g^\star/\sigma - h^\star)\frac{\cosh(z^\star + h^\star)}{\cosh h^\star} + (h^\star + z^\star)\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\Big)\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi}$$
$$+\mathrm{i}4\gamma_4 B^2\frac{\cosh 2(z^\star + h^\star)}{\cosh 2h^\star}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Bigg), \tag{C.23}$$

$$\frac{\partial^2 \phi_1^\star}{\partial x_1^\star \partial t^\star} = \frac{1}{2}\Big(\frac{1}{c_g^\star}\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) \quad \text{and} \tag{C.24}$$

$$\frac{\partial^2 \phi_1^\star}{\partial x^\star \partial t_1^\star} = \frac{1}{2}\Big(\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big). \tag{C.25}$$

For 2nd order the vertical local acceleration is

$$\frac{\partial w}{\partial t} = \epsilon\frac{\omega_c^2}{k_c}\Big(\frac{\partial w^\star}{\partial t^\star} + \epsilon\frac{\partial w^\star}{\partial t_1^\star}\Big) = a_c\omega_c^2\big(\frac{\partial^2 \phi_1^\star}{\partial z^\star \partial t^\star} + \epsilon\frac{\partial^2 \phi_2^\star}{\partial z^\star \partial t^\star} + \epsilon\frac{\partial^2 \phi_1^\star}{\partial z^\star \partial t_1^\star}\big) + O(\epsilon^2) \tag{C.26}$$

where

$$\frac{\partial^2 \phi_1^\star}{\partial z^\star \partial t^\star} = -\frac{1}{2}\big(B\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\big), \tag{C.27}$$

$$\frac{\partial^2 \phi_2^\star}{\partial z^\star \partial t^\star} = \frac{1}{2}\Bigg(\mathrm{i}\frac{1}{c_g^\star}\Big((h - c_g^\star/\sigma)\frac{\sinh(z^\star + h^\star)}{\cosh h^\star} - (h^\star + z^\star)\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}$$
$$-\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\Big)\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + 4\gamma_4 B^2\frac{\sinh 2(z^\star + h^\star)}{\cosh 2h^\star}\mathrm{e}^{2\mathrm{i}\chi} + \mathrm{c.c}\Bigg) \quad \text{and} \tag{C.28}$$

$$\frac{\partial^2 \phi_1^\star}{\partial z^\star \partial t_1^\star} = -\frac{1}{2}\Big(\mathrm{i}\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\frac{1}{\epsilon}\frac{\partial B}{\partial t^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big). \tag{C.29}$$

The space derivatives of the velocities are

$$\frac{\partial u}{\partial x} = \epsilon\frac{\omega_c k_c}{k_c}\Big(\frac{\partial^2 \phi_1^\star}{\partial x^\star \partial x^\star}\Big) + O(\epsilon^2) \quad \text{with} \quad \frac{\partial^2 \phi_1^\star}{\partial x^\star \partial x^\star} = \frac{1}{2}\big(\mathrm{i}B\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\big), \tag{C.30}$$

$$\frac{\partial u}{\partial z} = \epsilon\frac{\omega_c k_c}{k_c}\Big(\frac{\partial^2 \phi_1^\star}{\partial x^\star \partial z^\star}\Big) + O(\epsilon^2) \quad \text{with} \quad \frac{\partial^2 \phi_1^\star}{\partial x^\star \partial z^\star} = \frac{1}{2}\big(B\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\big), \tag{C.31}$$

$$\frac{\partial w}{\partial x} = \epsilon\frac{\omega_c k_c}{k_c}\Big(\frac{\partial^2 \phi_1^\star}{\partial z^\star \partial x^\star}\Big) + O(\epsilon^2) \quad \text{with} \quad \frac{\partial^2 \phi_1^\star}{\partial z^\star \partial x^\star} = \frac{1}{2}\Big(B\frac{\sinh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big) \quad \text{and} \tag{C.32}$$

$$\frac{\partial w}{\partial z} = \epsilon\frac{\omega_c k_c}{k_c}\Big(\frac{\partial^2 \phi_1^\star}{\partial z^\star \partial z^\star}\Big) + O(\epsilon^2) \quad \text{with} \quad \frac{\partial^2 \phi_1^\star}{\partial z^\star \partial z^\star} = -\frac{1}{2}\Big(\mathrm{i}B\frac{\cosh(z^\star + h^\star)}{\sinh h^\star}\mathrm{e}^{\mathrm{i}\chi} + \mathrm{c.c}\Big). \tag{C.33}$$

Then the horizontal convective acceleration is

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \epsilon\frac{\omega_c}{k_c}u^\star\epsilon\frac{\omega_c k_c}{k_c}\frac{\partial u^\star}{\partial x^\star} + \epsilon\frac{\omega_c}{k_c}w^\star\epsilon\frac{\omega_c k_c}{k_c}\frac{\partial u^\star}{\partial z^\star} = \epsilon^2\frac{\omega_c^2 k_c}{k_c^2}(u^\star\frac{\partial u^\star}{\partial x^\star} + w^\star\frac{\partial u^\star}{\partial z^\star}) \tag{C.34}$$

where

$$\begin{aligned}
u^\star\frac{\partial u^\star}{\partial x^\star}+w^\star\frac{\partial u^\star}{\partial z^\star} &= \frac{\partial \phi_1^\star}{\partial x^\star}\frac{\partial^2\phi_1^\star}{\partial x^\star\partial x^\star}+\frac{\partial \phi_1^\star}{\partial z^\star}\frac{\partial^2\phi_1^\star}{\partial x^\star\partial z^\star}+O(\epsilon)\\
&=\left(\frac{1}{4}\mathrm{i}\frac{\cosh^2(z^\star+h^\star)}{\cosh^2 h^\star}\Big(B^2\mathrm{e}^{2\mathrm{i}\chi}+\mathrm{c.c}\Big)-\frac{1}{4}\mathrm{i}\frac{\sinh^2(z^\star+h^\star)}{\cosh^2 h^\star}\Big(B^2\mathrm{e}^{2\mathrm{i}\chi}+\mathrm{c.c}\Big)\right)+O(\epsilon)\\
&=\left(\frac{\mathrm{i}}{4\cosh^2 h^\star}\Big(B^2\mathrm{e}^{2\mathrm{i}\chi}+\mathrm{c.c}\Big)\right)+O(\epsilon). \qquad \text{(C.35)}
\end{aligned}$$

And the vertical convective acceleration is

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=\epsilon\frac{\omega_c}{k_c}u^\star\epsilon\frac{\omega_c k_c}{k_c}\frac{\partial w^\star}{\partial x^\star}+\epsilon\frac{\omega_c}{k_c}w^\star\epsilon\frac{\omega_c k_c}{k_c}\frac{\partial w^\star}{\partial z^\star}=\epsilon^2\frac{\omega_c^2 k_c}{k_c^2}\Big(u^\star\frac{\partial w^\star}{\partial x^\star}+w^\star\frac{\partial w^\star}{\partial z^\star}\Big) \qquad \text{(C.36)}$$

where

$$\begin{aligned}
u^\star\frac{\partial w^\star}{\partial x^\star}+w^\star\frac{\partial w^\star}{\partial z^\star} &= \frac{\partial \phi_1^\star}{\partial x^\star}\frac{\partial^2\phi_1^\star}{\partial z^\star\partial x^\star}+\frac{\partial \phi_1^\star}{\partial z^\star}\frac{\partial^2\phi_1^\star}{\partial z^\star\partial z^\star}+O(\epsilon)\\
&=\left(\frac{\sinh 2(z^\star+h^\star)}{8\cosh^2 h^\star}\Big(2|B|^2+(B^2\mathrm{e}^{2\mathrm{i}\chi}+\mathrm{c.c})\Big)\right.\\
&\left.-\frac{\sinh 2(z^\star+h^\star)}{8\cosh^2 h^\star}\Big(-2|B|^2+(B^2\mathrm{e}^{2\mathrm{i}\chi}+\mathrm{c.c})\Big)\right)+O(\epsilon)\\
&=\left(\frac{\sinh 2(z^\star+h^\star)}{2\cosh^2 h^\star}|B|^2\right)+O(\epsilon). \qquad \text{(C.37)}
\end{aligned}$$

Appendix D

D.1 Horizontal velocity profiles

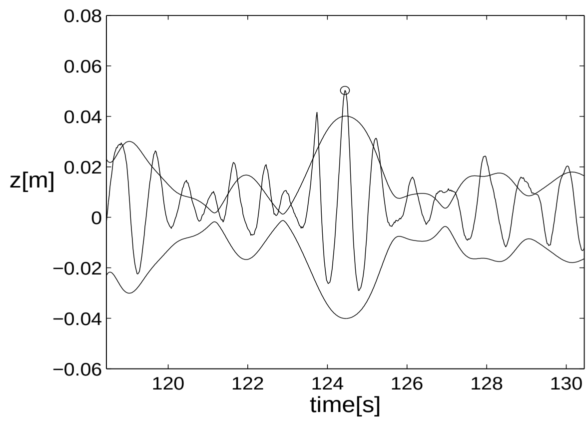
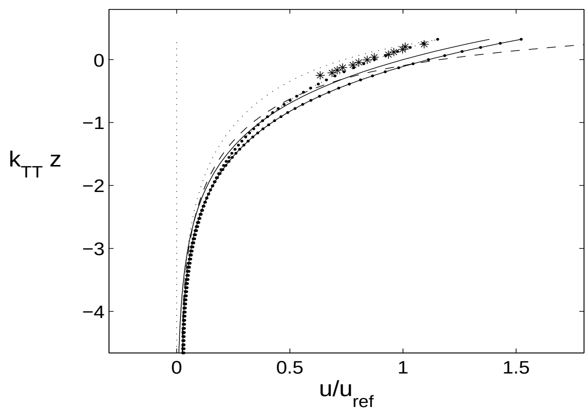
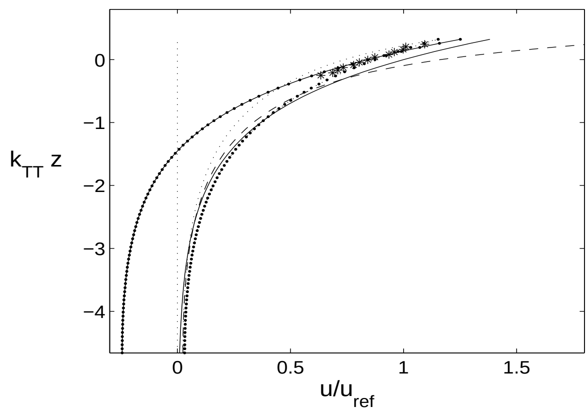


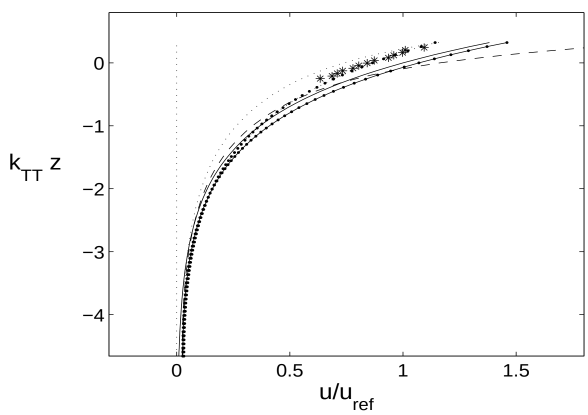
Figure D.1: η and the envelope



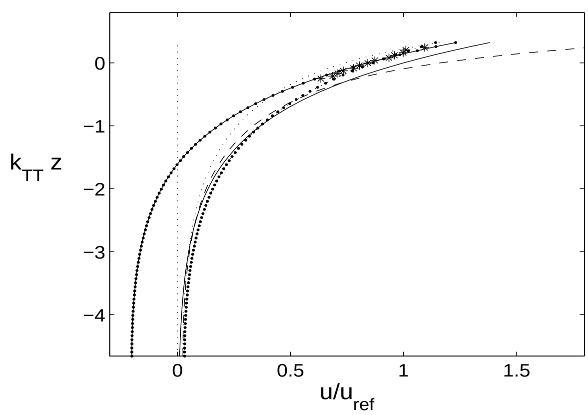
(a) $\omega_{LP} = 1.5\omega_c$ and U_{f0}



(b) $\omega_{LP} = 1.5\omega_c$ and U_{fm}



(c) $\omega_{LP} = 1.75\omega_c$ and U_{f0}



(d) $\omega_{LP} = 1.75\omega_c$ and U_{fm}

Figure D.2: Horizontal velocity at $t = 124.45$. Broken line (- -): Airy theory. Dotted line ($\cdots$): Wheeler stretching. Extra dotted line ($\cdot\bullet\cdot$): 1st order Schrödinger. Dotted solid line (-•-): 2nd order Schrödinger. Asterisk ($*$): measurements.

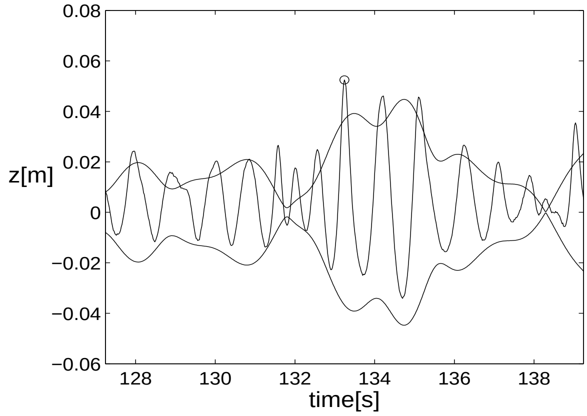
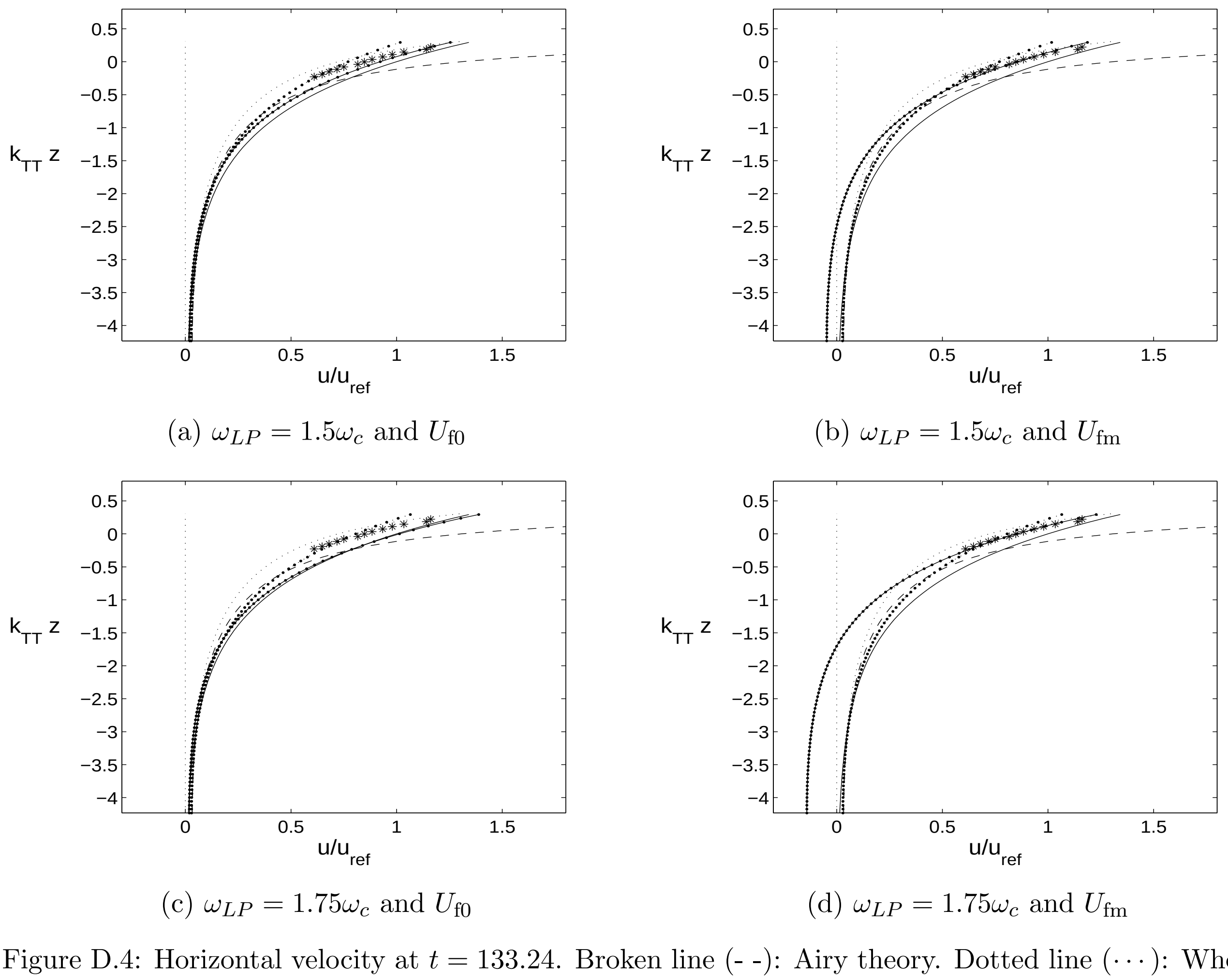


Figure D.3: η and the envelope



(a) $\omega_{LP} = 1.5\omega_c$ and U_{f0}

(b) $\omega_{LP} = 1.5\omega_c$ and U_{fm}

(c) $\omega_{LP} = 1.75\omega_c$ and U_{f0}

(d) $\omega_{LP} = 1.75\omega_c$ and U_{fm}

Figure D.4: Horizontal velocity at $t = 133.24$. Broken line (- -): Airy theory. Dotted line ($\cdots$): Wheeler stretching. Extra dotted line ($\cdot\bullet\cdot$): 1st order Schrödinger. Dotted solid line (-$\bullet$-): 2nd order Schrödinger. Asterisk ($*$): measurements.

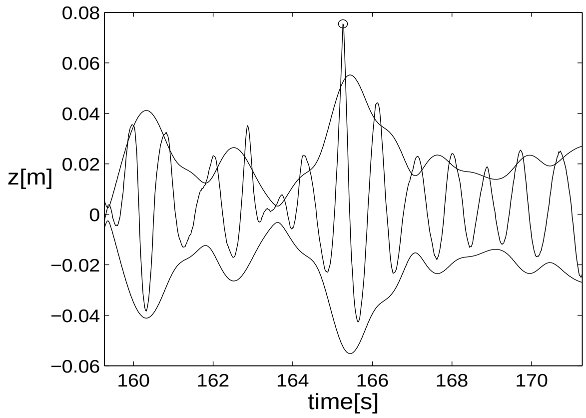
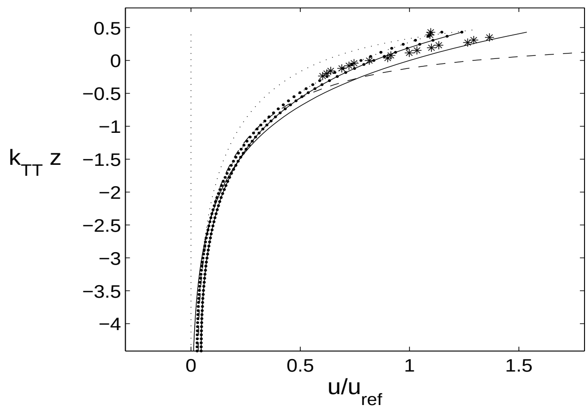
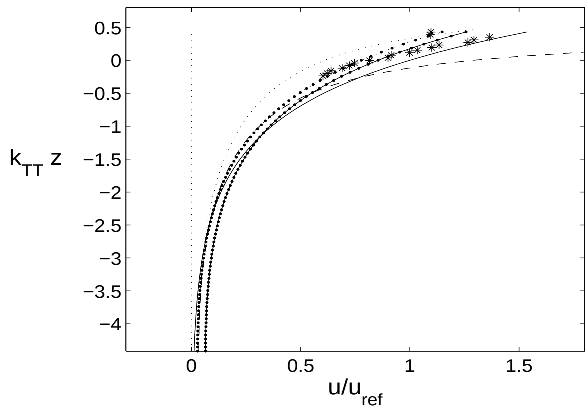


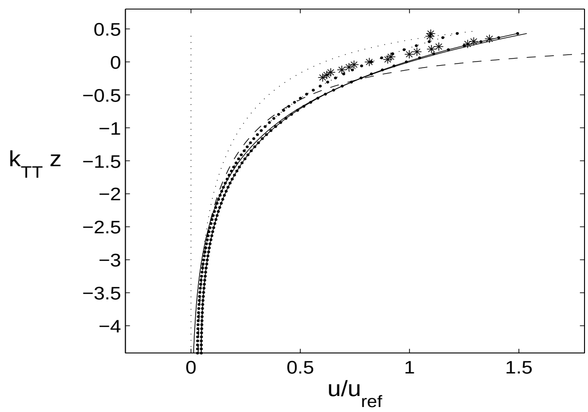
Figure D.5: η and the envelope



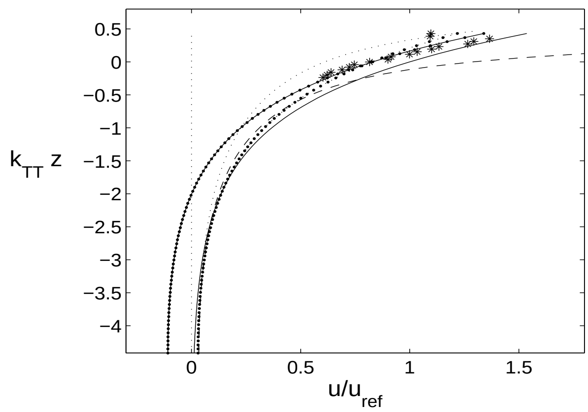
(a) $\omega_{LP} = 1.5\omega_c$ and U_{f0}



(b) $\omega_{LP} = 1.5\omega_c$ and U_{fm}



(c) $\omega_{LP} = 1.75\omega_c$ and U_{f0}



(d) $\omega_{LP} = 1.75\omega_c$ and U_{fm}

Figure D.6: Horizontal velocity at $t = 165.27$. Broken line (- -): Airy theory. Dotted line ($\cdots$): Wheeler stretching. Extra dotted line ($\cdot\bullet\cdot$): 1st order Schrödinger. Dotted solid line (-•-): 2nd order Schrödinger. Asterisk ($*$): measurements.

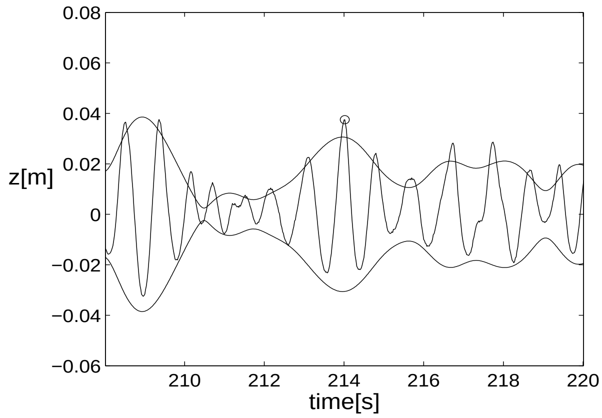
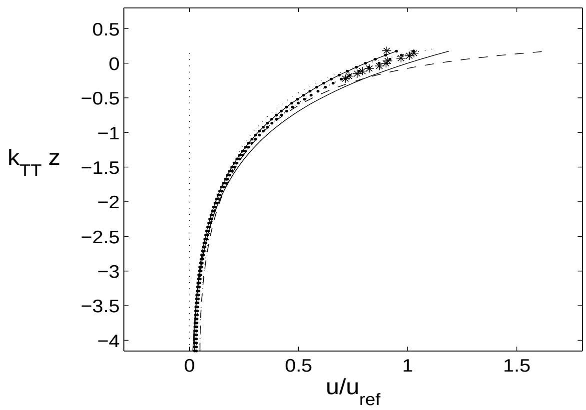
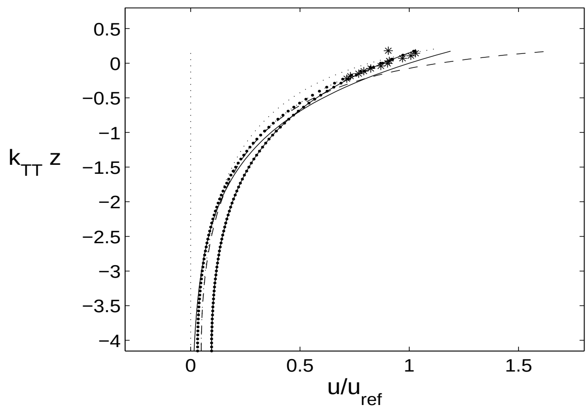


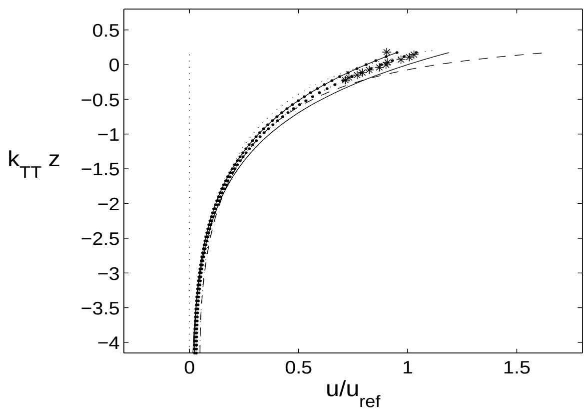
Figure D.7: η and the envelope



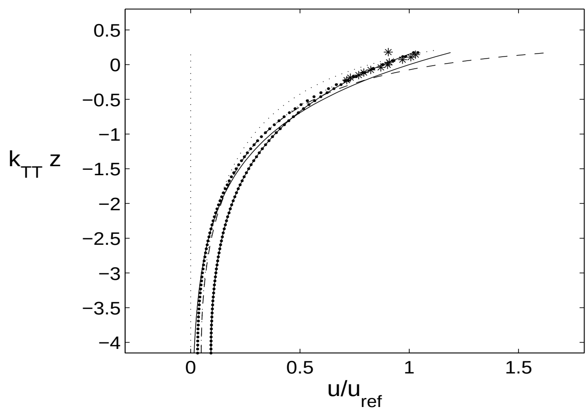
(a) $\omega_{LP} = 1.5\omega_c$ and U_{f0}



(b) $\omega_{LP} = 1.5\omega_c$ and U_{fm}



(c) $\omega_{LP} = 1.75\omega_c$ and U_{f0}



(d) $\omega_{LP} = 1.75\omega_c$ and U_{fm}

Figure D.8: Horizontal velocity at $t = 214.01$. Broken line (- -): Airy theory. Dotted line ($\cdots$): Wheeler stretching. Extra dotted line ($\cdot\bullet\cdot$): 1st order Schrödinger. Dotted solid line (-•-): 2nd order Schrödinger. Asterisk ($*$): measurements.

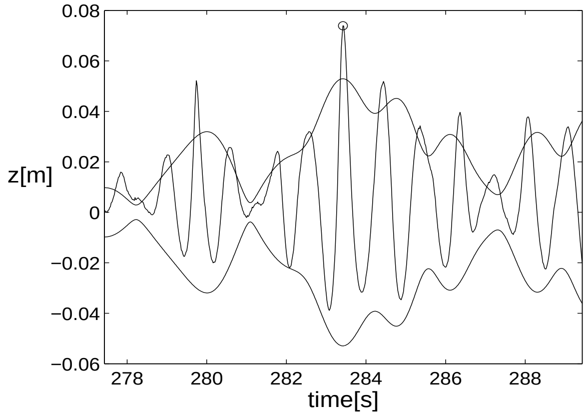
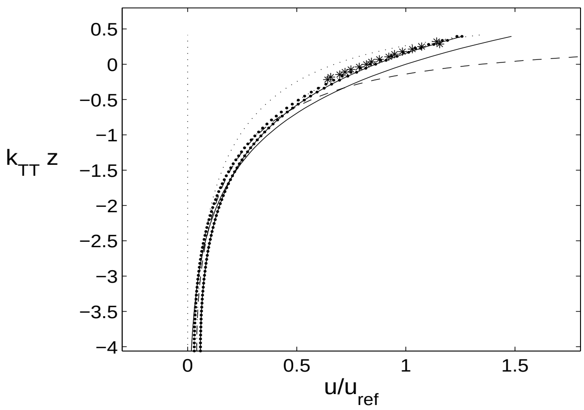
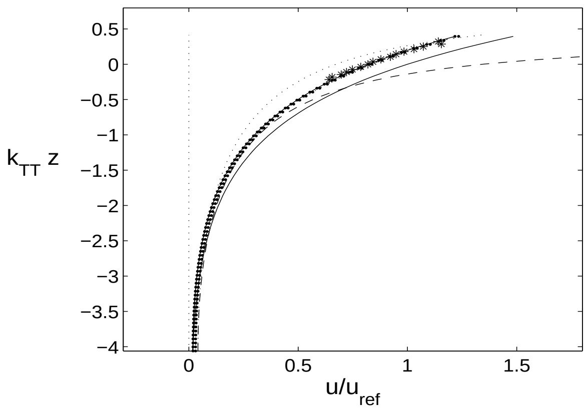


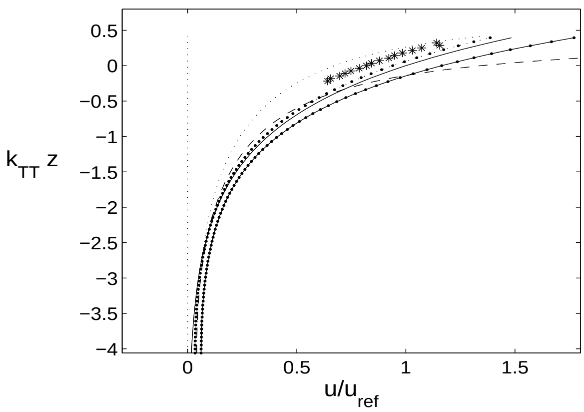
Figure D.9: η and the envelope



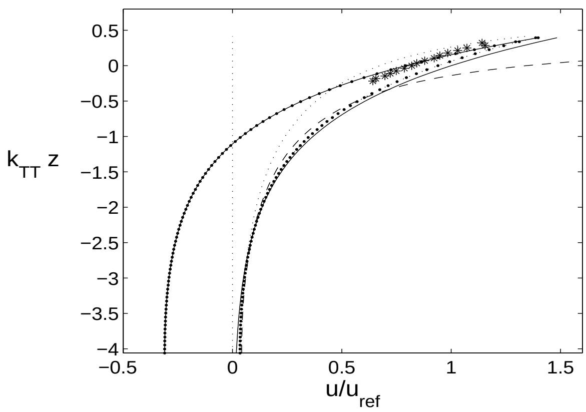
(a) $\omega_{LP} = 1.5\omega_c$ and U_{f0}



(b) $\omega_{LP} = 1.5\omega_c$ and U_{fm}



(c) $\omega_{LP} = 1.75\omega_c$ and U_{f0}



(d) $\omega_{LP} = 1.75\omega_c$ and U_{fm}

Figure D.10: Horizontal velocity at $t = 283.43$. Broken line (- -): Airy theory. Dotted line ($\cdots$): Wheeler stretching. Extra dotted line ($\cdot\bullet\cdot$): 1st order Schrödinger. Dotted solid line (-$\bullet$-): 2nd order Schrödinger. Asterisk ($*$): measurements.

References

[1] R. G. Dean and R. A. Dalrymple. *Water Wave Mechanics for Engineers and Scientists.* Prentice-Hall, Inc, 1984.

[2] Det Norske Veritas. *DNV Recommended Practice C205, Environmental Conditions and Environmental Loads*, April 2007.

[3] K. B. Dysthe. Note on a modification to the nonlinear schrödinger equation for application to deep water waves. *Proceedings of the Royal Society of London*, A(369):105–114, 1979.

[4] J. D. Fenton. A fifth-order stokes theory for steady waves. *Journal of Waterway, Port, Coastel and Ocean Engineering*, 111(2):216–234, March 1985.

[5] S. Gran. *A Course in Ocean Engineering.* Elsevier Science Publishers B.V, 1992.

[6] J. Grue, D. Clamond, M. Huseby, and A. Jensen. Kinematics of extreme waves in deep water. *Applied Ocean Research*, 25:355–366, 2003.

[7] J. R. Morison, M. P. OBrien, J. W. Johnson, and S. A. Schaaf. The force exerted by surface wave on piles. *Petroleum Transactions, AIME*, 189:149–154, 1950.

[8] J. N. Newman. *Marine Hydrodynamics.* The Massachusetts Institute of Technology, fourth edition, 1977.

[9] L. Skjelbreia and J. Hendrickson. Fifth order gravity wave theory. *Proceedings of Seventh Conference on Coastal Engineering*, pages 184–196, August 1960.

[10] K. Trulsen. Wave kinematics computed with the nonlinear schrödinger method for deep water. *Journal of Offshore Mechanics and Artic Engineering*, 121:126–130, May 1999.

[11] K. Trulsen, O. T. Gudmestad, and M. G. Velarde. The nonlinear schrödinger method for water wave kinematics on finite depth. *Wave Motion*, 33:379–395, 2001.

[12] K. Trulsen, A. Jensen, and J. Grue. Water wave kinematics of steep irregular waves - systematic pertubation approach, empirical law, piv measurements and egineering practice. *Proceedings 20th International Workshop on Water Waves and Floating Bodies*, 2005.

[13] J. D. Wheeler. Method for calculating forces produced by irregular waves. *Journal of Petroleum Technology*, 22(3):359–367, March 1970.